

Towards the Centenary of Sheffer Polynomial Sequences: Old and Recent Results

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Abstract: Sheffer's work is about to turn 100 years after its publication. In reporting this important event, we recall some interesting old and recent results, aware of the incompleteness of the wide existing literature. Particularly, we recall Sheffer's approach, the theory of Rota and his collaborators, the isomorphism between the group of Sheffer polynomial sequences and the so-called Riordan matrices group. This inspired the most recent approaches based on elementary matrix calculus. The interesting problem of orthogonality in the context of Sheffer sequences is also reported, recalling the results of Sheffer, Meixner, Shohat, and the very recent one of Galiffa et al., and of Costabile et al.

Keywords: sheffer sequence; recurrence relation; polynomial sequences; generating functions; umbral calculus

MSC: 11B37; 11B83; 05A40; 05E35



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1. Introduction

I.M. Sheffer's 1939 seminal work [1] was inspired by Pincherle's paper [2] on the study of the difference equation

$$\sum_{n=1}^k c_n \phi(x + h_n) = f(x). \quad (1)$$

For the solution of this equation, Pincherle considered a set of Appell [3] polynomials, and wrote the solutions as an infinite series of them.

Previously, in 1936, Sheffer [4] had studied the solution of the same equation by means of a different Appell set. In [5] he treated the more general equation

$$L[y(x)] := a_0 y(x) + a_1 y'(x) + \dots = f(x),$$

and found a solution, under suitable conditions on L and f . As a tribute to Sheffer, we reproduce part of his introduction in full.

"Here, too, it was possible to relate the equation to a corresponding problem of expanding functions in series of Appell polynomials.

As is well known, Appell sets $\{P_n\}_{n \in \mathbb{N}}$ are characterized by one of the equivalent conditions

$$P'_n(x) = P_{n-1}(x), \quad (P_n \text{ a polynomial of degree } n); \quad (2)$$

$$A(t)e^{xt} = \sum_{n=0}^{\infty} P_n(x)t^n, \quad (3)$$

where $A(t) = \sum_{n=0}^{\infty} a_n t^n$ is a formal power series, and where the product on the left of (3) is formally extended in a power series in accordance with the Cauchy rule. We shall say that the series $A(t)$ is the *determining series* for the set $\{P_n\}_{n \in \mathbb{N}}$.

For the particular equation

$$y(x+1) - y(x) = f(x),$$

Pincherle used the Appell set with $A(t) = \frac{1}{e^t - 1}$, getting essentially the Bernoulli polynomials.

We used $A(t) = e^t - 1$, so that $n!P_n(x) = (x+1)^n - x^n$.

Now this equation is also associated with the important set of Newton polynomials

$$N_0(x) = 1, \quad N_n(x) = \frac{x(x-1) \cdots (x-n+1)}{n!}, \quad n \geq 1,$$

which is not an Appell set. Yet, it has properties analogous to those ((2) and (3)) of Appell polynomials. In fact,

$$\Delta N_n(x) := N_n(x+1) - N_n(x) = N_{n-1}(x),$$

$$(1+t)^x = 1 \cdot e^{x \log(1+t)} = \sum_{n=0}^{\infty} N_n(x) t^n.$$

It is thus suggested that we define a class of difference polynomial sets, of which $\{N_n\}_{n \in \mathbb{N}}$ is a particular set, by means of the relations

$$\Delta P_n(x) = P_{n-1}(x), \quad n \geq 0.$$

And more generally, we can use other operators than $\frac{d}{dx}$ and Δ , to define further sets. We thus obtain all polynomial sets of *type zero* (as we denote them). The definition of sets of type zero generalizes readily to give sets of type one, two, ... and of infinite type" [1].

After the publication of Sheffer's paper [1], a wide class of related works have been written, many of which are quite recent. Some of these works also develop a basic-type of characterization and relationship by modern umbral calculus (see for example [6–16]).

Other papers contain applications in various disciplines. For example:

- probability theory [17–22];
- number theory [23,24];
- linear recurrence [7,25];
- general linear interpolation [26–29];
- operators approximation theory [29–34];
- specific A-type zero orthogonal polynomial sequences [35–47];
- extension of Sheffer sequences also in the multidimensional case [48,49].

Moreover we point out that a sufficiently comprehensive bibliography up to 1995 is in [50].

According to Galiffa et al. [51] "Indeed, research on the Sheffer sequence is an active area and important in its own right".

The present paper is structured in two parts and in six main Sections, some of which contain subsections:

- Part 1: 1939–2000
 - Section 2: Sheffer's approach
 - Section 3: Rota's et al. contributions
- Part 2: 2001–2022
 - Section 4: The Riordan group and the Sheffer group
 - Section 5: Elementary matrix calculus approach to umbral calculus
 - Section 6: Sheffer A-type zero orthogonal polynomial sequences
 - Section 7: Relationship between Sheffer A-type zero sequences and monomiality principle.

We will use the following notations, unless otherwise specified:

- $\mathcal{P}$ is the set of polynomials in the variable $x \in \mathbb{K}$ (usually $\mathbb{R}$ or $\mathbb{C}$); for any $n \in \mathbb{N}$, $\mathcal{P}_n$ is the set of polynomials of degree $\leq n$;
- $\{p_n\}_{n \in \mathbb{N}}$ is a polynomial sequence (p.s.), that is, for any $n \in \mathbb{N}$, p_n is a polynomial of degree exactly n ;
- $(a_i)_{i \in \mathbb{N}}$ is a numerical sequence with elements a_i ;
- $A = (a_{i,j})_{i,j \in \mathbb{N}}$ is an infinite lower triangular matrix with entries $a_{i,j}$; A_n is the leading submatrix of order n .

Every time we insert a bibliographical citation we also intend to refer to the references therein.

PART 1.

2. Sheffer'S Approach

"... and more generally, we can use other operators than $\frac{d}{dx}$ and Δ , to define further sets. We thus obtain all polynomial sets of type zero (as we denote them)." [1].

Hence, in his paper, I.M. Sheffer had the goal of determining more general polynomial sequences than those of P. Appell [3] and of binomial type [6,8,9,52]. However, preliminarily, he introduced a classification of polynomial sequences into different types.

2.1. Sheffer Classification

Let $\phi = \{\phi_n\}_{n \in \mathbb{N}}$ be a p.s., that is, for any $n \in \mathbb{N}$, ϕ_n is a polynomial of degree exactly n . We define the set of polynomials (not necessarily a p.s.) v_n , $n \geq 0$, by recurrence

$$\begin{cases} v_0(x)D\phi_1(x) = \phi_0(x) \\ v_n(x)D^{n+1}\phi_{n+1}(x) = \phi_n(x) - \sum_{k=0}^{n-1} v_k(x)D^{k+1}\phi_{n+1}(x), \quad n > 0. \end{cases} \quad (4)$$

Remark 1. We note that, being ϕ_n of degree exactly n , for any $n \in \mathbb{N}$, it follows that v_n is uniquely defined and has degree $\leq n$.

Let us define $V_\phi = \{v_n \mid \forall n \in \mathbb{N}, v_n \text{ is polynomial as in (4)}\}$.

Theorem 1 ([53]). For the p.s. $\{\phi_n\}_{n \in \mathbb{N}}$ there exists a unique differential operator of the form

$$J := J_\phi(x, D) = \sum_{k=0}^{\infty} v_k(x)D^{k+1}, \quad v_k \in V_\phi, \quad \forall k \in \mathbb{N}, \quad (5)$$

such that

$$J[\phi_0] = 0, \quad J[\phi_n] = \phi_{n-1}, \quad n \geq 1. \quad (6)$$

It is said that the p.s. $\{\phi_n\}_{n \in \mathbb{N}}$ belongs to the operator J and that J is the operator associated with the p.s. $\{\phi_n\}_{n \in \mathbb{N}}$.

Remark 2 ([53]). Not every operator of the form (5) is associated with a p.s. in the sense previously defined. An operator J of the form (5) is associated with a p.s. if and only if J maps x^n into $n x^{n-1}$. Hence it maps each polynomial of degree exactly n into a polynomial of degree exactly $n - 1$.

There is only one operator associated with a given p.s., but there are infinitely many polynomial sequences belonging to the same operator.

Theorem 2 ([1]). To each operator J of the form (5) correspond infinitely many polynomial sequences for which (6) holds. In particular, one and only one of these polynomial sequences, which we call the basic sequence and denote by $\{b_n\}_{n \in \mathbb{N}}$, is such that

$$b_0(x) = 1, \quad b_n(0) = 0, \quad n \geq 1.$$

Corollary 1 ([1]). Necessary and sufficient condition for $\{\psi_n\}_{n \in \mathbb{N}}$ to be a p.s. belonging to J is that there exists a numerical sequence $(a_n)_{n \in \mathbb{N}}$ such that

$$\psi_n(x) = \sum_{i=0}^n a_i b_{n-i}(x), \quad a_0 \neq 0, \quad \forall n \in \mathbb{N}, \quad (7)$$

where $\{b_n\}_{n \in \mathbb{N}}$ is the basic sequence for J .

Definition 1 ([1]). If no polynomial v_k in the set V_ϕ is of degree greater than m , but at least one is of degree m , the p.s. $\{\phi_n\}_{n \in \mathbb{N}}$ is of A-type m . If the degrees of the polynomials v_k are unbounded, then $\{\phi_n\}_{n \in \mathbb{N}}$ is of infinite type.

From Theorem 1 the following corollary holds.

Corollary 2 ([1]). There are infinitely many polynomial sequences for every A-type (finite or infinite).

Example 1 ([1,54]). Any Appell p.s. is of A-type zero. In fact, in this case, $J = D$.

Example 2 ([54]). Let $(a_i)_{i \in \mathbb{N}}$ be a numerical sequence with $a_0 \neq 0$. It is proved [54] that the p.s. $\{a_k\}_{k \in \mathbb{N}}$ defined as

$$a_k(x) = \sum_{i=0}^k \binom{k}{i} a_{k-i} \frac{x^i}{i!}$$

is of A-type 1 and the corresponding operator is

$$J_\phi(x, D) = D + xD^2.$$

In order to characterize the polynomial sequences of A-type m , $m > 0$, we remember the result of Huff and Rainville [55]. They showed, among other things, that a p.s. with generating function

$$\phi(t)f(xt) = \phi(t) {}_0F_m(-; \beta_1, \beta_2, \dots, \beta_m; \sigma xt) = \sum_{n=0}^{\infty} y_n(x)t^n,$$

with σ constant and ϕ analytic and not zero at $t = 0$, is of A-type m .

The Sheffer paper [1] contains a study on polynomial sequences of any A-type, but the most satisfying results are those related to polynomial sequences of A-type zero.

2.2. Polynomial Sequences of A-Type Zero

Sheffer found several characterizations of polynomial sequences of A-type zero. It will be convenient to restate the conditions for a p.s. to be of A-type zero as follows: $\{\phi_n\}_{n \in \mathbb{N}}$ is a p.s. of A-type zero if

$$J[\phi_n] = \phi_{n-1}, \quad n > 0,$$

where J is the operator

$$J[y] = \sum_{k=1}^{\infty} c_k y^{(k)}, \quad c_1 \neq 0, \quad c_k \in \mathbb{K}, \quad \forall k \in \mathbb{N}. \quad (8)$$

With the operator (8) Sheffer associated the formal power series

$$J(t) = \sum_{k=1}^{\infty} c_k t^k, \quad (9)$$

called the *generating function* of the operator (8).

The formal power series (9), being $c_1 \neq 0$, is invertible. Following Roman and Rota [8] we call it a δ -series and we call its inverse *compositional inverse*. It is denoted by

$$H(t) = \sum_{k=1}^{\infty} s_k t^k, \quad s_1 \neq 0, \quad s_k \in \mathbb{K}, \quad \forall k \in \mathbb{N}, \quad (10)$$

and verifies

$$J(H(t)) = H(J(t)) = t. \quad (11)$$

Remark 3. Sheffer's paper [1] (p. 596) in a footnote describes a recurrence procedure for the calculation of the coefficients $s_i = s_i(c_j)$, $i \geq 1$. Indeed they can be numerically generated by means of a formal algorithm [6] (pp. 6–8).

Then Sheffer gave the first characterization for polynomial sequences of A-type zero.

Theorem 3. A necessary and sufficient condition for $\{\phi_n\}_{n \in \mathbb{N}}$ to be of A-type zero corresponding to the operator J as in (8) is that a numerical sequence $(a_i)_{i \in \mathbb{N}}$, $a_0 \neq 0$, exists such that, setting

$$A(t) = \sum_{i=0}^{\infty} a_i t^i,$$

we get

$$A(t)e^{xH(t)} = \sum_{n=0}^{\infty} \phi_n(x) t^n, \quad (12)$$

with $H(t)$ as in (10).

Proof. The proof follows after observing that

$$e^{xH(t)} = \sum_{n=0}^{\infty} b_n(x) t^n,$$

where $\{b_n\}_{n \in \mathbb{N}}$ is the basic p.s. for J . Hence both the necessary and sufficient parts follow by (7) in Corollary 1. $\square$

The formal power series

$$A(t) = \sum_{i=0}^{\infty} a_i t^i.$$

is called by Sheffer [1] *determining series* of the p.s. $\{\phi_n\}_{n \in \mathbb{N}}$.

It is observed, also, that every p.s. satisfies infinitely many linear functional equations. One of the simplest equations for polynomial sequences of A-type zero is given in the following theorem.

Theorem 4 ([1]). Let $\{\phi_n\}_{n \in \mathbb{N}}$ be a p.s. of A-type zero corresponding to operator J , and let $A(t)$ be its determining series. Then $\{\phi_n\}_{n \in \mathbb{N}}$ satisfies the equation

$$L[y(x)] := \sum_{k=1}^{\infty} (q_{k,0} + x q_{k,1}) J^k[y] = \lambda y, \quad (13)$$

where $\lambda = n$, for $y = \phi_n$. The coefficients q are defined by

$$\frac{A'(t)}{A(t)} = \sum_{k=0}^{\infty} q_{k+1,0} t^k, \quad (14)$$

$$H'(t) = \sum_{k=0}^{\infty} q_{k+1,1} t^k. \quad (15)$$

Proof. The proof is based on (12). $\square$

From (13) in Theorem 4 a further characterization of polynomial sequences of A-type zero, expressed only in terms of the elements of the p.s. itself, follows.

Theorem 5 ([1]). A necessary and sufficient condition for a p.s. $\{\phi_n\}_{n \in \mathbb{N}}$ to be of A-type zero is that constants $q_{k,0}$, $q_{k,1}$ exist so that

$$\sum_{k=1}^{\infty} (q_{k,0} + x q_{k,1}) \phi_{n-k}(x) = n \phi_n(x), \quad \phi_{n-k}(x) = 0 \text{ for } k > n. \quad (16)$$

The operator J and the determining series A for $\{\phi_n\}_{n \in \mathbb{N}}$ are related to the coefficients q by (14)–(15).

By differentiating both sides of (12) with respect to x , by equating coefficients of the same powers of t , we obtain

$$\phi'_n(x) = s_1 \phi_{n-1}(x) + s_2 \phi_{n-2}(x) + \dots + s_n \phi_0(x), \quad n \geq 1. \quad (17)$$

This identity generates a further characterization.

Theorem 6 ([1]). A necessary and sufficient condition for a p.s. $\{\phi_n\}_{n \in \mathbb{N}}$ to be of A-type zero is that a numerical sequence $(s_n)_{n \in \mathbb{N}}$ exists for which (17) holds. In this case the operator J corresponding to $\{\phi_n\}_{n \in \mathbb{N}}$ is determined through $(s_n)_{n \in \mathbb{N}}$ by means of (10) and (11).

Another important topic in Sheffer's paper concerns the orthogonality of the polynomial sequences of A-type zero. It is described in the following Subsection.

2.3. A-Type Zero Polynomial Sequences That Are Orthogonal Polynomials

J. Shohat [56] proved that Hermite polynomials [6] (p. 134) are an Appell p.s., hence of A-type zero, but are also orthogonal polynomials [57,58]. Another orthogonal p.s. of A-type zero is Laguerre p.s. [6] (p. 184). This suggests the problem of determining all A-type zero polynomial sequences that are, also, orthogonal. J. Meixner [59] treated this problem by using Laplace transformation and taking

$$A(t)e^{xH(t)} = \sum_{n=0}^{\infty} \phi_n(x)t^n \quad (18)$$

as definition of A-type zero.

Sheffer in [1] gave a quite different treatment by means of the known properties of A-type zero polynomial sequences and the three-term recurrence relation for monic orthogonal polynomial sequences [57,58]

$$\phi_n(x) = (x + \lambda_n)\phi_{n-1}(x) - \mu_n\phi_{n-2}(x), \quad n \geq 1, \quad (19)$$

with λ_n, μ_n real constants, $\mu_n > 0, n > 1$.

Combining (16) and (19), the basic result of Sheffer is the following theorem.

Theorem 7 ([1]). A necessary and sufficient condition for an A-type zero p.s. $\{\phi_n\}_{n \in \mathbb{N}}$ to satisfy (19) is that

$$\lambda_n = \alpha + b n, \quad \text{and} \quad \mu_n = (n-1)(c + d n), \quad \alpha, b, c, d \in \mathbb{K},$$

with $c + d n \neq 0$ for $n > 1$.

With further analysis Sheffer characterized the A-type zero polynomial sequences that are, also, orthogonal, by the following result.

Theorem 8 ([1]). *A p.s. $\{\phi_n\}_{n \in \mathbb{N}}$ is of A-type zero and orthogonal if and only if the generating function (18) is expressed in one of the following forms*

$$A(t)e^{xH(t)} = \mu(1-bt)^c e^{\frac{d+atx}{1-bt}}, \quad a, b, c, \mu \neq 0; \quad (20)$$

$$A(t)e^{xH(t)} = \mu e^{t(b+ax) + ct^2}, \quad a, c, \mu \neq 0; \quad (21)$$

$$A(t)e^{xH(t)} = \mu e^{ct}(1-bt)^{d+ax}, \quad a, b, c, \mu \neq 0; \quad (22)$$

$$A(t)e^{xH(t)} = \mu \left(1 - \frac{t}{c}\right)^{d_1 + \frac{x}{a}} \left(1 - \frac{t}{b}\right)^{d_2 - \frac{x}{a}}, \quad a, b, c, \mu \neq 0, \quad b \neq c. \quad (23)$$

By properly choosing each of the parameters in (20)–(23) we can obtain all the Sheffer A-type zero orthogonal polynomial sequences (see [60]).

3. Rota's et al. Contributions

In 1970 G.C. Rota and his pupils [10,11,22,52] began to construct a completely rigorous theory of the “classical modern” umbral calculus. Classical umbral calculus, as it was from 1850 to 1970, consists of a symbolic technique for the manipulation of the sequences, whose mathematical rigor leaves much to be desired. Just remember Eric Temple Bell's failed attempt, in 1940, to persuade the mathematical community to accept umbral calculus as a legitimate mathematical tool. The theory of Rota et al. is based on the ideas of linear functional, linear operator and adjoint. In 1977 the authors were lucky enough to join in on formal theory, that can be called modern umbral calculus. A full exposition of this theory can be found in Roman's book [9]. The second chapter of this book contains definitions and general properties of the Sheffer sequences (i.e., A-type zero polynomial sequences) that are the main object of the study.

Sheffer Polynomial Sequences

Let $\mathcal{P}$ be the algebra of polynomials in the single variable x over the field $\mathbb{K}$ of characteristic zero. Let $\mathcal{P}^*$ be the vector space of all linear functionals on $\mathcal{P}$. The authors use the notation

$$\langle L | p \rangle$$

to denote the action of a linear functional L on a polynomial p . The formal power series

$$f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \quad (24)$$

defines a linear functional on $\mathcal{P}$ by setting

$$\langle f | x^n \rangle = a_n.$$

If the notation t^k is used for the k -th derivative operator on $\mathcal{P}$, that is,

$$t^k x^n = \begin{cases} (n)_k x^{n-k} & k \leq n \\ 0 & k > n, \end{cases}$$

where $(n)_k = n(n-1) \cdots (n-k+1)$, then any power series (24) is a linear operator on $\mathcal{P}$. It is defined as

$$f(t)x^n = \sum_{k=0}^n \binom{n}{k} a_k x^{n-k}.$$

So $f(t)p(x)$ denotes the action of the operator $f(t)$ on the polynomial $p(x)$. Thus a formal power series plays three roles in the umbral calculus theory of Rota et al.: a formal power series, a linear functional and a linear operator. S. Roman seems to encourage the young reader: “A little familiarity should remove any discomfort that may be felt by the use of this trinity” [9] (p. 12).

Let g be an invertible power series, that is,

$$g(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}, \quad a_0 \neq 0,$$

and let f be a δ -power series, that is

$$f(t) = \sum_{k=1}^{\infty} b_k \frac{t^k}{k!}, \quad b_1 \neq 0.$$

Theorem 9 ([9] (p. 17)). *There exists a unique p.s. $\{s_n\}_{n \in \mathbb{N}}$ satisfying the orthogonality conditions*

$$\langle g(t)f(t)^k | s_n(x) \rangle = n! \delta_{n,k}, \quad (25)$$

for all $n, k \geq 0$.

Proof. The uniqueness is based on the order of the power series $g(t)f(t)^k$, that is, $o(g(t)f(t)^k) = k$, for any $k > 0$. The existence of the solution s_n , for any $n \in \mathbb{N}$, is obtained from the solution of a nonsingular triangular system. $\square$

Following Roman's book [9], we say that the p.s. $\{s_n\}_{n \in \mathbb{N}}$ in (25) is the Sheffer sequence for the pair $(g(t), f(t))$, or that $\{s_n\}_{n \in \mathbb{N}}$ is Sheffer for $(g(t), f(t))$.

There are two important special cases of Sheffer sequences:

- (a) the Sheffer sequence for $(1, f(t))$ is the associated sequence for $f(t)$ (or the binomial p.s. [6] (p. 24) or the basic p.s. for the operator $f(t)$ [1]);
- (b) the Sheffer sequence for $(g(t), t)$ is the Appell p.s. [3] for $g(t)$.

The term *Appell sequence* in other sources [1] can differ for the factor $n!$.

Roman gave some characterization of Sheffer sequences.

Theorem 10 ([9] (p.18)). *The p.s. $\{s_n\}_{n \in \mathbb{N}}$ is Sheffer for $(g(t), f(t))$ if and only if*

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} s_k(y) \frac{t^k}{k!} \quad (26)$$

for all $y \in \mathbb{K}$, where $\bar{f}$ is the compositional inverse of f .

Remark 4. If we use for a formal power series the two variables x and t , we could write (26) as

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{k=0}^{\infty} s_k(x) \frac{t^k}{k!}. \quad (27)$$

It is the usual form for an exponential generating function.

Remark 5. Sheffer [1] characterized A -type zero polynomial sequences by the generating function

$$A(t)e^{xH(t)} = \sum_{k=0}^{\infty} u_k(x) t^k,$$

where $A(t)$ is an invertible power series and $H(t)$ is a δ -series.

From the comparison with (27) it follows that $\{s_n\}_{n \in \mathbb{N}}$ is a Sheffer sequence in the sense of Roman if and only if $\{\frac{1}{n!} s_n\}_{n \in \mathbb{N}}$ is a sequence of Sheffer A-type zero.

Remark 6. Just for historical record, polynomial sequences of Sheffer A-type zero are called poweroids by Steffensen [61] and sequences of generalized Appell-type by Erdelyi [62]. Although in Boas and Buck [63] the latter term is used for a more general set of polynomial sequences.

The generating function (27) provides a representation of Sheffer sequences in classical monomials.

Theorem 11 ([9] (p.19)). The p.s $\{s_n\}_{n \in \mathbb{N}}$ is Sheffer for $(g(t), f(t))$ if and only if

$$s_n(x) = \sum_{k=0}^n \frac{1}{k!} \left\langle g(\bar{f}(t))^{-1} f(t)^k \mid x^n \right\rangle x^k, \quad \forall n \in \mathbb{N}. \quad (28)$$

Proof. The thesis follows by applying both sides of (26) to x^n . $\square$

Remark 7. For an explicit calculation of the coefficients in (28) see [6] (pp. 11–13).

Theorem 12 ([9] (p.20)). A p.s $\{s_n\}_{n \in \mathbb{N}}$ is Sheffer for $(g(t), f(t))$ if and only if

$$f(t)s_n(x) = n s_{n-1}(x).$$

Interesting is the characterization of Sheffer polynomial sequences that generalizes the binomial formula.

Theorem 13 ([9] (p.21)). A p.s $\{s_n\}_{n \in \mathbb{N}}$ is Sheffer for $(g(t), f(t))$ if and only if

$$s_n(x+y) = \sum_{k=0}^n \binom{n}{k} p_k(y) s_{n-k}(x), \quad \forall n \in \mathbb{N}, y \in \mathbb{K}, \quad (29)$$

where $\{p_n\}_{n \in \mathbb{N}}$ is the (binomial) p.s. associated (basic) with the linear operator f .

Corollary 3. By interchanging x and y in (29) and setting $y = 0$, we get

$$s_n(x) = \sum_{k=0}^n \binom{n}{k} p_k(x) s_{n-k}(0), \quad \forall n \in \mathbb{N}.$$

Thus, given a p.s. $\{p_n\}_{n \in \mathbb{N}}$ associated (basic) with f , each Sheffer p.s. $\{s_n\}_{n \in \mathbb{N}}$ that uses f as its δ -power series is uniquely determined by the numerical sequence $(s_n(0))_{n \in \mathbb{N}}$.

Another important topic introduced by Rota et al. is the umbral composition.

Theorem 14 ([9] (p.44)). The set of Sheffer sequences is a group under umbral composition. In particular, if $\{s_n\}_{n \in \mathbb{N}}$ is Sheffer for $(g(t), f(t))$ and $\{r_n\}_{n \in \mathbb{N}}$ is Sheffer for $(h(t), l(t))$, then $\{r_n \circ s_n\}_{n \in \mathbb{N}}$ is Sheffer for $(g(t)h(f(t)), l(f(t)))$. The identity under umbral composition is the Sheffer p.s. $\{x^n\}_{n \in \mathbb{N}}$ and the inverse of $\{s_n\}_{n \in \mathbb{N}}$ is the Sheffer sequence for $(g(t)(\bar{f}(t))^{-1}, \bar{f}(t))$.

Remark 8. In Roman's book [9] the problem of orthogonal polynomial sequences which are also Sheffer is mentioned [9] (p.156–159), but in a more general way.

PART 2.

In the following Sections we want to discuss some of the more interesting results that have appeared in the last thirty years or so.

4. The Riordan Group and the Sheffer Group

L. Shapiro et al. in 1991 [64] found a new group of infinite lower triangular matrices. They called this group *Riordan group*. This name seems appropriate because due to J. Riordan's then recent death. Later the concept of Riordan matrices was generalized to exponential Riordan matrices by many authors [65,66]. An exponential Riordan matrix is an infinite lower triangular matrix whose j -th column (being the first indexed with 0) has the generating function

$$\frac{1}{j!}g(t)f(t)^j,$$

where $g(t)$ is an invertible power series and $f(t)$ is a δ -series, that is,

$$g(t) = \sum_{i=0}^{\infty} a_i \frac{t^i}{i!}, \quad a_0 \neq 0, \quad f(t) = \sum_{i=1}^{\infty} b_i \frac{t^i}{i!}, \quad b_1 \neq 0.$$

The exponential Riordan matrix generated by the formal power series $g(t)$, $f(t)$ is denoted by $[g(t), f(t)]$.

Remark 9. As mentioned in [67], “the concept of representing columns of an infinite matrix by formal power series is not new and goes back to Shur’s paper and Faber polynomials in 1945”.

In the set of exponential Riordan matrices, given $A = [g(t), f(t)]$ and $B = [h(t), l(t)]$, the matrix multiplication can be defined:

$$A \cdot B = [g(t), f(t)] \cdot [h(t), l(t)] = [g(t)h(f(t)), l(f(t))]. \quad (30)$$

The reader will have no difficulty in proving by himself that

- the identity is $I = [1, t]$;
- the inverse matrix is

$$[g(t), f(t)]^{-1} = \left[\frac{1}{g(\bar{f}(t))}, \bar{f}(t) \right],$$

where $\bar{f}(t)$ is the compositional inverse, hence is such that $f(\bar{f}(t)) = \bar{f}(f(t)) = t$.

Theorem 15. [64] The set of exponential Riordan matrices is a group with respect to the operation of matrix multiplication as defined in (30), called Riordan group.

In 2007 T.X. He et al. [68] considered the Sheffer group and proved that it is isomorphic to the Riordan group.

Using the notation of He, a Sheffer p.s. $\{p_n\}_{n \in \mathbb{N}}$ is defined as

$$g(t)e^{xf(t)} = \sum_{n=0}^{\infty} p_n(t)t^n. \quad (31)$$

This definition differs from Roman definition [9] by a constant $n!$, but it coincides with that in [1].

In the set of Sheffer polynomial sequences, if $\{p_n\}_{n \in \mathbb{N}}$ and $\{q_n\}_{n \in \mathbb{N}}$ are Sheffer polynomial sequences, He [68] defined the umbral composition “#” in this way:

$$p_n \# q_n = r_n, \quad \forall n \in \mathbb{N}, \quad (32)$$

where, if $p_n(x) = \sum_{k=0}^n p_{n,k}x^k$ and $q_n(x) = \sum_{k=0}^n q_{n,k}x^k$, then

$$r_n(x) = \sum_{k=0}^n r_{n,k}x^k,$$

with

$$r_{n,k} = \sum_{j=k}^n j! p_{n,j} q_{j,k}, \quad n \geq j \geq k.$$

Theorem 16 ([68]). *The set of all Sheffer polynomial sequences defined as in (31) with the operation “#” defined as in (32) is a group called the Sheffer group and denoted by $(\{p_n\}_{n \in \mathbb{N}}, \#)$. The identity of the group is $\left\{\frac{x^n}{n!}\right\}_{n \in \mathbb{N}}$. The inverse of $\{p_n\}_{n \in \mathbb{N}}$ generated by $g(t)e^{xf(x)}$ is the Sheffer p.s. generated by*

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)},$$

being $\bar{f}$ the compositional inverse of f .

Remark 10. *The result in the previous theorem, up to the factor $n!$, is conceptually identical to Roman’s one [9] (p. 44).*

Finally, the isomorphism between the groups is given by mapping

$$\{p_n\}_{n \in \mathbb{N}} \longrightarrow [g(t), f(t)],$$

that is, by associating with the Sheffer p.s. the exponential Riordan matrix whose rows are the coefficients of the polynomial p_n for any n .

For details we refer to [68], where there are also many examples and some applications.

5. Elementary Matrix Calculus Approach to Umbral Calculus

After isomorphism between Sheffer polynomials and exponential Riordan matrices in [6,26–28,69–73], there is an attempt to construct the modern umbral calculus through elementary matrix calculus. This is in contrast with the previous approaches considered very formal [74]. Furthermore S. Khan et al. wrote “The simplicity of the algebraic approach to the Appell and Sheffer sequences established in [69,72], allows several applications” [49].

Let $\{p_n\}_{n \in \mathbb{N}}$ be a p.s. with

$$p_n(x) = \sum_{k=0}^n t_{n,k}x^k, \quad t_{n,k} \in \mathbb{K}, \quad t_{n,n} \neq 0 \quad \forall n \in \mathbb{N}. \quad (33)$$

Setting

$$T_n = (t_{i,k})_{i,k \in \mathbb{N}}, \quad k \leq i, \quad i = 0, \dots, n, \quad \forall n \in \mathbb{N},$$

$$X_n = [1, x, x^2, \dots, x^n]^T,$$

we have

$$P_n = T_n X_n, \quad (34)$$

where

$$P_n = [p_0(x), p_1(x), \dots, p_n(x)]^T.$$

For $n \rightarrow \infty$, with obvious meaning of the symbols, we can write

$$P = T X, \quad (35)$$

where $T = (t_{ik})_{ik \in \mathbb{N}}$ is an infinite, nonsingular, lower triangular matrix [75] and T_n is the leader submatrix of order n , for any $n \in \mathbb{N}$. Formulas (34) and (35) are called *matrix forms* of the p.s. $\{p_n\}_{n \in \mathbb{N}}$.

The concept of representing polynomial sequences by lower triangular matrices is not new and goes back to G. Polya [76] and I. Shur [77]. In fact, Polya gave a solution of the Cauchy-Bellman functional equation for matrices

$$M(x)M(y) = M(x + y) \quad (36)$$

in the form

$$\begin{bmatrix} 1 & 0 & 0 & \dots & \dots & 0 \\ x & 1 & \ddots & & & \\ x^2 & 2x & 1 & \ddots & & \\ x^3 & 3x^2 & 3x & 1 & & \\ \vdots & & & & \ddots & 0 \\ x^n & \binom{n}{n-1}x^{n-1} & \binom{n}{n-2}x^{n-2} & \dots & \dots & 1 \end{bmatrix}.$$

Then Vein [12] observed that the $(n + 1)$ -th row of the matrix contains the terms of the polynomial expansion of $(1 + x)^n$, and the elements in the $(n + 1)$ -th column are the terms of the infinite series expansion of $(1 - x)^{-(n+1)}$.

Moreover, for the solution of (36), Vein in [13] proved the relation

$$M(x) = e^{xQ}, \quad (37)$$

where Q is an infinite triangular matrix with constant elements, and observed that

$$Q = M'(0). \quad (38)$$

From this relation Vein proved some identities among triangular matrices and inverse relations. Thereafter, he determined two sets of triangular matrices. The elements of one set are related to the terms of Laguerre, Hermite, Bernoulli, Euler and Bessel polynomials, whereas the elements of the other set consist of Stirling numbers of both kinds, the two-parameter Eulerian numbers and the numbers introduced by Touchard [78]. Hence it has been shown that these matrices are related by a number of identities. Some known and lesser known pairs of inverse scalar relations that arise in combinatorial analysis have been shown to be derivable from simple and obviously inverse pairs of matrix relations. Vein in [13] wrote "The referee has pointed out that this work is an explicit matrix version of the umbral calculus as presented by Rota et al. [9–11]".

In [6] the author aims to find well-known results on the umbral calculus and also new identities and properties of Sheffer sequences, by means of elementary matrix calculus. The approach is very different from Vein's. In fact, the starting point are the relations (33)–(35) and not (36)–(38), which indeed will be never considered.

Now we make a mention of the methods used in [6], through an historical and constructive path.

5.1. Appell Polynomial Sequences

Let $(a_i)_{i \in \mathbb{N}}$ be a numerical sequence with $a_0 \neq 0$. The infinite lower triangular matrix $A = (a_{i,j})_{i,j \in \mathbb{N}}$ with

$$a_{i,j} = \binom{i}{j} a_{i-j}$$

is called *Appell-type matrix* [6] (pp. 9–10). Then we consider the p.s. $\{a_n\}_{n \in \mathbb{N}}$ with

$$a_n(x) = \sum_{j=0}^n a_{n,j} x^j = \sum_{j=0}^n \binom{n}{j} a_{n-j} x^j, \quad \forall n \in \mathbb{N}, \quad (39)$$

called *Appell p.s.*

It's easy to verify that the matrix form

$$A X = \mathfrak{A}, \quad \text{and} \quad A_n X_n = \mathfrak{A}_n, \quad \forall n \in \mathbb{N}, \quad (40)$$

holds, where

$$\mathfrak{A} = [a_0(x), a_1(x), \dots, a_n(x), \dots]^T, \quad X = [1, x, x^2, \dots, x^n, \dots]^T$$

and

$$\mathfrak{A}_n = [a_0(x), a_1(x), \dots, a_n(x)]^T, \quad X_n = [1, x, x^2, \dots, x^n]^T.$$

The Appell-type matrix is nonsingular and its inverse is also an Appell-type matrix [6]. Moreover, if we set

$$g(t) = \sum_{i=0}^{\infty} a_i \frac{t^i}{i!}, \quad a_0 \neq 0,$$

$g(t)$ is an invertible power series and its inverse is

$$\frac{1}{g(t)} = \sum_{i=0}^{\infty} \bar{a}_i \frac{t^i}{i!},$$

where

$$\sum_{k=0}^n \binom{n}{k} a_k \bar{a}_{n-k} = \delta_{n,0}, \quad \forall n \in \mathbb{N}.$$

Then, if we set $A^{-1} = \bar{A} = (\bar{a}_{i,j})_{i,j \in \mathbb{N}}$, we have

$$\bar{a}_{i,j} = \binom{i}{j} \bar{a}_{i-j}. \quad (41)$$

The matrix $\bar{A}$ generates the p.s. $\{\bar{a}_n\}_{n \in \mathbb{N}}$ such that

$$\bar{a}_n(x) = \sum_{j=0}^n \binom{n}{j} \bar{a}_{n-j} x^j.$$

Moreover

$$\bar{A} X = \bar{\mathfrak{A}}, \quad \text{and} \quad \bar{A}_n X_n = \bar{\mathfrak{A}}_n, \quad \forall n \in \mathbb{N}.$$

For details we refer to [6] (p. 14).

The p.s. $\{\bar{a}_n\}_{n \in \mathbb{N}}$ is hence an Appell p.s., called *conjugate* of the p.s. $\{a_n\}_{n \in \mathbb{N}}$. The reader will have no difficulty in proving by himself the known identities

$$a'_n(x) = n a_{n-1}(x), \quad n \geq 1 \quad (42)$$

$$g(t) e^{xt} = \sum_{n=0}^{\infty} a_n(x) \frac{t^n}{n!}$$

that characterize an Appell p.s. [3].

From the matrix form (40) we get

$$X_n = \bar{A}_n \mathfrak{A}_n \quad (43)$$

and

$$x^n = \sum_{j=0}^n \bar{a}_{n,j} a_j(x), \quad \forall n \in \mathbb{N}. \quad (44)$$

From the latest formula we can derive a recurrence formula and a determinant form.

Theorem 17 ([6]). For the Appell p.s. $\{a_n\}_{n \in \mathbb{N}}$ defined as in (39) the following identities hold: $a_0(x) = \frac{1}{\bar{a}_0}$ and, for any $n \geq 1$,

$$a_n(x) = \frac{1}{\bar{a}_0} \left[x^n - \sum_{k=0}^{n-1} \binom{n}{k} \bar{a}_{n,k} a_k(x) \right]; \quad (45)$$

$$a_n(x) = \frac{(-1)^n}{\bar{a}_0^{n+1}} \begin{vmatrix} 1 & x & \cdots & x^{n-1} & x^n \\ \bar{a}_0 & \bar{a}_1 & \cdots & \bar{a}_{n-1} & \bar{a}_n \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \bar{a}_0 & \binom{n}{n-1} \bar{a}_1 \end{vmatrix}. \quad (46)$$

Proof. Relation (45) follows from (43). The (46) follows by Cramer's rule applied to the linear system (44), with $n = 0, \dots, M$, for any $M \in \mathbb{N}$.

The details can be found in [6] (pp. 83–84). $\square$

Remark 11. We note that the determinant form (46) is in [69]. Almost in the same period a similar form has been given by Yang et al. [79], but with very different and more sophisticated techniques.

An analogous result as in Theorem 17 holds for the conjugate sequence $\{\bar{p}_n\}_{n \in \mathbb{N}}$.

For an Appell p.s. a second recurrence relation and determinant form hold.

Theorem 18 ([6]). With the previous hypothesis and relations, for an Appell p.s. the following relations hold

$$a_{n+1}(x) = (x + b_0)a_n(x) + \sum_{k=0}^{n-1} \binom{n}{k} b_{n-k} a_k(x), \quad (47)$$

where $(b_i)_{i \in \mathbb{N}}$ is the numerical sequence given by

$$\frac{g'(t)}{g(t)} = \sum_{i=0}^{\infty} b_i \frac{t^i}{i!}; \quad (48)$$

and

$$a_0(x) = 1, \quad a_{n+1}(x) = \begin{vmatrix} x + b_0 & -1 & 0 & \cdots & \cdots & 0 \\ b_1 & x + b_0 & -1 & 0 & \cdots & 0 \\ b_2 & \binom{2}{1} b_1 & x + b_0 & -1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ b_n & \binom{n}{1} b_{n-1} & \cdots & \cdots & \binom{n}{n-1} b_1 & x + b_0 \end{vmatrix} \quad (49)$$

with b_i as in (48).

Proof. Cfr. [6] (pp. 83–86). $\square$

Remark 12. For the numerical sequence $(b_i)_{i \in \mathbb{N}}$ as in (48) we have the representation

$$b_n = \sum_{k=0}^n \binom{n}{k} a_{k+1} \bar{a}_{n-k}, \quad \forall n \in \mathbb{N}.$$

Remark 13. We observe that if

$$\sum_{k=0}^{n-2} \binom{n}{k} b_{n-k} a_k(x) = 0, \quad \forall x \in \mathbb{R}, \forall n \geq 2,$$

then the recurrence relation (47) becomes a three-term relation and, consequently, in suitable hypothesis, the sequence $\{a_n\}_{n \in \mathbb{N}}$ is also orthogonal [57,80]. It is known [56] that among classical orthogonal polynomials, only the Hermite sequence is, also, an Appell p.s. We will consider this topic afterwards.

Remark 14. The second recurrence relation and the second determinant form for Appell polynomial sequences are, also, in [81], but they are determined by a more general and complicate procedure.

The previous recurrence relations generate some differential equations for Appell polynomial sequences. For this, firstly we observe that from (42) we have

$$a_{n-k}(x) = \frac{a_n^{(k)}(x)}{n(n-1) \cdots (n-k+1)}, \quad \forall k = 0, \dots, n. \quad (50)$$

Then, using this relation we can prove the following theorem.

Theorem 19 ([6]). Let $\{a_n\}_{n \in \mathbb{N}}$ be the A.p.s. associated with matrix $A = (a_{i,j})_{i,j \in \mathbb{N}}$. Then $\{a_n\}_{n \in \mathbb{N}}$ satisfies the following differential equation

$$\frac{\bar{a}_n}{n!} y^{(n)}(x) + \frac{\bar{a}_{n-1}}{(n-1)!} y^{(n-1)}(x) + \cdots + \bar{a}_0 y(x) = x^n.$$

Proof. The proof is obtained by putting (50) in the first recurrence relation. $\square$

Combining (50) and (47), a second differential equation for Appell polynomial sequences can be obtained.

The first determinant form (46) allows to calculate the numerical value of $a_n(x)$ for every fixed value of the variable x . In fact, it is known that Gauss elimination without pivoting for an Hessenberg matrix is stable [82]. Moreover, it allows to prove the following orthogonality property.

Let L be a linear functional on $\mathcal{P}$. If we set

$$L(x^i) = \bar{a}_i, \quad i \geq 0,$$

the relation (46) allows to define the Appell p.s. denoted by $\{a_n^L\}_{n \in \mathbb{N}}$. Then we consider the $n+1$ linear functionals $L_i, i = 0, \dots, n$ such that

$$L_0(x^i) = L(x^i), \quad L_j(x^i) = L(D^j x^i), \quad j \leq i, i = 0, \dots, n. \quad (51)$$

Theorem 20 ([26,27]). For the Appell p.s. $\{a_n\}_{n \in \mathbb{N}}$ the following relations hold

$$L_i(a_n^L) = n! \delta_{i,n}, \quad i = 0, \dots, n.$$

Proof. The proof follows from (46). $\square$

Corollary 4. The Appell p.s. $\{a_n^L\}_{n \in \mathbb{N}}$ is the solution of the general linear interpolation problem

$$L_i(a_n(x)) = n! \delta_{i,n}, \quad i = 0, \dots, n, \quad (52)$$

Remark 15. We note that (52) is equivalent to Theorem 2.3.1 in [8] for Appell polynomial sequences.

Theorem 21 ([28]). (Representation theorem) With the previous hypothesis and notations, for any $P_n(x) \in \mathcal{P}_n$ we have

$$P_n(x) = \sum_{k=0}^n \frac{L(P_n^{(k)})}{k!} a_k^L(x). \quad (53)$$

Relation (53) is a natural generalization of the classic Taylor polynomials.

The previous theorem is extensible to the linear space X of real continuous functions defined in the interval $[a, b]$, with continuous derivatives of all necessary orders.

Theorem 22 ([28]). For any $f \in X$ the polynomial

$$P_n[f](x) = \sum_{i=0}^n \frac{L(f^{(i)})}{i!} a_i^L(x) \quad (54)$$

is the unique polynomial of degree $\leq n$ such that

$$L(P_n[f]^{(i)}) = L(f^{(i)}), \quad i = 0, \dots, n. \quad (55)$$

The polynomial (54) is called *Appell* or *umbral interpolant* for the function f . In [28] the estimation of the remainder

$$R_n[f](x) = f(x) - P_n[f](x)$$

can be found.

The second determinant form (49) allows to say that any Appell polynomial is the characteristic polynomial of a suitable Hessenberg matrix. In fact it has been proved [6] (p. 86) that if $\{a_n\}_{n \in \mathbb{N}}$ is the Appell p.s. with matrix A and related conjugate matrix $\bar{A}$, then every $a_n(x)$ is the characteristic polynomial of the production matrix [6] (p. 18) of $\bar{A}$, that is,

$$R_n = \bar{A}_n \hat{A}_n,$$

where $\hat{A}_n$ is the matrix A_n with its first row and last column removed. Hence the roots of an Appell polynomial $a_n(x)$ are the eigenvalues of matrix R_n .

For other properties we refer to a wide literature (see [6]).

5.2. Binomial-Type Polynomial Sequences

Roman and Rota [8] observed: “It remains a mystery why so many polynomial sequences occurring in various mathematical circumstances turn out to be of binomial-type”. They said, also, that “the notion of polynomial sequences of binomial-type goes back to E.T. Bell” [83,84], and we add to Aitken [85], “Steffens was the first to observe that the sequence associated with delta operators in the way as D is to x^n are of the binomial-type, but failed to notice the converse of this fact, which was first stated and proved by Mullin and Rota [52]”. In 1970 Mullin and Rota gave the first systematic theory, using operators methods instead of the less efficient generating functions methods [1], that had been exclusively used until then.

Garsia [74] observed: “Unfortunately, the notions and the proofs in that very original paper ([52], A/N) in some instances leave something to be desired, and even tend to obscure the remarkable simplicity and beauty of the results”.

An algebraic approach to Rota-Mullin theory has been considered in [14].

In the following we will use a matrix-calculus based approach.

Let $(b_i)_{i \in \mathbb{N}}$, $b_0 = 0$, $b_1 \neq 0$, $b_i \in \mathbb{K}$, $i \geq 0$, be a numerical sequence. We define the matrix $P = (p_{n,k})_{n,k \in \mathbb{N}}$ [6] (pp. 7–8) such that

$$\begin{cases} p_{n,0} = \delta_{n,0} & n \geq 0 \\ p_{n,1} = b_n & n \geq 1 \\ p_{n,k} = \frac{1}{k} \sum_{i=1}^{n-k+1} \binom{n}{i} p_{i,1} p_{n-i,k-1} & n \geq 2; k = 2, \dots, n \\ p_{n,k} = 0 & k > n. \end{cases} \quad (56)$$

P is called *binomial-type matrix* [6]. It is a non singular, infinite lower triangular matrix. Then we can consider the polynomial sequence

$$\begin{cases} p_0(x) = 1 \\ p_1(x) = p_{1,0} + p_{1,1}(x) \\ \dots \\ p_n(x) = p_{n,0} + p_{n,1}(x) + \dots + p_{n,n}x^n \\ \dots \end{cases}$$

It will be called *binomial-type polynomial sequence* (b.p.s. in the following).

The following characterization explains the construction of a binomial-type matrix P as in (56).

Theorem 23 ([6] (pp. 24,26)). *Let $\{p_n\}_{n \in \mathbb{N}}$ be a polynomial sequence. It is a b.p.s. if and only if there exists a numerical sequence $(b_i)_{i \in \mathbb{N}}$, with $b_0 = 0$, $b_1 \neq 0$, such that, for any $n \in \mathbb{N}$,*

$$\begin{cases} p'_n(x) = \sum_{i=1}^n \binom{n}{i} b_i p_{n-i}(x) = \sum_{i=0}^{n-1} \binom{n}{i} b_{n-i} p_i(x) \\ p_n(0) = 0, p_0(x) = 1 \end{cases}$$

and

$$e^{xf(t)} = \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!}, \quad (57)$$

$$\text{where } f(t) = \sum_{i=0}^{\infty} b_i \frac{t^i}{i!}.$$

Proof. Let $\{p_n\}_{n \in \mathbb{N}}$ be a b.p.s.. Hence there exists a numerical sequence $(b_i)_{i \in \mathbb{N}}$ with $b_0 = 0$, $b_1 \neq 0$ such that $p_n(x) = \sum_{k=0}^n p_{n,k} x^k$, where $p_{n,k}$ are defined as in (56). Then we have

$$p'_n(x) = \sum_{k=1}^n p_{n,k} k x^{k-1} = \sum_{k=1}^n \left(\sum_{i=1}^{n-k+1} \binom{n}{i} b_i p_{n-i,k-1} \right) x^{k-1} = \sum_{k=1}^n \binom{n}{k} b_k p_{n-k}(x). \quad (58)$$

With the reverse procedure, after integration, the opposite implication follows.

Property (57) follows by easy manipulations [6] (pp. 26–27). $\square$

Proposition 1 ([6] (pp. 6–9)). *With the previous notations and hypothesis we get*

$$f(\bar{f}(t)) = \bar{f}(f(t)) = t$$

if and only if

$$\bar{f}(t) = \sum_{i=0}^{\infty} \bar{b}_i \frac{t^i}{i!},$$

where $(\bar{b}_i)_{i \in \mathbb{N}}$ is defined by

$$\sum_{k=1}^n p_{n,k} \bar{b}_k = \delta_{n,1}, \quad n \geq 1,$$

being $p_{n,k}$ as in (56).

Power series $f(t)$ and $\bar{f}(t)$ are the compositional inverse of each other. From the numerical sequence $(\bar{b}_k)_{k \in \mathbb{N}}$ we can construct the matrix $\bar{P} = (\bar{p}_{n,k})_{n,k \in \mathbb{N}'}$ called the *conjugate binomial matrix* of P . It is proved [6] (p. 14) that $\bar{P} = P^{-1}$. The matrix $\bar{P}$ allows considering the p.s. $\{\bar{p}_n\}_{n \in \mathbb{N}'}$ called the *conjugate p.s.* of $\{p_n\}_{n \in \mathbb{N}'}$ with elements

$$\begin{cases} \bar{p}_0(x) = 1 \\ \bar{p}_1(x) = \bar{p}_{1,0} + \bar{p}_{1,1}(x) \\ \dots \\ \bar{p}_n(x) = \bar{p}_{n,0} + \bar{p}_{n,1}(x) + \dots + \bar{p}_{n,n}x^n \\ \dots \end{cases}$$

Proposition 2. *If $\{\bar{p}_n\}_{n \in \mathbb{N}}$ and $\{p_n\}_{n \in \mathbb{N}}$ are conjugate b.p.s., we have*

$$(p_n \circ \bar{p}_n)(x) = p_n(\bar{p}_n(x)) = \bar{p}_n(p_n(x)) = x^n. \quad (59)$$

Theorem 24. *Let $\mathcal{B}$ be the set of binomial polynomial sequences and “ $\circ$ ” the umbral composition [86] defined in $\mathcal{B}$. Then the algebraic structure $(\mathcal{B}, \circ)$ is a group.*

Remark 16. *An analogous result holds for the set $\mathcal{A}$ of Appell polynomial sequences. That is, the algebraic structure $(\mathcal{A}, \circ)$ is a group.*

For the conjugate b.p.s. $\{\bar{p}_n\}_{n \in \mathbb{N}}$ and $\{p_n\}_{n \in \mathbb{N}'}$ if we set

$$\hat{P}(x) = [p_0(x), \dots, p_n(x), \dots], \quad \bar{\bar{P}}(x) = [\bar{p}_0(x), \dots, \bar{p}_n(x), \dots],$$

we get the matrix forms

$$\begin{aligned} \hat{P} &= P X, & \text{and} & \quad \hat{P}_n = P_n X_n, \quad \forall n \in \mathbb{N}, \\ \bar{\bar{P}} &= \bar{P} X, & \text{and} & \quad \bar{\bar{P}}_n = \bar{P}_n X_n, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Theorem 25 ([6]). *For the conjugate b.p.s. $\{\bar{p}_n\}_{n \in \mathbb{N}}$ and $\{p_n\}_{n \in \mathbb{N}}$ the following identities hold*

$$\begin{aligned} \bullet \quad p_n(x) &= \frac{1}{\bar{p}_{n,n}} \left[x^n - \sum_{k=0}^{n-1} \bar{p}_{n,k} p_k(x) \right]; \\ \bullet \quad \bar{p}_n(x) &= \frac{1}{p_{n,n}} \left[x^n - \sum_{k=0}^{n-1} p_{n,k} \bar{p}_k(x) \right]; \end{aligned}$$

- $\hat{c}_0 p_{n+1}(x) = -(n\hat{c}_1 - x)p_n(x) - \sum_{k=2}^n \hat{c}_k p_{n-k+1}(x)$, where

$$\sum_{k=0}^n \binom{n}{k} \hat{c}_k b_{n-k+1} = \delta_{n,0}, \quad \forall n \in \mathbb{N},$$

with initial conditions $\hat{c}_0 = 1$, $b_1 = 1$, $p_0(x) = x$;

$$p_n(x) = \frac{(-1)^{n+1}}{\prod_{k=0}^n \bar{p}_{k,k}} \begin{vmatrix} x & x^2 & \cdots & x^{n-1} & x^n \\ \bar{p}_{1,1} & \bar{p}_{2,1} & \cdots & \bar{p}_{n-1,1} & \bar{p}_{n,1} \\ 0 & \bar{p}_{2,2} & \cdots & \bar{p}_{n-1,2} & \bar{p}_{n,2} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & & \bar{p}_{n-1,n-1} & \bar{p}_{n,n-1} \end{vmatrix}, \quad n \geq 1$$

and

$$\bar{p}_n(x) = \frac{(-1)^{n+1}}{\prod_{k=0}^n p_{k,k}} \begin{vmatrix} x & x^2 & \cdots & x^{n-1} & x^n \\ p_{1,1} & p_{2,1} & \cdots & p_{n-1,1} & p_{n,1} \\ 0 & p_{2,2} & \cdots & p_{n-1,2} & p_{n,2} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & & p_{n-1,n-1} & p_{n,n-1} \end{vmatrix}, \quad n \geq 1;$$

- referring to Sheffer's approach, the operator $J := J(x, D) = \sum_{i=1}^{\infty} \bar{b}_i D^{(i)}$, is the corresponding operator to the b.p.s. $\{p_n\}_{n \in \mathbb{N}}$ and this is the basic sequence for J , that is, $J[p_n] = n p_{n-1}$ [6] (p. 36).

For further details and properties we refer to [6] (pp. 24–45).

5.3. Sheffer Polynomial Sequences

In order to give an appropriate matrix-calculus based approach of Sheffer A-type zero polynomial sequences we consider

- two numerical sequences

$$(a_n)_{n \in \mathbb{N}}, \text{ with } a_0 \neq 0, a_i \in \mathbb{K}, i \in \mathbb{N},$$

$$(b_n)_{n \in \mathbb{N}} \text{ with } b_0 = 0, b_1 \neq 0, b_i \in \mathbb{K}, i \in \mathbb{N};$$

- the Appell-type matrix $A = (a_{i,j})_{i,j \in \mathbb{N}}$ with $a_{i,j} = \binom{i}{j} a_{i-j}$;
- the binomial-type matrix $P = (p_{i,j})_{i,j \in \mathbb{N}}$ with $p_{i,j}$ defined as in (56);
- the Appell p.s. $\{a_n\}_{n \in \mathbb{N}}$ with

$$a_n(x) = \sum_{k=0}^n \binom{n}{k} a_{n-k} x^k, \quad \forall n \in \mathbb{N};$$

- the binomial p.s. $\{p_n\}_{n \in \mathbb{N}}$ with

$$p_n(x) = \sum_{k=0}^n p_{n,k} x^k, \quad \forall n \in \mathbb{N};$$

- the formal power series

$$g(t) = \sum_{i=0}^{\infty} a_i \frac{t^i}{i!}, \quad f(t) = \sum_{i=0}^{\infty} b_i \frac{t^i}{i!}$$

and the related inverse and compositional inverse

$$\frac{1}{g(t)} = \sum_{i=0}^{\infty} \bar{a}_i \frac{t^i}{i!}, \quad \bar{f}(t) = \sum_{i=0}^{\infty} \bar{b}_i \frac{t^i}{i!};$$

- the linear operator

$$J[y] = \sum_{i=1}^{\infty} \bar{b}_i \frac{y^{(i)}}{i!},$$

where $(\bar{b}_i)_{i \in \mathbb{N}}$ is defined as in Proposition 1.

Then we consider the umbral composition of the polynomial sequences $\{a_n\}_{n \in \mathbb{N}}$ and $\{p_n\}_{n \in \mathbb{N}}$, denoted by $\{s_n\}_{n \in \mathbb{N}}$, that is [86],

$$s_n(x) = \sum_{i=0}^n \binom{n}{i} a_{n-i} p_i(x), \quad \forall n \in \mathbb{N}. \quad (60)$$

We call *Sheffer* p.s. the sequence $\{s_n\}_{n \in \mathbb{N}}$ related to the numerical sequences $(a_i)_{i \in \mathbb{N}}$, $(b_i)_{i \in \mathbb{N}}$ as defined above [72].

We denote by $S = (s_{i,j})_{i,j \in \mathbb{N}}$ the infinite lower triangular matrix associated with the p.s. $\{s_n\}_{n \in \mathbb{N}}$, that is,

$$s_n(x) = \sum_{k=0}^n s_{n,k} x^k, \quad \forall n \in \mathbb{N}. \quad (61)$$

Proposition 3. *With the previous notations and hypothesis we get*

$$S = A P. \quad (62)$$

Proof. See [6,72]. $\square$

As for Appell and binomial-type polynomial sequences, we will define the conjugate sequence of a Sheffer p.s. $\{s_n\}_{n \in \mathbb{N}}$, that is, the p.s. with matrix S^{-1} (see [6] (p. 14, pp. 150–151)).

For that, we consider the p.s. $\{\bar{s}_n\}_{n \in \mathbb{N}}$ such that

$$\bar{s}_n(x) = \sum_{i=0}^n \bar{s}_{n,i} x^i, \quad \forall n \in \mathbb{N},$$

where

$$\bar{s}_{n,j} = \sum_{k=0}^n \binom{n}{k} g_k \bar{p}_{n-k,j}, \quad j = 0, \dots, n, \quad n \in \mathbb{N}$$

being $(g_i)_{i \in \mathbb{N}}$ defined by

$$\sum_{i=0}^k \binom{k}{i} \hat{s}_i g_{k-i} = \delta_{k,0}, \quad \text{with} \quad \hat{s}_k = \sum_{j=1}^k a_j \bar{p}_{k,j}.$$

Let $\bar{S} = (\bar{s}_{i,j})_{i,j \in \mathbb{N}}$ be the infinite lower triangular matrix associated with the p.s. $\{\bar{s}_n\}_{n \in \mathbb{N}}$.

Proposition 4 ([6] (p. 14)). *With the previous notations and hypothesis we get*

$$\bar{S} = S^{-1}. \quad (63)$$

Remark 17. By construction the matrix $\bar{S}$ is the product of an Appell matrix and a binomial-type matrix. This is not evident from (62).

Hence the p.s. $\{\bar{s}_n\}_{n \in \mathbb{N}}$, being the umbral composition of an Appell and a binomial-type p.s., is a Sheffer p.s., called conjugate of the p.s. $\{s_n\}_{n \in \mathbb{N}}$.

Now we give the matrix form of a Sheffer p.s. We set

$$\begin{aligned} S(x) &= [s_0(x), \dots, s_n(x), \dots], & \text{and} & \quad S_n(x) = [s_0(x), \dots, s_n(x)], \quad \forall n \in \mathbb{N}, \\ \bar{S}(x) &= [\bar{s}_0(x), \dots, \bar{s}_n(x), \dots], & \text{and} & \quad \bar{S}_n(x) = [\bar{s}_0(x), \dots, \bar{s}_n(x)], \quad \forall n \in \mathbb{N}, \\ P(x) &= [p_0(x), \dots, p_n(x), \dots], & \text{and} & \quad P_n(x) = [p_0(x), \dots, p_n(x)], \quad \forall n \in \mathbb{N}. \end{aligned}$$

Therefore we have

$$S(x) = S X = (A P) X = A (P X) \quad (64)$$

and

$$S_n(x) = S_n X_n = (A_n P_n) X_n = A_n (P_n X_n).$$

From these matrix forms we can derive recurrence relations, determinant forms and differential equations. For details we refer to [6] (pp. 149–165).

Theorem 26 ([6] (pp. 158–159)). For the Sheffer p.s. $\{s_n\}_{n \in \mathbb{N}}$ the following identities hold

$$\frac{1}{k!} s_n^{(k)}(x) = \sum_{i=k}^n \binom{n}{i} p_{i,k} s_{n-i}(x), \quad \forall n \in \mathbb{N}, k = 0, \dots, n.$$

Particularly, for $k = 1$,

$$s'_n(x) = \sum_{i=0}^{n-1} \binom{n}{i} b_{n-i} s_i(x).$$

Proof. The proof follows by differentiation of (60), taking into account (64) and (56). $\square$

Proposition 5 ([6] (p. 159)). The following recurrence relation for the columns of the matrix S holds

$$s_{n,k+1} = \frac{1}{k+1} \sum_{i=k}^n \binom{n}{i} b_{n-i} s_{i,k} = \frac{1}{k+1} \sum_{i=1}^{n-k} \binom{n}{i} b_i s_{n-i,k}, \quad k = 0, \dots, n-1,$$

with boundary conditions

$$s_{n,0} = a_n, \quad s_{n,n} = a_0 b_1^n, \quad n \geq 0.$$

In order to determine the relationship with Sheffer A-type zero polynomial sequences we get the following theorem.

Theorem 27 ([6] (p. 153)). For a Sheffer p.s. $\{s_n\}_{n \in \mathbb{N}}$ defined as in (60) or (61) we have

$$g(t) e^{xf(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!},$$

$$J s_n = n s_{n-1}.$$

Now we can say that the p.s. $\{s_n\}_{n \in \mathbb{N}}$, defined above, is a Sheffer A-type zero p.s. For other properties we refer to a wide existing literature (see, for example [6]).

Finally we can observe that the previous construction allows us to write an explicit algorithm (Algorithm 1) for the numerical generation of a Sheffer p.s.

Algorithm 1 Appell, binomial-type, Sheffer polynomial sequences.1: Initialization: $N, a_i, b_i, i = 0, \dots, N$;

2: Appell-type matrix:

$$A = (a_{i,j})_{i,j=0,\dots,N'} \quad a_{i,j} = \binom{i}{j} a_{i-j}, \quad i = 0, \dots, N, \quad j = 0, \dots, i;$$

3: binomial-type matrix:

$$P = (p_{n,k})_{n,k=0,\dots,N'}$$

 $p_{n,k}$ as in (56);

4: Appell p.s.:

$$a_n(x) = \sum_{k=0}^n \binom{n}{k} a_{n-k} x^k, \quad n = 0, \dots, N;$$

plot a_n ;

5: binomial-type p.s.:

$$p_n(x) = \sum_{k=0}^n p_{n,k} x^k, \quad n = 0, \dots, N;$$

plot p_n ;

6: Sheffer-type matrix:

$$S = (s_{i,j})_{i,j=0,\dots,N'} \quad s_{i,j} = \sum_{k=0}^j \binom{i}{k} a_{i-k} p_{k,j}, \quad i = 0, \dots, N, \quad j = 0, \dots, i;$$

7: Sheffer p.s.:

$$s_n(x) = \sum_{k=0}^n \binom{n}{k} a_{n-k} p_k(x), \quad n = 0, \dots, N;$$

plot s_n ;

8: end.

6. Sheffer A-Type Zero Orthogonal Polynomial Sequences

We have observed that I.M. Sheffer in his work [1] characterized the A-type zero polynomial sequences which satisfy, also, an orthogonal condition.

This problem has been considered before by Meixner [59], Sholat [56], but with a different analysis. Recently, D.J. Galiffa et al. [51] showed that all Sheffer A-type zero orthogonal polynomial sequences can be characterized by using only the generating function that defines this class and a monic three-term recurrence relation. They, therefore, simplified Sheffer's analysis.

6.1. Galiffa et al. Analysis

It is suitable to use the same definitions and notations as in [51].

Definition 2. A p.s. $\{P_n\}_{n \in \mathbb{N}}$ is classified as A-type zero if there exist two numerical sequences $(a_i)_{i \in \mathbb{N}}$ and $(h_i)_{i \in \mathbb{N}}$ such that

$$A(t)e^{xH(t)} = \sum_{n=0}^{\infty} P_n(x)t^n, \quad (65)$$

with

$$A(t) = \sum_{i=0}^{\infty} a_i t^i, \quad a_0 = 1,$$

$$H(t) = \sum_{i=1}^{\infty} h_i t^i, \quad h_1 = 1.$$

To determine which orthogonal set satisfy (65), Sheffer used a monic three-term recurrence relation of the form [57,58]

$$P_{n+1}(x) = (x + \lambda_{n+1})P_n(x) - \mu_{n+1}P_{n-1}(x), \quad n \geq 0, \quad (66)$$

with $\mu_n > 0$, for any $n \in \mathbb{N}$ and $P_{-1} = 0$.

The idea of Galiffa et al. [51] consists in obtaining some coefficients of P_n by (65). They observed that

$$\begin{aligned} \sum_{n=0}^{\infty} a_n t^n e^{x(t+h_2t^2+h_3t^3+\dots)} &= \sum_{n=0}^{\infty} a_n t^n e^{xt} e^{h_2xt^2} e^{h_3xt^3} \dots \\ &= \sum_{k_0=0}^{\infty} a_{k_0} t^{k_0} \sum_{k_1=0}^{\infty} \frac{(xt)^{k_1}}{k_1!} \sum_{k_2=0}^{\infty} \frac{(h_2xt^2)^{k_2}}{k_2!} \sum_{k_3=0}^{\infty} \frac{(h_3xt^3)^{k_3}}{k_3!} \dots \end{aligned}$$

The general term in each of the products above is

$$a_{k_0} t^{k_0} \frac{x^{k_1} t^{k_1}}{k_1!} \frac{h_2^{k_2} x^{k_2} t^{2k_2}}{k_2!} \frac{h_3^{k_3} x^{k_3} t^{3k_3}}{k_3!} \dots$$

Thus, discovering the coefficient of $x^r t^r$ is equivalent to determining all of the nonnegative integer solutions $\{k_0, k_1, k_2, \dots\}$ of the linear Diophantine equations

$$k_1 + k_2 + k_3 + \dots = r \quad (67)$$

$$k_0 + k_1 + 2k_2 + 3k_3 + \dots = s, \quad (68)$$

where (67) represents the x -exponents and (68) the t -exponents.

In order to satisfy (66) we have to observe the coefficients $x^n t^n$, $x^{n-1} t^{n-1}$ and $x^{n-2} t^{n-2}$. We omit the calculation, for which we refer to [51].

Lemma 1. For the Sheffer A-type zero polynomial $P_n(x) = c_{n,0}x^n + c_{n,1}x^{n-1} + c_{n,2}x^{n-2} + O(x^{n-3})$ as in (66) we have

$$\begin{aligned} c_{n,0} &= \frac{1}{n!}, & c_{n,1} &= \frac{a_1}{(n-1)!} + \frac{h_2}{(n-2)!} \\ c_{n,2} &= \frac{a_2}{(n-2)!} + \frac{a_1 h_2 + h_3}{(n-3)!} + \frac{h_2^2}{2!(n-4)!}. \end{aligned}$$

Interestingly enough, the coefficients $c_{n,0}, c_{n,1}, c_{n,2}$ above are expressed in terms of only the first two nonunitary coefficients of t in $A(t)$ as $H(t)$, that is a_1, a_2, h_2, h_3 .

Then the authors in [51] showed that Sheffer A-type zero polynomial sequences satisfy a monic three-term recurrence relation if and only if

$$\begin{aligned} \lambda_{n+1} &= a_1 + 2h_2n \\ \mu_{n+1} &= (a_1^2 - 2a_2 + 2a_1h_2 - 4h_2^2 + 3h_3)n + (4h_2^2 - 3h_3)n^2. \end{aligned}$$

Hence the following orthogonal polynomial sequences all necessarily belong to the Sheffer A-type zero classes

$$\{(-1)^n n! L_n^{(\alpha)}(x)\}, \quad \{2^{-n} H_n(x)\}, \quad \{(-a)^n C_n(x; a)\},$$

$$\left\{ \frac{c^n(\beta)_n}{(c-1)^n} M_n(x; \beta, c) \right\}, \quad \{(2 \sin \phi)^{-n} n! P_n^{(\lambda)}(x; \phi)\}, \quad \{(-N)_n p^n K_n(x; p, N)\}.$$

These are respectively the monic forms of the Laguerre, Hermite, Charlier, Meixner, Meixner–Pollaczek and Krawtchouk polynomials, as defined in [51,60].

6.2. A Further Note on the Orthogonality of Sheffer A-Type Zero Polynomial Sequences

The analysis of Galiffa et al. in [51] can be further improved, by using an equivalent definition of Sheffer A-type zero polynomial sequences. In fact, assuming as definition a differential relation by using Theorem 26 and Proposition 5, we get

Theorem 28 ([54]). *For monic Sheffer A-type zero polynomial sequences we have*

$$\begin{aligned} s_{n,n} &= 1, & s_{n,n-1} &= na_1 + \frac{n(n-1)}{2}b_2, \\ s_{n,n-2} &= \frac{n(n-1)}{2}a_2 + \frac{n(n-1)(n-2)}{2}a_1b_2 + \frac{n(n-1)(n-2)}{6}b_3 \\ &\quad + \frac{n(n-1)(n-2)(n-3)}{8}b_2^2. \end{aligned}$$

Proof. Cfr. [54]. $\square$

Theorem 29 ([54]). *A monic Sheffer p.s. satisfies the three-term recurrence relation if and only if*

$$\begin{aligned} \lambda_{n+1} &= -(a_1 + nb_2) \\ \mu_{n+1} &= n\left(a_1^2 - a_2 + a_1b_2 - b_2^2 + \frac{1}{2}b_3\right) + n^2\left(b_2^2 - \frac{1}{2}b_3\right). \end{aligned}$$

Proof. From the comparison between (66) and (61) we have

$$\begin{aligned} \lambda_{n+1} &= s_{n,n-1} - s_{n+1,n} \\ \mu_{n+1} &= -\lambda_{n+1}s_{n,n-1} + s_{n,n-2} - s_{n+1,n-1}. \end{aligned}$$

From Theorem 28 we get the result. $\square$

We observe that these results coincide with the ones in [51].

Remark 18. *In the set of the Appell polynomial sequences we have the family of the monic orthogonal polynomials with*

$$\lambda_{n+1} = -a_1, \quad \mu_{n+1} = (a_1^2 - a_2)n, \quad a_1^2 - a_2 > 0.$$

In particular, among the classic orgonal polynomials only Hermite polynomial sequences [6] (p. 134) are also Appell polynomial sequences.

7. Relationship between Sheffer A-Type Zero Sequences and Monomiality Principle

The idea of monomiality goes back to J. Steffenson [61] but only in the last thirty years this idea has been systematically used by other authors (see [87–89]).

Definition 3. *A polynomial sequence $\{p_n\}_{n \in \mathbb{N}}$ is quasi monomial if and only if there exist two linear operators $\hat{P}$, $\hat{M}$, independent on n , called derivative and multiplicative operators, respectively, verifying the identities*

$$\hat{P}(p_n(x)) = np_{n-1}(x), \quad n \geq 1, \quad (69)$$

$$\hat{M}(p_n(x)) = p_{n+1}(x), \quad n \geq 0. \quad (70)$$

Hence $\hat{P}$ and $\hat{M}$ play an analogous role to that of derivative and multiplicative operators, respectively, on classic monomials.

The operators $\hat{P}$ and $\hat{M}$ satisfy the following commutative property

$$[\hat{P}, \hat{M}] = \hat{P}\hat{M} - \hat{M}\hat{P} = \hat{1},$$

so they display a Veyl group structure.

Let the p.s. $\{p_n\}_{n \in \mathbb{N}}$ be quasi monomial with respect to the operators $\hat{P}$, $\hat{M}$. Then some of its properties can be easily derived from those of the operators themselves:

1. if $\hat{P}$, $\hat{M}$ have a differential representation, that is, $\hat{P} = \hat{P}(D_x)$, $\hat{M} = \hat{M}(x, D_x)$, then for any $n \in \mathbb{N}$ the polynomial p_n satisfies the differential equations

$$\begin{aligned}\hat{M}\hat{P}(p_n(x)) &= n p_n(x), \\ \hat{P}\hat{M}(p_n(x)) &= (n+1)p_n(x);\end{aligned}$$

2. assuming $p_0(x) = 1$, p_n can be explicitly constructed as

$$p_n(x) = \hat{M}^n(1);$$

3. from the above identity it follows that the exponential generating function of $\{p_n\}_{n \in \mathbb{N}}$ is given by

$$e^{t\hat{M}}(1) = \sum_{n=0}^{\infty} \frac{(t\hat{M})^n}{n!}(1),$$

and therefore

$$e^{t\hat{M}}(1) = \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!}.$$

Let $\{s_n\}_{n \in \mathbb{N}}$ be a Sheffer A-type zero p.s. with exponential generating function [6] (p. 153)

$$g(t)e^{xf(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!},$$

where

$$\begin{aligned}g(t) &= \sum_{i=0}^{\infty} a_i \frac{t^i}{i!}, \quad a_0 \neq 0, \quad a_i \in \mathbb{K}, \\ f(t) &= \sum_{i=1}^{\infty} b_i \frac{t^i}{i!}, \quad b_1 \neq 0, \quad b_i \in \mathbb{K}.\end{aligned}$$

It has been showed [90,91] that a Sheffer A-type zero p.s. is quasi monomial with respect to the differential operators

$$\begin{aligned}\hat{P} &= \bar{f}(D_x) \\ \hat{M} &= x f'(\bar{f}(D_x)) + \frac{g'(\bar{f}(D_x))}{g(\bar{f}(D_x))}.\end{aligned}$$

Conversely, if $\{s_n\}_{n \in \mathbb{N}}$ is a p.s. satisfying (69), (70), with $\hat{M} = \hat{M}(x, D_x)$, $\hat{P} = \hat{P}(x, D_x)$, then necessarily it is of Sheffer A-type zero.

8. Conclusions

As the centenary of the publication of I.M. Sheffer's famous paper approaches, we wanted to honor his memory by recalling some old and recent results. In particular we recalled the idea of the classification of polynomials by means of suitable linear differential operators and Sheffer's method for the study of A-type zero polynomials.

Later Rota et al., in 1970, framed the study of A-type zero polynomials with the umbral calculus. Indeed, after the theory of Rota et al., modern umbral calculus was essentially confused with the study of polynomials of A-type zero. Another relevant idea was the isomorphism between the group of A-type zero polynomials and the Riordan group of exponential-type matrices introduced at the end of the last century. This gave a different vision to the subject and allowed the development of algebraic methods. For example, the

attempt to set modern umbral calculus on elementary matrix calculus. The simplicity of this result has allowed numerous theoretical and computational applications.

The constant proliferation of new ideas, theoretical and applicative, involving polynomial sets, make us believe that the Sheffer sequences are an active and important research area in its own right.

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