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Correction

Correction: Smirnov, S. A Guaranteed Deterministic Approach to Superhedging—The Case of Convex Payoff Functions on Options. *Mathematics* 2019, *7*, 1246

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The author is sorry to report that the statement in Proposition 1 in the paper [1] is not valid, unless some additional assumptions concerning "no arbitrage", price dynamics and trading constraints are admitted. The inherited convexity property of the solutions of Bellman–Isaacs equations holds for the generalised Kolokoltsov model [2], satisfying the NDAO (no arbitrage opportunities) condition. The author would like therefore to make the following corrections to his paper [1].

1. In the original paper, the formulation and the proof of Proposition 1 are the following.

Proposition 1. *If the compacts $K_t(\cdot)$, $t = 0, \ldots, N$ are convex and the payoff functions g_t, $t = 1, \ldots, N$ are convex, then the Bellman–Isaacs functions v_t^*, $t = 0, \ldots, N$, defined by (1), are also convex.*

Proof. Let us prove the lemma by induction, using the Bellman–Isaacs Equations (1). For $s = N$, function v_s^* is convex because $v_N^*(\cdot) = g_N(\cdot)$. Let v_s be convex for $s = N, \ldots, t+1$; now, we show that v_t is also convex. Indeed, the function $(\overline{x}_t, h) \mapsto v_{t+1}^*(\overline{x}_t, x_t + y) - hy$ is convex for any $y \in K_t(\cdot)$, so the function

$$\varphi_t(\overline{x}_t, h) = \sup_{y \in K_t(\cdot)} [v_{t+1}^*(\overline{x}_t, x_t + y) - hy]$$

is convex with respect to $(\bar{x}_t, h)$. Let us denote

$$\psi_t(\overline{x}_t) = \inf_{h \in D_t(\cdot)} \varphi_t(\overline{x}_t, h).$$

Consider $\overline{x}_t^1$ and $\overline{x}_t^2$ of $(\mathbb{R}^n)^{t-1}$ and their convex combination $\overline{x}_t = q_1\overline{x}_t^1 + q_2\overline{x}_t^2$ with weights q_1, $q_2 \geq 0$, $q_1 + q_2 = 1$. For any given $\varepsilon > 0$, there are $h_1 = h_1(\varepsilon, \overline{x}_t^1) \in D_t(\cdot)$ and $h_2 = h_2(\varepsilon, \overline{x}_t^2) \in D_t(\cdot)$ such that $\varphi_t(\overline{x}_t^1, h_1) \leq \psi_t(\overline{x}_t^1) + \varepsilon$ and $\varphi_t(\overline{x}_t^2, h_2) \leq \psi_t(\overline{x}_t^2) + \varepsilon$. Due to the convexity of $D_t(\cdot)$, we obtain $q_1h_1 + q_2h_2 \in D_t(\cdot)$. The following inequalities can be written:

$$\begin{aligned} q_1\psi_t(\overline{x}_t^1) + q_2\psi_t(\overline{x}_t^2) - \varepsilon &\geq q_1\varphi_t(\overline{x}_t^1, h_1) + q_2\varphi_t(\overline{x}_t^2, h_2) \geq \\ &\geq \varphi_t(\overline{x}_t, q_1h_1 + q_2h_2) \geq \inf_{h \in D_t(\cdot)} \varphi_t(\overline{x}_t, h) = \psi_t(\overline{x}_t). \end{aligned}$$

As $\varepsilon > 0$ is arbitrary, we obtain

$$\psi_t(q_1\overline{x}_t^1 + q_2\overline{x}_t^2) \leq q_1\psi_t(\overline{x}_t^1) + q_2\psi_t(\overline{x}_t^2).$$

Since

$$v_t^*(\cdot) = g_t(\cdot) \vee \psi_t(\cdot),$$

check for updates

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the function $v_t^*(\cdot)$ is convex as the maximum of convex functions. $\square$

It should be corrected as follows.

Proposition 1. *Let the (deterministic) price dynamics be of Markov type, representable in the multiplicative form: the recurrent relation (R1) applies to the (discounted) price at time t of one unit of risky asset* $i \in \{1, \dots, n\}$

$$X_t^i = M_t^i X_{t-1}^i,\ M_t = (M_t^1, \dots, M_t^n) \in \check{C}_t,\ t = 1, \dots, N, \tag{R1}$$

where $\check{C}_t$ *are convex compact sets with non-void interior such that* $\check{C}_t \subseteq (0, \infty)^n$*. Suppose that there are no trading constraints, i.e.,* $D_t(\cdot) \equiv \mathbb{R}^n$*, and the NDAO condition (no arbitrage opportunities) is satisfied. If the payoff functions on the American option* $\bar{x}_t \mapsto g_t(\bar{x}_t),\ t = 1, \dots, N$ *are convex, then the solutions of Bellman–Isaacs equations* $\bar{x}_t \mapsto v_t^*(\bar{x}_t),\ t = N, \dots, 1$ *are also convex.*

Proof. Note first, that for the considered model the NDAO condition is tantamount to $e \in \check{C}_t$, where $e = (1, \dots, 1) \in \mathbb{R}^n$. Denote by $\Lambda(m)$ the diagonal matrix with main diagonal entries equal to $m^1, \dots, m^n$. Considering X_t as the vector-column of n prices of risky assets, we can rewrite (R1) as follows:

$$X_t = \Lambda(M_t) X_{t-1},\ M_t \in \check{C}_t,\ t = 1, \dots, N. \tag{R2}$$

Using the representation (R2) of price dynamics, let us prove the assertion by induction. The convexity is immediate for $s = N$. By induction, suppose that v_s^* are convex for $s = N, \dots, t \geq 1$. The convexity of v_{t-1}^* follows from the formula (R3), which is a direct consequence of Theorem 2 in [6]:

$$v_{t-1}^*(\bar{x}_{t-1}) = g_{t-1}(\bar{x}_{t-1}) \vee \sup_{P \in \mathcal{P}^*(\check{C}),\, \int m\, P(dm) = e} \left[\int v_t^*((\bar{x}_{t-1}, \Lambda(m) x_{t-1})\, P(dm) \right], \tag{R3}$$

where $\mathcal{P}^*(E)$ stands for the class of all the probability measures, concentrated on finite subsets of a (non-void) set E. The convexity is conserved when taking integral (in fact, convex combinations) and supremum in (R3), whence the required result. $\square$

2. In the original paper, Remark 1 is the following.

Remark 1.

(1) *If the conditions of Proposition 1 are fulfilled, then the functions* $v_t^*(\cdot)$ *are convex everywhere on* $\mathbb{R}^n$ *and, therefore, continuous (the same applies to the payoff functions* $g_t(\cdot)$*). Therefore, the condition (USC) from Reference [5], that is, the upper semicontinuity of* $v_t^*(\cdot)$*,* $t = 0, \dots, N$*, is fulfilled, so (Reference [5], Proposition 3.3) is applicable, and hence, there is an equilibrium with mixed extension* $\mathcal{P}(K_t(\cdot))$*; moreover,* $\mathcal{P}_t^{opt}(\cdot) \neq \emptyset$*.*

(2) *If the maximizer in expression (2.6) from [8] is unique, that is,* $\mathcal{P}_t^{opt}(\cdot) = \{Q_t^*(\cdot)\}$ *is a one-point set, then by Lemma 1,* $\mathrm{supp}(Q_t^*(\cdot)) \subseteq \mathrm{ext}(K_t(\cdot))$*. Applying a two-stage optimization defined by the relations (11) and (12) in Reference [7] and taking into account Item 2 of Remark 3.2 in Reference [7], we conclude that the number of points in support* $|\,\mathrm{supp}(Q_t^*(\cdot))| \leq n + 1$*. Moreover, if the conditions of Theorem 2.1 from Reference [8] are fulfilled, the mapping* $x \mapsto Q_t^*(x)$ *is (weakly) continuous and* $x \mapsto \mathrm{supp}(Q_t^*(\cdot))$ *is a lower semicontinuous multivalued mapping.*

(3) *In the case when* $K_t(\cdot)$ *are convex polyhedra, that is, can be represented as a convex hull of a finite number of points (According to Theorem 19.1 in Reference [29], the polyhedrality of a convex set is equivalent to its finite generation; in the case of compactness, such a set coincides with the convex hull of a finite number of points; see also (Reference [28] Definition 2.2).), the set of extreme points* $\mathrm{ext}(K_t(\cdot))$ *is finite and* $m = |\,\mathrm{ext}(K_t(\cdot))| \geq n + 1$*; so* $n + 1$ *of these* m *points constitute the optimal mixed strategy's support.*

It should be corrected as follows.

Remark 1.
Assume that NDSAUP holds and the functions $v_t^*(\cdot)$, $t = 0, \dots, N$ are convex everywhere (since the functions $v_t^*(\cdot)$ are convex everywhere on $(\mathbb{R}^n)^t$, they are continuous (see, for example, [29], Corollary 10.1.1) on $(\mathbb{R}^n)^t$).

(1) Therefore, the condition (USC) from [5], i.e., the upper semicontinuity of $v_t^*(\cdot)$, $t = 0, \dots, N$, is fulfilled, so [5], Proposition 3.3 is applicable, and hence, there is an equilibrium with mixed extension $\mathcal{P}(K_t(\cdot))$; moreover, $\mathcal{P}_t^{opt}(\cdot) \neq \emptyset$.

(2) If the maximizer in expression (2.6) from [8] is unique, i.e., $\mathcal{P}_t^{opt}(\cdot) = \{Q_t^*(\cdot)\}$ is a one-point set, then by Lemma 1, $\operatorname{supp}(Q_t^*(\cdot)) \subseteq \operatorname{ext}(K_t(\cdot))$. Applying a two-stage optimization defined by the relations (11) and (12) in [7] and taking into account Item 2 of Remark 3.2 in [7], we conclude that the number of points in support $|\operatorname{supp}(Q_t^*(\cdot))| \leq n+1$. Moreover, if the conditions of Theorem 2.1 from [8] are fulfilled, the mapping $x \mapsto Q_t^*(x)$ is (weakly) continuous and $x \mapsto \operatorname{supp}(Q_t^*(\cdot))$ is a lower semicontinuous multivalued mapping.

(3) In the case that $K_t(\cdot)$ are convex polyhedra, i.e., can be represented as a convex hull of a finite number of points (according to Theorem 19.1 in [29], the polyhedrality of a convex set is equivalent to its finite generation; in the case of compactness, such a set coincides with the convex hull of a finite number of points; see also [28], Definition 2.2), the set of extreme points $\operatorname{ext}(K_t(\cdot))$ is finite and $m = |\operatorname{ext}(K_t(\cdot))| \geq n+1$; so $n+1$ of these m points constitute the optimal mixed strategy support.

3. In the original paper Proposition 2 is formulated as follows.

Proposition 2. *Let the compact-valued mappings $K_t(\cdot)$ be continuous, convex-valued mappings, $D_t(\cdot)$ be weakly continuous (That is, lower semicontinuous and closed (see the terminology in § 14 in Reference [30])), functions, and $g_t(\cdot)$ be convex (since the functions $g_t(\cdot)$ are convex everywhere on $\mathbb{R}^n$, they are continuous (see, for example, Reference [29], Corollary 10.1.1).), $t = 1, \dots, N$. Suppose that the robust condition of no sure arbitrage with unbounded profit $RNDSAUP$ and one of two following conditions hold:*

(1) set $K_t(\cdot)$ is strictly convex $t = 1, \dots, N$;

(2) $K_t(x)$ is a convex polyhedron with a constant (independent of x) number of vertices (the set of vertices of a compact convex polyhedron coincides with the set of its extreme points.), $t = 1, \dots, N$.

Then, the multi-valued mapping $x \mapsto \mathcal{P}_t^{opt}(x) \cap \mathcal{P}^n(\operatorname{ext}(K_t(x))) \neq \emptyset$ is upper semicontinuous.

The statement of Proposition 2 should be reformulated as follows.

Proposition 2. *Assume that the functions $v_t^*(\cdot)$ $t = 0, \dots, N$ are convex everywhere on $(\mathbb{R}^n)^t$. Let the compact-valued mappings $K_t(\cdot)$ be continuous, convex-valued mappings, $D_t(\cdot)$ be weakly continuous (i.e., lower semicontinuous and closed (see the terminology in § 14 in [30])), and functions $g_t(\cdot)$ be convex, $t = 1, \dots, N$. Suppose that the robust condition of no sure arbitrage with unbounded profit $RNDSAUP$ and one of two following conditions hold:*

(1) set $K_t(\cdot)$ is strictly convex $t = 1, \dots, N$;

(2) $K_t(x)$ is a convex polyhedron with a constant (independent of x) number of vertices (the set of vertices of a compact convex polyhedron coincides with the set of its extreme points), $t = 1, \dots, N$.

Then, the multivalued mapping $x \mapsto \mathcal{P}_t^{opt}(x) \cap \mathcal{P}^n(\operatorname{ext}(K_t(x))) \neq \emptyset$ is upper semicontinuous.

4. In the original paper Proposition 3 is formulated as follows.

Proposition 3. *Let there be no trading constraints; the condition of no sure arbitrage $NDSA$ be fulfilled. Suppose that the functions $g_t(\cdot)$ are convex and $K_t(\cdot)$ are convex polyhedra, $t = 1, \dots, N$.*

(1) *If the condition*

$$0 \notin \mathrm{ri}(\mathrm{conv}(A)) \text{ for any } A \subseteq \mathrm{ext}(K_t(\cdot)), \text{ such as } |A| \leq n, \tag{11}$$

is fulfilled, then we have the following:

- *there is an optimal mixed strategy* $Q_t^*(\cdot)$ *with zero mean and* $\mathrm{supp}(Q_t^*(\cdot)) \subseteq \mathrm{ext}(K_t(\cdot))$ *satisfying the condition of maximum cardinality of support, that is,* $|\mathrm{supp}(Q_t^*(\cdot))| \equiv n+1$;
- *compacts* $K_t(\cdot)$ *are full-dimensional, that is,* $\dim K_t(\cdot) = n$*; and*
- *the robust condition of no arbitrage opportunities RNDAO is fulfilled.*

(2) *If, in addition,* $\mathcal{P}_t^{opt}(\cdot)$ *contains a single element, that is,* $\mathcal{P}_t^{opt}(\cdot) = \{Q_t^*(\cdot)\}$*, the compact-valued mappings* $K_t(\cdot)$ *are continuous,* $t = 1, \ldots, N$*, then multi-valued mapping* $x \mapsto \mathrm{supp}(Q_t^*(\cdot))$ *can be decomposed into n non-coincident continuous everywhere branches, each of which is a vertex of one of the* $K_t(\cdot)$ *n-simplex (the n-simplex is a solid polyhedron in* $\mathbb{R}^n$ *with* $n+1$ *vertices (which are the extreme points of this polyhedron)) containing* 0*. (There can be several such n-simplexes.)*

The statement of Proposition 3 should be reformulated as follows.

Proposition 3. *Assume that the functions* $v_t^*(\cdot)$ $t = 0, \ldots, N$ *are convex everywhere on* $(\mathbb{R}^n)^t$*. Let there be no trading constraints; the condition of no sure arbitrage NDSA is fulfilled. Suppose that the functions* $g_t(\cdot)$ *are convex and* $K_t(\cdot)$ *are convex polyhedra,* $t = 1, \ldots, N$*.*

(1) *If the condition*

$$0 \notin \mathrm{ri}(\mathrm{conv}(A)) \text{ for any } A \subseteq \mathrm{ext}(K_t(\cdot)), \text{ such as } |A| \leq n, \tag{11}$$

is fulfilled, then we have the following:

- *there is an optimal mixed strategy* $Q_t^*(\cdot)$ *with zero mean and* $\mathrm{supp}(Q_t^*(\cdot)) \subseteq \mathrm{ext}(K_t(\cdot))$ *satisfying the condition of maximum cardinality of support, i.e.,* $|\mathrm{supp}(Q_t^*(\cdot))| \equiv n+1$;
- *compacts* $K_t(\cdot)$ *are full-dimensional, i.e.,* $\dim K_t(\cdot) = n$*; and*
- *the robust condition of no arbitrage opportunities RNDAO is fulfilled.*

(2) *If, in addition,* $\mathcal{P}_t^{opt}(\cdot)$ *contains a single element, i.e.,* $\mathcal{P}_t^{opt}(\cdot) = \{Q_t^*(\cdot)\}$*, the compact-valued mappings* $K_t(\cdot)$ *are continuous,* $t = 1, \ldots, N$*, then multivalued mapping* $x \mapsto \mathrm{supp}(Q_t^*(\cdot))$ *can be decomposed into n non-coincident continuous everywhere branches, each of which is a vertex of one of the* $K_t(\cdot)$ *n-simplex (the n-simplex is a solid polyhedron in* $\mathbb{R}^n$ *with* $n+1$ *vertices (which are the extreme points of this polyhedron)). containing* 0*. (There can be several such n-simplexes).*

5. In the original paper reference no.6 is as follows:
Smirnov, S.N. Guaranteed deterministic approach to superhedging: Equilibrium in the case of no trading constraints. *J. Math. Sci.* **2020**. accepted.
It should be updated as follow:
Smirnov, S.N. Guaranteed deterministic approach to superhedging: Equilibrium in the case of no trading constraints. *J. Math. Sci.* **2020**, *248*, 105–115.

The author would like to apologize for any inconvenience caused to the readers by this mistake and the changes to the text.

References

1. Smirnov, S. A Guaranteed Deterministic Approach to Superhedging—The Case of Convex Payoff Functions on Options. *Mathematics* **2019**, *7*, 1246. [CrossRef]
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