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The Groups of Isometries of Metric Spaces over Vector Groups

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Abstract: In this paper, we consider the groups of isometries of metric spaces arising from finitely generated additive abelian groups. Let A be a finitely generated additive abelian group. Let $R = \{1, \varrho\}$ where ϱ is a reflection at the origin and $T = \{t_a : A \to A, t_a(x) = x + a, a \in A\}$. We show that (1) for any finitely generated additive abelian group A and finite generating set S with $0 \notin S$ and $-S = S$, the maximum subgroup of Isom$X(A, S)$ is RT; (2) $D \trianglelefteq RT$ if and only if $D \leq T$ or $D = RT'$ where $T' = \{h^2 : h \in T\}$; (3) for the vector groups over integers with finite generating set $S = \{u \in \mathbb{Z}^n : |u| = 1\}$, Isom X$(\mathbb{Z}^n, S) = O_n(\mathbb{Z})\mathbb{Z}^n$. The paper also includes a few intermediate technical results.

Keywords: abelian group; automorphism; isometry; vector group

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1. Cayley Graphs as Metric Spaces

The metric dimensions of the three classical geometric spaces and of Riemann surfaces were determined in [1]. This was followed up by the work in [2,3] where metric dimensions of metric manifolds were determined. Our work of the present paper is on the groups of isometries of metric spaces of vector groups over the integers.

As in [4] (p. 34), let G be a group and $S \subseteq G$ be a generating set of G such that $1 \notin S$ and $S^{-1} = S$. Define the *Cayley graph* $X = X(G, S)$ through the specification of its sets of vertices and edges

$$V(X) = G, \ E(X) = \{gh : g, h \in G, gh^{-1} \in S\}.$$

The condition $S^{-1} = S$ implies that the resulting graph is undirected, and the condition $1 \notin S$ implies that the graph has no loops. The condition that S is a generating set of G is to ensure that X is connected. The connectivity is imposed for the simple reason that our interest is in the metric properties of metric spaces. A graph X is a metric space X with its intrinsic path metric.

Let X and Y be metric spaces with distance functions ρ_X and ρ_Y, respectively. If a bijective mapping $f : X \to Y$ preserces distances then it is called an *isometry*. Namely, if for any $x, y \in X$,

$$\rho_Y(f(x), f(y)) = \rho_X(x, y)$$

then the bijective mapping f is called an *isometry*. For basic concepts and results not explicitly defined or presented in this note, the reader is referred to [1,5]. The *group of isometries* of a metric space is the set of all isometries of the space, with the composition of functions as group operation.

Let X be a metric space with distance function ρ. Determining the group of isometries of metric space (X, ρ) is an interesting problem [6]. Let $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. The distance function d_p is defined by

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$$d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}.$$

If $p = \infty$, then the distance function d_∞ is defined by

$$d_\infty(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}.$$

Hence $(\mathbb{R}^n, d_2)$ is the well-known Euclidean metric space. As in [7], let $p \neq 2$ be a real number, $p \geq 1$ or $p = \infty$. Then, the group of isometries of metric space $(\mathbb{R}^n, d_p)$ is $O_n(\mathbb{Z})\mathbb{R}^n$, where $O_n(\mathbb{Z})$ denote the orthogonal group in dimension n over $\mathbb{Z}$.

An *automorphism* of a graph $X = (V, E)$ is a permutation mapping f of the vertex set V, such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(G)$. The set of automorphisms of a graph forms a group under the composition of functions. This group is called the *automorphism group* of the graph X. The aim of this note is to study the group of isometries of metric spaces of vector groups over the integers.

A finitely generated additive abelian group over the integers is abbreviated as a *vector group* over $\mathbb{Z}$. In this note, among other results, the following two theorems are obtained.

Let A be a finitely generated additive abelian group and suppose that $\varrho : A \to A$ satisfies $\varrho(x) = -x$. Let $R = \{1, \varrho\}$ and $T = \{t_a : A \to A, t_a(x) = x + a, a \in A\}$. Then, RT is the maximum subgroup of Isom$X(A,S)$ (Theorem 3).

$D \trianglelefteq RT$ if and only if $D \leq T$ or $D = RT'$ where $T' = \{h^2 : h \in T\}$ (Theorem 4).

If $S = \{u \in \mathbb{Z}^n : |u| = 1\}$, then Isom X$(\mathbb{Z}^n, S) = O_n(\mathbb{Z})\mathbb{Z}^n$ (Theorem 6).

2. Abelian Groups

Let X be a metric space with distance function ρ, and let Isom X denote the group of isometries of X. If X is also a graph, then denote by Aut X the group of automorphisms of X.

Let X be a metric space and $x \in X$. Let r be a positive real number. The *sphere* centered at $x \in X$ with radius r is denoted $S(x,r) = \{y \in X : \rho(x,y) = r\}$.

Theorem 1. *Let X be a graph. Then, Isom X = Aut X.*

Proof. For any $f \in$ Isom X and $x, y \in X$, we have

$$\rho(x,y) = \rho(f(x), f(y)).$$

Hence, if $\rho(x,y) = 1$ then $\rho(f(x), f(y)) = 1$ and if $\rho(x,y) > 1$ then $\rho(f(x), f(y)) > 1$. Hence, $xy \in E(X)$ if and only if $f(x)f(y) \in E(X)$. Therefore, $f \in$ Aut X.

For any $g \in$ Aut X, we have $g^{-1} \in$ Aut X. Let $x, y \in X$. Suppose that $a = \rho(x,y)$ and $b = \rho(g(x), g(y))$. Let $x_0 = x$ and $x_a = y$. Then, the shortest path exists $[x_0, x_1, x_2, \ldots, x_{a-1}, x_a]$ of length a connecting x_0 and x_a. Since $g \in$ Aut X, for any $i \in \{0, 1, \ldots, a-1\}$, $x_i x_{i+1} \in E(X)$ if and only if $g(x_i)g(x_{i+1}) \in E(X)$. Hence, there is a path $[g(x_0), g(x_2), g(x_3), \ldots, g(x_{a-1}), g(x_a)]$ of length a connecting $g(x_0)$ and $g(x_a)$. Then, $\rho(g(x), g(y)) \leq a$. Hence, $b \leq a$. Let $y_0 = g(x)$ and $y_b = g(y)$. Since $\rho(g(x), g(y)) = b$, the shortest path exists $[y_0, y_1, y_2, \ldots, y_{b-1}, y_b]$ of length b connecting y_0 and y_b. Since $g^{-1} \in$ Aut X, for any $i \in \{0, 1, \ldots, b-1\}$, $y_i y_{i+1} \in E(X)$ if and only if $g^{-1}(y_i)g^{-1}(y_{i+1}) \in E(X)$. Hence, there is a path $[g^{-1}(y_0), g^{-1}(y_1), g^{-1}(y_2), \ldots, g^{-1}(y_{b-1}), g^{-1}(y_b)]$ of length b connecting $g^{-1}(y_0)$ and $g^{-1}(y_b)$. Then, $\rho(x,y) \leq b$. Hence, $a \leq b$. In summary, $\rho(x,y) = \rho(g(x), g(y))$ and $g \in$ Isom X. Therefore, we have that the set of Isom X is equal to the set of Aut X.

Since the group operation in Isom X is composition and the same is true for the group operation in Aut X, Isom X = Aut X. $\square$

Lemma 1. *Let X be a metric space with distance function ρ and $f : X \to X$ be a bijective mapping. Then, f is an isometry of X if and only if for any $x \in X$ and any real number $r \geq 0$, $f(S(x,r)) = S(f(x),r)$.*

Proof. Suppose that f is an isometry of X. Then, for any $x \in X$, $r \geq 0$ and $y \in S(x,r)$, we have $\rho(x,y) = r = \rho(f(x), f(y))$. Hence, $f(y) \in S(f(x),r)$. Then, $f(S(x,r)) \subseteq S(f(x),r)$. Let $u \in S(f(x),r)$ with $r \geq 0$. By the definition of a sphere, $\rho(u, f(x)) = r$. Let $v = f^{-1}(u)$. Since f is an isometry of X, f^{-1} is also an isometry of X. Hence, $\rho\big(f^{-1}(u), f^{-1}f(x)\big) = r = \rho(v,x)$. Hence, $v \in S(x,r)$. Since f is a mapping, $u = f(v) \in f(S(x,r))$. Hence, we have $S(f(x),r) \subseteq f(S(x,r))$. Therefore, $f(S(x,r)) = S(f(x),r)$.

Suppose that for any $x \in X$ and every real number $r \geq 0$, $f(S(x,r)) = S(f(x),r)$. Let $y \in X$. Then, $y \in S(x, \rho(x,y))$. Since f is a mapping, $f(y) \in f(S(x,\rho(x,y)))$. Since $f(S(x,r)) = S(f(x),r)$, $f(y) \in S(f(x), \rho(x,y))$. Therefore, $\rho(f(x), f(y)) = \rho(x,y)$. □

Corollary 1. *Let A be A finitely generated additive abelian group and S be a finite generating set of A with $0 \notin S$ and $-S = S$. Let $X = X(A,S)$ and $\varrho : X \to X$ be a bijective mapping with $\varrho(x) = -x$. Then, ϱ is an isometry of $X = X(A,S)$ that fixes 0.*

Proof. Let r be a nonnegative real number. By the definition of ϱ, we have $\varrho^2 = 1$, that is, $\varrho(\varrho(x)) = x$. Let $y \in S(\varrho(x),r)$. Since $\varrho(x) = -x$, $S(\varrho(x),r) = S(-x,r)$. Hence, $y \in S(\varrho(x),r)$ if and only if $y \in S(-x,r)$. Suppose that $b = \rho(x,-y)$. Let $x_0 = -x$ and $x_r = y$. Since $\rho(-x,y) = r$, the shortest path exists $[x_0, x_1, x_2, \ldots, x_{r-1}, x_r]$ of length r connecting x_0 and x_r. Since $-S = S$, for any $i \in \{0,1,\ldots,r-1\}$, $-x_{i+1} + x_i \in S$ if and only if $x_{i+1} - x_i \in S$. Then, a path exists $[-x_0, -x_1, -x_2, \ldots, -x_{r-1}, -x_r]$ of length r connecting $-x_0$ and $-x_r$. Then, $\rho(x,-y) \leq r$. Hence, $b \leq r$. Let $y_0 = x$ and $y_b = -y$. Since $b = \rho(x,-y)$, the shortest path exists $[y_0, y_1, y_2, \ldots, y_{b-1}, y_b]$ of length b connecting y_0 and y_b. Since $-S = S$, for any $i \in \{0,1,\ldots,b-1\}$, $-y_{i+1} + y_i \in S$ if and only if $y_{i+1} - y_i \in S$. Then, a path exists $[-y_0, -y_1, -y_2, \ldots, -y_{b-1}, -y_b]$ of length b connecting $-y_0$ and $-y_b$. Since $y = -y_b$, $\rho(-x,y) \leq b$. Hence, $r \leq b$. In summary, $b = r$. Since $\rho(x,-y) = b = r = \rho(-x,y)$, $y \in S(-x,r)$ if and only if $-y \in S(x,r)$. Therefore, $\varrho(y) \in S(x,r)$. Since ϱ is a mapping, $\varrho(y) \in S(x,r)$ if and only if $y = \varrho(\varrho(y)) \in \varrho(S(x,r))$. Therefore, $\varrho(S(x,r)) = S(\varrho(x),r)$. By Lemma 1, ϱ is an isometry of X. Since $-0 = 0$, we have $\varrho(0) = 0$. □

If for every $x \in A$, $\varrho(x) = -x$, then ϱ is called a *reflection* at 0. It is clear that $\varrho^2 = 1$. Denote $R = \{1, \varrho\}$.

Lemma 2. *Let A be a finitely generated additive abelian group and S be a finite generating set of A with $0 \notin S$ and $-S = S$. Then, $R = \{1, \varrho\} \leq$ Isom $X(A,S)$.*

Proof. By Corollary 1, $\varrho \in \operatorname{Isom} X(A,S)$. Since $\varrho^2 = 1$, $\{1,\varrho\} \leq \operatorname{Isom} X(A,S)$. □

Let $a \in A$ be fixed. Define $t_a : A \to A$ by $t_a(x) = x + a$. The mapping t_a is called the *translation* by a. By this definition, $t_0 = 1$. Denote $T = \{t_a : a \in A\}$.

Corollary 2. *Let A be a finitely generated additive abelian group and S be a finite generating set of A with $0 \notin S$ and $-S = S$. Then, $T \subseteq$ Isom$X(A,S)$.*

Proof. Let r be a nonnegative real number and $y \in S(t_a(x),r)$. Since $t_a(x) = x + a$, $S(t_a(x),r) = S(x+a,r)$. Hence, $y \in S(t_a(x),r)$ if and only if $y \in S(x+a,r)$. Suppose that $b = \rho(x, y-a)$. Let $x_0 = x+a$ and $x_r = y$. Since $y \in S(x+a,r)$, we have $\rho(x+a,y) = r$. Hence, the shortest path exists $[x_0, x_1, x_2, \ldots, x_{r-1}, x_r]$ of length r connecting x_0 and x_r. Since $(x_{i+1} - a) - (x_i - a) = x_{i+1} - x_i \in S$, $(x_i - a)(x_{i+1} - a) \in E(X)$ if and only if $x_i x_{i+1} \in E(X)$. Then, a path exists $[x_0 - a, x_1 - a, x_2 - a, \ldots, x_{r-1} - a, x_r - a]$ of length r connecting $x_0 - a$ and $x_r - a$. Since $x = x_0 - a$ and $y = x_r$, $\rho(x, y-a) \leq r$. Hence, $b \leq r$. Let $y_0 = x$ and $y_b = y - a$. Since $b = \rho(x, y-a)$, the shortest path exists $[y_0, y_1, y_2, \ldots, y_{b-1}, y_b]$ of length b

connecting y_0 and y_b. Since $(x_{i+1}+a)-(x_i+a) = x_{i+1}-x_i \in S$, $(x_i+a)(x_{i+1}+a) \in E(X)$ if and only if $x_i x_{i+1} \in E(X)$. Then, a path exists $[y_0+a, y_1+a, y_2+a, \ldots, y_{b-1}+a, y_b+a]$ of length b connecting y_0+a and y_b+a. Then, $\rho(x+a,y) \leq b$. Hence, $r \leq b$. Therefore, $r = b$. Hence, $y \in S(x+a,r)$ if and only if $y-a \in S(x,r)$. Since t_a is a mapping, $y-a \in S(x,r)$ if and only if $y = t_a(y-a) \in t_a(S(x,r))$. Therefore, $t_a(S(x,r)) = S(t_a(x),r)$. By Lemma 1, $t_a \in \operatorname{Isom} X(A,S)$. $\square$

Lemma 3. *Let A be a finitely generated additive abelian group and S be a finite generating set of A with $0 \notin S$ and $-S = S$. Then, $T \leq \operatorname{Isom} X(A,S)$.*

Proof. Since the composition of two translations is a translation, the set of all translations is closed under composition. Hence, by Corollary 2, $T \leq \operatorname{Isom} X(A,S)$. $\square$

Theorem 2. *Let A be a finitely generated additive abelian group and S be a finite generating set of A with $0 \notin S$ and $-S = S$. Let T denote the group of all translations of $X(A,S)$ and let $R = \{1, \varrho\}$ where ϱ is the reflection at 0. Then, $RT \leq \operatorname{Isom} X(A,S)$.*

Proof. By Lemma 2 and Lemma 3, we have both R and T are subgroups of $\operatorname{Isom} X(A,S)$. Let $h \in RT$. Then, $a \in A$, $f \in R$ exist and $g \in T$ such that $h = fg$ and $g = t_a$. If $f = 1$, then $h = fg = g = gf \in TR$. If $f \neq 1$, then $f = \varrho$. Since $g^{-1} = t_{-a}$ and $f = \varrho$, $\varrho g = \varrho t_a$ and $g^{-1}\varrho = t_{-a}\varrho$. For every $x \in X$,

$$\varrho g(x) = \varrho t_a(x) = \varrho(x+a) = -x-a = t_{-a}(-x) = t_{-a}\varrho(x) = g^{-1}\varrho(x).$$

Hence, $\varrho g = g^{-1}\varrho$. Since $g^{-1} \in T$, $h = fg = g^{-1}f \in TR$. Hence, $RT \subseteq TR$. Similarly, we may showed that $TR \subseteq RT$. Hence, we have $TR = RT$. Therefore, $RT \leq \operatorname{Isom} X(A,S)$. $\square$

Lemma 4. *Let A be a finitely generated additive abelian group and $f : A \to A$ be a bijective mapping with $f(0) = 0$. If for any finite generating set S of A with $0 \notin S$ and $-S = S$, $f \in \operatorname{Isom} X(A,S)$, then $f \in R$.*

Proof. Let $f : A \to A$ be a bijective mapping with $f(0) = 0$. Suppose that $f \neq 1$ and $f \neq \varrho$. Suppsoe first that there exists $x \in A$ such that $f(x) \neq x$ and $f(x) \neq -x$. Let $y = 0$. Then, $f(y) - f(x) = -f(x) \neq y - x$ and $f(y) - f(x) = -f(x) \neq x - y$.

Suppose, therefore, that for any $x \in A$, $f(x) = x$ or $f(x) = -x$. Since $f \neq 1$, there $y \in A$ exists with $y \neq 0$ such that $f(y) = -y$. Since $f \neq \varrho$, $x \in A$ exists with $x \neq 0$ such that $f(x) = x$. Hence, $f(y) - f(x) = -y - x \neq y - x$ and $f(y) - f(x) = -y - x \neq x - y$. Therefore, $x, y \in A$ exist such that $f(y) - f(x) - x + y \neq 0$ and $f(y) - f(x) - y + x \neq 0$.

Since A is a group and $f : A \to A$ is a mapping, $f(y) - f(x) - x + y$ and $f(y) - f(x) - y + x \in A$. Since A is finitely generated, there is a finite generating set W of A with $0 \notin W$ and $-W = W$. Hence, a generating set S exists such that $S = W \setminus \{f(y) - f(x), f(x) - f(y)\} \cup \{y - x, x - y, f(y) - f(x) - x + y, f(x) - f(y) - y + x\}$. Hence, a generating set S exists such that $y - x \in S$ and $f(y) - f(x) \notin S$. Therefore, $\rho(x,y) \neq \rho(f(x), f(y))$ and hence $f \notin \operatorname{Isom} X$ with $f(0) = 0$. $\square$

Theorem 3. *Let A be a finitely generated additive abelian group and ϱ be the reflection at 0. Suppose that $R = \{1, \varrho\}$, $T = \{t_a : t_a(x) = x + a, a \in A\}$ and $G \geq RT$. If for any finite generating set S of A with $0 \notin S$ and $-S = S$, $G \leq \operatorname{Isom} X(A,S)$, then $G = RT$.*

Proof. Let G be a group with $G \geq RT$ and $G \neq RT$. Then, $g \in G \setminus RT$ exists. Let h be defined by $h(x) = g(x) - g(0)$. Then, h is a mapping with $h(0) = 0$. Since $1 \in T$, we have $h \notin R$. By Lemma 4, a finite generating set S of A exists with $0 \notin S$ and $-S = S$ such that $h \notin \operatorname{Isom} X(A,S)$. Since $g(x) = h(x) + g(0)$, $g \notin \operatorname{Isom} X(A,S)$. Therefore, $G \nleq \operatorname{Isom} X(A,S)$. $\square$

Theorem 4. *Let A be a finitely generated additive abelian group and ϱ be the reflection at 0. Suppose that $R = \{1, \varrho\}$, $T = \{t_a : t_a(x) = x + a, a \in A\}$ and $G \geq RT$. $D \trianglelefteq RT$ if and only if $D \leq T$ or $D = RT'$ where $T' = \{h^2 : h \in T\}$.*

Proof. If $A = 1$, then $T = 1 = R$ and the conclusion of the theorem is true. Suppose that $|A| \geq 2$. Since T is an abelian group, $T' \leq T$. Let $D \trianglelefteq RT$, ϱ be the reflection at 0 and t_a be the translation by a. Then, $R' \leq R$ and $T' \leq T$ exists such that $D = R'T'$. Hence, for any $f \in R, g \in T$ $f' \in R'$ and $g' \in T'$, we have $fgf'g'(fg)^{-1} = fgf'g'g^{-1}f^{-1} \in R'T'$. If $R' = 1$, then $D = T' \leq T$.

Suppose, therefore, that $R' = \{1, \varrho\}$. If $f' = \varrho$, then $fg\varrho g'g^{-1}f^{-1} \in D$. Suppose that $g = t_a \in T$ and $g' = t_b \in T'$, $a, b \in A$. If $f = 1$, then for any $x \in X$,

$$\begin{aligned} fg\varrho g'g^{-1}f^{-1}(x) &= g\varrho g'g^{-1}(x) = g\varrho g'(x-a) = g\varrho(x-a+b) \\ &= g(-x+a-b) = -x+2a-b = t_{2a-b}\varrho(x). \end{aligned}$$

Hence, $t_{2a-b}\varrho \in D$. If $f = \varrho$, then for any $x \in X$,

$$\begin{aligned} fg\varrho g'g^{-1}f^{-1}(x) &= fg\varrho g'g^{-1}(-x) = fg\varrho g'(-x-a) = fg\varrho(-x-a+b) \\ &= fg(x+a-b) = f(x+2a-b) = -x-2a+b = t_{-2a+b}\varrho(x). \end{aligned}$$

Hence, $t_{-2a+b}\varrho \in D$. Since $\varrho \in D$, we have $t_{-2a+b} \in D$ and $t_{2a-b} \in D$. Since $t_{-2a+b}, t_{2a-b} \in T$, we have $t_{-2a+b}, t_{2a-b} \in D \cap T$. At the beginning of the proof we have concluded $D = R'T'$. By our assumption, $R' = \{1, \varrho\}$. Hence, $D = R'T' = \{1, \varrho\}T' = T' \cup \varrho T'$. Therefore, $D \cap T = (T' \cup \varrho T') \cap T = (T' \cap T) \cup (\varrho T' \cap T)$. Suppose that there is $f \in \varrho T' \cap T$. Then, $f \in \varrho T'$ and $f \in T$. Hence, $a, b \in A$ exist (a, b are fixed elements) such that $f = t_a$ and $f = \varrho t_b$. Hence, for arbitrary $x \in X$, $x + a = -(x + b)$. That is, $2x = -(a + b)$, a contradiction to the assumption that $|A| \geq 2$. Hence, $\varrho T' \cap T = \emptyset$. Since $T' \leq T$, $D \cap T = (T' \cap T) \cup \emptyset = T' \cap T = T'$. Hence, $t_{2a-b}, t_{-2a+b} \in T'$. Since $t_b, t_{-b} \in T'$, we have $t_{2a}, t_{-2a} \in T'$. Hence, for any $h \in T$, $h^2 \in T'$.

If $D \leq T$, then for any $f \in R$, $g \in T$ and $g' \in D$, we have $(fg)g'(fg)^{-1} = fgg'g^{-1}f^{-1}$. Since A is abelian, T is abelian. Hence, $fgg'g^{-1}f^{-1} = fg'f^{-1}$. Suppose that $g' = t_a$. If $f = 1$, then $fg'f^{-1} = g'$. If $f = \varrho$, then

$$fg'f^{-1}(x) = fg'(-x) = f(-x+a) = x-a = (g')^{-1}(x).$$

Since $g', (g')^{-1} \in D$, we have $D \trianglelefteq RT$.

Suppose, conversely, that for $T' = \{h^2 : h \in T\}$, $D = RT'$. Then, for any $f, f' \in R$, $g \in T$ and $g' \in T'$, we have $(fg)f'g'(fg)^{-1} = fgf'g'g^{-1}f^{-1}$. Suppose that $f' = 1$. Then, $fgf'g'g^{-1}f^{-1} = fgg'g^{-1}f^{-1}$. Since T is abelian, $fgg'g^{-1}f^{-1} = fg'f^{-1}$. Suppose that $g' = t_a$. If $f = 1$, then $fg'f^{-1} = g'$. If $f = f$, then

$$fg'f^{-1}(x) = fg'(-x) = f(-x+a) = x-a = (g')^{-1}(x).$$

Since both $g', (g')^{-1} \in D$, we have $D = RT' \trianglelefteq RT$. Suppose that $f' = \varrho$, $g = t_a \in T$ and $g' = t_b \in T'$, $a, b \in A$.

If $f = 1$, then

$$\begin{aligned} fgf'g'g^{-1}f^{-1}(x) &= gf'g'g^{-1}(x) = gf'g'(x-a) = gf'(x-a+b) \\ &= g(-x+a-b) = -x+2a-b = \varrho t_{-2a+b}(x). \end{aligned}$$

If $f = \varrho$, then

$$\begin{aligned} fgf'g'g^{-1}f^{-1}(x) &= fgf'g'g^{-1}(-x) = fgf'g'(-x-a) = fgf'(-x-a+b) \\ &= fg(x+a-b) = f(x+2a-b) = -x-2a+b = \varrho t_{2a-b}(x). \end{aligned}$$

Since $T' = \{h^2 : h \in T\}$, $t_{2a} \in T'$. Since $t_b \in T'$, we have $t_{2a-b}, t_{-2a+b} \in T'$. Hence, $\varrho t_{-2a+b}, \varrho t_{2a-b} \in RT' = D$. Therefore, $D \trianglelefteq RT$. $\square$

3. Vector Groups

In this section, we consider the vector groups over $\mathbb{Z}$ with finite generating set $S = \{u \in \mathbb{Z}^n : |u| = 1\}$, that is, the set of all unit vectors.

The following Lemma is straightforward from the definition of $X(\mathbb{Z}^n, S)$.

Lemma 5. *Let $p = (p_1, p_2, \ldots, p_n)$, $q = (q_1, q_2, \ldots, q_n) \in X(\mathbb{Z}^n, S)$. Then, $\rho(p, q) = \sum_{i=1}^{n} |q_i - p_i|$.*

Let P, Q be nonempty subsets of X with finite or countably infinite cardinality. Thus, we may write $P = \{v_1, v_2, \ldots, v_n, \ldots\}$. If for every $x, y \in Q$, $x \neq y$ implies $\rho(v_i, x) \neq \rho(v_i, y)$ for at least one index i; then, P is said to *resolve* Q and is called a *resolving set* or briefly a *resolver* for Q.

Lemma 6. *Let $P, Q \subseteq X$ and $f, g \in$ Isom X. Suppose that P resolves Q, $f(P) = P = g(P)$ and $f(Q) = Q = g(Q)$. If $f|_P = g|_P$, then $f|_Q = g|_Q$.*

Proof. Suppose that there exists $x \in Q$ such that $f(x) \neq g(x)$. Since P resolves Q, there exists $v \in P$ such that $\rho(f(v), f(x)) \neq \rho(f(v), g(x))$. Since $f|_P = g|_P$, $\rho(f(v), f(x)) \neq \rho(g(v), g(x))$. Since $f, g \in \text{Isom}\, X$, $\rho(v, x) = \rho(f(v), f(x)) \neq \rho(g(v), g(x)) = \rho(v, x)$. This is a contradiction. Therefore, $f|_Q = g|_Q$. $\square$

Lemma 7. *Let $X = X(\mathbb{Z}^n, S)$ and r be a positive integer. Then, $S(0, r)$ resolves $S(0, r+1)$.*

Proof. Let r be a positive integer, $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$ be two different points in $S(0, r+1)$. Then, $|p_1| + |p_2| + \cdots + |p_n| = |q_1| + |q_2| + \cdots + |q_n|$. Suppose that $i \in \{1, 2, \ldots, n\}$ exists such that $|p_i| < |q_i|$. Let

$$x = \left(p_1, p_2, \ldots, p_{i-1}, \frac{p_i(|p_i| - 1)}{|p_i|}, p_{i+1}, p_{i+2}, \ldots, p_n\right).$$

Then, $x \in S(0, r)$. Since $|p_i| < |q_i|$ and $\left|\frac{p_i(|p_i| - 1)}{|p_i|}\right| < |p_i|$, $\rho(x, p) = 1 < \rho(x, q)$. Suppose, therefore, that for any $i \in \{1, 2, \ldots, n\}$, $|p_i| = |q_i|$. Then, $i \in \{1, 2, \ldots, n\}$ exists such that $p_i = -q_i \neq 0$. Let $j \in \{1, 2, \ldots, n\}$ with $j \neq i$ and

$$x = \left(p_1, p_2, \ldots, p_{j-1}, \frac{p_j(|p_j| - 1)}{|p_j|}, p_{j+1}, p_{j+2}, \ldots, p_n\right).$$

Then, $x \in S(0, r)$. Since $\left|\frac{p_j(|p_j| - 1)}{|p_j|} - p_j\right| = 1$ and $|p_i - q_i| > 1$, $\rho(x, p) = 1 < \rho(x, q)$. Therefore, $S(0, r)$ resolves $S(0, r+1)$. $\square$

Theorem 5. *Let $X = X(\mathbb{Z}^n, S)$ and $f \in$ Isom $X(\mathbb{Z}^n, S)$. Then, f is a linear mapping.*

Proof. Consider the standard orthonormal basis $e_1, e_2, \ldots, e_n$. For $i = 1, 2, \ldots, n$, let $s_i = e_i$ and $s_{n+i} = -e_i$. By the definition of S, we have $S = \{s_1, s_2, \ldots, s_{2n}\}$. Since $f \in \text{Isom}\, X(\mathbb{Z}^n, S)$, $f(S) = S$. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$. Then, $x = x_1 s_1 + x_2 s_2 + \cdots + x_n s_n$. Hence, $f(x) = f(x_1 s_1 + x_2 s_2 + \cdots + x_n s_n)$.

Let $g(x) = x_1 f(s_1) + x_2 f(s_2) + \cdots + x_n f(s_n)$. Since $f(S) = S$, for $x, y \in \mathbb{Z}^n$, we have $\rho(g(x), g(y)) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n| = \rho(x, y)$. Hence, $g \in \operatorname{Isom} X(\mathbb{Z}^n, S)$. Let $u, v \in \mathbb{Z}^n$ and $l, m \in \mathbb{Z}$. Then,

$$\begin{aligned} g(lu + mv) &= g(lu_1 + mv_1, lu_2 + mv_2, \ldots, lu_n + mv_n) \\ &= (lu_1 + mv_1) f(s_1) + (lu_2 + mv_2) f(s_2) + \cdots + (lu_n + mv_n) f(s_n) \\ &= l(u_1 f(s_1) + \cdots + u_n f(s_n)) + m(v_1 f(s_1) + \cdots + v_n f(s_n)) \\ &= lg(u) + mg(v). \end{aligned}$$

Hence g is a linear mapping.

We now show that $f(-s_i) = -f(s_i)$. Suppose that $f(-s_i) \neq -f(s_i)$. Then, $j, k \in \{1, 2, \ldots, 2n\}$ exist with $j \neq k - n$, $j \neq k$ and $j \neq k + n$ such that $s_j = f(-s_i)$ and $s_k = f(s_i)$. Since $s_j \neq -s_k$, we have $(s_j + s_k) \in S(0, 2)$ and $(f(-s_i) + f(s_i)) \in S(0, 2)$. Hence, $\rho(f(-s_i) + f(s_i), f(-s_i)) = |f(s_i)| = |s_k| = 1$ and $\rho(f(-s_i) + f(s_i), f(s_i)) = |f(-s_i)| = |s_j| = 1$. Since $f \in \operatorname{Isom} X(\mathbb{Z}^n, S)$, we have $f^{-1}(f(-s_i) + f(s_i)) \in S(0, 2)$, $\rho(f^{-1}(f(-s_i) + f(s_i)), -s_i) = 1$ and $\rho(f^{-1}(f(-s_i) + f(s_i)), s_i) = 1$. Let $x = f^{-1}(f(-s_i) + f(s_i))$. Then, we have $x \in S(0, 2)$, $\rho(x, -s_i) = 1$ and $\rho(x, s_i) = 1$. We show that there is no such x. Since $x \in S(0, 2)$ and $\rho(x, s_i) = 1$, $l \in \{1, 2, \ldots, 2n\}$ exists with $l \neq i$ such that $x = s_i + s_l$ or $x = 2s_i$. Hence, $\rho(x, -s_i) > 1$. This contradicts $\rho(x, -s_i) = 1$. Hence, there is no such x. Therefore, we have $f(-s_i) = -f(s_i)$.

We now show that $f = g$. If $r = 1$, then $S(0, r) = S(0, 1) = S$. Hence, for any $x \in S$, $i \in \{1, 2, \ldots, n\}$ exists such that $x = s_i$ or $x = -s_i$. Since $f(-s_i) = -f(s_i)$, we have $f(x) = f(s_i) = 1 f(s_i) = g(x)$ or $f(x) = f(-s_i) = -f(s_i) = g(x)$.

Suppose that $r \geq 1$ and any $x \in S(0, r)$, $f(x) = g(x)$. By Lemma 7, $S(0, r)$ resolves $S(0, r + 1)$. Since $f, g \in \operatorname{Isom} X(\mathbb{Z}^n, S)$, $f(S(0, r)) = g(S(0, r)) = S(0, r)$ and $f(S(0, r + 1)) = g(S(0, r + 1)) = S(0, r + 1)$. By Lemma 6, for any $y \in S(0, r + 1)$, $f(y) = g(y)$. Hence, $f = g$ and f is a linear mapping. □

As in [8] (p. 39), the orthogonal group in dimension n over $\mathbb{Z}$ is denoted $O_n(\mathbb{Z})$.

Lemma 8. *$M \in O_n(\mathbb{Z})$ if and only if each column and row of M has exactly one non-zero entry, that is 1 or -1.*

Proof. For any $M \in O_n(\mathbb{Z})$, we have $MM^T = I = M^T M$. Let R be a row of M and C be a column of M. Since every entry of the main diagonal of I is 1, we have $RR^T = 1$ and $C^T C = 1$. Since entries of R and C are integers, R has exactly one non-zero element, which is 1 or -1. The same is true for C.

Suppose that every row and every column has exactly one non-zero entry that is 1 or -1. Then, for each row R and each column C, $RR^T = 1 = C^T C$. If R' is a row of M and $R' \neq R$, then $R' R^T = 0 = R R'^T$. The same is true for the columns of M. Hence, $MM^T = I = M^T M$. □

We now determine the group of isometries of metric space $X(\mathbb{Z}^n, S)$. We first consider the isometries that fixes 0.

Lemma 9. *Let M be an $n \times n$ matrix over $\mathbb{R}$. Then, $M \in O_n(\mathbb{Z})$ if and only if $f = Mx$ is an isometry of $X(\mathbb{Z}^n, S)$ that fixes 0.*

Proof. Let $M = (m_{ij})_{n \times n}$. Suppose that $f(x) = Mx$ is an isometry of $X(\mathbb{Z}^n, S)$ that fixes 0. Consider the standard orthonormal basis $e_1, e_2, \ldots, e_n$. Let $s_i = e_i$ and $s_{n+i} = -e_i$, $i \in \{1, 2, \ldots, n\}$. By the definition of S, we have $S = \{s_1, s_2, \ldots, s_{2n}\}$. Then, for any $s_i \in S$, $f(s_i) \in S$. Hence $Ms_i \in S$. Therefore, for any $j \in \{1, 2, \ldots, n\}$, we have $|m_{1j}| + |m_{2j}| + \cdots + |m_{nj}| = 1$. Hence, each column of M has exactly one non-zero entry that is equal to 1 or -1.

We now show that each row of M has exactly one non-zero entry that is 1 or -1. Since f is a mapping, $f(\mathbb{Z}^n) \subseteq \mathbb{Z}^n$. Suppose that $i, j \in \{1, 2, \ldots, n\}$ exist such that $m_{ij} \notin \mathbb{Z}$. Then,

$f(e_j) = Me_j \notin \mathbb{Z}$. Hence, if f is a mapping and $f(x) = Mx$, then every entry of M is an integer. Since f is an isometry of $X(\mathbb{Z}^n, S)$, $f(S) = S$. Suppose that $i, j \in \{1, 2, \ldots, n\}$ exist such that $m_{ij} \notin \{-1, 0, 1\}$. Then, the absolute value of the i-th entry of Me_j is at least 2. By Lemma 5, $\rho(0, f(e_j)) = \rho(0, Me_j)$ is equal to the sum of the absolute values of the entries of Me_j. Hence, $\rho(0, f(e_j)) \geq 2$. This contradicts $f(S) = S$. Hence, for every $i, j \in \{1, 2, \ldots, n\}$, $m_{ij} \in \{-1, 0, 1\}$. Suppose that M has a zero row. Then M is singular. This contradicts the assumption that f is bijective. Suppose that a row of M exists with at least two nonzero entries. Let m_{ij} and m_{ik} be nonzero. Then, $|m_{ij}| = |m_{ik}| = 1$. If $m_{ij} = m_{ik}$, then $Ms_j = Ms_k$. This contradicts the assumption that f is a bijective mapping. If $m_{ij} = -m_{ik}$, then $Ms_j = Ms_{n+k}$. This contradicts the assumption that f is a bijective mapping. Therefore, each row of M has exactly one non-zero element that is 1 or -1. By Lemma 8, $M \in O_n(\mathbb{Z})$.

Suppose, conversely, that $M \in O_n(\mathbb{Z})$. Let

$$x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{Z}^n.$$

By Lemma 5, $\rho(x, y) = \sum_{i=1}^{n} |x_i - y_i|$. By Lemma 5,

$$\begin{aligned}\rho(Mx, My) &= \sum_{i=1}^{n}\left|\sum_{j=1}^{n} m_{ij}x_j - \sum_{j=1}^{n} m_{ij}y_j\right| = \sum_{i=1}^{n}\left|\sum_{j=1}^{n}(m_{ij}x_j - m_{ij}y_j)\right| \\ &= \sum_{i=1}^{n}\left|\sum_{j=1}^{n} m_{ij}(x_j - y_j)\right|.\end{aligned}$$

Let $i, k \in \{1, 2, \ldots, n\}$ be fixed. Suppose that $m_{ik} \neq 0$. By Lemma 8, each row of M has exactly one non-zero entry which is -1 or 1. Hence,

$$\left|\sum_{j=1}^{n} m_{ij}(x_j - y_j)\right| = |x_k - y_k|.$$

and hence

$$\sum_{i=1}^{n}\left|\sum_{j=1}^{n} m_{ij}(x_j - y_j)\right| = \sum_{k=1}^{n}|x_k - y_k| = \rho(x, y).$$

Hence, $f(x) = Mx$ defines an isometry f. □

Since the group $\{f : f = Mx,\ M \in O_n(\mathbb{Z})\}$ isomorphic $O_n(\mathbb{Z})$, we denote $O_n(\mathbb{Z}) = \{f : f = Mx, M \in O_n(\mathbb{Z})\}$. Since the group of all translations of $X(\mathbb{Z}^n, S)$ is isomorphic to $\mathbb{Z}^n$, we may write $\mathbb{Z}^n = \{f : f = x + a, a \in \mathbb{Z}^n\}$.

Theorem 6. *Isom* $X(\mathbb{Z}^n, S) = O_n(\mathbb{Z})\mathbb{Z}^n$.

Proof. By Theorem 5 and Lemma 9, the group of isometries of $X(\mathbb{Z}^n, S)$ that fixes 0 is $O_n(\mathbb{Z})$. Hence, $O_n(\mathbb{Z}) \leq \text{Isom}\, X(\mathbb{Z}^n, S)$. By Lemma 3, all translations of $X(\mathbb{Z}^n, S)$ form a subgroup of Isom $X(\mathbb{Z}^n, S)$. That is, $\mathbb{Z}^n \leq \text{Isom}\, X(\mathbb{Z}^n, S)$. Let $f \in O_n(\mathbb{Z})$ and $g \in \mathbb{Z}^n$. Then, by Lemma 9, $M \in O_n(\mathbb{Z})$ exists such that $f(x) = Mx$. We also have, $a \in \mathbb{Z}^n$ such that $g(x) = x + a$. Let $h(x) = x + Ma$. Then, $h \in \mathbb{Z}^n$. Hence, $hf(x) = h(Mx) = Mx + Ma = M(x + a) = f(x + a) = fg(x)$. Hence, $\mathbb{Z}^n O_n(\mathbb{Z}) \subseteq O_n(\mathbb{Z})\mathbb{Z}^n$. Let $p(x) = x + M^{-1}a$. Then, $p \in \mathbb{Z}^n$. Hence, $fp(x) = f(x + M^{-1}a) = M(x + M^{-1}a) = Mx + a = g(Mx) = gf(x)$. Hence, $O_n(\mathbb{Z})\mathbb{Z}^n \subseteq \mathbb{Z}^n O_n(\mathbb{Z})$. Therefore, $O_n(\mathbb{Z})\mathbb{Z}^n = \mathbb{Z}^n O_n(\mathbb{Z})$ and $O_n(\mathbb{Z})\mathbb{Z}^n \leq \text{Isom}\, X(\mathbb{Z}^n, S)$.

Let $g \in \text{Isom}\, X(\mathbb{Z}^n, S)$. Let $t(x) = g(x) - g(0)$. Then, t is an isometry of $X(\mathbb{Z}^n, S)$ with $t(0) = 0$. By Theorem 5 and Lemma 9, $t \in O_n(\mathbb{Z})$. Since $g(x) = t(x) + g(0)$, $g \in O_n(\mathbb{Z})\mathbb{Z}^n$. Therefore, Isom $X(\mathbb{Z}^n, S) = O_n(\mathbb{Z})\mathbb{Z}^n$. □

Let $S = \{u \in \mathbb{Z}^2 : |u| = 1\} = \{(0,1),(0,-1),(1,0),(-1,0)\}$. Then, the group of isometries of metric spaces arising from Gaussian integers Isom $X(\mathbb{Z}^2, S)$ is $O_2(\mathbb{Z})\mathbb{Z}^2$, where

$$O_2(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\}.$$

4. Conclusions

For any finitely generated additive abelian group A and finite generating set S with $0 \notin S$ and $-S = S$, the maximum subgroup of Isom$X(A,S)$ is RT, where $R = \{1, \varrho\}$ and $T = \{t_a : t_a(x) = x + a, a \in A\}$. $D \trianglelefteq RT$ if and only if $D \leq T$ or $D = RT'$ where $T' = \{h^2 : h \in T\}$. For the vector groups over integers with finite generating set $S = \{u \in \mathbb{Z}^n : |u| = 1\}$, Isom $\mathrm{X}(\mathbb{Z}^n, S) = O_n(\mathbb{Z})\mathbb{Z}^n$.

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