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Poincaré Map for Discontinuous Fractional Differential Equations

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Abstract: We work with a perturbed fractional differential equation with discontinuous right-hand sides where a discontinuity function crosses a discontinuity boundary transversally. The corresponding Poincaré map in a neighbourhood of a periodic orbit of an unperturbed equation is found. Then, bifurcations of periodic boundary solutions are analysed together with a concrete example.

Keywords: fractional differential equation; periodic boundary condition; discontinuous system; Poincaré map; bifurcations

MSC: 26A33; 34A08

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1. Introduction

Many applications of fractional calculus and for discontinuous systems can be found in biophysics, quantum mechanics, group theory, robotics or economics. The main reason for the applications of fractional calculus is that fractional-order models have more degrees of freedom and are more flexible than the integer-order ones. On the other hand, discontinuous systems model phenomena with non-smooth forces or dry frictions. As basic sources are books such as [1–6], various analytical and numerical methods are applied as they are demonstrated in [7–11]. Moreover, the theory and applications of fractional systems are rapidly developing, supported by many recent papers and Special Issues such as [12–16], involving stability, asymptotic periodicity, synchronization, memory effect and several other important and interesting behaviours of fractional models.

We work with a perturbed fractional differential equation with globally Lipschitz right-hand sides and which change their forms, i.e., they are discontinuous and transversally cross a discontinuity boundary. We are looking for a solution of a perturbed system with a periodic boundary condition in a neighbourhood of a periodic orbit of an unperturbed system. That means a solution of an unperturbed equation is periodic. We consider Caputo fractional derivatives of an order less than one. Our approach is based on the well-known method of a Poincaré map constructed near an investigated periodic orbit widely applied in dynamical systems as either smooth or non-smooth [17,18]. Since our studied problem has a integral-differential structure due of using together integer and Caputo fractional derivatives, the extension of Poincaré method is not elementary, but rather technical. We also believe that it will have a good reason to continue in this study.

The paper is organised as follows. In Section 2, we introduce our problem and prove the existence of a global solution. In Section 3, we show the existence of a Poincaré map in a neighbourhood of a periodic orbit of an unperturbed equation, which allows a bifurcation analysis of periodic boundary solutions given in Section 4. Section 5 demonstrates our method on a concrete example, and it discusses a possible scenario of qualitative behaviour of our example problem. Section 6 summarises our achievements and proposes our future task.

2. Existence of Solutions

Let us consider the following equation:

$$x'' + \mu_1 {}^C D_0^q x + g(x, x') = \mu_2 h(x), \tag{1}$$

where $x \in \mathbb{R}$, $q \in (0,1)$, $\mu_1, \mu_2 \in \mathbb{R}$ are small parameters, and function g is defined by

$$g(x, x') = \begin{cases} g_+(x) & \text{for } x' > 0, \\ g_-(x) & \text{for } x' < 0. \end{cases}$$

We suppose that functions $g_\pm$ and h are globally Lipschitz continuous functions on $\mathbb{R}$, and g transversally crosses the discontinuity boundary $x' = 0$, i.e., $g_-(x)g_+(x) > 0$.

We say that $x(t)$ is a solution of the Equation (1) if it is continuous, piecewise C^2 and satisfies (1). The Equation (1) is equivalent to the following system:

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -\mu_1 {}^C D_0^q x_1 - g(x_1, x_2) + \mu_2 h(x_1), \end{aligned} \tag{2}$$

where the second equation can be rewritten to the form

$$x_2' = -\mu_1 \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} x_2(s) ds - g(x_1, x_2) + \mu_2 h(x_1).$$

Initial conditions are as follows:

$$x_1(0) = u_0, \quad x_2(0) = v_0.$$

We work with $v_0 > 0$.

Theorem 1. *The solution $x = (x_1(u_0, v_0, \mu_1, \mu_2, t), x_2(u_0, v_0, \mu_1, \mu_2, t))$ of the system (2) with fixed $g(x_1, x_2) = g_\pm(x_1)$ exists on $\mathbb{R}_+ = [0, \infty)$, and it is expressed as*

$$\begin{aligned} x_1 &= u_0 + \int_0^t x_2(s) ds, \\ x_2 &= v_0 - \mu_1 \frac{1}{\Gamma(1-q)} \int_0^t \int_0^s (s-z)^{-q} x_2(z) dz ds + \int_0^t (-g_\pm(x_1(s)) + \mu_2 h(x_1(s))) ds. \end{aligned}$$

Proof. The existence of a solution is shown by using the Banach fixed point theorem on a Banach space $X = C(\mathbb{R}_+, \mathbb{R})^2$ with a norm

$$\|x\|_a = \max_{[\mathbb{R}_+} \|x(t)\| e^{-at},$$

where $a > 0$ is specified below and $\|x\| = \max\{|x_1|, |x_2|\}$ is a norm on $\mathbb{R}^2$. When $x \in C(\mathbb{R}_+, \mathbb{R})$, we use the same norm $\|x\|_a = \max_{[\mathbb{R}_+} |x(t)| e^{-at}$.

Consider $F = (F_1, F_2)$ as a functional $F(x_1, x_2; u_0, v_0, \mu_1, \mu_2)$, where

$$\begin{aligned} F_1(x_1, x_2; u_0, v_0, \mu_1, \mu_2)(t) &= u_0 + \int_0^t x_2(s) ds, \\ F_2(x_1, x_2; u_0, v_0, \mu_1, \mu_2)(t) &= v_0 - \mu_1 \frac{1}{\Gamma(1-q)} \int_0^t \int_0^s (s-z)^{-q} x_2(z) dz ds \\ &\quad + \int_0^t (-g_\pm(x_1(s)) + \mu_2 h(x_1(s))) ds. \end{aligned}$$

We show that there is a unique point x such that $x = F(x)$ for $x = (x_1, x_2)$. We now verify that F is contractive on X for a suitable $a > 0$:

$$\begin{aligned}
&|F_1(x_1,x_2;u_0,v_0,\mu_1,\mu_2)(t) - F_1(y_1,y_2;u_0,v_0,\mu_1,\mu_2)(t)| \\
&= \left| u_0 + \int_0^t x_2(s)ds - u_0 - \int_0^t y_2(s)ds \right| \le \int_0^t |x_2(s) - y_2(s)|ds \\
&\le \int_0^t \|x-y\|_a e^{as} ds \le \|x-y\|_a \frac{e^{at}}{a}
\end{aligned} \tag{3}$$

for $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Multiplying (3) by e^{-at}, we get:

$$\|F_1(x_1,x_2;u_0,v_0,\mu_1,\mu_2) - F_1(y_1,y_2;u_0,v_0,\mu_1,\mu_2)\|_a \le \frac{1}{a}\|x-y\|_a.$$

The mapping F_1 is contractive if

$$a > 1. \tag{4}$$

Next, we compute

$$\begin{aligned}
&|F_2(x_1,x_2;u_0,v_0,\mu_1,\mu_2)(t) - F_2(y_1,y_2;u_0,v_0,\mu_1,\mu_2)(t)| \\
&= \Bigg| \mu_1 \frac{1}{\Gamma(1-q)} \int_0^t \int_0^s (s-z)^{-q}(y_2(z) - x_2(z))dzds + \int_0^t (g_\pm(y_1(s)) - \mu_2 h(y_1(s)) \\
&- g_\pm(x_1(s)) + \mu_2 h(x_1(s)))ds \Bigg| \le \left| \mu_1 \frac{1}{\Gamma(1-q)} \int_0^t \int_0^s (s-z)^{-q}(y_2(z) - x_2(z))dzds \right| \\
&+ \left| \int_0^t (g_\pm(y_1(s)) - \mu_2 h(y_1(s)) - g_\pm(x_1(s)) + \mu_2 h(x_1(s)))ds \right|.
\end{aligned} \tag{5}$$

First, the integral in (5) is evaluated as follows:

$$\begin{aligned}
\left| \int_0^t \int_0^s (s-z)^{-q}(y_2(z) - x_2(z))dzds \right| &\le \int_0^t \int_0^s (s-z)^{-q}|y_2(z) - x_2(z)|dzds \\
&\le \int_0^t \int_0^s (s-z)^{-q}\|x-y\|_a e^{az} dzds
\end{aligned} \tag{6}$$

Now, we first use substitution $s - z = r$, then $ar = u$.

$$\begin{aligned}
\int_0^s (s-z)^{-q} e^{az} dz &\le \int_0^\infty r^{-q} e^{a(s-r)} dr = e^{as} \int_0^\infty r^{-q} e^{-ar} dr = \frac{e^{as}}{a} \int_0^\infty \left(\frac{u}{a}\right)^{-q} e^{-u} du = \\
&= \frac{e^{as}}{a^{1-q}} \int_0^\infty u^{-q} e^{-u} du = \frac{e^{as}}{a^{1-q}} \Gamma(1-q)
\end{aligned}$$

Thus, integral (6) is bounded by

$$\int_0^t \int_0^s (s-z)^{-q}\|x-y\|_a e^{az} dzds \le \frac{\|x-y\|_a}{a^{1-q}} \Gamma(1-q) \frac{e^{at}}{a}. \tag{7}$$

Now we use Lipschitz continuity of functions $g_\pm$ and h. Let L_G be a Lipschitz constant of function $G_\pm(z)$, where $G_\pm(z) = g_\pm(z) - \mu_2 h(z)$.

$$\begin{aligned}
\left| \int_0^t (G_\pm(y_1(s)) - G_\pm(x_1(s)))ds \right| &\le \int_0^t \left| G_\pm(y_1(s)) - G_\pm(x_1(s)) \right| ds \\
&\le \int_0^t L_G \|x-y\|_a e^{as} ds = L_G \|x-y\|_a \frac{e^{at}}{a}
\end{aligned} \tag{8}$$

Using (5), (7), and (8) can be bounded by:

$$|F_2(x_1, x_2; u_0, v_0, \mu_1, \mu_2)(t) - F_2(y_1, y_2; u_0, v_0, \mu_1, \mu_2)(t)| \le \left(L_G + \frac{1}{a^{1-q}\mu_1}\right) \|x - y\|_a \frac{e^{at}}{a},$$

which implies

$$\|F_2(x_1, x_2; u_0, v_0, \mu_1, \mu_2) - F_2(y_1, y_2; u_0, v_0, \mu_1, \mu_2)\|_a \le \left(L_G + \frac{1}{a^{1-q}\mu_1}\right) \frac{1}{a} \|x - y\|_a$$

This shows that F_2 is contractive if (4) holds along with

$$\left(L_G + \frac{1}{a^{1-q}\mu_1}\right) \frac{1}{a} < 1$$

a map $F = (F_1, F_2)$ has a unique fixed point. The proof is completed. □

3. Poincaré Map

The solution x_2 varies depending on $g_\pm$. If x_2 is non-negative, we work with g_+, otherwise we work with g_-. We will therefore look for times t_1 and t_2 where x_2 is equal to zero, where g_+ changes to g_- and vice versa.

Let us discuss a case, where parameters μ_1 and μ_2 are equal to 0. System (2) will then be system of ordinary differential equations.

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= - g_\pm(x_1) \end{aligned} \tag{9}$$

For T_0-periodic solution of system (9) is

$$x(0) = x(T_0)$$

fulfilled. The fixed T_0-periodic solution of system (9) will be denoted by x^0. The initial condition can be written as

$$x^0(0) = (u, v).$$

The solution x^0 can be written in the terms of x_1 and x_2:

$$\begin{aligned} x_1^0(t) &= x_1(x_1(0), x_2(0), 0, 0, t) = x_1(u, v, 0, 0, t) \\ x_2^0(t) &= x_2(x_1(0), x_2(0), 0, 0, t) = x_2(u, v, 0, 0, t). \end{aligned} \tag{10}$$

The function x_2^0 is of class C^1 in all its variables.

Let us suppose the existence of t_1 and t_2 that $0 < t_1 < t_2 < T_0$ and $x_2^0(t_1) = x_2^0(t_2) = 0$. At these points, function $g_\pm$ changes its form from g_+ to g_- and vice versa. We work with $v > 0$, therefore $t \in [0, t_1]$ will be $g_\pm = g_+$, which means that x_2 will be of the following form:

$$x_2(u, v, 0, 0, t) = v - \int_0^t g_+(x_1(s)) ds.$$

It can be seen that t_1 can be identified as a function $t_1 = t_1(u, v, 0, 0)$ that

$$x_2(u, v, 0, 0, t_1(u, v, 0, 0)) = 0.$$

In the case where parameters μ_1 and μ_2 are not equal to 0, we obtain for $t \in [0, t_1]$

$$x_2(u,v,\mu_1,\mu_2,t) = v - \frac{\mu_1}{\Gamma(1-q)} \int_0^t \int_0^s (s-z)^{-q} x_2(z) dz ds \quad (11)$$
$$+\mu_2 \int_0^t h(x_1(s)) ds - \int_0^t g_+(x_1(s)) ds.$$

Similarly, for t_2

$$x_2(u,v,0,0,t) = v - \int_0^{t_1} g_+(x_1(s)) ds - \int_{t_1}^t g_-(x_1(s)) ds,$$

where t_2 can be seen as a function $t_2 = t_2(u,v,0,0)$, that

$$x_2(u,v,0,0,t_2(u,v,0,0)) = 0.$$

If μ_1 and μ_2 are not equal to 0, we obtain for $t \in [t_1,t_2]$

$$x_2(u,v,\mu_1,\mu_2,t) = v - \frac{\mu_1}{\Gamma(1-q)} \int_0^{t_1} \int_0^s (s-z)^{-q} x_2(z) dz ds \quad (12)$$
$$-\frac{\mu_1}{\Gamma(1-q)} \int_{t_1}^t \int_0^{t_1} (s-z)^{-q} x_2(z) dz ds - \frac{\mu_1}{\Gamma(1-q)} \int_{t_1}^t \int_{t_1}^s (s-z)^{-q} x_2(z) dz ds$$
$$+\mu_2 \int_0^{t_1} h(x_1(s)) ds + \mu_2 \int_{t_1}^t h(x_1(s)) ds - \int_0^{t_1} g_+(x_1(s)) ds - \int_{t_1}^t g_-(x_1(s)) ds.$$

Similarly, for interval $[t_2,T]$:

$$x_2(u,v,\mu_1,\mu_2,t) = v - \frac{\mu_1}{\Gamma(1-q)} \int_0^{t_2} \int_0^s (s-z)^{-q} x_2(z) dz ds \quad (13)$$
$$-\frac{\mu_1}{\Gamma(1-q)} \int_{t_2}^t \int_0^{t_2} (s-z)^{-q} x_2(z) dz ds - \frac{\mu_1}{\Gamma(1-q)} \int_{t_2}^t \int_{t_2}^s (s-z)^{-q} x_2(z) dz ds$$
$$+\mu_2 \int_0^{t_1} h(x_1(s)) ds + \mu_2 \int_{t_1}^{t_2} h(x_1(s)) ds + \mu_2 \int_{t_2}^t h(x_1(s)) ds$$
$$- \int_0^{t_1} g_+(x_1(s)) ds - \int_{t_1}^{t_2} g_-(x_1(s)) - \int_{t_2}^t g_+(x_1(s)) ds.$$

First, let us look at the solution x_2 for μ_1=μ_2=0. The solution can be written in the following way:

$$x_2(t) = \begin{cases} x_{2,1}(t) & \text{for } t \in [0,t_1], \\ x_{2,2}(t) & \text{for } t \in [t_1,t_2], \\ x_{2,3}(t) & \text{for } t \in [t_2,T]. \end{cases} \quad (14)$$

It can be easily seen, that

$$x_{2,1}(t_1) = x_{2,2}(t_1),$$
$$x_{2,2}(t_2) = x_{2,3}(t_2),$$
$$x_{2,3}(T) = x_{2,1}(0),$$

where the last equation is result from T-periodicity.

The solution for μ_1 and μ_2 not equal to 0 can be written as $x_{2,1}$, $x_{2,2}$ and $x_{2,3}$ on corresponding intervals $[0,t_1]$, $[t_1,t_2]$ and $[t_2,T]$, where these solutions are from (11)–(13).

It can be seen, that derivatives of function x_2 with respect to t are non-zero for $t = t_1$ and $t = t_2$.

The solution $x_1(t)$ can be written similarly:

$$x_1(t) = \begin{cases} x_{1,1}(t) & \text{for } t \in [0, t_1], \\ x_{1,2}(t) & \text{for } t \in [t_1, t_2], \\ x_{1,3}(t) & \text{for } t \in [t_2, T]. \end{cases} \tag{15}$$

$$\begin{aligned} x_{1,1}(t_1) &= x_{1,2}(t_1), \\ x_{1,2}(t_2) &= x_{1,3}(t_2), \\ x_{1,3}(T) &= x_{1,1}(0). \end{aligned}$$

Let us recall $x^0(t)$, a solution to an unperturbed Equation (1), i.e., for $\mu_1 = \mu_2 = 0$. $x^0(t)$ therefore indicates solutions (14) and (15). Now, we will look for a T-periodic boundary solution of system (2) near $x^0(t)$, which means that in system (2) we take μ_1 and μ_2 as small, close to 0. To reduce the number of variables, take μ_1 and μ_2 in the form $\varepsilon\hat{\mu}_1$ and $\varepsilon\hat{\mu}_2$, while ε will be taken small close to 0, and $\hat{\mu}_1$ and $\hat{\mu}_2$ can be any. For simplicity, we will rename the variables $\hat{\mu}_1$ and $\hat{\mu}_2$ to μ_1 and μ_2.

The system (2) can be written in the form

$$x' = F_{\pm}(x) + \varepsilon G(x), \tag{16}$$

that $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $F_{\pm}(x) = \begin{pmatrix} x_2 \\ -g_{\pm}(x_1) \end{pmatrix}$ and $\varepsilon G(x) = \begin{pmatrix} 0 \\ \varepsilon(-\hat{\mu}_1 {}^C D_0^q x_2 + \hat{\mu}_2 h(x_1)) \end{pmatrix}$. Now we define Poincaré map. We choose $u_0 = 0$ and $v_0 = \xi > 0$.

t_3 is, similarly to t_1 and t_2, function of variables ξ and ϵ, that defines as $x_1(t_3) = 0$. t_3 determines the time, when solution meets its initial value, which can be seen in Figure 1 The Poincaré map is defined as

$$\tilde{P}(\xi, \varepsilon) = x_+(x_-(t_2), t_3, \varepsilon), \tag{17}$$

where $x_+(\xi, \varepsilon, t)$ denotes the solution of (16) with $F_+(x)$ right-hand side and $x_-(\xi, \varepsilon, t)$ the solution of (16) with $F_-(x)$ right-hand side. To find periodic boundary solution we solve the equation

$$\xi - \tilde{P}(\xi, \varepsilon) = 0. \tag{18}$$

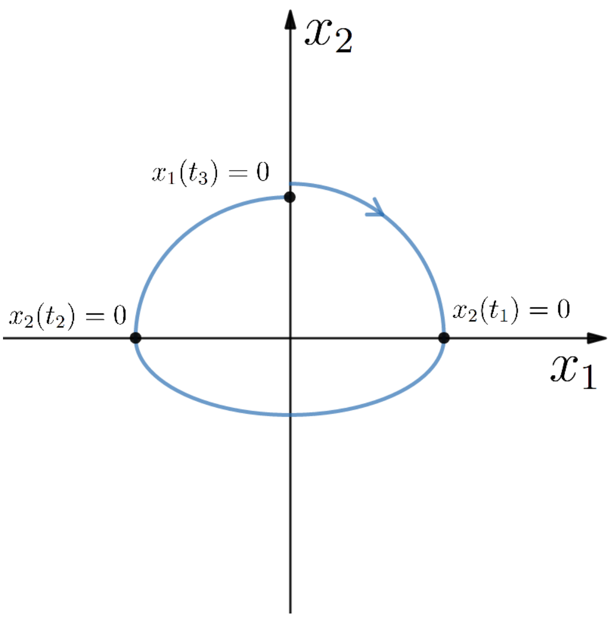

Figure 1. Graphical representation of the behaviour of the solution at times t_1, t_2 and t_3.

Differentiating the Equation (11) with respect to ξ at the point $(\xi, \varepsilon) = (x_0, 0)$ we obtain

$$x_2(x_0,0,t)_\xi = 1 - \int_0^t g'_+(x_1(s))x_{1\xi}(s)(s)ds. \tag{19}$$

In this case, we have $t \in [0, t_1]$, which gives us g_+ on the right-hand side. Differentiating the Equation (11) with respect to ε at the point $(\xi, \varepsilon) = (x_0, 0)$, we have

$$\begin{aligned} x_2(x_0,0,t)_\varepsilon = &- \frac{\hat{\mu}_1}{\Gamma(1-q)} \int_0^t \int_0^s (s-z)^{-q} x_2(z)dzds + \\ &+ \hat{\mu}_2 \int_0^t h(x_1(s))ds - \int_0^t g'_+(x_1)x_{1\varepsilon}(s)ds. \end{aligned} \tag{20}$$

Denote the first point, where g_+ changes to g_- as $\hat{x}_1$. $t_{1\xi}(x_0, 0)$ can be obtained by differentiating (11) at $t = t_1(\xi, \varepsilon)$ with respect to ξ as

$$t_{1\xi}(x_0,0) = \frac{1}{g_+(\hat{x}_1)} \left(1 + \int_0^{t_1} g'_+(x_1(s))x_{1\xi}(s)ds\right). \tag{21}$$

Similarly, by differentiating with respect to ε, we can express $t_{1\varepsilon}$ as

$$\begin{aligned} t_{1\varepsilon}(x_0,0) = &\frac{1}{g_+(\hat{x}_1)} \Bigg(- \frac{\hat{\mu}_1}{\Gamma(1-q)} \int_0^{t_1} \int_0^s (s-z)^{-q} x_2(z)dzds + \\ &+ \hat{\mu}_2 \int_0^{t_1} h(x_1(s))ds - \int_0^{t_1} g'_+(x_1)x_{1\varepsilon}(s)ds \Bigg). \end{aligned} \tag{22}$$

Now, we take time $t \in [t_1, t_2]$, which gives us g_- on the right-hand side. Differentiating (12) with respect to ξ at the point $(\xi, \varepsilon) = (\hat{x}_1, 0)$, we obtain

$$\begin{aligned} x_2(\hat{x}_1,0,t)_\xi = &1 - \Bigg(t_{1\xi} g_+(\hat{x}_1) + \int_0^{t_1} g'_+(x_1(s))x_{1\xi}(s)ds - t_{1\xi} g_-(\hat{x}_1) + \\ &+ \int_{t_1}^t g'_-(x_1(s))x_{1\xi}(s)ds \Bigg). \end{aligned} \tag{23}$$

Differentiating (12) with respect to and ε at the point $(\xi, \varepsilon) = (\hat{x}_1, 0)$, we obtain

$$\begin{aligned} x_2(\hat{x}_1,0,t)_\varepsilon = &- \frac{\hat{\mu}_1}{\Gamma(1-q)} \Bigg(\int_0^{t_1} \int_0^s (s-z)^{-q} x_2(z)dzds \\ &+ \int_{t_1}^t \int_0^{t_1} (s-z)^{-q} x_2(z)dzds + \int_{t_1}^t \int_{t_1}^s (s-z)^{-q} x_2(z)dzds \Bigg) \\ &+ \hat{\mu}_2 \int_0^t h(x_1(s))ds - \Bigg(g_+(\hat{x}_1)t_{1\varepsilon} + \int_0^{t_1} g'_+(x_1)x_{1\varepsilon}(s)ds \\ &- g_-(\hat{x}_1)t_{1\varepsilon} + \int_{t_1}^t g'_-(x_1)x_{1\varepsilon}(s)ds \Bigg). \end{aligned} \tag{24}$$

For $t = t_2$ function g_- changes to g_+. Denote by $\hat{x}_2$ the point, when the change happens. Differentiating (12) at $t = t_2(\xi, \epsilon)$ with respect to ξ and ε, respectively, we have:

$$\begin{aligned} t_{2\xi}(x_0,0) = &\frac{1}{g_-(\hat{x}_2)} \Bigg(1 - \int_0^{t_1} g'_+(x_1(s))x_{1\xi}(s)ds - \int_{t_1}^{t_2} g'_-(x_1(s))x_{1\xi}(s)ds \\ &- \frac{1}{g_+(\hat{x}_1)} \left(1 + \int_0^{t_1} g'_+(x_1(s))x_{1\xi}(s)ds\right) \Big(g_+(x_1) - g_-(x_1)\Big) \Bigg), \end{aligned} \tag{25}$$

$$
\begin{aligned}
t_{2\varepsilon}(x_0,0) =& \frac{1}{g_-(\hat{x}_2)}\Bigg(\frac{\hat{\mu}_1}{\Gamma(1-q)}\bigg(\int_0^{t_1}\int_0^s (s-z)^{-q}x_2(z)dzds \\
&+\int_{t_1}^{t_2}\int_0^{t_1}(s-z)^{-q}x_2(z)dzds+\int_{t_1}^{t_2}\int_{t_1}^{s}(s-z)^{-q}x_2(z)dzds\bigg) \\
&-\hat{\mu}_2\int_0^{t_2}h(x_1(s))ds+\int_0^{t_1}g'_+(x_1)x_{1\varepsilon}(s)ds+\int_{t_1}^{t_2}g'_-(x_1)x_{1\varepsilon}(s)ds \\
&+\Big(g_+(x_1)-g_-(x_1)\Big)\frac{1}{g_+(\hat{x}_1)}\bigg(-\frac{\hat{\mu}_1}{\Gamma(1-q)}\int_0^{t_1}\int_0^s(s-z)^{-q}x_2(z)dzds \\
&+\hat{\mu}_2\int_0^{t_1}h(x_1(s))ds-\int_0^{t_1}g'_+(x_1)x_{1\varepsilon}(s)ds\bigg)\Bigg).
\end{aligned}
\tag{26}
$$

Now, take $t \in [t_2, t_3]$, and on the right-hand side of the equation is function g_+. Differentiating (13) with respect to ξ and ε at the point $(\xi,\varepsilon) = (\hat{x}_2, 0)$, respectively, we have:

$$
\begin{aligned}
x_2(\hat{x}_2,0,t)_\xi = 1-&\bigg(t_{1\xi}g_+(x_1)+\int_0^{t_1}g'_+(x_1(s))x_{1\xi}(s)ds+t_{2\xi}g_-(x_1)-t_{1\xi}g_-(x_1) \\
&+\int_{t_1}^{t_2}g'_-(x_1(s))x_{1\xi}(s)ds-t_{2\xi}g_+(x_1)+\int_{t_2}^{t}g'_+(x_1(s))x_{1\xi}(s)ds\bigg),
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
x_2(\hat{x}_2,0,t)_\varepsilon =-&\bigg(\frac{\hat{\mu}_1}{\Gamma(1-q)}\bigg(\int_0^{t_2}\int_0^s(s-z)^{-q}x_2(z)dzds \\
&+\int_{t_2}^{t}\int_0^{t_2}(s-z)^{-q}x_2(z)dzds+\int_{t_2}^{t}\int_{t_2}^{s}(s-z)^{-q}x_2(z)dzds\bigg) \\
&+\bigg(\hat{\mu}_2\int_0^t h(x_1(s))ds\bigg) \\
&-\bigg(g_+(x_1(t_1))t_{1\varepsilon}+\int_0^{t_1}g'_+(x_1)x_{1\varepsilon}(s)ds+g_-(x_1)t_{2\varepsilon}-g_-(x_1)t_{1\varepsilon} \\
&+\int_{t_1}^{t_2}g'_-(x_1)x_{1\varepsilon}(s)ds-t_{2\varepsilon}g_+(x_1)+\int_{t_2}^{t}g'_+(x_1)x_{1\varepsilon}(s)ds\bigg).
\end{aligned}
\tag{28}
$$

4. Bifurcation Analysis

Poincaré map fulfils:

$$
P(x_0,\varepsilon) = \begin{pmatrix} 0 \\ Q(\xi,\varepsilon)\end{pmatrix},
\tag{29}
$$

where $x_0 = \begin{pmatrix} 0 \\ \xi \end{pmatrix}$. Denote by $\overline{\xi}$ the point, for which the periodic orbit x^0 exists for $\varepsilon = 0$. Then, we consider

$$
\xi \in (\overline{\xi}-\delta, \overline{\xi}+\delta), \quad \xi \mapsto Q(\xi,\varepsilon),
$$

for $\delta > 0$. In our particular case, the function Q equals $x_{2,3}(t_3)$, where t_3 is the point, where $x_{1,1}(0) = x_{1,3}(t_3) = 0$.

Let $\varepsilon = 0$. It is easy to see that $\overline{\xi}$ solves

$$
Q(\xi,0)-\xi = 0,
$$

hence

$$
Q(\overline{\xi},0)-\overline{\xi} = 0.
$$

By differentiating the equation with respect to ξ at the point $\xi = \overline{\xi}$, two options can occur:

$$Q_{\xi}(\overline{\xi}, 0) \neq 1$$

or

$$Q_{\xi}(\overline{\xi}, 0) = 1.$$

In the first option, we obtain a isolated periodic boundary orbit, which means that the equation has a unique solution $\xi = \Psi(\varepsilon)$ that fulfils $\overline{\xi} = \Psi(0)$.

In the second option, the bifurcation may occur. Let us look at the special, degenerated, case of the second option:

$$Q(\xi, 0) \equiv \xi. \tag{30}$$

In this case, we obtain several solutions, the one-parameter system of periodic solutions of the unperturbed system. Assume that (30) is fulfilled. Using Hadamard's Lemma express $Q(\xi, \varepsilon)$ in the form:

$$Q(\xi,\varepsilon) = Q(\xi,0) + Q(\xi,\varepsilon) - Q(\xi,0) = Q(\xi,0) + \varepsilon Q_1(\xi,\varepsilon) = \xi + \varepsilon Q_1(\xi,\varepsilon)$$

The equation

$$\varepsilon Q_1(\xi, \varepsilon) = 0$$

is obtained. After multiplying the equation by $\frac{1}{\varepsilon}$ and giving $\varepsilon = 0$, we have:

$$Q_1(\xi, 0) = 0. \tag{31}$$

According to the roots of Equation (31), the existence of periodic boundary orbits can be discussed. If Equation (31) has a unique non-degenerated solution $\xi = \overline{\xi}$ for $\xi \in (\xi - \delta, \xi + \delta)$, we obtain a local unique periodic boundary solution.

If the equation has no solution, a periodic boundary orbit does not exist. If the equation has several solutions, we obtain several periodic boundary orbits. In our particular case of Equation (1), $Q_1(\xi, \varepsilon)$ can be found by differentiating (13) with respect to ε. We use the notation $\mu_1 = \varepsilon\hat{\mu}_1$ and $\mu_2 = \varepsilon\hat{\mu}_2$, as introduced earlier in this chapter. For $\varepsilon = 0$:

$$\begin{aligned} Q_1(\xi,0) = &-\frac{\hat{\mu}_1}{\Gamma(1-q)}\left(\int_0^{t_2}\int_0^{s}(s-z)^{-q}x_2(z)dzds\right. \\ &+\left.\int_{t_2}^{t_3}\int_0^{t_2}(s-z)^{-q}x_2(z)dzds + \int_{t_2}^{t_3}\int_{t_2}^{s}(s-z)^{-q}x_2(z)dzds\right) \\ &+\left(\hat{\mu}_2\int_0^{t_3}h(x_1(s))ds\right) \\ &-\left(g_+(\hat{x}_1)t_{1\varepsilon} + \int_0^{t_1}g_+'(x_1(s))x_{1\varepsilon}(s)ds + g_-(\hat{x}_2)t_{2\varepsilon} - g_-(\hat{x}_1)t_{1\varepsilon}\right. \\ &+\left.\int_{t_1}^{t_2}g_-'(x_1(s))x_{1\varepsilon}(s)ds + t_{3\varepsilon}g_+(\hat{x}_3) - t_{2\varepsilon}g_+(\hat{x}_2) + \int_{t_2}^{t_3}g_+'(x_1(s))x_{1\varepsilon}(s)ds\right). \end{aligned} \tag{32}$$

Since $x_1(t_3) = 0$, $t_{3\varepsilon}$ can be expressed by differentiating x_1 with respect to ε.

$$t_{3\varepsilon} = -\frac{1}{x_2(t_3)}\int_0^{t_3}x_{2\varepsilon}(s)ds. \tag{33}$$

Using (22), (26) and (33) we can express $Q_1(\xi, \varepsilon)$ for $\varepsilon = 0$:

$$
\begin{aligned}
Q_1(\xi,0) = & -\frac{\hat{\mu}_1}{\Gamma(1-q)}\bigg(\int_0^{t_2}\int_0^{s}(s-z)^{-q}x_2(z)dzds \\
& + \int_{t_2}^{t_3}\int_0^{t_2}(s-z)^{-q}x_2(z)dzds + \int_{t_2}^{t_3}\int_{t_2}^{s}(s-z)^{-q}x_2(z)dzds\bigg) \\
& + \bigg(\hat{\mu}_2\int_0^{t_3}h(x_1(s))ds\bigg) \\
& - \bigg(\Big(g_+(\hat{x}_1)-g_-(\hat{x}_1)\Big)\frac{1}{g_+(\hat{x}_1)}\bigg(-\frac{\hat{\mu}_1}{\Gamma(1-q)}\int_0^{t_1}\int_0^{s}(s-z)^{-q}x_2(z)dzds \\
& + \hat{\mu}_2\int_0^{t_1}h(x_1(s))ds - \int_0^{t_1}g_+'(x_1(s))x_{1\varepsilon}(s)ds\bigg) + \\
& + \Big(g_-(\hat{x}_2)-g_+(\hat{x}_2)\Big)\frac{1}{g_-(\hat{x}_2)}\bigg(\frac{\hat{\mu}_1}{\Gamma(1-q)}\bigg(\int_0^{t_1}\int_0^{s}(s-z)^{-q}x_2(z)dzds \\
& + \int_{t_1}^{t_2}\int_0^{t_1}(s-z)^{-q}x_2(z)dzds + \int_{t_1}^{t_2}\int_{t_1}^{s}(s-z)^{-q}x_2(z)dzds\bigg) \\
& - \hat{\mu}_2\int_0^{t_2}h(x_1(s))ds + \int_0^{t_1}g_+'(x_1(s))x_{1\varepsilon}(s)ds + \int_{t_1}^{t_2}g_-'(x_1(s))x_{1\varepsilon}(s)ds \\
& + \Big(g_+(x_1)-g_-(x_1)\Big)\frac{1}{g_+(\hat{x}_1)}\bigg(-\frac{\hat{\mu}_1}{\Gamma(1-q)}\int_0^{t_1}\int_0^{s}(s-z)^{-q}x_2(z)dzds \\
& + \hat{\mu}_2\int_0^{t_1}h(x_1(s))ds - \int_0^{t_1}g_+'(x_1(s))x_{1\varepsilon}(s)ds\bigg)\bigg)\bigg) - g_+(\hat{x}_3)\frac{1}{x_2(t_3)}\int_0^{t_3}x_{2\varepsilon}(s)ds \\
& + \int_0^{t_1}g_+'(x_1(s))x_{1\varepsilon}(s)ds + \int_{t_1}^{t_2}g_-'(x_1(s))x_{1\varepsilon}(s)ds + \int_{t_2}^{t_3}g_+'(x_1(s))x_{1\varepsilon}(s)ds\bigg).
\end{aligned} \tag{34}
$$

5. Example of the Specific Equation

In further research, we will solve the equation, where the degenerated case (30) occurs. Using methods described in a previous chapter, we will solve specific equations in the form of (1). Let us solve an Equation (1) in unperturbed form.

$$x'' + g_\pm(x) = 0. \tag{35}$$

Equation (35) is equivalent to the system

$$
\begin{aligned}
x_1' &= x_2 \\
x_2' &= -g_\pm(x_1).
\end{aligned} \tag{36}
$$

Let the function $g_\pm$ be such that there exists a periodic solution.

$$g_\pm(x_1) = \begin{cases} x_1, & x_2 \geq 0 \\ 4x_1, & x_2 < 0 \end{cases} \tag{37}$$

It is obvious that the periodic solution of system (36) exists.

Figure 2 shows that the solution of the system (36) is periodic. For x_2 positive $g_\pm = x_1$, therefore, the solutions above the x_1 axis are circles. The solutions under the x_1 axis are ellipses, and the reason is that $g_\pm = 4x_1$. We can look for the times t_1 and t_2, in which the function $g_\pm$ changes from g_+ to g_- and vice versa. The solution of the system (36) for $t \in [0, t_1]$ equals

$$
\begin{aligned}
x_1(t) &= c_1\cos(t) + c_2\sin(t), \\
x_2(t) &= c_2\cos(t) - c_1\sin(t).
\end{aligned}
$$

t_1 is the first time in which the solution $x_2(t_1) = 0$. Because the time t_1 changes for different values of c_1 and c_2, we take the initial conditions as we used in the previous chapter, namely $x(0) = \begin{pmatrix} 0 \\ v_0 \end{pmatrix}$. In this case, the solution will be in the form of:

$$\begin{aligned} x_1(t) =& v_0 \sin(t), \\ x_2(t) =& v_0 \cos(t). \end{aligned} \tag{38}$$

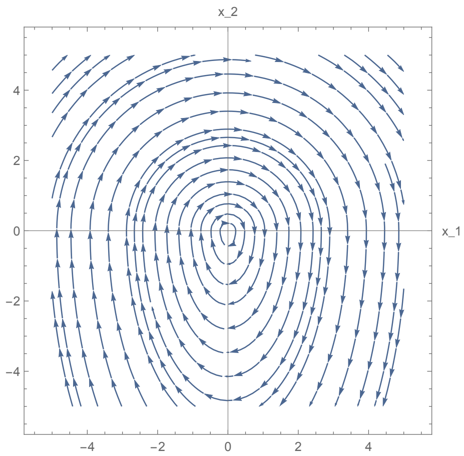

Figure 2. Local phase portrait of system (36).

Time t_1 will be the time when $x_2(t_1) = 0$. It is obvious that $t_1 = \frac{\pi}{2}$. The system has on the interval $[\frac{\pi}{2}, t_2]$ initial conditions $x_1(\frac{\pi}{2}) = v_0$ a $x_2(\frac{\pi}{2}) = 0$. Therefore, the solution is in the form

$$\begin{aligned} x_1(t) =& - v_0 \cos(2t), \\ x_2(t) =& 2v_0 \sin(2t). \end{aligned} \tag{39}$$

Hence, the time t_2, in which the function g_- changes to g_+, is $t_2 = \pi$. Let us look for a T-periodic solution, which means, that the solutions $x_1(T) = x_1(0)$ and $x_2(T) = x_2(0)$. The solution for $t \in [\pi, T]$ equals

$$\begin{aligned} x_1(t) =& v_0 \cos(t), \\ x_2(t) =& - v_0 \sin(t). \end{aligned} \tag{40}$$

Therefore, $T = \frac{3\pi}{2}$. Using the method described in the previous chapter, it is possible to find a Poincaré map for a perturbed equation

$$x'' + g_\pm(x) + \varepsilon(\mu_1 {}^C D_0^q x - \mu_2 h(x)) = 0, \tag{41}$$

where $g_\pm(x)$ is determined by the relation (37) and $h(x)$ can be chosen as x, so $h(x) = x$. Solution x can be written as

$$\begin{aligned} x_1(t) &= \int_0^t x_2(s) ds \\ x_2(t) &= \xi - \varepsilon\mu_1 \frac{1}{\Gamma(1-q)} \int_0^t \int_0^s (s-z)^{-q} x_2(z) dz ds + \int_0^t (-g_\pm(x_1(s)) + \varepsilon\mu_2 x_1(s) ds) \end{aligned}$$

In this case, we can express Q_1 as

$$
\begin{aligned}
Q_1(\xi,0) = &-\frac{\hat{\mu}_1}{\Gamma(1-q)}\Bigg(\int_0^{t_2}\int_0^{s}(s-z)^{-q}x_2(z)dzds \\
&+\int_{t_2}^{t_3}\int_0^{t_2}(s-z)^{-q}x_2(z)dzds+\int_{t_2}^{t_3}\int_{t_2}^{s}(s-z)^{-q}x_2(z)dzds\Bigg) \\
&+\left(\hat{\mu}_2\int_0^{t_3}x_1(s)ds\right) \\
&-\Bigg(\left(\hat{x}_1-4\hat{x}_1\right)\frac{1}{\hat{x}_1}\Bigg(-\frac{\hat{\mu}_1}{\Gamma(1-q)}\int_0^{t_1}\int_0^{s}(s-z)^{-q}x_2(z)dzds \\
&+\hat{\mu}_2\int_0^{t_1}x_1(s)ds-\int_0^{t_1}4x_{1\varepsilon}(s)ds\Bigg)+ \\
&+\left(4\hat{x}_2-\hat{x}_2\right)\frac{1}{4\hat{x}_2}\Bigg(\frac{\hat{\mu}_1}{\Gamma(1-q)}\Bigg(\int_0^{t_1}\int_0^{s}(s-z)^{-q}x_2(z)dzds \\
&+\int_{t_1}^{t_2}\int_0^{t_1}(s-z)^{-q}x_2(z)dzds+\int_{t_1}^{t_2}\int_{t_1}^{s}(s-z)^{-q}x_2(z)dzds\Bigg) \\
&-\hat{\mu}_2\int_0^{t_2}x_1(s)ds+\int_0^{t_1}x_{1\varepsilon}(s)ds+\int_{t_1}^{t_2}4x_{1\varepsilon}(s)ds \\
&+\left(\hat{x}_1-4\hat{x}_1\right)\frac{1}{\hat{x}_1}\Bigg(-\frac{\hat{\mu}_1}{\Gamma(1-q)}\int_0^{t_1}\int_0^{s}(s-z)^{-q}x_2(z)dzds \\
&+\hat{\mu}_2\int_0^{t_1}x_1(s)ds-\int_0^{t_1}x_{1\varepsilon}(s)ds\Bigg)\Bigg)-\hat{x}_3\frac{1}{\hat{x}_3}\int_0^{t_3}x_{2\varepsilon}(s)ds \\
&+\int_0^{t_1}x_{1\varepsilon}(s)ds+\int_{t_1}^{t_2}4x_{1\varepsilon}(s)ds+\int_{t_2}^{t_3}x_{1\varepsilon}(s)ds\Bigg).
\end{aligned}
\tag{42}
$$

where $\hat{x}_1$, $\hat{x}_2$ and $\hat{x}_3$ are the first, second and third intersection points, respectively.

$$
\begin{aligned}
\hat{x}_1 &= \begin{pmatrix} x_1(t_1) \\ x_2(t_1) \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}, a \in \mathbb{R}. \\
\hat{x}_2 &= \begin{pmatrix} x_1(t_2) \\ x_2(t_2) \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}, b \in \mathbb{R}. \\
\hat{x}_3 &= \begin{pmatrix} x_1(t_3) \\ x_2(t_3) \end{pmatrix} = \begin{pmatrix} 0 \\ c \end{pmatrix}, c \in \mathbb{R}.
\end{aligned}
$$

Times t_1, t_2 and t_3 satisfy following equations:

$$
\begin{aligned}
0 = x_2(t_1) =&\xi-\varepsilon\mu_1\frac{1}{\Gamma(1-q)}\int_0^{t_1}\int_0^{s}(s-z)^{-q}x_2(z)dzds+(\varepsilon\mu_2-1)\int_0^{t_1}x_1(s)ds \\
0 = x_2(t_2) =&\xi-\frac{\mu_1}{\Gamma(1-q)}\int_0^{t_1}\int_0^{s}(s-z)^{-q}x_2(z)dzds \\
&-\frac{\mu_1}{\Gamma(1-q)}\int_{t_1}^{t_2}\int_0^{t_1}(s-z)^{-q}x_2(z)dzds-\frac{\mu_1}{\Gamma(1-q)}\int_{t_1}^{t_2}\int_{t_1}^{s}(s-z)^{-q}x_2(z)dzds \\
&+(\varepsilon\mu_2-1)\int_0^{t_1}x_1(s)ds+(\varepsilon\mu_2-4)\int_{t_1}^{t_2}x_1(s)ds \\
0 = x_1(t_3) =&\int_0^{t_3}x_2(s)ds
\end{aligned}
$$

To use (34) for our specific equation, we have to find functions $x_{1\varepsilon}(t)$ and $x_{2\varepsilon}$. Function $x_{2\varepsilon}$ was defined (20), (24) and (28) on intervals $[0,t_1]$,$[t_1,t_2]$ and $[t_2,t_3]$, respectively. Knowing that $x_1' = x_2$ (2) (symbol $'$ symbolizes the time derivative), we can formulate ODE system to find functions $x_{1\varepsilon}(t)$ and $x_{2\varepsilon}$ on each interval using functions (38), (39) and (40) as $x_1(t)$ and $x_2(t)$.

On interval $[0,t_1] = [0,\frac{\pi}{2}]$, we have

$$x'_{1\varepsilon} = x_{2\varepsilon} \tag{43}$$
$$x'_{2\varepsilon} = -\frac{\mu_1}{\Gamma(1-q)}\int_0^t (t-z)^{-q}\xi\cos z dz + \mu_2\xi\sin t - x_{1\varepsilon}, \tag{44}$$

with initial conditions $x_{1\varepsilon}(0) = x_{2\varepsilon}(0) = 0$. which, using a variation of parameters method, give the solution in the form:

$$\begin{pmatrix} x_{1\varepsilon} \\ x_{2\varepsilon} \end{pmatrix} = e^{At}\int_0^t e^{-As} f(s)ds, \tag{45}$$

where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $f(t) = \begin{pmatrix} 0 \\ -\frac{\mu_1}{\Gamma(1-q)}\int_0^t (t-z)^{-q}\xi\cos z dz + \mu_2\xi\sin t \end{pmatrix}$. Similarly, the ODE system on the second interval $[\frac{\pi}{2},\pi]$:

$$x'_{1\varepsilon} = x_{2\varepsilon} \tag{46}$$
$$x'_{2\varepsilon} = -\frac{\mu_1}{\Gamma(1-q)}\left(\int_0^{\pi/2} (t-z)^{-q}\xi\cos z dz + \int_{\pi/2}^t (t-z)^{-q} 2\xi\sin 2z dz\right) - \mu_2\xi\cos 2t + 4x_{1\varepsilon}. \tag{47}$$

Its initial conditions can be calculated from Equation (45) at $t = \frac{\pi}{2}$. The solution on interval $[\frac{\pi}{2},\pi]$ using a variation of parameters method is in the form:

$$\begin{pmatrix} x_{1\varepsilon} \\ x_{2\varepsilon} \end{pmatrix} = e^{At}K + e^{At}\int_0^t e^{-As} f(s)ds, \tag{48}$$

where $A = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$, $K = e^{A\frac{\pi}{2}}\begin{pmatrix} x_{1\varepsilon}(\frac{\pi}{2}) \\ x_{2\varepsilon}(\frac{\pi}{2}) \end{pmatrix}$

and $f(t) = \begin{pmatrix} 0 \\ -\frac{\mu_1}{\Gamma(1-q)}\left(\int_0^{\pi/2} (t-z)^{-q}\xi\cos z dz + \int_{\pi/2}^t (t-z)^{-q} 2\xi\sin 2z dz\right) - \mu_2\xi\cos 2t \end{pmatrix}$.

On the last interval $[\pi,\frac{3}{2}\pi]$ is the ODE system in the form

$$x'_{1\varepsilon} = x_{2\varepsilon} \tag{49}$$
$$\begin{aligned} x'_{2\varepsilon} = &-\frac{\mu_1}{\Gamma(1-q)}\left(\int_0^{\pi/2} (t-z)^{-q}\xi\cos z dz + \int_{\pi/2}^{\pi} (t-z)^{-q} 2\xi\sin 2z dz\right. \\ &\left. + \int_{\pi/2}^t (t-z)^{-q}(-\xi)\sin z dz\right) + \mu_2\xi\cos t - x_{1\varepsilon}, \end{aligned} \tag{50}$$

with initial conditions equal to (48) at time $t = \pi$. The solution on interval $[\pi,\frac{3}{2}\pi]$ is in the form:

$$\begin{pmatrix} x_{1\varepsilon} \\ x_{2\varepsilon} \end{pmatrix} = e^{At}K + e^{At}\int_0^t e^{-As} f(s)ds, \tag{51}$$

where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $K = e^{A\pi}\begin{pmatrix} x_{1\varepsilon}(\pi) \\ x_{2\varepsilon}(\pi) \end{pmatrix}$

and $f(t) = \begin{pmatrix} 0 \\ -\frac{\mu_1}{\Gamma(1-q)}\left(\int_0^{\pi/2} (t-z)^{-q}\xi\cos z dz + \int_{\pi/2}^{\pi} (t-z)^{-q} 2\xi\sin 2z dz + \right. \\ \left. + \int_{\pi/2}^t (t-z)^{-q}(-\xi)\sin z dz\right) + \mu_2\xi\cos t \end{pmatrix}$. We do not express analytical solutions (45), (48) and (51) in another form, because the given formulas lead to hypergeometric functions. For specific parameter choices, it is possible to find a numerical solution, but we do not go for details. We focus in this paper on the analytical theory of the Poincaré mapping method rather than its numerical investigation, which is interesting but postponed to another of our studies based on [19–22].

On the other hand, to understand the dynamics of (41), we first consider its limit case for $q = 1$ (see [23]), so we have

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{x_1(5 - 3\,\mathrm{sgn}\, x_2)}{2} - \varepsilon(\mu_1 x_2 - \mu_2 h(x_1)), \end{aligned} \tag{52}$$

We see on Figures 3–5 different dynamics depending on h. The perturbation $h(x_1) = x_1$ on Figure 3 keeps the symmetry along the origin. It seems that the origin is a global attractor. The perturbation $h(x_1) = x_1 + 10$ on Figure 4 breaks the symmetry along the origin. The perturbation $h(x_1) = x_1 + x_1^3$ on Figure 5 keeps the symmetry along the origin, but it is nonlinear, and the dynamics are more interesting and complex.

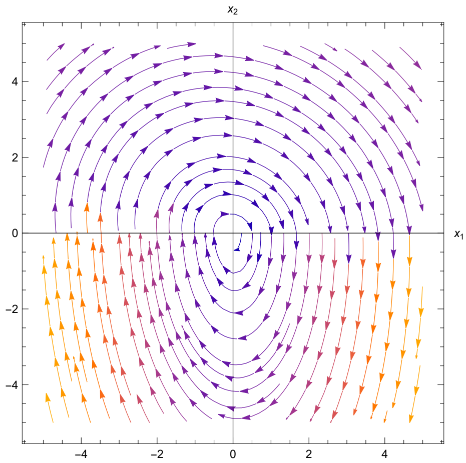

Figure 3. Local phase portrait of (52) for $\varepsilon = 0.1$, $\mu_1 = \mu_2 = 1$ and $h(x_1) = x_1$.

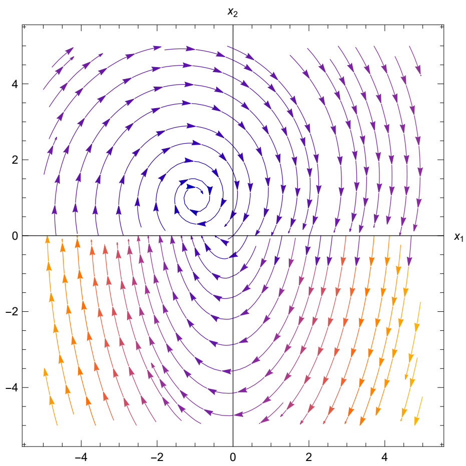

Figure 4. Local phase portrait of (52) for $\varepsilon = 0.1$, $\mu_1 = 1$, $\mu_2 = -1$ and $h(x_1) = x_1 + 10$.

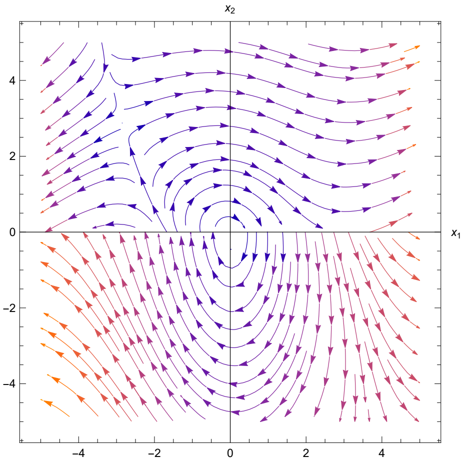

Figure 5. Local phase portrait of (52) for $\varepsilon = 0.1$, $\mu_1 = \mu_2 = 1$ and $h(x_1) = x_1 + x_1^3$.

Next, we consider a limit case of (41) for $q = 0$, so we have

$$\begin{aligned} x_1' =& x_2 \\ x_2' =& -\frac{x_1(5 - 3\,\mathrm{sgn}\, x_2)}{2} - \varepsilon(\mu_1(x_1 - x_1(0)) - \mu_2 h(x_1)) \end{aligned} \tag{53}$$

(53) is no longer an ODE. For instance, if $\mu_1 = \mu_2 = 1$ and $h(x_1) = x_1$, we obtain

$$\begin{aligned} x_1' =& x_2 \\ x_2' =& -\frac{x_1(5 - 3\,\mathrm{sgn}\, x_2)}{2} + \varepsilon x_2(0), \end{aligned} \tag{54}$$

The solutions of (54) seem to be repelled by Figure 6. Summarizing, the study of qualitative property of (41) is challenging by varying $q \in (0, 1)$.

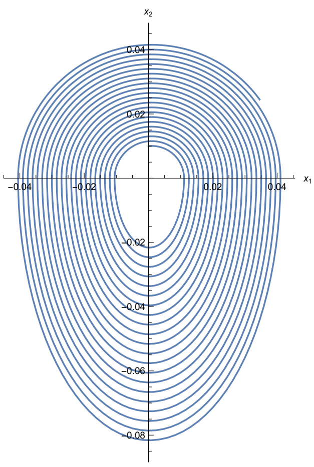

Figure 6. Solution of (54) for $\varepsilon = 0.1$ and $x_1(0) = 0$, $x_2(0) = 0.01$.

6. Conclusions

We work with a fractional differential equation with a discontinuous right-hand side, which is equivalent to the system (2). Suppose that functions $g_\pm$ are globally Lipschitz continuous, which change their form according to the sign of x_2 and transversally cross the discontinuity boundary.

We look for a periodic boundary solution of the system in a neighbourhood of the periodic orbit of an unperturbed system. That means the solution of the unperturbed equation is periodic.

We have shown the existence of the solution of the studied equation and found the corresponding Poincaré map in a neighbourhood of the periodic orbit of the unperturbed equation. We also present a bifurcation analysis of periodic boundary solutions.

We demonstrate how to apply our found formula to a concrete problem.

In the forthcoming work, we intend to generalize this theory to higher dimensions. Some of possible directions are outlined above.

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References

1. Di Bernardo, M. *Piecewise-Smooth Dynamical Systems: Theory and Applications*; Springer: London, UK, 2008.
2. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Science: Amsterdam, The Netherlands, 2006.
3. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Differential Equations*; John Wiley & Sons, Inc.: New York, NY, USA, 1993.
4. Olejnik, P.; Awrejcewicz, J.; Fečkan, M. *Modeling, Analysis and Control of Dynamical Systems: With Friction and Impacts*; World Scientific Series on Nonlinear Science Series A: Volume 92; World Scientific Publishing Co. Pte. Ltd.: Singapore, 2018.
5. Podlubny, I. *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications*; Academic Press: San Diego, CA, USA, 1999.
6. Samko, S.; Kilbas, A.A.; Marichev, O. *Fractional Integrals and Derivatives, Theory and Applications*; Gordon and Breach Science Publishers, Amsterdam, The Netherlands, 1993.
7. Kexue, L.; Jigen, P. Laplace transform and fractional differential equations. *Appl. Math. Lett.* **2011**, *24*, 2019–2023. [CrossRef]
8. Diethelm, K. Monotonicity of functions and sign changes of their Caputo derivatives. *Fract. Calc. Appl. Anal.* **2016**, *19*, 561–566. [CrossRef]
9. Evans, R.M.; Katugampola, U.N.; Edwards, D.A. Applications of fractional calculus in solving Abel-type integral equations: Surface–volume reaction problem. *Comput. Math. Appl.* **2017**, *73*, 1346–1362. [CrossRef]
10. Podlubny, I. *The Laplace Transform Method for Linear Differential Equations of the Fractional Order*; Ústav Experimentálnej Fyziky SAV: Košice, Slovakia, 1994.
11. Fečkan, M.; Pospíšil, M. On the bifurcation of periodic orbits in discontinuous systems. *Commun. Math. Anal.* **2010**, *8*, 87–108.
12. Fečkan, M.; Sathiyaraj, T.; Wang, J.R. Synchronization of butterfly fractional order chaotic system. *Mathematics* **2020**, *8*, 446. [CrossRef]
13. Fečkan, M.; Danca, M.-F. Stability, Periodicity, and Related Problems in Fractional-Order Systems. *Mathematics* **2022**, *10*, 2040. [CrossRef]
14. Tarasov, V.E. General fractional vector calculus. *Mathematics* **2021**, *9*, 2816. [CrossRef]
15. Tarasov, V.E. Exact solutions of Bernoulli and logistic fractional differential equations with power law coefficients. *Mathematics* **2020**, *8*, 2231. [CrossRef]
16. Tarasov, V.E. Mathematical Economics: Application of Fractional Calculus. *Mathematics* **2020**, *8*, 660. [CrossRef]
17. Fečkan, M.; Pospíšil, M. *Poincaré-Andronov-Melnikov Analysis for Non-Smooth Systems*; Elsevier: Amsterdam, The Netherlands, 2016.
18. Guckenheimer, J.; Holmes, P. *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*; Springer: New York, NY, USA, 1983.
19. Fečkan, M.; Kelemen, S. Discretization of Poincaré map. *Electron. J. Qual. Theory Differ. Equ.* **2013**, *60*, 1–33. [CrossRef]
20. Fečkan, M. Note on a Poincaré map. *Math. Slovaca* **1991**, *41*, 83–87.
21. Cek, M.K.; Marek, M. *Computational Methods in Bifurcation Theory and Dissipative Structures*; Springer: New York, NY, USA, 1983.
22. Henon, M. On the numerical computation of Poincaré maps. *Phys. Nonlinear Phenom.* **1982**, *5*, 412–414. [CrossRef]
23. Fečkan, M.; Pospíšil, M.; Wang, J.R. Note on weakly fractional differential equations. *Adv. Differ. Equ.* **2019**, *2019*, 143. [CrossRef]