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Mathematical Modeling of COVID-19 Transmission in the Form of System of Integro-Differential Equations

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Abstract: The model of the spread of the coronavirus pandemic in the form of a system of integro-differential equations is studied. We focus our consideration on the number of hospitalized patients, i.e., on the needs of the system regarding hospital beds that can be provided for hospitalization and the corresponding medical personnel. Traditionally, in such models, the number of places needed was defined as a certain percentage of the number of infected at the moment. This is not quite adequate, since it takes a certain period of time for the development of the disease to the stage at which hospitalization is required. This will be especially evident at the start of new waves of the epidemic, when there is a large surge in the number of infected people, but the need for hospitalization places and additional medical personnel will appear later. Taking this circumstance into account using integral terms in the model allows us to conclude in corresponding additional to existing cases that the wave of disease will attenuate after some time. In others, it will relieve unnecessary panic, because the healthcare system has a certain period to create additional hospitalization places, order medicines and mobilize the necessary medical personnel. We obtain estimates of reproduction number in the case of the model described by a system of integro-differential equations. Results on the exponential stability of this integro-differential system are obtained. It is demonstrated that the condition of the exponential stability coincides with the fact that the reproduction number of the spread of the pandemic is less than one.

Keywords: integro-differential equations system; stability of integro-differential system; Cauchy matrix; COVID-19; infectious disease; reproduction number

MSC: 34k20; 34k21

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1. Introduction

The COVID-19 pandemic is considered the biggest global thread worldwide because of the millions of confirmed infections, which have been accompanied by hundreds of thousands of deaths all over the world. This global problem has attracted the interest of researchers of different areas, giving rise to a number of proposals to analyze and predict the evolution of a pandemic (see, for example, [1,2]).

There were many publications in which various models of COVID-19 transmission were considered [3–11]. However, in all of them, only the case where the number of hospitalized patients is a percent of all infected people at the current time was considered. This seems to be an unnatural assumption. The need for beds for hospitalization, drugs and personnel can be filled with corresponding delay (in this model, it is approximately one week). Our paper, as far as we know, is the first one which takes into account this fact. Equations with fractional derivatives can be also considered as models taking this effect of delay into account. An approach to equations with fractional derivatives was proposed, for example in the recent papers [12–14]. This topic can be considered in future work.

The coronavirus pandemic has stimulated numerous studies in microbiology, immunology, pharmacology, and the so-called statistical epidemiology. It is statistical epidemiology that we will talk about in the proposed article. Specific properties in geographic [15] and social behavior aspects [16,17] can be also considered in modeling. Concerning super-spreaders (see, for example, ref. [18] and references therein), it can be noted that the authors of that paper proposed a mathematical model for the spread of COVID-19 disease with emphasis on the transmissibility of super-spreader individuals. We see the importance of [18] in that it signaled the directions of future study: the local stability of the disease-free equilibrium in terms of the basic reproduction number and the sensitivity of the model with respect to the variation of its parameters were studied. Let us try to look at all this epidemiological topic from a mathematical point of view and find out what properties of mathematical models are associated with completely natural medical statistical questions. The reproduction number, which we hear about every day from television screens, and for whose the calculation a rather intricate technique is used, turns out to be simply a sufficient condition for the exponential stability of a system linearized in the neighborhood of a free equilibrium (stationary solution). Thus, there is the problem of estimating the reproduction number to a classical stability problem in the theory of differential equations.

Our task is prediction, i.e., according to the number of infected and seriously ill patients at the moment, to predict their number in the future. However, at the moment, in technological and medical statistical processes, we know only a certain measurement accuracy inherent in the instrument itself for measuring in the technical process, and of course, there is a certain statistical error in medical statistical problems. What predictions can be made if there is no stability with respect to initial values? Is there at least a continuous dependence on the initial conditions?

When building any mathematical model, we abstract from certain factors that seem insignificant to us. It seems that their influence on the process should be small. However, this is just our guess. The weather, for example, may suddenly change and affect the spread of the epidemic (heavy rains wash away viruses, and in addition, people stay at home because of them). In Israel, there was a case when a severe influenza epidemic was stopped by a teachers' strike, which left children, the main spreaders of influenza, at home. Thus, completely unexpected factors can, under certain conditions, also affect the spread of the epidemic. Summarizing what has been said, we see that the model must be stable with respect to the right-hand side in order for our prediction to reflect the possible reality. The stability with respect to the right-hand side, as the classical theory of stability of differential equations teaches us, is equivalent to the exponential stability of the system. All coefficients in the mathematical model are the result of some manipulations with the process data in the past. They naturally carry errors in measurements, which have already been discussed above. In addition, these ratios may change over time. Will these measurement errors seriously affect our predictions about the course of the simulated process? The stability of solutions to the variation of coefficients should also be considered in the analysis of systems. Seasonality can be considered as one of the factors leading to periodic coefficients in epidemiological models. When thinking about the model, there are also questions about the adequacy of the model itself. Does it reflect the process in principle? Let us take as an example the problem of planning the number of beds and calculating the number of necessary drugs and medical personnel.

In real models in epidemiology, the mathematical models with the delay feedback control, when the control is based on the values of the process at certain points in the past, and not just the current values of that process, can naturally appear [19]. When discussing the model of the spread of the coronovirus epidemic, we will focus on the number of hospitalized patients, i.e., on the needs of the system regarding hospital beds that can be provided for in hospitalization and in the corresponding medical personnel. Traditionally, in such models, the number of places needed was defined as a certain percentage of the number of infected at the current moment t. It is clear, however, that this is not quite adequate. After all, it takes a certain incubation period and time from the development of

the disease to the stage at which hospitalization is required. For a coronavirus infection, this process can take about a week before the onset of a serious stage of disease. This will be especially evident at the start of new waves of the epidemic, when there is a large surge in the number of infected people, but the need for hospitalization places and additional medical personnel will appear later. In some cases, this circumstance allows us to conclude that the wave of diseases will attenuate after some time. In others, it will relieve unnecessary panic, because the health-care system has a certain period to create additional hospitalization places, order medicines and mobilize the necessary medical personnel. Thus, to describe the spread of an epidemic with such an applied aspect as the assessment of the necessary beds, the ordering of medicines and the mobilization of medical personnel, the system has to be in a form of differential equations with memory (with a delay, to use the terminology established in differential equations). Of course, we need statistics of a different kind than those that are traditionally used now, namely, the percentage of seriously ill patients by every day after infection. Note that there are also so-called post-coronavirus complications, which to some extent require consideration. Thus, the modeling of COVID-19 transmission leads us to an analysis of systems of functional differential equations in the forms of delay or integro-differential ones. In order to predict the process, we have to study the stability properties of these systems. We need to study representations of solutions of these systems. Estimates of the Cauchy matrix will be one of the important parts of this study. In this article, we will consider a system of integro-differential equations as a model. We calculate the reproduction number. In appropriate cases, it turns out to be lower than that obtained for the traditional model, where planning was carried out according to the number of patients at the current time only. This demonstrates that the health-care system has some additional margin of safety.

2. COVID-19: Traditional Compartment Model

In this paper, we start with the model of COVID-19 pandemic spreading which was proposed in [15]. The total population N is subdivided into eight epidemiological classes: susceptible class S, exposed class E, symptomatic and infectious class I, super-spreaders class P, infectious but asymptomatic class A, hospitalized class H, recovery class R, and fatality class F. The model was constructed in the following form:

$$\begin{cases} \frac{dS(t)}{dt} &= -\beta\frac{I(t)}{N}S(t) - \beta l\frac{H(t)}{N}S(t) - \beta'\frac{P(t)}{N}S(t)\,, \\ \frac{dE(t)}{dt} &= \beta\frac{I(t)}{N}S(t) + \beta l\frac{H(t)}{N}S(t) + \beta'\frac{P(t)}{N}S(t) - kE(t)\,, \\ \frac{dI(t)}{dt} &= k\rho_1 E(t) - (\gamma_a+\gamma_i)I(t) - \delta_i I(t)\,, \\ \frac{dP(t)}{dt} &= k\rho_2 E(t) - (\gamma_a+\gamma_i)P(t) - \delta_p P(t)\,, \\ \frac{dA(t)}{dt} &= k(1-\rho_1-\rho_2)E(t)\,, \\ \frac{dH(t)}{dt} &= \gamma_a(I(t)+P(t)) - \gamma_r H(t) - \delta_h H(t)\,, \\ \frac{dR(t)}{dt} &= \gamma_i(I(t)+P(t)) + \gamma_r H(t)\,, \\ \frac{dF(t)}{dt} &= \delta_i I(t) + \delta_p P(t) + \delta_h H(t)\,. \end{cases} \tag{1}$$

where β quantifies the human-to-human transmission coefficient per unit time (days) per person, β' quantifies a high transmission coefficient due to super-spreaders, and l quantifies the relative transmissibility of hospitalized patients. Here, k is the rate at which an individual leaves the exposed class by becoming infectious (symptomatic, super- spreaders

or asymptomatic); ρ_1 is the proportion of progression from exposed class E to symptomatic infectious class I; ρ_2 is a relative very low rate at which exposed individuals become super-spreaders, while $(1-\rho_1-\rho_2)$ is the progression from exposed to asymptomatic class; γ_a is the average rate at which symptomatic and super-spreaders individuals become hospitalized; γ_i is the recovery rate without being hospitalized; γ_r is the recovery rate of hospitalized patients; and δ_i , δ_p , and δ_h are the disease-induced death rates due to infected, super-spreaders, and hospitalized individuals, respectively. Values of the model parameters are corresponding to the situation of Wuhan, as discussed in [18] (see Table 1).

Table 1. Values of parameters presented in [18] according to situation in Wuhan.

Parameters	Description	Value
N	Constant of population	N
β	Transmission coefficient from infected individuals	2.55
β'	Transmission coefficient due to super-spreaders	7.65
l	Relative transmissibility of hospitalized patients	1.56
k	Rate at which exposed people become infectious	0.25
ρ_1	Rate at which exposed people become infected I	0.58
ρ_2	Rate at which exposed people become super-spreaders P	0.001
γ_a	Rate of being hospitalized	0.94
γ_i	Recovery rate without being hospitalized	0.27
γ_r	Recovery rate of hospitalized patients	0.5
δ_i	Disease-induced death rate due to infected class I	3.5
δ_p	Disease-induced death rate due to super-spreaders P	1
δ_h	Disease-induced death rate due to hospitalized class H	0.3

3. Developments of COVID-19 Traditional Model

In this paper, we consider the following modification of the COVID-19 model.

$$\begin{cases} \frac{dS(t)}{dt} &= -\beta\frac{I(t)}{N}S(t) - \beta l\frac{H(t)}{N}S(t) - \beta'\frac{P(t)}{N}S(t)\,, \\ \frac{dE(t)}{dt} &= \beta\frac{I(t)}{N}S(t) + \beta l\frac{H(t)}{N}S(t) + \beta'\frac{P(t)}{N}S(t) - kE(t)\,, \\ \frac{dI(t)}{dt} &= k\rho_1 E(t) - (\gamma_a+\gamma_i)I(t) - \delta_i I(t)\,, \\ \frac{dP(t)}{dt} &= k\rho_2 E(t) - (\gamma_a+\gamma_i)P(t) - \delta_p P(t)\,, \\ \frac{dA(t)}{dt} &= k(1-\rho_1-\rho_2)E(t)\,, \\ \frac{dH(t)}{dt} &= \gamma_a\int_0^t \mathbf{Q_I}(s,t)I(s)\,ds + \gamma_a\int_0^t \mathbf{Q_P}(s,t)P(s)\,ds - \gamma_r H(t) - \delta_h H(t)\,, \\ \frac{dR(t)}{dt} &= \gamma_i(I(t)+P(t)) + \gamma_r H(t)\,, \\ \frac{dF(t)}{dt} &= \delta_i I(t) + \delta_p P(t) + \delta_h H(t)\,. \end{cases} \tag{2}$$

$$\mathbf{Q_I}(t,s) = \kappa_I(t,s)e^{-\eta_i(t-s)},\ \mathbf{Q_P}(t,s) = \kappa_P(t,s)e^{-\eta_p(t-s)}, \tag{3}$$

where $\kappa_I(t,s)$, $\kappa_P(t,s)$ satisfy standard assumptions, for examle, $\kappa_I(t,s)$, $\kappa_P(t,s)$ and $\int_0^t \kappa_I(t,s)\,ds$, $\int_0^t \kappa_P(t,s)\,ds$ are essentially bounded for $0 \leq t < \infty$. We start our study with the kernels.

$$\mathbf{K_I}(t,s) = e^{-\eta_i(t-s)},\ \mathbf{K_P}(t,s) = e^{-\eta_p(t-s)}. \tag{4}$$

It is clear for coming to a more adequate model in which, for example, the incubation period of the disease is taken into account, we can take $\kappa_I(t,s) = 0$, $\kappa_P(t,s) = 0$ for $t-h < s < t$, $t \in (0,\infty)$ (see Figure 1).

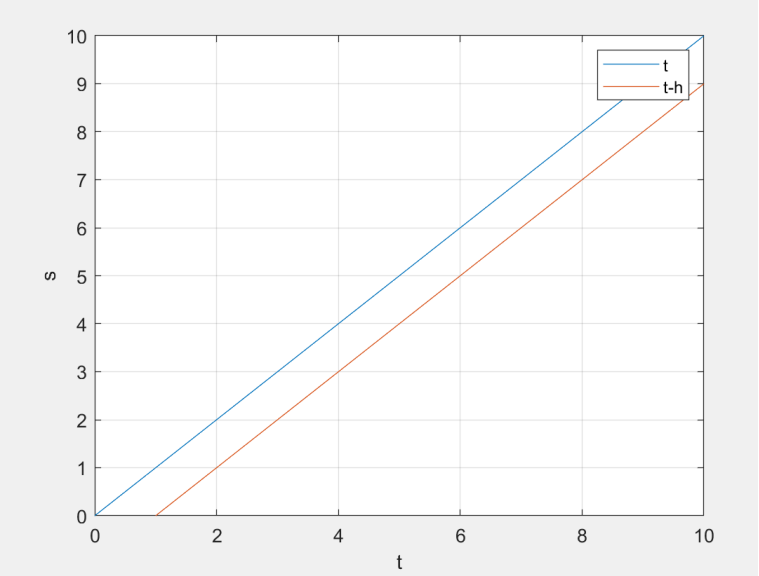

Figure 1. To description of kernels in the integral terms.

We come to the system

$$\begin{cases} \frac{dS(t)}{dt} &= -\beta\frac{I(t)}{N}S(t) - \beta l\frac{H(t)}{N}S(t) - \beta'\frac{P(t)}{N}S(t)\,, \\ \frac{dE(t)}{dt} &= \beta\frac{I(t)}{N}S(t) + \beta l\frac{H(t)}{N}S(t) + \beta'\frac{P(t)}{N}S(t) - kE\,, \\ \frac{dI(t)}{dt} &= k\rho_1 E(t) - (\gamma_a+\gamma_i)I(t) - \delta_i I(t)\,, \\ \frac{dP(t)}{dt} &= k\rho_2 E(t) - (\gamma_a+\gamma_i)P(t) - \delta_p P(t)\,, \\ \frac{dA(t)}{dt} &= k(1-\rho_1-\rho_2)E(t)\,, \\ \frac{dH(t)}{dt} &= \gamma_a\int_0^t e^{-\eta_i(t-s)}I(s)\,ds + \gamma_a\int_0^t e^{-\eta_p(t-s)}P(s)\,ds - \gamma_r H(t) - \delta_h H(t)\,, \\ \frac{dR(t)}{dt} &= \gamma_i(I(t)+P(t)) + \gamma_r H(t)\,, \\ \frac{dF(t)}{dt} &= \delta_i I(t) + \delta_p P(t) + \delta_h H(t)\,, \end{cases} \tag{5}$$

where integral terms with such kernels are applied.

Let us explain why this step can be considered as a natural one. It was described in the Introduction that development of the disease requires a corresponding period of time (for example, incubation period and time, in which symptoms are weak), which is usually measured by 6–7 days. In addition, post-corona cases can lead to hospitalization. It is also clear that manifestations of a severe form of the disease leading to hospitalization

and/or death do not appear immediately in statistics but rather after a few days. It can be described by a corresponding sum in the sixth equation (and of course in the last one, but we do not consider it, since we are concentrating our study only on numbers of beds and personnel needed) of system (5). The number of hospitalized patients is described in the sixth equation of system (1), which can be developed by applying integral terms in the sixth equation, which makes the model, in our opinion, more reasonable. It can be noted that adding integral terms of the forms $\int_0^t q_I(t,s)I'(s)ds$ and $\int_0^t q_P(t,s)P'(s)ds$ in the sixth equation of system (2) would look very reasonable. For "simple" and sufficiently smooth kernels, for example, for the kernels of the form (4), one can integrate by parts and obtain the sixth equation in approximately the same form as in system (2). Let us restrict ourselves to system (2) to start somewhere.

Hospitalization is used in the case of a severe form of the course of the disease, which means that here also, the dependence should not be on the state at the current moment of time but on the states at some previous moments, which are separated from the current by certain time intervals due to the dynamics of the development of the disease.

It should be stressed that an integro-differential system with the kernels of considered type appears in other mathematical models in medicine. See for example the recent paper [20], where integral terms in the similar form are used in the analysis of dynamics of tumor-immune interactions.

It is clear that it is worth considering more population groups, at least also the division according to the principle vaccinated/not vaccinated, and of course, statistical data for each of these groups should be collected, taking into account the development of disease in each of them. Such models could be considered in the future.

The basic reproduction number, denoted R_0, is 'the expected number of secondary cases produced, in a completely susceptible population, by a typical infective individual' [21]. If $R_0 < 1$, then on average, an infected individual produces less than one new infected individual over the course of its infectious period, and the infection cannot grow. Conversely, if $R_0 > 1$, then each infected individual produces, on average, more than one new infected individual, and the disease can invade the population. For the case of a single infected compartment, R_0 is simply the infection rate. One of the purposes of our paper is to obtain estimates of the basic reproduction number R_0. This allows us to estimate the number of needed hospital beds.

We can come to the following integro-differential system.

$$\begin{cases} \frac{dE(t)}{dt} &= \beta\frac{I(t)}{N}S(t) + \beta l\frac{H(t)}{N}S(t) + \beta'\frac{P(t)}{N}S(t) - kE(t)\,, \\ \frac{dI(t)}{dt} &= k\rho_1 E(t) - (\gamma_a+\gamma_i)I(t) - \delta_i I(t)\,, \\ \frac{dP(t)}{dt} &= k\rho_2 E(t) - (\gamma_a+\gamma_i)P(t) - \delta_p P(t)\,, \\ \frac{dH(t)}{dt} &= \gamma_a\int_0^t e^{-\eta_i(t-s)}I(s)\,ds + \gamma_a\int_0^t e^{-\eta_p(t-s)}P(s)\,ds - \gamma_r H(t) - \delta_h H(t)\,, \end{cases} \tag{6}$$

as it was outlined in [18].

Let us denote the new coefficients:

$$\begin{aligned} \omega_i &= \gamma_a+\gamma_i+\delta_i\,, \\ \omega_p &= \gamma_a+\gamma_i+\delta_p\,, \\ \omega_h &= \gamma_r+\delta_h\,. \end{aligned} \tag{7}$$

Considering system (6), we propose an approach reducing it to a system of ordinary differential equations. Substituting $x_1 = E, x_2 = I, x_3 = P, x_6 = H$, and linearizing the

system in a neighborhood of the stationary point (0,0,0,0), we obtain the corresponding linear system

$$\begin{cases} \dfrac{dx_1(t)}{dt} &= \beta x_2(t) + \beta l x_6(t) + \beta' x_3(t) - k x_1(t)\,, \\ \dfrac{dx_2(t)}{dt} &= k\rho_1 x_1(t) - \omega_i x_2(t)\,, \\ \dfrac{dx_3(t)}{dt} &= k\rho_2 x_1(t) - \omega_p x_3(t)\,, \\ \dfrac{dx_6(t)}{dt} &= \gamma_a \displaystyle\int_0^t e^{-\eta_i(t-s)} x_2(s)\,ds + \gamma_a \int_0^t e^{-\eta_p(t-s)} x_3(s)\,ds - \omega_h x_6(t)\,. \end{cases} \tag{8}$$

Note that the original system of [18] in the coordinates denoted above can be written in the form:

$$\begin{cases} \dfrac{dx_1(t)}{dt} &= \beta x_2(t) + \beta l x_6(t) + \beta' x_3(t) - k x_1(t)\,, \\ \dfrac{dx_2(t)}{dt} &= k\rho_1 x_1(t) - \omega_i x_2(t)\,, \\ \dfrac{dx_3(t)}{dt} &= k\rho_2 x_1(t) - \omega_p x_3(t)\,, \\ \dfrac{dx_6(t)}{dt} &= \gamma_a x_2(t) + \gamma_a x_3(t) - \omega_h x_6(t)\,. \end{cases} \tag{9}$$

The reproduction number of the system (9), calculated as in [18] is

$$R_0 = \frac{\beta\rho_1}{\omega_i} + \frac{\beta'\rho_2}{\omega_p} + \frac{\beta l(\omega_i\gamma_a\rho_2 + \omega_p\gamma_a\rho_1)}{\omega_h\omega_i\omega_p}\,. \tag{10}$$

Consider the system

$$\begin{cases} \dfrac{dx_1(t)}{dt} &= \beta x_2(t) + \beta l x_6(t) + \beta' x_3(t) - k x_1(t)\,, \\ \dfrac{dx_2(t)}{dt} &= k\rho_1 x_1(t) - \omega_i x_2(t)\,, \\ \dfrac{dx_3(t)}{dt} &= k\rho_2 x_1(t) - \omega_p x_3(t)\,, \\ \dfrac{dx_4(t)}{dt} &= -\eta_i x_4(t) + x_2(t)\,, \\ \dfrac{dx_5(t)}{dt} &= -\eta_p x_5(t) + x_3(t)\,, \\ \dfrac{dx_6(t)}{dt} &= \gamma_a x_4(t) + \gamma_a x_5(t) - \omega_h x_6(t)\,. \end{cases} \tag{11}$$

Our approach is based on the following assertion connecting integro-differential system (8) and system of ordinary differential equations (11).

Lemma 1. *The vector-solution* $col(x_1, x_2, x_3, x_6)$ *of the fourth-order system (8) and the components* x_1, x_2, x_3 *and* x_6 *of the vector-solution* $col(x_1, x_2, x_3, x_4, x_5, x_6)$ *of the sixth-order system (11) with the initial conditions* $x_4(0) = 0, \quad x_5(0) = 0$, *coincide.*

Proof of Lemma 1. The proof of Lemma 1 follows from the formula of presentation of the general solution of the scalar linear differential equations of the first order.

$$\frac{dx_4(t)}{dt} = -\eta_i x_4(t) + x_2(t), \quad \frac{dx_5(t)}{dt} = -\eta_p x_5(t) + x_3(t), \quad t \in [0, \infty).$$

Actually, the solution of $x_4(0) = 0,\ x_5(0) = 0$ can be written in the form

$$x_4(t) = \int_0^t C_4(t,s)x_2(s)\,ds,\ x_5(t) = \int_0^t C_5(t,s)x_3(s)\,ds\,.$$

Substituting these presentations into the sixth equation of system (11), we obtain system (8). □

4. Reproduction Number for Modified Integro-Differential System

In this section, we calculate the reproduction number using the matrix of parameters, repeating the same steps as in the papers [18,21]. Let us write the following matrices defined in these papers:

$$\mathbf{F} = \begin{pmatrix} 0 & \beta & \beta' & 0 & 0 & \beta l \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{V} = \begin{pmatrix} k & 0 & 0 & 0 & 0 & 0 \\ -k\rho_1 & \omega_i & 0 & 0 & 0 & 0 \\ -k\rho_2 & 0 & \omega_p & 0 & 0 & 0 \\ 0 & -1 & 0 & \eta_i & 0 & 0 \\ 0 & 0 & -1 & 0 & \eta_p & 0 \\ 0 & 0 & 0 & -\gamma_a & -\gamma_a & \omega_h \end{pmatrix}, \tag{12}$$

The inverse matrix of V:

$$\mathbf{V^{-1}} = \begin{pmatrix} \frac{1}{k} & 0 & 0 & 0 & 0 & 0 \\ \frac{\rho_1}{\omega_i} & \frac{1}{\omega_i} & 0 & 0 & 0 & 0 \\ \frac{\rho_2}{\omega_p} & 0 & \frac{1}{\omega_p} & 0 & 0 & 0 \\ \frac{\rho_1}{\omega_i\eta_i} & \frac{1}{\omega_i\eta_i} & 0 & \frac{1}{\eta_i} & 0 & 0 \\ \frac{\rho_2}{\omega_p\eta_p} & 0 & \frac{1}{\omega_p\eta_p} & 0 & \frac{1}{\eta_p} & 0 \\ \frac{\gamma_a(\rho_1\eta_p\omega_p+\rho_2\eta_i\omega_i)}{\omega_h\omega_i\omega_p\eta_i\eta_p} & \frac{\gamma_a}{\omega_h\omega_i\eta_i} & \frac{\gamma_a}{\omega_h\omega_p\eta_p} & \frac{\gamma_a}{\omega_h\eta_i} & \frac{\gamma_a}{\omega_h\eta_p} & \frac{1}{\omega_h} \end{pmatrix}. \tag{13}$$

After calculating the matrix $F \cdot V^{-1}$, we obtain:

$$\begin{pmatrix} \frac{\beta\rho_1}{\omega_i} + \frac{\beta'\rho_2}{\omega_p} + \frac{\beta l(\omega_i\eta_i\gamma_a\rho_2+\omega_p\eta_p\gamma_a\rho_1)}{\omega_h\omega_i\omega_p\eta_i\eta_p} & \frac{\beta}{\omega_i} + \frac{\beta\gamma_a l}{\omega_h\omega_i\eta_i} & \frac{\beta'}{\omega_p} + \frac{\beta\gamma_a l}{\omega_h\omega_p\eta_p} & \frac{\beta\gamma_a l}{\omega_h\eta_i} & \frac{\beta\gamma_a l}{\omega_h\eta_p} & \frac{\beta l}{\omega_h} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{14}$$

Repeating the steps described in [18], we come to the reproduction number $\overline{R_0}$ for system (11):

$$\overline{R_0} = \frac{\beta\rho_1}{\omega_i} + \frac{\beta'\rho_2}{\omega_p} + \frac{\beta l(\omega_i\eta_i\gamma_a\rho_2 + \omega_p\eta_p\gamma_a\rho_1)}{\omega_h\omega_i\omega_p\eta_i\eta_p}. \tag{15}$$

Let us compare this reproduction number $\overline{R_0}$ with the reproduction number obtained in [18] (see (10)). To achieve the inequality $\overline{R_0} < R_0$, we have to obtain the following inequality

$$\frac{\beta l(\omega_i\eta_i\gamma_a\rho_2 + \omega_p\eta_p\gamma_a\rho_1)}{\omega_h\omega_i\omega_p\eta_i\eta_p} < \frac{\beta l(\omega_i\gamma_a\rho_2 + \omega_p\gamma_a\rho_1)}{\omega_h\omega_i\omega_p}. \tag{16}$$

After simple algebraic manipulation, we can write:

$$\frac{\gamma_a\rho_2}{\omega_p\eta_p} + \frac{\gamma_a\rho_1}{\omega_i\eta_i} < \frac{\gamma_a\rho_2}{\omega_p} + \frac{\gamma_a\rho_1}{\omega_i},$$

and finally, in the case of $\eta_p > 1$ and $\eta_i > 1$, we obtain the final reproduction number $\overline{R_0}$ satisfying the inequality $\overline{R_0} < R_0$.

Remark 1. *We can see that in many cases, our $\overline{R_0}$ can be essentially less than the reproduction number in [18]. Values of the model parameters corresponding to the situation of Wuhan, as discussed in [18], are $\beta = 2.55$, $\beta' = 7.65$, $l = 1.56$, $k = 0.25$, $\rho_1 = 0.58$, $\rho_2 = 0.001$, $\gamma_a = 0.94$, $\gamma_i = 0.27$, $\gamma_r = 0.5$, $\delta_i = 3.5$, $\delta_p = 1$, $\delta_h = 0.3$. In this case $R_0 = 0.945$. In our model, we obtain dependence of the reproduction number on the parameters of the integral terms (see Table 2).*

Table 2. Dependence of the reproduction number on parameters of the integral terms.

η_i	η_p	$\overline{R_0}$
2	2	0.6063
2	3	0.6060
2	4	0.6058
3	2	0.5104
4	2	0.4624

Of course, to obtain these parameters, we need additional statistics that take into account the influence of newly infected persons every day on the level of hospitalization.

5. Stability of the Integro-Differential System

Let us analyze the stability of the following system

$$\begin{cases} \frac{dx_1(t)}{dt} + kx_1(t) &= \beta x_2(t) + \beta l x_6(t) + \beta' x_3(t)\,, \\ \frac{dx_2(t)}{dt} + \omega_i x_2(t) &= k\rho_1 x_1(t)\,, \\ \frac{dx_3(t)}{dt} + \omega_p x_3(t) &= k\rho_2 x_1(t)\,, \\ \frac{dx_4(t)}{dt} + \eta_i x_4(t) &= x_2(t)\,, \\ \frac{dx_5(t)}{dt} + \eta_p x_5(t) &= x_3(t)\,, \\ \frac{dx_6(t)}{dt} + \omega_h x_6(t) &= \gamma_a x_4(t) + \gamma_a x_5(t)\,. \end{cases} \tag{17}$$

Theorem 1. *If all coefficients in system (17) are positive, and the inequality*

$$\frac{\beta\rho_1}{\omega_i} + \frac{\beta'\rho_2}{\omega_p} + \frac{\beta l(\omega_i\eta_i\gamma_a\rho_2 + \omega_p\eta_p\gamma_a\rho_1)}{\omega_h\omega_i\omega_p\eta_i\eta_p} < 1 \tag{18}$$

is true, then system (8) is exponentially stable.

Proof of Theorem 1. Consider also the corresponding non- homogeneous system

$$\begin{cases} \frac{dx_1(t)}{dt} + kx_1(t) &= \beta x_2(t) + \beta l x_6(t) + \beta' x_3(t) + f_1(t)\,, \\ \frac{dx_2(t)}{dt} + \omega_i x_2(t) &= k\rho_1 x_1(t) + f_2(t)\,, \\ \frac{dx_3(t)}{dt} + \omega_p x_3(t) &= k\rho_2 x_1(t) + f_3(t)\,, \\ \frac{dx_4(t)}{dt} + \eta_i x_4(t) &= x_2(t) + f_4(t)\,, \\ \frac{dx_5(t)}{dt} + \eta_p x_5(t) &= x_3(t) + f_5(t)\,, \\ \frac{dx_6(t)}{dt} + \omega_h x_6(t) &= \gamma_a x_4(t) + \gamma_a x_5(t) + f_6(t)\,, \end{cases} \tag{19}$$

Here, $f_1, f_2, f_3, f_4, f_5, f_6$ are essentially bounded functions. Using the representation of solution of the diagonal system

$$\begin{cases} \frac{dx_1(t)}{dt} + kx_1(t) &= \phi_1(t)\,, \\ \frac{dx_2(t)}{dt} + \omega_i x_2(t) &= \phi_2(t)\,, \\ \frac{dx_3(t)}{dt} + \omega_p x_3(t) &= \phi_3(t)\,, \\ \frac{dx_4(t)}{dt} + \eta_i x_4(t) &= \phi_4(t)\,, \\ \frac{dx_5(t)}{dt} + \eta_p x_5(t) &= \phi_5(t)\,, \\ \frac{dx_6(t)}{dt} + \omega_h x_6(t) &= \phi_6(t)\,, \end{cases} \tag{20}$$

Here, $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6$ are essentially bounded functions. We can find the Cauchy functions of every one of the scalar diagonal equations of system (20):

$$\begin{cases} C_1(t,s) = e^{-k(t-s)} \quad \text{and} \quad \int_0^t C_1(t,s) = \frac{1}{k}\,, \\ C_2(t,s) = e^{-\omega_i(t-s)} \quad \text{and} \quad \int_0^t C_2(t,s) = \frac{1}{\omega_i}\,ds\,, \\ C_3(t,s) = e^{-\omega_p(t-s)} \quad \text{and} \quad \int_0^t C_3(t,s) = \frac{1}{\omega_p}\,ds\,, \\ C_4(t,s) = e^{-\eta_i(t-s)} \quad \text{and} \quad \int_0^t C_4(t,s) = \frac{1}{\eta_i}\,ds\,, \\ C_5(t,s) = e^{-\eta_p(t-s)} \quad \text{and} \quad \int_0^t C_5(t,s) = \frac{1}{\eta_p}\,ds\,, \\ C_6(t,s) = e^{-\omega_h(t-s)} \quad \text{and} \quad \int_0^t C_6(t,s) = \frac{1}{\omega_h}\,ds\,. \end{cases} \tag{21}$$

We have the following representations of the components x_1, x_2, x_3, x_4, x_5, and x_6 of the solution vector of system (17):

$$\begin{cases} x_1(t) &= \int_0^t C_1(t,s)[\beta x_2(s) + \beta l x_6(s) + \beta' x_3(s)]\,ds + e^{-kt}x_1(0)\,, \\ x_2(t) &= \int_0^t C_2(t,s)k\rho_1 x_1(s)\,ds + e^{-\omega_i t}x_2(0)\,, \\ x_3(t) &= \int_0^t C_3(t,s)k\rho_2 x_1(s)\,ds + e^{-\omega_p t}x_3(0)\,, \\ x_4(t) &= \int_0^t C_4(t,s)x_2(s)\,ds\,, \\ x_5(t) &= \int_0^t C_5(t,s)x_3(s)\,ds\,, \\ x_6(t) &= \int_0^t C_6(t,s)[\gamma_a x_4(s) + \gamma_a x_5(s)]\,ds + e^{-\omega_h t}x_6(0)\,. \end{cases} \tag{22}$$

The components $x_4(t)$ and $x_5(t)$ can be written as

$$\begin{cases} x_4(t) &= \int_0^t C_4(t,s)\left[\int_0^s C_2(s,\xi)k\rho_1 x_1(\xi)\,d\xi + e^{-\omega_i s}x_2(0)\right]ds\,, \\ x_5(t) &= \int_0^t C_5(t,s)\left[\int_0^s C_3(s,\xi)k\rho_2 x_1(\xi)\,d\xi + e^{-\omega_p s}x_3(0)\right]ds\,. \end{cases} \tag{23}$$

The component $x_6(t)$ can be obtained

$$\begin{aligned} x_6(t) = \int_0^t C_6(t,s)\Big\{\gamma_a \int_0^s C_4(s,\xi)\left[\int_0^\xi C_2(\xi,\theta)k\rho_1 x_1(\theta)\,d\theta + e^{-\omega_i\xi}x_2(0)\right]d\xi \\ +\gamma_a \int_0^s C_5(s,\xi)\left[\int_0^\xi C_3(\xi,\theta)k\rho_2 x_1(\theta)\,d\theta + e^{-\omega_p\xi}x_3(0)\right]d\xi\Big\}ds + e^{-\omega_h t}x_6(0)\,. \end{aligned} \tag{24}$$

The component $x_1(t)$ can be represented as

$$\begin{aligned} x_1(t) = e^{-kt}x_1(0) + \int_0^t C_1(t,s)\Big\{\beta\left[\int_0^s C_2(s,\xi)k\rho_1 x_1(\xi)\,d\xi + e^{-\omega_i s}x_2(0)\right] \\ +\beta'\left[\int_0^s C_3(s,\xi)k\rho_2 x_1(\xi)\,d\xi + e^{-\omega_p s}x_3(0)\right] \\ +\beta l\left[\int_0^s C_6(s,\xi)\gamma_a\left(\int_0^\xi C_4(\xi,\theta)[\int_0^\theta C_2(\theta,\mu)k\rho_1 x_1(\mu)\,d\mu + e^{-\omega_i\theta}x_2(0)]\,d\theta\right.\right. \\ \left.\left. + \int_0^\xi C_5(\xi,\theta)[\int_0^\theta C_3(\theta,\mu)k\rho_2 x_1(\mu)\,d\mu + e^{-\omega_p\theta}x_3(0)]\,d\theta\right)d\xi + e^{-\omega_h s}x_6(0)\right]\Big\}ds. \end{aligned} \tag{25}$$

After passing to the module, we obtain:

$$\begin{aligned} |x_1(t)| \le \max_{0\le s\le t}|x_1(s)|\Big\{\int_0^t C_1(t,s)\Big[\beta\int_0^s C_2(s,\xi)k\rho_1\,d\xi + \beta'\int_0^s C_3(s,\xi)k\rho_2\,d\xi \\ \beta l\{\int_0^s C_6(s,\xi)\gamma_a\Big[\int_0^\xi C_4(\xi,\theta)\int_0^\theta C_2(\theta,\mu)k\rho_1\,d\mu\,d\theta \\ +\int_0^\xi C_5(\xi,\theta)\int_0^t C_3(\theta,\mu)k\rho_2\,d\mu\,d\theta\Big]\,d\phi\}\Big]ds + e^{-kt}\Big\} \\ +\int_0^t C_1(t,s)\Big[\beta l\Big\{\int_0^s C_6(s,\xi)\gamma_a\Big[\int_0^\xi C_4(\xi,\theta)e^{-\omega_i\theta}|x_2(0)|\,d\theta\,d\xi \\ +\int_0^\xi C_5(\xi,\theta)e^{-\omega_p\theta}|x_3(0)|\,d\theta\Big]\,d\xi\Big\} + \beta l e^{-\omega_h s}|x_6(0)| + \beta e^{-\omega_i s}|x_2(0)| + \beta' e^{-\omega_p s}|x_3(0)|\Big\}ds. \end{aligned} \tag{26}$$

Let us write the inequality for $\sup_{t\ge0}|x_1(t)|$:

$$\sup_{t\ge0}|x_1(t)| \le \sup_{t\ge0}|x_1(t)|\frac{1}{k}\{\beta\frac{k\rho_1}{\omega_i} + \beta l\frac{\gamma_a}{\omega_h}[\frac{1}{\eta_i}\frac{k\rho_1}{\omega_i} + \frac{1}{\eta_p}\frac{k\rho_2}{\omega_p}] + \beta'\frac{k\rho_2}{\omega_p}\}. \tag{27}$$

Thus, the following inequality is true:

$$1 \leq \frac{\beta\rho_1}{\omega_i} + \frac{\beta'\rho_2}{\omega_p} + \frac{\beta l(\omega_i\eta_i\gamma_a\rho_2 + \omega_p\eta_p\gamma_a\rho_1)}{\omega_h\omega_i\omega_p\eta_i\eta_p}. \tag{28}$$

The opposite inequality

$$\frac{\beta\rho_1}{\omega_i} + \frac{\beta'\rho_2}{\omega_p} + \frac{\beta l(\omega_i\eta_i\gamma_a\rho_2 + \omega_p\eta_p\gamma_a\rho_1)}{\omega_h\omega_i\omega_p\eta_i\eta_p} < 1 \tag{29}$$

excludes the existence of nontrivial solutions of system (17) with the zero initial conditions $x_1(0) = x_2(0) = x_3(0) = x_4(0) = x_5(0) = x_6(0) = 0$. It follows from here that system (19) has the unique solution for every bounded $f_1(t)$, $f_2(t)$, $f_3(t)$, $f_4(t)$, $f_5(t)$, $f_6(t)$ and this solution is bounded on the semiaxis $t > 0$. According to Bohl–Perron theorem (see for example, [22], p. 500), it follows that system (17) is exponentially stable. Now, according to Lemma 1, system (6) is exponentially stable. □

Remark 2. *We see that the conditions of the exponential stability and of the fact that the reproduction number is less than one coincide. Are these two facts equivalent?*

It is clear that the attenuation of the epidemic, characterized by inequality $\overline{R} < 1$ for the reproduction number, will follow from the fact of exponential stability. Is it true that the exponential stability of the system (8) will follow from the inequality $\overline{R} < 1$ In general, no, as the following examples show.

Example 1.

$$\begin{cases} x_1'(t) + x_1(t) = 0 \\ x_2'(t) - x_2(t) = 0 \end{cases}, \quad \begin{cases} x_1(t) = x_1(0)e^{-t} \\ x_2(t) = x_2(0)e^{t} \end{cases} \tag{30}$$

Example 2. *The case of the first component tending to zero and the second tending to infinity for t tending to infinity is possible not only in the case of separated components but in systems with connected components, as the following example demonstrates.*

$$\begin{cases} x_1'(t) + x_1(t) = 0 \\ x_2'(t) + x_1(t) - x_2(t) = 0 \end{cases} \tag{31}$$

$$\begin{cases} x_1(t) = x_1(0)e^{-t} \\ x_2(t) = -\frac{1}{2}x_1(0)e^{-t} + (x_2(0) + \frac{1}{2}x_1(0))e^{t} \end{cases}$$

It is important to clearly state sufficient reasons to justify the novelty of the study.

6. Stability and Estimates for System of Integro-Differential Equations

Theorem 2. *Let all coefficients in system (11) be non-negative; then, two assertions are equivalent:*

1. *The solution-vector* $z = col(z_1, \ldots, z_n)$ *of the system*

$$\begin{cases} kz_1 - \beta z_2 - \beta l z_6 - \beta' z_3 &= 1 \\ -k\rho_1 z_1 + \omega_i z_2 &= 1 \\ -k\rho_2 z_1 + \omega_p z_3 &= 1 \\ -z_2 + \eta_i z_4 &= 1 \\ -z_3 + \eta_p z_5 &= 1 \\ -\gamma_a z_4 - \gamma_a z_5 + \omega_h z_6 &= 1 \end{cases}, \tag{32}$$

   has all positive components z_j, $j = 1, \ldots, n$,
2. *The system (11) is exponentially stable.*

Proof of Theorem 2. The proof of this assertion follows from [23]. □

It is clear from Lemma 1 that the exponential stability of system (11) implies the exponential stability of system (8). Thus, the necessary and sufficient conditions of the exponential stability of system (11) include the positivity of all components of the solution-vector z. Finding its solution for the coefficients decribing in Remark 4, we obtain

$$\begin{cases} z_1 &= \frac{40(277{,}133{,}103\eta_i + 231{,}308{,}103\eta_p + 407{,}197{,}450\eta_i\eta_p)}{3(92{,}869{,}000\eta_i\eta_p - 287781\eta_i - 78{,}317{,}980\eta_p)} \\ z_2 &= \frac{20(51{,}162{,}696\eta_i + 35{,}243{,}091\eta_p + 84{,}087{,}650\eta_i\eta_p)}{9(92{,}869{,}000\eta_i\eta_p - 287781\eta_i - 78{,}317{,}980\eta_p)} \\ z_3 &= \frac{14{,}676{,}831\eta_i - 1{,}789{,}545{,}069\eta_p + 2{,}174{,}453{,}650\eta_i\eta_p}{51(92{,}869{,}000\eta_i\eta_p - 287781\eta_i - 78{,}317{,}980\eta_p)} \\ z_4 &= \frac{2{,}517{,}574{,}000\eta_p + 1{,}020{,}663{,}891}{9(92{,}869{,}000\eta_i\eta_p - 287{,}781\eta_i - 78{,}317{,}980\eta_p)} \\ z_5 &= \frac{6{,}910{,}772{,}650\eta_i - 5{,}783{,}762{,}049}{51(92{,}869{,}000\eta_i\eta_p - 287{,}781\eta_i - 78{,}317{,}980\eta_p)} \\ z_6 &= \frac{25(972{,}217{,}419\eta_i + 1{,}412{,}409{,}079\eta_p + 710{,}447{,}850\eta_i\eta_p)}{153(92{,}869{,}000\eta_i\eta_p - 287781\eta_i - 78{,}317{,}980\eta_p)} \end{cases}. \tag{33}$$

Remark 3. *Substituting* η_i *and* η_p *we obtain dependence of the values of* $z_1, \ldots, z_6$ *on the parameters of the integral terms (see Table 3).*

Table 3. Dependence of $z_1, \ldots, z_6$ on parameters of the integral terms .

η_i	η_p	z_1	z_2	z_3	z_4	z_5	z_1
2	2	164.6359	5.2807	0.4711	3.1404	0.7356	5.8042
2	3	153.0019	4.9226	0.4698	2.9613	0.4899	5.3052
2	4	147.1927	4.7437	0.4691	2.8719	0.3673	5.0560
3	2	124.6623	4.0501	0.4666	1.6834	0.7333	4.0896
4	2	110.0255	3.5995	0.4649	1.1499	0.7325	3.4618

Consider the system with error on the right-hand side and the values of the parameters

$$x'(t) + A(t)x(t) = f(t) + \Delta f(t), \tag{34}$$

where $A(t)$ is the matrix of the coefficients in the right-hand side of (11). Consider also the system

$$AY = I, \tag{35}$$

where I is the unit (6×6) matrix and $Y = \{y_{ij}\}_{i,j=1}^{6}$.

Theorem 3. *If all coefficients in (11) are non-negative and all components of the solution-vector $z = col(z_1, z_2, z_3, z_4, z_5, z_6)$ are positive, then*

$$\lim_{t\to\infty} \int_0^t \sum_{j=1}^{6} C_{ij}(t,s)\, ds = z_i\,, \quad i = 1, \ldots, 6, \tag{36}$$

$$\lim_{t\to\infty} \int_0^t C_{ij}(t,s)\, ds = y_{ij}\,, \quad i,j = 1, \ldots, 6, \tag{37}$$

where $Y = \{y_{ij}\}_{i,j=1}^{6}$ is the solution of (35), and

$$|\Delta x_i(t)| \equiv |x_i(t) - X_i(t)| \le \sum_{j=1}^{6} y_{ij} |\Delta f_j^*|\,, \quad i = 1, \ldots, 6, \tag{38}$$

where

$$|\Delta f_j^*| = \operatorname*{ess\,sup}_{t\ge 0} |\Delta f_j(t)|, \tag{39}$$

and the vectors-solution $x(t) = col(x_1(t), \ldots, x_6(t))$, $X(t) = col(X_1(t), \ldots, X_6(t))$ are solutions of (34) and of the system

$$X'(t) + A(t)X(t) = f(t) \tag{40}$$

respectively.

Proof of Theorem 3. The representation of solution of system (34) in the case of $x_i(0) = 0$, $i = 1, \ldots, 6$, is

$$x(t) = \int_0^t C(t,s) f(s)\, ds + \int_0^t C(t,s) \Delta f(s)\, ds. \tag{41}$$

The Cauchy matrix of system (11) is non-negative (see Corollary 16.1 ([22], p. 404)). Equalities (36) and (37) follow from Corollary 16.2 [22]. It is clear that

$$\int_0^t C(t,s) |\Delta f(s)|\, ds \le \operatorname*{ess\,sup}_{t\ge 0} |\Delta f(t)| \int_0^t C(t,s)\, ds. \tag{42}$$

Inequality (38) follows from representation (41) and (37). □

Now, let us estimate the influence of the errors in the parameters. Consider the system

$$x'(t) + A(t)x(t) + (\Delta A x)(t) = f(t) \tag{43}$$

where $\Delta A : C \longrightarrow L_\infty$ is a bounded operator (C is the space of continuous functions and L_∞ is the space of essentially bounded functions $x : [0, \infty) \longrightarrow (-\infty, \infty)$). It is clear that

$$x'(t) + Ax(t) = f(t) - (\Delta A x)(t). \tag{44}$$

The representation of solution with the initial condition $x_1(0) = 0, \ldots, x_6(0) = 0$ is

$$x(t) = \int_0^t C(t,s) f(s)\, ds - \int_0^t C(t,s) \Delta A x(s)\, ds. \tag{45}$$

Let us introduce the operator $\mathcal{K} : L_\infty \longrightarrow L_\infty$ by the equality

$$(\mathcal{K}x)(t) = \int_0^t C(t,s)(\Delta A x)(s)\, ds. \tag{46}$$

If $\|\mathcal{K}\| < 1$, then our errors in "coefficients" cannot disturb the stability of the system (40), i.e., (43) stays exponentially stable if (11) is exponentially stable. The influence of the errors in "coefficients" on the solution $x(t) = col(x_1(t), \ldots, x_6(t))$ can be estimated as

$$|\Delta x_i(t)| \leq \frac{\|\Delta A\|}{1 - \|\mathcal{K}\|}, \quad i = 1, \ldots, 6. \tag{47}$$

Substituting the estimate of $C(t,s)$ given by (36), we obtain

$$|\Delta x_i(t)| \leq \frac{\|\Delta A\|}{1 - z^*\|\Delta A\|}, \quad i = 1, \ldots, 6, \quad t \in [0, \infty), \tag{48}$$

where $z^* = \max_{1 \leq i \leq 6}\{z_i\}$ and z_i are defined by (33) and Table 3. We noted that the case of the kernels described by (3) with $\mathbf{Q_I}(t,s) = 0$, $\mathbf{Q_P}(t,s) = 0$ for $t - h < s < t$ looks more natural in the COVID-19 model. It leads us to the operator (46).

If we take instead of the kernels (4) the kernels (3), where $\kappa_I(t,s) = 0$, $\kappa_P(t,s) = 0$ for $t - h < s < t$, we arrive at

$$\|\Delta A\| \leq \gamma_a\left(\frac{1 - e^{-\eta_i h}}{\eta_i} + \frac{1 - e^{-\eta_p h}}{\eta_p}\right). \tag{49}$$

If h is "small", we have that $\|\Delta A\|$ is also "small", and our system (5) is a "sufficiently exact" model of system (2).

7. Conclusions

We start our paper with the explanation of why the traditional model of COVID-19 transmission seems unnatural because of the assumption that the number of infected people acts immediately on the number of beds for hospitalization. The incubation period of COVID-19 is about 7 days. In the beginning of new waves, the traditional model is not exact, since we have so many infected people, but we do not need such a large number of beds in hospitals. We have at least a week for organizing places of hospitalization medical personnel and drugs. We propose to add the integral terms catching this fact. We also pay attention to the fact that all coefficients, even in the traditional model, are the results of calculating. Everyone makes small errors in these calculations. We suppose that it is important to estimate the influence of these errors. We make it when we obtain explicit formulas for these influences. When every model is constructed, we do not take into account the corresponding connections of the process with reality. We estimate the possible influence of this "separation" on solution. It should be noted that the idea of the reducing integro-differential system (2) to the system of the order 10 of an ordinary differential equation allows solving this new system numerically. The main trouble in the numerical solution of systems with integral terms is the following. Integral terms accumulate errors of calculation. Our reducing avoids these troubles. Standard numerical procedures for solving ordinary differential equations systems can be used.

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