mathematics

Article

Sub-Super Solutions Method Combined with Schauder's Fixed Point for Existence of Positive Weak Solutions for Anisotropic Non-Local Elliptic Systems

Rafik Guefaifia [1], Gelson Concei¸cao G. dos Santos [2], Tahar Bouali [3,4], Rashid Jan [5], Salah Boulaaras [1,*] and Asma Alharbi [1]

1 Department of Mathematics, College of Sciences and Arts in ArRass, Qassim University, Buraydah 51452, Saudi Arabia
2 Faculty of Mathematics, Federal University of Pará, Belém 66075110, Brazil
3 Department of Mathematics, College of Science, Jazen University, Jazen 277, Saudi Arabia
4 Department of Mathematics and Computer Science, Larbi Tebessi University, Tebessa 12002, Algeria
5 Department of Mathematics, University of Swabi, Swabi 23561, Pakistan
* Correspondence: s.boularas@qu.edu.sa

Abstract: In this research, we investigate the presence of weak positive solutions for a family of anisotropic non-local elliptic systems in bounded domains using the sub-super solutions approach in conjunction with Schauder's fixed point.

Keywords: anisotropic non-local elliptic systems; existence; positive solutions; sub-supersolution; Schauder's fixed point

MSC: 35J60; 35B30; 35B40

Citation: Guefaifia, R.; dos Santos, G.C.G.; Bouali, T.; Jan, R.; Boulaaras, S.; Alharbi, A. Sub-Super Solutions Method Combined with Schauder's Fixed Point for Existence of Positive Weak Solutions for Anisotropic Non-Local Elliptic Systems. *Mathematics* **2022**, *10*, 4479. https://doi.org/10.3390/math10234479

Academic Editor: Salvatore Sessa

Received: 8 November 2022
Accepted: 25 November 2022
Published: 27 November 2022

1. Introduction

A partial differential equation is a function of more than one variable, which is the key distinction between PDEs and ordinary differential equations. Many factors in nature are changing at once and manifest as phenomena. We end up with a PDE since they demand several variables rather than simply one. It has been observed that elliptic equations appear when dealing with physical laws. Therefore, elliptic equations are mostly associated with real-world problems and are important to study. In the literature, several such problems have been presented and solved with different assumptions [1–5].

We concentrate our attention on the following system in this paper:

$$\begin{cases} -M_1\left(\sum_{i=1}^{N}\left\|\frac{\partial v_1}{\partial y_i}\right\|_{p_i}^{p_i}\right)\Delta_{\overrightarrow{p}} v_1 = F_1(y, v_1, v_2) & \text{in } \Phi, \\ -M_2\left(\sum_{i=1}^{N}\left\|\frac{\partial v_2}{\partial y_i}\right\|_{p_i}^{p_i}\right)\Delta_{\overrightarrow{p}} v_2 = F_2(y, v_1, v_2) & \text{in } \Phi, \\ v_1 = v_2 = 0 & \text{on } \partial\Phi, \end{cases} \tag{1}$$

where $N > 2$, $\partial\Phi$ is the smooth boundary of the open bounded domain Φ of $\mathbf{R}^N$, $\partial\Phi$ $\Delta_{\overrightarrow{p}} v := -\sum_{i=1}^{N} \frac{\partial}{\partial y_i}(|\frac{\partial v_i}{\partial y_i}|^{p_i-2}\frac{\partial v_i}{\partial y_i})$ is the anisotropic operator and $p_i, i = 1, ..., N$ are real numbers with $2 \le p_1 \le p_2 \le ... \le p_N < +\infty$, M_j, $j = 1, 2$, are continuous positive functions on $\mathbf{R}^+$ and $F_j : \overline{\Phi} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are continuous maps. The Anisotropic Kirchhoff type system (1) is a generalized variant of the basic Kirchhoff system. Problems with population dynamics and some physical events are examples of such systems. In general, such a problem can be solved in one of two ways: variational or topological. When employing variational approaches to solve a problem, the requirements on the second item of system (1) are frequently more restrictive. We will employ a topological method here, specifically the sub-super solution method (combined with Schauder's fixed point) which is based on the comparison principle.

It is demonstrated in [6] that the comparison principle fails in this situation, making the conventional sub-super solution approach ineffective for Kirchhoff equations. Therefore, the results of the p-Kirchhoff type system's existence based on the original application of the sub-super solution approach were unsuccessful.

In this study, we show that under weaker criteria than those suggested in the preceding papers, we can prove the desired result in more general circumstances. In order to do this, we refer back to [6] in which the researchers proposed a unique sub-super solution strategy for the classical Kirchhoff system, based on the Schauder Fixed-Point Theorem rather than the comparison principle, with $\overrightarrow{p} = p = 2$. However, the proof seems to be ambiguous. As a matter of fact, the embedding of $(L^{\infty}(\Phi))^2 \to (L^{\infty}(\Phi))^2$ is not compact, and the function examined from $H_0^1(\Phi)$ in $L^{\infty}(\Phi)$ is not compact.

Based on the concepts in [6] and the information in [7], we demonstrate that the new version of the sub-super solution approach performs admirably on the non-local issue and can be extended to nonlinear non-local systems. As a result, we establish the existence of several never-before-considered nonlinearities Fj in a positive solution to the $\overrightarrow{p}$-Kirchhoff problem (1).

The research work is divided into sections as follows: We demonstrate the equivalence of a $\overrightarrow{p}$-Kirchhoff system to a non-local $\overrightarrow{p}$-Laplacian system, moreover, the viability of the novel non-local $\overrightarrow{p}$-Laplacian sub-super solution approach $\overrightarrow{p}$-Laplacian system and $\overrightarrow{p}$-Kirchhoff system. In Section 3, we demonstrate how the modified sub-super solution technique suggested in [6] may be applied to the nonlinear problem. In Section 4, we demonstrate the method's effectiveness for the issue (1.1) and present the fundamental theory underlying the existence of a positive solution. Finally, we give an explanation of the results obtained.

2. Preliminaries

Here, we indicate the bounded domain by Φ in $\mathbf{R}^N$, $N \geq 2$. Further, take the real numbers $1 < p_1 < p_2 < ... < p_N$ and take a vector $\overrightarrow{p}$ such that $\overrightarrow{p} = (p_1, p_2, ..., p_N) \in \mathbf{R}^N$. In the next step, we take the Sobolev space $W^{1,\overrightarrow{p}}(\Phi)$ given by

$$W^{1,\overrightarrow{p}}(\Phi) := \left\{ v \in W^{1,1}(\Phi); \frac{\partial v}{\partial y_i} \in L^{p_i}(\Phi), i = 1, ..., N \right\},$$

which is a Banach space equipped with the norm

$$\|v\|_{1,\overrightarrow{p}} := \|v\|_{L^1(\Phi)} + \sum_{i=1}^{N} \left\| \frac{\partial v}{\partial y_i} \right\|_{L^{p_i}(\Phi)}, \tag{2}$$

where $\|v\|_{L^{p_i}(\Phi)}$ denotes the usual norm of $L^{p_i}(\Phi)$. The closure of $C_0^{\infty}(\Phi)$ in $W^{1,\overrightarrow{p}}(\Phi)$ with regard to the norm $\|.\|_{1,\overrightarrow{p}}$ defines the Banach space, which will be designated as $W_0^{1,\overrightarrow{p}}(\Phi)$. Think about the harmonic mean $\overline{p}$ of p_i, $i = 1, ..., N$, provided by

$$\overline{p} := \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}.$$

Suppose that $\overline{p} < N$ and define the symbol $\overline{p}^* := \frac{N\overline{p}}{N-\overline{p}}$. If $p_N < \overline{p}^*$ then there exists an embedding $W_0^{1,\overrightarrow{p}}(\Phi) \hookrightarrow L^q(\Phi)$ which is compact for $q \in [1, p^*)$ and is continuous in the case $q \in [1, \overline{p}^*]$ see [7]. As a consequence of this, the norm is

$$\|v\| := \sum_{i=1}^{N} \left\| \frac{\partial v}{\partial y_i} \right\|_{L^{p_i}(\Phi)}, v \in W_0^{1,\overrightarrow{p}}(\Phi)$$

is equivalent to the norm mentioned in the above (2).

Lemma 1 ([7])**.** *Let $a \in L^\infty(\Phi)$. Then the problem*

$$\begin{cases} -\Delta_{\overrightarrow{p}} v = a & \text{in } \Phi, \\ v = 0 & \text{on } \partial\Phi, \end{cases}$$

has a solution in $W_0^{1,\overrightarrow{p}}(\Phi)$.

Lemma 2 ([7])**.** *Assume that bounded domain Φ and take $v, w \in W_0^{1,\overrightarrow{p}}(\Phi)$ fulfilling*

$$\begin{cases} -\Delta_{\overrightarrow{p}} v \leq -\Delta_{\overrightarrow{p}} w & \text{in } \Phi, \\ v \leq w & \text{on } \partial\Phi, \end{cases}$$

where $v \leq w$ on $\partial\Phi$ implies that $(v-w)^+ := \max\{0, v-w\} \in W_0^{1,\overrightarrow{p}}(\Phi)$. Then $v \leq w$ in Φ.

Lemma 3 ([7])**.** *Take a bounded and admissible $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ such that one can find a continuous embedding $W_0^{1,1}(\Phi) \hookrightarrow L^{\frac{N}{N-1}}(\Phi)$ and the best constant of the mentioned embedding is indicated by C_0. Then we have that*

$$\|v\|_{W_0^{1,1}(\Phi)} \leq C_0 \|v\|_{L^{\frac{N}{N-1}}(\Phi)}, \ \forall v \in W_0^{1,1}(\Phi), \tag{3}$$

where $\|v\|_{W_0^{1,1}(\Phi)} := \||\nabla v|\|_{L^1}, v \in W_0^{1,1}(\Phi)$.

Numerous ideas about anisotropic has been presented in the literature [8–11]. Adapting the ideas of [7] to the anisotropic setting, we obtain the result below which is significant for the current work.

Lemma 4 ([7])**.** *Let $\lambda > 0$ and consider $v \in W_0^{1,\overrightarrow{p}}(\Phi)$ the unique solution of*

$$\begin{cases} -\Delta_{\overrightarrow{p}} v = \lambda & \text{in } \Phi, \\ v = 0 & \text{on } \partial\Phi. \end{cases} \tag{4}$$

Define $h := \frac{p_1}{2|\Phi|^{\frac{1}{N}}} C_0$. In the case $h \leq \lambda$ then $v \in L^\infty(\Phi)$ with $\|v\|_{L^\infty(\Phi)} \leq C^* \lambda^{\frac{1}{p_1 - 1}}$ and $\|v\|_{L^\infty(\Phi)} \leq C_* \lambda^{\frac{1}{p_N - 1}}$. If $h > \lambda$, where C^* and C_* are positive constants that rely only on $p_1, p_N, |\Phi|$ and C_0, with C_0 given in (3).

3. Kirchhoff and Non-Local Problem

In non-local issues, such as the Kirchhoff problem [6,12], the comparison principle and, as a result, the approach of sub-super solution cannot be applied.

As a result, we transform issue (1) into a non-local system that can be resolved by a modified sub-super solution strategy. For this reason, we apply the same concepts discussed in [6] to the linear example, where $\overrightarrow{p} = p = 2$. Further, we take the invertible function $N_j(t) = M_j(t)t$ and put $R_j(t) = N_j^{-1}(t)$. We introduced the non-local operator $\mathcal{R}_j : L^\infty(\Phi) \times L^\infty(\Phi) \to \mathbb{R}$ defined by

$$\mathcal{R}_j(v_1, v_2) = R_j(\int_\Phi F_j(y, v_1, v_2) v_j).$$

Multiplying (1) by v_j and integrating, we get

$$
\begin{aligned}
M_j\left(\sum_{i=1}^{N}\|\frac{\partial v_j}{\partial y_i}\|_{p_i}^{p_i}\right)\left(\sum_{i=1}^{N}\|\frac{\partial v_j}{\partial y_i}\|_{p_i}^{p_i}\right) &= \int_{\Phi} F_j(y,\, v_1,\, v_2)v_j \\
&\Rightarrow \left(\sum_{i=1}^{N}\|\frac{\partial v_j}{\partial y_i}\|_{p_i}^{p_i}\right) \\
&= R_j(\int_{\Phi} F_j(y,\, v_1,\, v_2)v_j) \\
&= \mathcal{R}_j(v_1,\, v_2), j=1,2.
\end{aligned}
$$

The following non-local system is now introduced

$$
\begin{cases}
-\Delta_{\overrightarrow{p}} v_1 = \dfrac{F_1(y,v_1,v_2)}{M_1(\mathcal{R}_1(v_1,v_2))} & \text{in } \Phi, \\
-\Delta_{\overrightarrow{p}} v_2 = \dfrac{F_2(y,v_1,v_2)}{M_2(\mathcal{R}_2(v_1,v_2))} & \text{in } \Phi, \\
v_1 = v_2 = 0 & \text{on } \partial\Phi.
\end{cases}
\tag{5}
$$

Lemma 5. *Prove that (1) and (5) are equivalent systems.*

Proof. It is clear that each solution of (1) is also a solution of system (5). For the converse part, we will show that $\mathcal{R}_j(v_1,\, v_2) = \sum_{i=1}^{N}\left\|\frac{\partial v_j}{\partial y_i}\right\|_{p_i}^{p_i}, j=1,2$. Assume another solution $(v_1,\, v_2)$ of the system (5). After simple mathematical calculation, we have

$$
\sum_{i=1}^{N}\left\|\frac{\partial v_j}{\partial y_i}\right\|_{p_i}^{p_i} = \frac{\int_{\Phi} F_j(y,v_1,v_2)v_i}{M_j(R_j(\int_{\Phi} F_j(y,v_1,v_2)v_j))} = R_j(\int_{\Phi} F_j(y,\, v_1,\, v_2)vj) = \mathcal{R}_j(v_1,\, v_2),
$$

for $j=1,2$. Hence, the proof is completed. □

4. The Sub-Super Approach for Non-Local $\overrightarrow{p}$-Laplacian System

In the literature, the solution of different systems has been investigated via sub-super approach [13–16]. Here, we demonstrate how the linear example with $\overrightarrow{p} = p = 2$ in the sub-super solution approach [6] may be modified for the $\overrightarrow{p}$-Laplacian in the non-local situation. Consider the non-local system as

$$
\begin{cases}
-\Delta_{\overrightarrow{p}} v_1 = G_1(y,\, v_1,\, v_2,\, A_1(v_1,v_2)) & \text{in } \Phi, \\
-\Delta_{\overrightarrow{p}} v_2 = G_2(y,\, v_1,\, v_2,\, A_2(v_1,v_2)) & \text{in } \Phi, \\
v_1 = v_2 = 0 & \text{on } \partial\Phi,
\end{cases}
\tag{6}
$$

where $A_j : L^1(\Phi) \times L^1(\Phi) \to \mathbf{R}$ and $G_j : \overline{\Phi} \times \mathbf{R}^3 \to \mathbf{R}$.

Definition 1. *The pair of solutions* $(\underline{v}_1,\, \overline{v}_1), (\underline{v}_2,\, \overline{v}_2) \in (W^{1,\overrightarrow{p}}(\Phi) \cap L^{\infty}(\Phi)) \times (W^{1,\overrightarrow{p}}(\Phi) \cap L^{\infty}(\Phi))$ *is a sub-super solution of (6) if the following holds true*

$(H_1)\underline{v}_j \le \overline{v}_j$ a.e in Φ and $\underline{v}_j \le 0 \le \overline{v}_j$ on $\partial\Phi$ for $j=1,2$.
$(H_2) - \Delta_{\overrightarrow{p}}\underline{v}_1 - G_1(y,\, \underline{v}_1,\, w,\, A_1(v,w)) \le 0 \le -\Delta_{\overrightarrow{p}}\overline{v}_1 - G_1(y,\, \overline{v}_1,\, w,\, A_1(v,w))$,
$(H_3) - \Delta_{\overrightarrow{p}}\underline{v}_2 - G_2(y,\, v,\, \underline{v}_2,\, A_2(v,w)) \le 0 \le -\Delta_{\overrightarrow{p}}\overline{v}_2 - G_2(y,\, v,\, \overline{v}_2,\, A_2(v,w))$.

Inequalities in (H_2) and (H_3) are in the weak sense for all $(v,\, w) \in [\underline{v}_1,\, \overline{v}_1] \times [\underline{v}_2,\, \overline{v}_2]$, where, for $v \le w$ a.e. in $\Phi, [v,\, w]$ is defined by,

$$
[v,\, w] := \{x:\ v(y) \le x(y) \le w(y),\ a.e.\ y \in \Phi\}.
$$

In the upcoming theorem, we will prove the main result of this research work.

Theorem 1. *Let us assume the following*
D_1.A_j *is a continuous operator for* $j = 1, 2$ *which maps bounded sets to bounded sets*
D_2.G_j *is continuous for* $j = 1, 2$ *in* $\overline{\Phi} \times \mathbf{R}^3$.

Then, the system (6) has a solution $(v_1,\ v_2) \in [\underline{v}_1,\ \overline{v}_1] \times [\underline{v}_2,\ \overline{v}_2]$ *if a pair of sub-super solutions of (6) exists in the sense of Definition 1.*

Proof. First of all take the operators, $K_j : L^1(\Phi) \to L^1(\Phi)$ introduced in the following manner

$$K_j(x_j)(y) = \begin{cases} \underline{v}_j(y)\ if\ x_j(y) \le \underline{v}_j(v), \\ x_j(y)\ if\ \underline{v}_j(y) \le x_j(y) \le \overline{v}_j(y), \\ \overline{v}_j(y)\ if\ x_j(y)) \ge \overline{v}_j(y). \end{cases}$$

Remark that $\forall x_j \in L^1(\Phi)$, $K_j(x_j) \in \left[\underline{v}_j,\ \overline{v}_j\right]$ and as $\underline{v}_j, \overline{v}_j \in L^\infty(\Phi)$, there exist constants $\underline{b}_j, \overline{b}_j$ such that $K_j(x_j) \in [\underline{b}_j, \overline{b}_j]$ a.e. in Φ. Then, $|A_j(K_1(x_1),\ K_2(x_2))| \le c_j$. Through the continuity of G_j in $\overline{\Phi} \times [\underline{b}_1,\ \overline{b}_2] \times [\underline{b}_2,\ \overline{b}_2] \times [-c_j,\ c_j]$, we can find a positive constant of continuity C_j of $G_j, j = 1, 2$. Therefore, we have the following inequality:

$$|G_j(y,\ K_1(x_1)(y),\ K_2(x_2)(y),\ A_j(K_1(x_1), K_2(x_2)))| \le C_j.$$

Here, the Nemytskii operator is denoted by

$$H_j : L^1(\Phi) \times L^1(\Phi) \to L^1(\Phi)$$

and is given by

$$H_j(v_1,\ v_2)(y) = G_j(y,\ K_1(v_1)(y),\ K_2(v_2)(y),\ A_j(K_1(v_1),\ K_2(v_2)))\,.$$

the boundedness of Φ is obtained through the $L^\infty(\Phi)$-boundedness of H_j. At the end, through the continuity of G_j and dominated convergence theorem, we deduce the continuity of H_j and

$$\|H_j(v_1,\ v_2)\|_\infty \le C_j,\ \forall (v_1,\ v_2) \in L^1(\Phi) \times L^1(\Phi). \tag{7}$$

Here, we fix $(x_1,\ x_2) \in L^1(\Phi) \times L^1(\Phi)$ by the Minty–Browder Theorem, then a unique pair solution $(u_1,\ u_2) \in W^{1,\vec{p}_1}(\Phi) \times W^{1,\vec{p}_2}(\Phi))$ exists of

$$\begin{cases} -\Delta_{\vec{p}} u_1 = H_1(K_1(x_1),\ K_2(x_2)) & \text{in } \Phi, \\ -\Delta_{\vec{p}} u_2 = H_2(K_1(x_1),\ K_2(x_2)) & \text{in } \Phi, \\ u_1 = u_2 = 0 & \text{on } \partial\Phi. \end{cases} \tag{8}$$

In these circumstance, we introduce the operator $S : L^1(\Phi) \times L^1(\Phi) \to L^1(\Phi) \times L^1(\Phi)$ by $S(x_1,\ x_2) = (u_1,\ u_2)$ where $(u_1,\ u_2)$ is the unique solution of (8). □

Claim 1: Take a bounded sequence $(x_{1,n},\ x_{2,n})$ in $L^1(\Phi) \times L^1(\Phi)$ and $(u_{1,n},\ u_{2,n}) = S(x_{1,n},\ x_{2,n})$ while S is compact. After that

$$\int_\Phi \sum_{i=1}^N \left|\frac{\partial u_{j,n}}{\partial y_i}\right|^{p_i-2} \frac{\partial u_{j,n}}{\partial y_i}.\frac{\partial \phi_j}{\partial y_i} = \int_\Phi H_j(K_1(x_{1,n}),\ K_2(x_{2,n}))\phi_j\ \forall \phi_j \in W_0^{1,\vec{p}}(\Phi), j = 1, 2. \tag{9}$$

Using the test function $\phi_j = u_{j,n}$ in (9) and the inequality (7), we get

$$\sum_{i=1}^N \left\|\frac{\partial u_{j,n}}{\partial y_i}\right\|_{p_i}^{p_i} \le C_j \int_\Phi |u_{j,n}| \le \overline{C}_j \|u_{j,n}\|, \tag{10}$$

where $\overline{C}_j > 0$ are constants which have no rely on $n \in \mathbb{N}$. The inequality $|s|^{p_1} \leq 1 + |s|^{p_i}, s \in \mathbf{R}, i = 1, ..., N$, provides that

$$\sum_{i=1}^{N} \left\| \frac{\partial u_{j,n}}{\partial y_i} \right\|_{p_i}^{p_1} \leq \widetilde{C}(\|u_{j,n}\| + 1), j = 1, 2, \tag{11}$$

for all $n \in \mathbb{N}$. As $C\left(a_1^b + ... + a_N^b\right) \geq (a_1 + ... + a_N)^b$ for $a_i \geq 0, i = 1, ..., N$ and $1 \leq b$, where $0 < C$ rely only on b and N; this implies that

$$\left(\sum_{i=1}^{N} \left\| \frac{\partial u_{j,n}}{\partial y_i} \right\|_{p_i} \right)^{p_1} \leq L(\|u_{j,n}\| + 1), j = 1, 2, \tag{12}$$

for all $n \in \mathbb{N}$ with $L > 0$ being a constant that does not depend on $n \in \mathbb{N}$. From (12), it follows that the sequence $(u_{1,n}, u_{2,n})$ is bounded in $W_0^{1,\overrightarrow{p}}(\Phi) \times W_0^{1,\overrightarrow{p}}(\Phi)$. Since the embedding $W_0^{1,\overrightarrow{p}}(\Phi) \hookrightarrow L^1(\Phi)$ is compact, the boundness of $(u_{1,n}, u_{2,n})$ in $W_0^{1,\overrightarrow{p}}(\Phi) \times W_0^{1,\overrightarrow{p}}(\Phi)$ implies that there exists a convergent sub-sequence of $(u_{1,n}, u_{2,n})$ in $L^1(\Phi) \times L^1(\Phi)$. The compactness of S implies that $(u_{1,n},\ u_{2,n})$ is bounded in $W_0^{1,p_1}(\Phi) \times W_0^{1,p_2}(\Phi)$. It is easy to find a convergent sub-sequence of $(u_{1,n},\ u_{2,n})$ in $L^1(\Phi) \times L^1(\Phi)$ through compact embedding.

From (10), we conclude there exists $L_i > 0$ in a manner that

$$\|u_{i,n}\|_{W_0^{1,p_i}(\Phi)} \leq \overline{K}_i \Rightarrow \|u_{i,n}\|_{p_j} \leq L_i. \tag{13}$$

Remark that in (7), (10) and (13), the constants are independent of the choice of $(x_1,\ x_2)$. Then, for some $\overline{L} > 0$, we conclude that $S(L^1(\Phi) \times L^1(\Phi)) \subset B_{L^1(\Phi)\times L^1(\Phi)}(0, \overline{L})$. By the Schauder fixed point theorem, in $B_{L^1(\Phi)\times L^1(\Phi)}(0, \overline{L})$, one can find a unique $(v_1,\ v_2) \in L^1(\Phi) \times L^1(\Phi)$ in a manner that $S(v_1,\ v_2) = (v_1,\ v_2)$, that is

$$\begin{cases} -\Delta_{\overrightarrow{p}_1} v_1 = H_1(K_1(v_1),\ K_2(v_2)) & \text{in } \Phi, \\ -\Delta_{\overrightarrow{p}_2} v_2 = H_2(K_1(v_1),\ K_2(v_2)) & \text{in } \Phi, \\ v_1 = v_2 = 0 & \text{on } \partial\Phi. \end{cases} \tag{14}$$

Finally, $(v_1,\ v_2)$ is a solution of problem (1.1) if, and only if, $K_1(v_1) = v_1$ and $K_2(v_2) = v_2$, furthermore $\underline{v}_1 \leq v_1 \leq \overline{v}_1$ and $\underline{v}_2 \leq v_2 \leq \overline{v}_2$. We have to establish that $(\underline{v}_1 - v_1)^+ = 0, (v_1 - \overline{v}_1)^+ = 0, (\underline{v}_2 - v_2)^+ = 0$ and $(v_2 - \overline{v}_2)^+ = 0$. To establish the result, take $(\underline{v}_1 - v_1)^+ = 0$. The remaining cases can be supported by the same logic. Assume

$$\Phi^+ = \{y \in \Phi,\ \underline{v}_1(y) > v_1(y)\}.$$

As $(\underline{v}_1,\ \underline{v}_2)$ is a sub-solution then, for $\phi = (\underline{v}_1 - v_1)^+ \in W_0^{1,\overrightarrow{p}}(\Phi), v = K_1(v_1)$ and $w = K_2(v_2)$, we have

$$\int_\Phi \sum_{i=1}^{N} |\frac{\partial \underline{v}_1}{\partial y_i}|^{p_i-2} \frac{\partial \underline{v}_1}{\partial y_i} . \frac{\partial}{\partial y_i} (\underline{v}_1 - v_1)^+ \leq \int_\Phi G_1[y,\ \underline{v}_1,\ K_2(v_2), A_1(K_1(v_1),\ K_2(v_2))](\underline{v}_1 - v_1)^+$$

and as $(v_1,\ v_2)$ is a solution of (14), we have

$$\int_\Phi \sum_{i=1}^{N} |\frac{\partial v_1}{\partial y_i}|^{p_i-2} \frac{\partial v_1}{\partial y_i} . \frac{\partial}{\partial y_i} (\underline{v}_1 - v_1)^+ = \int_\Phi G_1[y,\ K_1(v_1),\ K_2(v_2), A_1(K_1(v_1),\ K_2(v_2))](\underline{v}_1 - v_1)^+.$$

Then

$$\begin{aligned} & \int_{\Phi} \sum_{i=1}^{N} \Big[\Big(|\frac{\partial \underline{v}_1}{\partial y_i}|^{p_i-2} \frac{\partial \underline{v}_1}{\partial y_i} - |\frac{\partial v_1}{\partial y_i}|^{p_i-2} \frac{\partial v_1}{\partial y_i} \Big) \Big] . \frac{\partial}{\partial y_i} (\underline{v}_1 - v_1)^+ \\ \leq \ & \int_{\Phi} [G_1(y, \underline{v}_1,\ K_2(v_2), A_1(K_1(v_1),\ K_2(v_2)) \\ - \ & G_1(y,\ K_1(v_1),\ K_2(v_2), A_1(K_1(v_1),\ K_2(v_2))](\underline{v}_1 - v_1)^+. \end{aligned}$$

In $\Phi^+, K_1(v_1) = \underline{v}_1$ through a remark, then we have

$$\begin{aligned} & \int_{\Phi^+} \sum_{i=1}^{N} \Big[\Big(|\frac{\partial \underline{v}_1}{\partial y_i}|^{p_i-2} \frac{\partial \underline{v}_1}{\partial y_i} - |\frac{\partial v_1}{\partial y_i}|^{p_i-2} \frac{\partial v_1}{\partial y_i} \Big) \Big] . \frac{\partial}{\partial y_i} (\underline{v}_1 - v_1)^+ \\ = \ & \int_{\Phi^+} (|\nabla \underline{v}_1|^{p_1-2} \nabla \underline{v}_1 - |\nabla v_1|^{p_1-2} \nabla v_1) . (\frac{\partial}{\partial y_i} \underline{v}_1 - \frac{\partial}{\partial y_i} v_1) \leq 0. \end{aligned}$$

As a result, $\underline{v}_1 - v_1 = 0$ in Φ^+ due to the monotonicity of the $\overrightarrow{p}$-Laplacian. Then, we have $\underline{v}_1 \leq v_1$ in Φ. In the same way, we also arrive at $v_1 \leq \overline{v}_1$ in Φ. We get to the conclusion that $\underline{v}_1 \leq v_1 \leq \overline{v}_1$ and $\underline{v}_2 \leq v_2 \leq \overline{v}_2$. This indicates that $(v_1,\ v_2)$, the solution of (14), is also a solution of the main problem (6), since $K_1(v_1) = v_1$ and $K_2(v_2) = v_2$.

5. The Sub-Super Solution for $\overrightarrow{p}$-Kirchhoff Systems

The solution of Kirchhoff systems is investigated with different assumptions [17–20]. As in [6] for anisotropic $\overrightarrow{p}$-Laplacian system, the sub-super solution of problem (2.1) is now defined, and the existence theorem for the main system is subsequently presented (1).

Definition 2. *The pair of sub-super solutions* $(\underline{v}_1,\ \overline{v}_1), (\underline{v}_2,\ \overline{v}_2) \in (W^{1,\overrightarrow{p}}(\Phi) \cap L^\infty(\Phi))^2$ *is a solution of (5) if the following holds:*

1. $\underline{v}_j \leq \overline{v}_j$ in Φ and $\underline{v}_j \leq 0 \leq \overline{v}_j$ on $\partial\Phi$ for $j = 1, 2$.
2. $-\mathcal{R}_1(v,\ w)\Delta_{\overrightarrow{p}} \underline{v}_1 - F_1(y,\ \underline{v}_1,\ w) \leq 0 \leq -\mathcal{R}_1(v,\ w)\Delta_{\overrightarrow{p}} \overline{v}_1 - F_1(y,\ \overline{v}_1,\ w)$.
3. $-\mathcal{R}_2(v,\ w)\Delta_{\overrightarrow{p}} \underline{v}_2 - F_2(y,\ v,\ \underline{v}_2) \leq 0 \leq -\mathcal{R}_2(v,\ w)\Delta_{\overrightarrow{p}} \overline{v}_2 - F_2(y,\ v,\ \overline{v}_2)$.

For all purposes, the last two inequalities are regarded in the weak sense for $(v,\ w) \in [\underline{v}_1,\ \overline{v}_1] \times [\underline{v}_2,\ \overline{v}_2]$.

Setting $A_j(v_1,\ v_2) = M_j(\mathcal{R}_i(v_1,\ v_2))$ and $G_i(y,\ r,\ s,\ t) = \frac{F_j(y,r,s)}{t}$ in (6), we get the below result.

Theorem 2. *Assume the following for* $j = 1, 2$*:* $(H_1)' N_j$ *is invertible,*
$(H_2)' M_j(t) \geq m_i > 0$ *in* $[0, +\infty[$,
$(H_3)' F_j$ *is continuous in* $\overline{\Phi} \times \mathbf{R}^2$.
In the sense of Definition 2, if a pair of sub-super solutions exists of (5), then one can find a solution $(v_1,\ v_2)$ *of (5) in a manner that* $(v_1,\ v_2) \in [\underline{v}_1,\ \overline{v}_1] \times [\underline{v}_2,\ \overline{v}_2]$.

Proof. Through the continuity of M_j, on $[0, +\infty[$, and $\mathcal{R}_j, H_j$ is satisfied. In addition to this, due to $(H_2)'$ and $(H_3)'$, hypothesis D_2 of Theorem 3.1 is also verified. From these, we can say that (5) has a solution $(v_1,\ v_2) \in [\underline{v}_1,\ \overline{v}_1] \times [\underline{v}_2,\ \overline{v}_2]$. Consequently, the pair $(v_1,\ v_2)$ is also has a solution of (1.1). Furthermore, recall $(H_1)'$ for the definition of the operator $\mathcal{R}_j$. □

6. Applications

Consider the $\overrightarrow{p}$-Kirchhoff system

$$\begin{cases} -M_1\left(\sum_{i=1}^{N}\left\|\frac{\partial v_1}{\partial y_i}\right\|_{p_i}^{p_i}\right)\Delta_{\overrightarrow{p}}v_1 &= \lambda v_1^{q_1}+v_2^{q_2} \quad \text{in } \Phi,\\ -M_2\left(\sum_{i=1}^{N}\left\|\frac{\partial v_2}{\partial y_i}\right\|_{p_i}^{p_i}\right)\Delta_{\overrightarrow{p}}v_2 &= \mu v_2^{r_2}+v_1^{r_1} \quad \text{in } \Phi,\\ v_1=v_2 &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{15}$$

where $\lambda,\mu,q_j,r_j\in\mathbb{R}$,

Proposition 1. *Assume that $0<q_j,r_j<p_1-1$ for $j=1,2$ and suppose that $M_j:[0,+\infty)\to\mathbb{R}$ satisfies one of the conditions below:*

(i) $M_j(t)\geq m_0>0$ *in* $[0,+\infty[$, *for some* $m_0>0$.
(ii) N_j *is invertible.*
(iii) $M_j(t)\leq m_1$ *in* $[0,+\infty[$, *for some* $m_1>0$.

Then, there exist $\lambda^*,\mu^*>0$ such that system (15) has a positive solution for all $\lambda\geq\lambda^*$ and $\mu\geq\mu^*$.

Proof. Let $x\in W_0^{1,\overrightarrow{p}}(\Phi)\cap L^\infty(\Phi)$ be the unique solution of the problem

$$\begin{cases} -\Delta_{\overrightarrow{p}}x=\rho & \text{in } \Phi,\\ x=0 & \text{on } \partial\Phi, \end{cases} \tag{16}$$

where $\rho>0$ will be chosen later. For $\rho>1$ large enough we have by Lemma 1.4 that there is a constant $T>0$ that does not depend on ρ, such that

$$0<x(y)\leq T\rho^{\frac{1}{p_1-1}} \quad \text{in } \Phi. \tag{17}$$

Since T does not depend on ρ and $0<q_j,r_j<p_1-1$ for $j=1,2$ we can choose $\rho>0$ sufficiently large such that

$$\begin{cases} \lambda T^{q_1}\rho^{\frac{q_1}{p_1-1}}+T^{q_2}\rho^{\frac{q_2}{p_1-1}} & \leq \rho m_0,\\ \mu T^{r_2}\rho^{\frac{r_2}{p_1-1}}+T^{r_1}\rho^{\frac{r_1}{p_1-1}} & \leq \rho m_0, \end{cases} \tag{18}$$

where m_0 is given in (i).

Note that by (i) we have $\mathcal{R}_j(v,w)\geq m_0$, hence defining $\overline{v}_1=\overline{v}_2=x$ and using (17) and (18), we obtain

$$\begin{cases} \lambda\overline{v}_1^{q_1}+w^{q_2}\leq-\mathcal{R}_1(v,w)\Delta_{\overrightarrow{p}}\overline{v}_1 & \text{in } \Phi,\\ \mu\overline{v}_2^{r_2}+v^{r_2}\leq-\mathcal{R}_2(v,w)\Delta_{\overrightarrow{p}}\overline{v}_2 & \text{in } \Phi,\\ \overline{v}_1=\overline{v}_2\ =\ 0 & \text{on } \partial\Phi, \end{cases}$$

for all $(v,w)\in[0,\overline{v}_1]\times[0,\overline{v}_2]$.

Now, let $0<u_\beta\in W_0^{1,\overrightarrow{p}}(\Phi)$ be a solution of the problem

$$\begin{cases} -\Delta_{\overrightarrow{p}}u=u^\beta & \text{in } \Phi,\\ u=0 & \text{on } \partial\Phi, \end{cases} \tag{19}$$

where $\beta\in(0,p_1-1)$. This solution can be obtained by a simple application of the Ekeland variational principle to the C^1 functional $J:W_0^{1,\overrightarrow{p}}(\Phi)\to\mathbf{R}$ given by

$$J(v) := \int_{\Phi} \sum_{i=1}^{N} \frac{1}{p_i} \left\| \frac{\partial v}{\partial y_i} \right\|_{p_i}^{p_i} dy - \frac{1}{\beta+1} \int_{\Omega} (v^+)^{\beta+1} dy,$$

where $v^+ = \max\{v, 0\}$.

By Bootstrap regularity we obtain $u_\beta \in L^\infty(\Phi)$. Hence, we can take $\rho > 0$ in (17) such that $\|u_\beta\|^{\beta}_{L^\infty(\Phi)} \leq \rho$, then $-\Delta_{\overrightarrow{p}} x \leq -\Delta_{\overrightarrow{p}} u_\beta$ in Φ. By the comparison principle we have $x \leq u_\beta$ in Φ.

From (iii), we have $\mathcal{R}_j(v, w) \leq m_1$, for all $(v, w) \in [0, \overline{v}_1] \times [0, \overline{v}_2]$. Hence, choosing $\lambda^* = m_1$ and $\mu^* = m_1 > 0$ such that for each $\lambda \geq \lambda^*$ and $\mu \geq \mu^*$, we have

$$\mathcal{R}_1(v, w) \leq \lambda \ \text{ and } \mathcal{R}_2(v, w) \leq \mu \ \text{ for all } \ (v, w) \in [0, \overline{v}_1] \times [0, \overline{v}_2]. \tag{20}$$

Thus, defining $\underline{v}_1 = u_{q_1}$ and $\underline{w}_1 = u_{r_2}$, from (19) and (20) we obtain,

$$\begin{cases} -\mathcal{R}_1(v, w)\Delta_{\overrightarrow{p}} \underline{v}_1 \leq \lambda \underline{v}_1^{q_1} + w^{q_2} & \text{in } \Phi, \\ -\mathcal{R}_2(v, w)\Delta_{\overrightarrow{p}} \underline{v}_2 \leq \mu \underline{v}_2^{r_2} + v^{r_1} & \text{in } \Phi, \\ \underline{v}_1 = \underline{v}_2 \ = \ 0 & \text{on } \partial\Phi, \end{cases}$$

for all $(v, w) \in [0, \overline{v}_1] \times [0, \overline{v}_2]$. Moreover, $\underline{v}_j \leq \overline{v}_j, j = 1, 2$. $\square$

7. Conclusions

In this work, we examined the presence of weak positive solutions for a family of anisotropic nonlocal elliptic systems in bounded domains using the sub-super solutions method combined with Schauder's fixed point theorem. The sub-super solution for $\overrightarrow{p}$-Kirchhoff systems is introduced and the existence results are established in this work. In the next work, by using the same method, we will give the existence result of weak positive solutions of the following particular case of the right hand side:

$$\begin{cases} F_1(y,\ v_1,\ v_2) = a(y) f_1(v_1, v_2), \\ \\ F_2(y,\ v_1,\ v_2) = b(y) f_2(v_1, v_2), \end{cases}$$

where $a(y)$, $b(y)$ are changing their sign in Φ and $v_1 = v_2 \ = \ 0$ on $\partial\Phi$.

Author Contributions: Conceptualization, R.G., S.B. and A.A.; methodology, R.G., R.J.; formal analysis, R.J., S.B. and T.B.; investigation, A.A. and T.B.; resources, S.B.; writing—original draft preparation, R.G., G.C.G.d.S. and R.J.; writing—review and editing, S.B.; visualization, A.A.; supervision, S.B.; project administration, R.J. and G.C.G.d.S.; funding acquisition, S.B. All authors have read and agreed to the published version of the manuscript.

Funding: The researchers would like to thank the Deanship of Scientific Research at Qassim University for funding the publication of this project.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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