mathematics

Article

On the BiHom-Type Nonlinear Equations

Hui Wu and Xiaohui Zhang *

School of Mathematical Sciences, Qufu Normal University, Qfnu 273165, China
* Correspondence: zhangxh2015@qfnu.edu.cn

Abstract: In this paper, the Heisenberg doubles and Long dimodules of a BiHom-Hopf algebra are introduced. Then, we discussed the relation between BiHom-Hopf equation and BiHom-pentagon equation, and we obtain the solutions of BiHom-Hopf equation from Heisenberg doubles. We also showed that the parametric generalized Long dimodules can provide the solutions of BiHom-Yang-Baxter equation and generalized $\mathcal{D}$-equation.

Keywords: BiHom-Hopf algebra; BiHom-Hopf equation; BiHom-pentagon equation; generalized $\mathcal{D}$-equation

MSC: 16T99; 16T25

check for updates

Citation: Wu, H.; Zhang, X. On the BiHom-Type Nonlinear Equations. *Mathematics* **2022**, *10*, 4360. https://doi.org/10.3390/math10224360

Academic Editor: Shuanhong Wang

Received: 25 October 2022
Accepted: 12 November 2022
Published: 19 November 2022

1. Introduction

The theory of Hom-type algebras arises from the q-deformations of Witt and Virasoro algebras (see [1,2]). Then, the theory of Hom-type algebras is rapidly developing into a hot topic in algebra theory ([3–8]). In 2008, Makhlouf and Silvestrov introduced the definition of Hom-associative algebras [4]. In 2012, Caenepeel and Goyvaerts introduced the monoidal Hom–Hopf algebras ([9]) in order to provide a categorical approach to Hom-type algebras. In 2015, as the generalization of both Hom-(co)algebras and monoidal Hom-(co)algebras, BiHom-(co)algebras and BiHom-bialgebras were investigated by Graziani, Makhlouf, Menini, and Panaite in [10]. Note that a BiHom-algebra is an algebra in which the identities defining the structure are twisted by two homomorphisms. This class of algebras was introduced from a categorical approach in [10] which can be viewed as an extension of the class of Hom-algebras. Further research on BiHom-type algebras could be found in [11–15] and so on.

It is well known that some classical nonlinear equations in Hopf algebra theory, such as the quantum Yang-Baxter equation, the Hopf equation, the pentagon equation, and the $\mathcal{D}$-equation. In [16], the algebraic solutions of Hopf equation and pentagon equation are discussed. In [17], Militaru proved that each Long dimodule gave rise to a solution for the $\mathcal{D}$-equation. Long dimodules are the building stones of the Brauer-Long group [18]. The discussion of solutions of BiHom-type Yang-Baxter equation can be seen in [11,19]. The natural consideration is to ask: does there exist algebraic solutions of BiHom-Hopf equation, BiHom-pentagon equation, and BiHom-type $\mathcal{D}$-equation? That is the motivation of our paper.

In order to obtain the algebraic solutions of the above BiHom-type nonlinear equations, we introduced the Heisenberg doubles and the parametric generalized BiHom-Long dimodules of a BiHom-Hopf algebra. We also generalized Theorem 3.1 and Theorem 5.10 in [20].

The paper is organized as follows. In Section 2, we first recall some notions of BiHom-Hopf algebras. In Section 3, we describe the BiHom-Hopf equation and BiHom-pentagon equation, and provide the algebraic solutions of BiHom-Hopf equation through Heisenberg doubles. In Section 4, we introduce the parametric generalized BiHom-Long dimodules, and provide the algebraic solutions of the BiHom–Yang-Baxter equation and generalized $\mathcal{D}$-equation.

2. Preliminaries

Throughout the paper, $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}, \cdots$ always mean integers in $\mathbb{Z}$. Let $\Bbbk$ be a fixed field and $char(\Bbbk) = 0$, and $Vec_{\Bbbk}$ be the category of finite dimensional $\Bbbk$-spaces. All algebras are supposed to be over $\Bbbk$. For the comultiplication Δ of a $\Bbbk$-space C, we use the Sweedler–Heyneman's notation (see [21]): $\Delta(c) = c_1 \otimes c_2$, for any $c \in C$. When we say a "BiHom-algebra" or a "BiHom-coalgebra", we mean the unital BiHom-algebra and counital BiHom-coalgebra. We always assume that the BiHom-structure maps are invertible.

2.1. BiHom-Hopf Algebras

In this section, we will review several definitions and notations related to BiHom-bialgebras.

Recall from [10] or [15] that a *BiHom-algebra A over $\Bbbk$* is a 5-tuple $(A, \mu_A, 1_H, \alpha_A, \beta_A)$, where A is a $\Bbbk$-linear space, $1_A \in A$ is an element (the *unit*), $\alpha_A, \beta_A : A \to A$ are both bijective linear maps, $\mu_A : A \otimes A \to A$ is a linear map, with notation $\mu(a \otimes b) = ab$, satisfying the following conditions, for all $a, b, c \in A$:

$$\alpha_A(1_A) = \beta_A(1_A) = 1_A, \ \ a1_A = \alpha_A(a), \ \ 1_A a = \beta_A(a), \ \ \alpha_A(a)(bc) = (ab)\beta_A(c),$$
$$\alpha_A \circ \beta_A = \beta_A \circ \alpha_A, \ \ \alpha_A(ab) = \alpha_A(a)\alpha_A(b), \ \ \beta_A(ab) = \beta_A(a)\beta_A(b).$$

Remark 1. *Note that the second line of the above identities can be derived from the first line. See ([15], Proposition 2.9).*

Example 1.

(1) *If $A = (A, \mu_A, 1_A)$ is an associative algebra, $\alpha, \beta : A \to A$ are both algebra isomorphisms, then $(A, \mu_A \circ (\alpha \otimes \beta), 1_A, \alpha, \beta)$ is a BiHom-algebra.*

(2) *If $\alpha = \beta$, then A becomes a Hom-algebra.*

A *BiHom-coalgebra C over $\Bbbk$* is a 5-tuple $(C, \Delta_C, \varepsilon_C, \phi_C, \psi_C)$, in which C is a linear space, $\phi_C, \psi_C : C \to C$ are linear isomorphisms, $\varepsilon_C : C \to k$ and $\Delta_C : C \to C \otimes C$ are linear maps, such that

$$c_1\varepsilon_C(c_2) = \phi_C(c), \ \ \varepsilon_C(c_1)c_2 = \psi_C(c),$$
$$\varepsilon_C(\phi_C(c)) = \varepsilon_C(\psi_C(c)) = \varepsilon_C(c), \ \ \phi_C(c_1) \otimes \Delta_C(c_2) = \Delta_C(c_1) \otimes \psi_C(c_2),$$
$$\phi_C \circ \psi_C = \psi_C \circ \phi_C, \ \ \Delta_C(\phi_C(c)) = \phi_C(c_1) \otimes \phi_C(c_2), \ \ \Delta_C(\psi_C(c)) = \psi_C(c_1) \otimes \psi_C(c_2).$$

Remark 2. *Note that the third line of the above identities can be derived from the first two lines. See ([15], Proposition 2.11).*

Example 2.

(1) *If $(C, \Delta_C, \varepsilon_C)$ is a coassociative coalgebra, $\phi, \psi : C \to C$ are both coalgebra isomorphisms, then $(C, (\phi \otimes \psi) \circ \Delta_C, \varepsilon_C, \phi, \psi)$ is a BiHom-coalgebra.*

(2) *If $\phi = \psi$, then C becomes a Hom-coalgebra.*

A *BiHom-bialgebra H over $\Bbbk$* is a 9-tuple $(H, \mu_H, 1_H, \Delta_H, \varepsilon_H, \alpha_H, \beta_H, \phi_H, \psi_H)$, with the property that $(H, \mu_H, 1_H, \alpha_H, \beta_H)$ is a BiHom-algebra, $(H, \Delta_H, \varepsilon_H, \phi_H, \psi_H)$ is a BiHom-coalgebra, and Δ_H, ε_H are all morphisms of BiHom-algebras preserving unit, i.e., for all $h, g \in H$,

$$\Delta_H(hg) = h_1g_1 \otimes h_2g_2, \ \varepsilon_H(hg) = \varepsilon_H(h)\varepsilon_H(g),$$
$$\Delta_H(1_H) = 1_H \otimes 1_H, \ \varepsilon_H(1_H) = 1_{\Bbbk}.$$

Moreover, it is easy to check that α_H, β_H are BiHom-coalgebra maps, ϕ_H, ψ_H are BiHom-algebra maps, and they commute with each other (see ([15], Proposition 2.14).

Example 3.

(1) *If* $(H, \mu_H, 1_H, \Delta_H, \varepsilon_H)$ *is a bialgebra,* $\alpha, \beta, \phi, \psi : H \to H$ *are all bialgebra isomorphisms, then* $H^{Bi} = (H, \mu_H \circ (\alpha \otimes \beta), 1_H, (\phi \otimes \psi) \circ \Delta_H, \varepsilon_H, \alpha, \beta, \phi, \psi)$ *is a BiHom-bialgebra.*

(2) *If* $H = (H, \mu_H, 1_H, \Delta_H, \varepsilon_H, \alpha, \beta, \phi, \psi)$ *is a finite dimensional BiHom-bialgebra,* $H^* = hom(H, \Bbbk)$*. Define the multiplication* $\star$*, the comultiplication* Δ_{H^*} *(with notation* $\Delta_{H^*}(p) = p_1 \otimes p_2$*) and* ε_{H^*} *by*

$$(p \star q)(h) = p(\alpha^{-1}\phi^{-1}(h_1))q(\beta^{-1}\psi^{-1}(h_2)), \ \ \varepsilon_{H^*}(p) = p(1_H),$$
$$(p_1 \otimes p_2)(h \otimes g) = p(\alpha^{-1}\psi^{-1}(h)\beta^{-1}\phi^{-1}(g)), \quad \text{where } p, q \in H^*, \ h, g \in H.$$

Define $\alpha_{H^*}, \beta_{H^*}, \phi_{H^*}, \psi_{H^*}$ *by*

$$\alpha_{H^*}(p) = p \circ \alpha^{-1}, \ \beta_{H^*}(p) = p \circ \beta^{-1}, \ \phi_{H^*}(p) = p \circ \psi^{-1}, \ \psi_{H^*}(p) = p \circ \phi^{-1}.$$

Then, $H^\star = (H^*, \star, \varepsilon_H, \Delta_{H^*}, \varepsilon_{H^*}, \alpha_{H^*}, \beta_{H^*}, \phi_{H^*}, \psi_{H^*})$ *is a BiHom-bialgebra.*

(3) *If* $\alpha = \beta = \phi = \psi$*, then H becomes a Hom-bialgebra. If* $\alpha^{-1} = \beta^{-1} = \phi = \psi$*, then H becomes a monoidal Hom-bialgebra.*

Recall from [22] that a *BiHom-Hopf algebra H over* $\Bbbk$ is a 10-tuple $(H, \mu_H, 1_H, \Delta_H, \varepsilon_H, S_H, \alpha_H, \beta_H, \phi_H, \psi_H)$, where $H = (H, \mu, 1_H, \Delta, \varepsilon, \alpha, \beta, \phi, \psi)$ is a BiHom-bialgebra, $S : H \to H$ (the *antipode*) commutes with $\alpha, \beta, \phi, \psi$, and satisfies, for any $h \in H$,

$$h_1 S(h_2) = S(h_1)h_2 = \varepsilon(h)1_H.$$

Proposition 1. *Recall from [22] that, if H is a BiHom-Hopf algebra, then for any* $a, b \in H$*, the antipode S satisfies*

$$S(ab) = S\alpha^{-1}\beta(b)S\alpha\beta^{-1}(a), \ \ S(1_H) = 1_H, \tag{1}$$
$$\Delta(S(a)) = S\phi\psi^{-1}(a_2) \otimes S\phi^{-1}\psi(a_1), \ \ \varepsilon \circ S = \varepsilon, \tag{2}$$
$$S\alpha^2\phi^2 = S\beta^2\psi^2. \tag{3}$$

Moreover, if S is a bijection, then

$$\alpha^2\phi^2 = \beta^2\psi^2, \tag{4}$$
$$S^{-1}(ab) = S^{-1}\alpha^{-1}\beta(b)S^{-1}\alpha\beta^{-1}(a), \ \ S^{-1}(1_H) = 1_H, \tag{5}$$
$$\Delta(S^{-1}(a)) = S^{-1}\phi\psi^{-1}(a_2) \otimes S^{-1}\phi^{-1}\psi(a_1), \ \ \varepsilon \circ S^{-1} = \varepsilon, \tag{6}$$
$$S^{-1}\alpha^{-2}\beta^2(a_2)a_1 = a_2 S^{-1}\alpha^2\beta^{-2}(a_1) = \varepsilon(a)1_H. \tag{7}$$

Example 4.

(1) *If* $(H, S, \mu_H, 1_H, \Delta_H, \varepsilon_H)$ *is a Hopf algebra,* $\alpha, \beta, \phi, \psi : H \to H$ *are all Hoppf algebra isomorphisms and satisfying* $S\alpha^2\phi^2 = S\beta^2\psi^2$*, then* $H^{BiH} = (H, S, \mu_H \circ (\alpha \otimes \beta), 1_H, (\phi \otimes \psi) \circ \Delta_H, \varepsilon_H, \alpha, \beta, \phi, \psi)$ *is a BiHom-Hopf algebra.*

(2) *If* $H = (H, S, \mu, 1_H, \Delta, \varepsilon, \alpha, \beta, \phi, \psi)$ *is a BiHom-Hopf algebra, under the consideration of Example 3 (2), we immediately obtain that* $H^{\star cop} = (H^*, \star, \varepsilon, \Delta_{H^*}^{cop}, \varepsilon_{H^*}, (S^{-1})^*, (\alpha^{-1})^*, (\beta^{-1})^*, (\psi^{-1})^*, (\phi^{-1})^*)$ *and* $H^{\star op} = (H^*, \star^{op}, \varepsilon, \Delta_{H^*}, \varepsilon_{H^*}, (S^{-1})^*, (\beta^{-1})^*, (\alpha^{-1})^*, (\phi^{-1})^*, (\psi^{-1})^*)$ *are all BiHom-Hopf algebras.*

(3) *If* $\alpha = \beta$ *and* $\phi = \psi$*, then H becomes the so-called monoidal BiHom-Hopf algebra (see ([10], Definition 6.4)). If* $\alpha = \beta = \phi = \psi$*, then H becomes the usual Hom–Hopf algebra. Similarly, if* $\alpha = \beta = \phi^{-1} = \psi^{-1}$*, then H becomes the usual monoidal Hom–Hopf algebra.*

2.2. BiHom-Modules and BiHom-Comodules of a BiHom-Bialgebra

Assume that $H = (H, \mu, 1_H, \Delta, \varepsilon, \alpha, \beta, \phi, \psi)$ is a BiHom-bialgebra. Recall that a $\Bbbk$-space M is called a *left BiHom-module of H* (in short, an *H-BiHom-module*) if there exist

$\Bbbk$-linear isomorphisms $\alpha_M, \beta_M, \phi_M, \psi_M : M \to M$ (the *Hom-structure maps*), and an H action $\theta_M : H \otimes M \to M$ (with notation $\theta_M(h \otimes m) = h \cdot m$), such that, for any $h, g \in H$, $m \in M$,

$$\alpha_M,\ \beta_M,\ \phi_M,\ \psi_M \text{ commute with each other,}$$
$$\alpha(h) \cdot \alpha_M(m) = \alpha_M(h \cdot m),\ \beta(h) \cdot \beta_M(m) = \beta_M(h \cdot m),\ \phi(h) \cdot \phi_M(m) = \phi_M(h \cdot m),$$
$$\psi(h) \cdot \psi_M(m) = \psi_M(h \cdot m),\ \alpha(h) \cdot (g \cdot m) = (hg) \cdot \beta_M(m),\ 1_H \cdot m = \beta_M(m).$$

If $(M, \alpha_M, \beta_M, \phi_M, \psi_M)$ and $(N, \alpha_N, \beta_N, \phi_N, \psi_N)$ are left H-BiHom-modules with H-actions θ_M and, respectively, θ_N, a *morphism of H-BiHom-modules* $f \in hom_{\Bbbk}(M, N)$ is an H-linear map satisfying the conditions

$$\alpha_N \circ f = f \circ \alpha_M,\ \beta_N \circ f = f \circ \beta_M,\ \phi_N \circ f = f \circ \phi_M,\ \psi_N \circ f = f \circ \psi_M.$$

The category of H-BiHom-modules and morphisms will be denoted by ${}_H\mathcal{BM}$.

Remark 3.

(1) *Obviously,* $H \in Obj({}_H\mathcal{BM})$.
(2) *The definition of right BiHom-module of H can be defined in a similar way.*
(3) *For any integers* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}, \mathfrak{f}, \mathfrak{g}, \mathfrak{h} \in \mathbb{Z}$, $M, N, P \in {}_H\mathcal{BM}$, ${}_H\mathcal{BM}$ *forms a monoidal category under the following structures:*
   - *the tensor product of* $(M, \alpha_M, \beta_M, \phi_M, \psi_M)$ *and* $(N, \alpha_N, \beta_N, \phi_N, \psi_N)$ *is* $(M \otimes N, \alpha_{M\otimes N}, \alpha_M \otimes \alpha_N, \beta_M \otimes \beta_N, \phi_M \otimes \phi_N, \psi_M \otimes \psi_N)$, *where the H-action on* $M \otimes N$ *is given by*
   $$h \cdot (m \otimes n) = \alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\phi^{\mathfrak{c}}\psi^{\mathfrak{d}}(h_1) \cdot m \otimes \alpha^{\mathfrak{e}}\beta^{\mathfrak{f}}\phi^{\mathfrak{g}}\psi^{\mathfrak{h}}(h_2) \cdot n, \quad \text{where } m \in M,\ n \in N,\ h \in H;$$
   - *the unit object is* $(\Bbbk, id_{\Bbbk}, id_{\Bbbk}, id_{\Bbbk}, id_{\Bbbk})$ *with the trivial module action;*
   - *for any* $m \in M$, $n \in N$, $p \in P$, *the the associativity and the unit constraints are given by*
   $$\mathbf{a}_{M,N,P}((m \otimes n) \otimes p) = \alpha_M^{-\mathfrak{a}}\beta_M^{-\mathfrak{b}}\phi_M^{-\mathfrak{c}-1}\psi_M^{-\mathfrak{d}}(m) \otimes (n \otimes \alpha_P^{\mathfrak{e}}\beta_P^{\mathfrak{f}}\phi_P^{\mathfrak{g}}\psi_P^{\mathfrak{h}+1}(p));$$
   $$\mathbf{l}_M(1_{\Bbbk} \otimes m) = \alpha_M^{-\mathfrak{e}}\beta_M^{-\mathfrak{f}}\phi_M^{-\mathfrak{g}}\psi_M^{-\mathfrak{h}-1}(m),\ \ \mathbf{r}_M(m \otimes 1_{\Bbbk}) = \alpha_M^{-\mathfrak{a}}\beta_M^{-\mathfrak{b}}\phi_M^{-\mathfrak{c}-1}\psi_M^{-\mathfrak{d}}(m).$$

   We write this monoidal category by ${}_H\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{e},\mathfrak{f},\mathfrak{g},\mathfrak{h}}$.

Dually, recall from ([15], Definition 5.3) that a *right H-BiHom-comodule* is a 5-tuple $(M, \alpha_M, \beta_M, \phi_M, \psi_M)$, where M is a linear space, $\alpha_M, \beta_M, \phi_M, \psi_M : M \to M$ are linear isomorphisms, and we have a linear map (called a *coaction*) $\rho : M \to M \otimes H$, with notation $\rho(m) = m_0 \otimes m_1$, for all $m \in M$, such that the following conditions are satisfied

$$\phi_M,\ \psi_M,\ \alpha_M,\ \beta_M \text{ commute with each other,}$$
$$(\alpha_M \otimes \alpha) \circ \rho = \rho \circ \alpha_M,\ (\beta_M \otimes \beta) \circ \rho = \rho \circ \beta_M,\ (\phi_M \otimes \phi) \circ \rho = \rho \circ \phi_M,$$
$$(\psi_M \otimes \psi) \circ \rho = \rho \circ \psi_M, \phi_M(m_0) \otimes m_{11} \otimes m_{12} = m_{00} \otimes m_{01} \otimes \psi(m_1), m_0\varepsilon(m_1) = \phi_M(m).$$

If $(M, \alpha_M, \beta_M, \phi_M, \psi_M)$ and $(N, \alpha_N, \beta_N, \phi_N, \psi_N)$ are right H-BiHom-comodules with coactions ρ_M and, respectively ρ_N, a *morphism of right H-BiHom-comodules* $f : M \to N$ is a linear map satisfying the conditions

$$\alpha_N \circ f = f \circ \alpha_M,\ \beta_N \circ f = f \circ \beta_M,\ \phi_N \circ f = f \circ \phi_M,$$
$$\psi_N \circ f = f \circ \psi_M,\ \rho_N \circ f = (f \otimes id_H) \circ \rho_M.$$

The category of H-BiHom-comodules and H-colinear morphisms will be denoted by $\mathcal{BM}^H$.

Remark 4.

(1) *Obviously,* $H \in Obj\mathcal{BM}^H$.
(2) *The definition of left BiHom-comodule of H can be defined in a similar way.*
(3) *For any integers* $\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l}, \mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q} \in \mathbb{Z}$, $\mathcal{BM}^H$ *forms a monoidal category under the following structures:*

- *the tensor product of H-BiHom-comodules* $(U, \alpha_U, \beta_U, \phi_U, \psi_U)$ *and* $(V, \alpha_V, \beta_V, \phi_V, \psi_V)$ *is* $(U \otimes V, \alpha_U \otimes \alpha_V, \beta_U \otimes \beta_V, \phi_U \otimes \phi_V, \psi_U \otimes \psi_V)$ *with the H-coaction* $\rho^{U \otimes V}$:

$$u \otimes v \mapsto u_{(0)} \otimes v_{(0)} \otimes \alpha^{\mathfrak{i}} \beta^{\mathfrak{j}} \phi^{\mathfrak{k}} \psi^{\mathfrak{l}}(u_{(1)}) \alpha^{\mathfrak{m}} \beta^{\mathfrak{n}} \phi^{\mathfrak{p}} \psi^{\mathfrak{q}}(v_{(1)});$$

- *the unit object is* $(\mathbb{k}, id_{\mathbb{k}}, id_{\mathbb{k}}, id_{\mathbb{k}}, id_{\mathbb{k}})$ *with the trivial coaction;*
- *the associativity constraint* **a** *and the unit constraint* **l** *and* **r** *are given by*

$$\mathbf{a}_{U,V,W}((u \otimes v) \otimes w) = \alpha_U^{\mathfrak{i}+1} \beta_U^{\mathfrak{j}} \phi_U^{\mathfrak{k}} \psi_U^{\mathfrak{l}}(u) \otimes (v \otimes \alpha_W^{-\mathfrak{m}} \beta_W^{-\mathfrak{n}-1} \phi_W^{-\mathfrak{p}} \psi_W^{-\mathfrak{q}}(w));$$
$$\mathbf{r}_U(u \otimes 1_{\mathbb{k}}) = \alpha^{\mathfrak{i}+1} \beta_U^{\mathfrak{j}} \phi_U^{\mathfrak{k}} \psi_{(}^{\mathfrak{l}} u), \ \ \mathbf{l}_U(1_{\mathbb{k}} \otimes u) = \alpha^{\mathfrak{m}} \beta_U^{\mathfrak{n}+1} \phi_U^{\mathfrak{p}} \psi_{(}^{\mathfrak{q}} u).$$

We denote this monoidal category by $(\mathcal{BM}^H)^{\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}}_{\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q}}$.

3. The BiHom-Type Heisenberg Doubles and the BiHom-Hopf Equation

In this section, we will discuss the algebraic solutions of the BiHom-Hopf equation and the BiHom-pentagon equation.

3.1. BiHom-Hopf Equation and BiHom-Pentagon Equation

In this subsection, we will discuss the relation between the BiHom-Hopf equation and BiHom-pentagon equation.

Definition 1. *Let* $A = (A, \mu_A, 1_A, \alpha_A, \beta_A)$ *be a BiHom-algebra over* $\mathbb{k}$, $\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$ *be an element in* $A \otimes A$ *and satisfy*

$$(\alpha_A \otimes \alpha_A)\mathcal{R} = \mathcal{R}, \ \ (\beta_A \otimes \beta_A)\mathcal{R} = \mathcal{R}. \tag{8}$$

(1) If $\mathcal{R}$ *satisfies* $\mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12} = \mathcal{R}^{12}\mathcal{R}^{23}$, *i.e.,*

$$\sum \beta_A(\mathcal{R}^{(1)})\mathcal{S}^{(1)} \otimes \mathcal{T}^{(1)}\mathcal{S}^{(2)} \otimes \mathcal{T}^{(2)}\alpha_A(\mathcal{R}^{(2)}) = \sum \alpha_A(\mathcal{R}^{(1)}) \otimes \mathcal{R}^{(2)}\mathcal{S}^{(1)} \otimes \beta_A(\mathcal{S}^{(2)}), \tag{9}$$

where $\mathcal{R} = \mathcal{S} = \mathcal{T}$, *then we say* $\mathcal{R}$ *is* a solution of the BiHom-Hopf equation.

(2) If $\mathcal{R}$ *satisfies* $\mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23} = \mathcal{R}^{23}\mathcal{R}^{12}$, *i.e.,*

$$\sum \mathcal{R}^{(1)}\alpha_A(\mathcal{S}^{(1)}) \otimes \mathcal{R}^{(2)}\mathcal{T}^{(1)} \otimes \beta_A(\mathcal{S}^{(2)})\mathcal{T}^{(2)} = \sum \beta_A(\mathcal{R}^{(1)}) \otimes \mathcal{S}^{(1)}\mathcal{R}^{(2)} \otimes \alpha_A(\mathcal{S}^{(2)}), \tag{10}$$

then we say $\mathcal{R}$ *is a solution of the BiHom-pentagon equation.*

Example 5.

(1) $1_A \otimes 1_A$ *is a solution of the BiHom-Hopf equation and the BiHom-pentagon equation.*
(2) *For any* $a \in A$, $a \otimes 1_A$ *is a solution of the BiHom-Hopf equation if and only if* $\alpha_A(a) = \beta_A(a)$ *and* $\alpha_A(a)a = \alpha_A(a)$, $1_A \otimes a$ *is a solution of the BiHom-Hopf equation if and only if* $\alpha_A(a) = \beta_A(a)$ *and* $a\alpha_A(a) = \alpha_A(a)$.
(3) *If* $\alpha_A = \beta_A = id_A$, *then A is the usual algebra, and the solution of the BiHom-Hopf equation becomes the solution of usual Hopf equation, the solution of the BiHom-pentagon equation becomes the solution of usual pentagon equation (see [[16], Definition 11] for details).*

Proposition 2.

(1) *If* $\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \in A \otimes A$ *is a solution of the BiHom-Hopf equation, then* $\mathcal{R}^{21} = \sum \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)} \in A \otimes A$ *is a solution of the BiHom-pentagon equation.*

(2) *If* $\mathcal{R} \in A \otimes A$ *is invertible, then* $\mathcal{R}$ *is a solution of the BiHom-Hopf equation if and only if* $\mathcal{R}^{-1}$ *is a solution of the BiHom-pentagon equation.*

Proof.

(1) Self evident.

(2) Note that $\mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12}$ and $\overline{\mathcal{R}}^{12}\overline{\mathcal{R}}^{13}\overline{\mathcal{R}}^{23}$ are inverse with each other, $\mathcal{R}^{12}\mathcal{R}^{23}$ and $\overline{\mathcal{R}}^{23}\overline{\mathcal{R}}^{12}$ are inverse with each other, where $\overline{\mathcal{R}}$ means the inverse of $\mathcal{R}$. Hence, the conclusion holds.

□

Proposition 3. *Let* $A = (A, \mu_A, 1_A, \alpha_A, \beta_A)$, $\mathcal{R} \in A \otimes A$ *be an invertible solution of the BiHom-Hopf equation. If we define a* $\Bbbk$*-linear map* $\Delta_L : A \to A \otimes A$, $a \mapsto a_1 \otimes a_2$, *by*

$$a_1 \otimes a_2 := (\mathcal{R}(1_A \otimes a))\mathcal{R}^{-1} = \sum \alpha_A(\mathcal{R}^{(1)})\overline{\mathcal{R}}^{(1)} \otimes (\mathcal{R}^{(2)}a)\overline{\mathcal{R}}^{(2)},$$

where $\mathcal{R}^{-1} = \sum \overline{\mathcal{R}}^{(1)} \otimes \overline{\mathcal{R}}^{(2)}$, $a \in A$, *then* Δ_L *is a BiHom-algebra morphism. Furthermore,* $(A, \Delta_L, \alpha_A \circ \beta_A)$ *forms a Hom-coalgebra without a counit.*

Proof. For any $a, b \in A$, we compute

$$\begin{aligned} &\Delta_L(a)\Delta_L(b) = \sum(\alpha_A(\mathcal{R}^{(1)})\overline{\mathcal{R}}^{(1)} \otimes (\mathcal{R}^{(2)}a)\overline{\mathcal{R}}^{(2)})(\alpha_A(\mathcal{S}^{(1)})\overline{\mathcal{S}}^{(1)} \otimes (\mathcal{S}^{(2)}b)\overline{\mathcal{S}}^{(2)}) \\ = \;& (\alpha_A(\mathcal{R}^{(1)})(\alpha_A^{-1}(\overline{\mathcal{R}}^{(1)})\alpha_A\beta_A^{-1}(\mathcal{S}^{(1)})))\beta_A(\overline{\mathcal{S}}^{(1)}) \otimes ((\mathcal{R}^{(2)}a)((\alpha_A^{-2}(\overline{\mathcal{R}}^{(2)})\beta_A^{-1}(\mathcal{S}^{(2)}))b))\beta_A(\overline{\mathcal{S}}^{(2)}) \\ \overset{(8)}{=} \;& \alpha_A^2(\mathcal{R}^{(1)})\overline{\mathcal{S}}^{(1)} \otimes (\alpha_A(\mathcal{R}^{(2)})(ab))\overline{\mathcal{S}}^{(2)} \overset{(3.1)}{=} \Delta_L(ab), \end{aligned}$$

which implies Δ_L is a BiHom-algebra morphism.

Moreover, since Proposition 2, $\mathcal{R}^{-1}$ is a solution of the BiHom-pentagon equation, then we have

$$\begin{aligned} &\Delta_L(a_1) \otimes \alpha_A\beta_A(a_2) \\ \overset{(8)}{=} \;& \sum \alpha_A(\mathcal{S}^{(1)})\beta_A(\overline{\mathcal{S}}^{(1)}) \otimes (\mathcal{S}^{(2)}\mathcal{R}^{(1)})(\overline{\mathcal{R}}^{(1)}\overline{\mathcal{S}}^{(2)}) \otimes (\alpha_A^{-1}\beta_A(\mathcal{R}^{(2)})\alpha_A\beta_A(a))\alpha_A(\overline{\mathcal{R}}^{(2)}) \\ \overset{(9)}{=} \;& \sum(\beta_A(\mathcal{R}^{(1)})\mathcal{T}^{(1)})\beta_A(\overline{\mathcal{S}}^{(1)}) \otimes (\mathcal{S}^{(1)}\mathcal{T}^{(2)})(\overline{\mathcal{R}}^{(1)}\overline{\mathcal{S}}^{(2)}) \otimes ((\alpha_A^{-1}(\mathcal{S}^{(2)})\mathcal{R}^{(2)})\alpha_A\beta_A(a))\alpha_A(\overline{\mathcal{R}}^{(2)}) \\ \overset{(10)}{=} \;& \sum(\beta_A(\mathcal{R}^{(1)})\mathcal{T}^{(1)})(\overline{\mathcal{R}}^{(1)}\alpha_A(\overline{\mathcal{S}}^{(1)})) \otimes (\mathcal{S}^{(1)}\mathcal{T}^{(2)})(\overline{\mathcal{R}}^{(2)}\overline{\mathcal{T}}^{(1)}) \\ & \quad \otimes ((\alpha_A^{-1}(\mathcal{S}^{(2)})\mathcal{R}^{(2)})\alpha_A\beta_A(a))(\beta_A(\overline{\mathcal{S}}^{(2)})\overline{\mathcal{T}}^{(2)}) \\ \overset{(8)}{=} \;& \sum \alpha_A^2\beta_A(\mathcal{R}^{(1)})\alpha_A\beta_A(\overline{\mathcal{S}}^{(1)}) \otimes \alpha_A(\mathcal{S}^{(1)})\beta_A(\overline{\mathcal{T}}^{(1)}) \otimes (\mathcal{S}^{(2)}((\mathcal{R}^{(2)}a)\overline{\mathcal{S}}^{(2)}))\beta_A(\overline{\mathcal{T}}^{(2)}) \\ \overset{(8)}{=} \;& \alpha_A\beta_A(a_1) \otimes \Delta_L(a_2), \end{aligned}$$

hence the conclusion holds. □

3.2. Heisenberg Doubles of a BiHom-Hopf Algebra

In this subsection, we will provide the algebraic solutions of BiHom-Hopf equation from Heisenberg doubles. From now on, we assume that $H = (H, S)$ is a finite dimensional BiHom-Hopf algebra, and S is bijective. Recall from Example 3 (2) that

$$H^\star = (H^*, \star, \varepsilon_H, \Delta_{H^*}, \varepsilon_{H^*}, (\alpha^{-1})^*, (\beta^{-1})^*, (\phi^{-1})^*, (\psi^{-1})^*)$$

is a BiHom-bialgebra. Then, we obtain the following definition.

Definition 2. *For any $\mathfrak{r}, \mathfrak{s}, \mathfrak{u}, \mathfrak{v} \in \mathbb{Z}$, the $\mathfrak{r}, \mathfrak{s}, \mathfrak{u}, \mathfrak{v}$-th* Heisenberg double $\mathfrak{H}_{\mathfrak{r},\mathfrak{s},\mathfrak{u},\mathfrak{v}}(H) = H \otimes H^\star$ *of H, in a form containing H and $H^\star$, is a BiHom-algebra with the following structure*

$$(a \otimes p)\sharp(b \otimes q) := a\phi^{-1}(b_1) \otimes p(\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\phi^{\mathfrak{u}}\psi^{\mathfrak{v}}(b_2)\beta^{-1}(?)) \star q, \quad 1_{\mathfrak{H}_{\mathfrak{r},\mathfrak{s},\mathfrak{u},\mathfrak{v}}(H)} := 1_H \otimes \varepsilon,$$
$$\alpha_{\mathfrak{H}_{\mathfrak{r},\mathfrak{s},\mathfrak{u},\mathfrak{v}}(H)} = \alpha \otimes (\alpha^{-1})^*, \ \beta_{\mathfrak{H}_{\mathfrak{r},\mathfrak{s},\mathfrak{u},\mathfrak{v}}(H)} = \beta \otimes (\beta^{-1})^*,$$

where $p, q \in H^\star$, $a, b \in H$.

Proof. For any $a, b, c, x \in H$, $p, q, f \in H^\star$, we have

$$\begin{aligned}
&(\alpha_{\mathfrak{H}_{\mathfrak{r},\mathfrak{s},\mathfrak{u},\mathfrak{v}}(H)}(a \otimes p))\sharp((b \otimes q)\sharp(c \otimes f))(x) \\
= \ & \alpha(a)(\phi^{-1}(b_1)\phi^{-2}(c_{11})) \otimes (p((\alpha^{\mathfrak{r}-1}\beta^{\mathfrak{s}}\phi^{\mathfrak{u}}\psi^{\mathfrak{v}}(b_2)\alpha^{\mathfrak{r}-1}\beta^{\mathfrak{s}}\phi^{\mathfrak{u}-1}\psi^{\mathfrak{v}}(c_{12}))\alpha^{-1}\beta^{-1}(?)) \\
&\quad \star (q(\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\phi^{\mathfrak{u}}\psi^{\mathfrak{v}}(c_2)\beta^{-1}(?)) \star f))(x) \\
= \ & (a\phi^{-1}(b_1))\beta\phi^{-1}(c_1) \otimes p(\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\phi^{\mathfrak{u}}\psi^{\mathfrak{v}}(b_2)(\alpha^{\mathfrak{r}-1}\beta^{\mathfrak{s}}\phi^{\mathfrak{u}-1}\psi^{\mathfrak{v}}(c_{21})\alpha^{-2}\beta^{-2}\phi^{-2}(x_{11}))) \\
&\quad q(\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\phi^{\mathfrak{u}}\psi^{\mathfrak{v}-1}(c_{22})\alpha^{-1}\beta^{-2}\phi^{-1}\psi^{-1}(x_{12}))f(\beta^{-2}\psi^{-1}(x_2)) \\
= \ & (a\phi^{-1}(b_1))\beta\phi^{-1}(c_1) \otimes (p(\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\phi^{\mathfrak{u}}\psi^{\mathfrak{v}}(b_2)\beta^{-1}(?)) \star q)(\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}+1}\phi^{\mathfrak{u}}\psi^{\mathfrak{v}}(c_2)\alpha^{-1}\beta^{-1}\phi^{-1}(x_1)) \\
&\quad f(\beta^{-2}\psi^{-1}(x_2)) \\
= \ & (a\phi^{-1}(b_1) \otimes p(\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\phi^{\mathfrak{u}}\psi^{\mathfrak{v}}(b_2)\beta^{-1}(?)) \star q)(\beta(c) \otimes f \circ \beta^{-1})(x) \\
= \ & ((a \otimes p)\sharp(b \otimes q))\sharp(\beta(c) \otimes f \circ \beta^{-1})(x),
\end{aligned}$$

which implies the BiHom-associative law. Obviously, we have

$$(a \otimes p)\sharp(1_H \otimes \varepsilon) = (\alpha \otimes (\alpha^{-1})^*)(a \otimes p), \ \ (1_H \otimes \varepsilon)\sharp(a \otimes p) = (\beta \otimes (\beta^{-1})^*)(a \otimes p),$$

which implies the BiHom-unit law. Hence, $(\mathfrak{H}_{\mathfrak{r},\mathfrak{s},\mathfrak{u},\mathfrak{v}}(H), \sharp, 1_H \otimes \varepsilon, \alpha \otimes (\alpha^{-1})^*, \beta \otimes (\beta^{-1})^*)$ is a BiHom-algebra. □

Theorem 1. $\sum(\alpha^{-\mathfrak{r}-2}\beta^{-\mathfrak{s}+1}\phi^{-\mathfrak{u}}\psi^{-\mathfrak{v}-1}(e_i) \otimes \varepsilon) \otimes (1_H \otimes e^i) \in \mathfrak{H}_{\mathfrak{r},\mathfrak{s},\mathfrak{u},\mathfrak{v}}(H) \otimes \mathfrak{H}_{\mathfrak{r},\mathfrak{s},\mathfrak{u},\mathfrak{v}}(H)$ *is a solution of the BiHom-Hopf equation, where e_i and e^i are dual bases of H and $H^\star$, respectively.*

Proof. For any $x, y, z \in H$, we have

$$\begin{aligned}
&\sum((\alpha^{-\mathfrak{r}-2}\beta^{-\mathfrak{s}+2}\phi^{-\mathfrak{u}}\psi^{-\mathfrak{v}-1}(e_i) \otimes \varepsilon)\sharp(\alpha^{-\mathfrak{r}-2}\beta^{-\mathfrak{s}+1}\phi^{-\mathfrak{u}}\psi^{-\mathfrak{v}-1}(a_i) \otimes \varepsilon))(x) \\
&\quad \otimes ((\alpha^{-\mathfrak{r}-2}\beta^{-\mathfrak{s}+1}\phi^{-\mathfrak{u}}\psi^{-\mathfrak{v}-1}(o_i) \otimes \varepsilon)\sharp(1_H \otimes a^i))(y) \otimes ((1_H \otimes o^i)\sharp(1_H \otimes e^i(\alpha^{-1}(?))))(z) \\
= \ & \sum(\alpha^{-\mathfrak{r}-2}\beta^{-\mathfrak{s}+2}\phi^{-\mathfrak{u}}\psi^{-\mathfrak{v}-1}(e_i)\alpha^{-\mathfrak{r}-2}\beta^{-\mathfrak{s}+1}\phi^{-\mathfrak{u}-1}\psi^{-\mathfrak{v}-1}(a_{i1})\varepsilon(a_{i2}) \otimes \varepsilon)(x) \\
&\quad \otimes (\alpha^{-\mathfrak{r}-1}\beta^{-\mathfrak{s}+1}\phi^{-\mathfrak{u}}\psi^{-\mathfrak{v}-1}(o_i) \otimes a^i(\beta^{-1}(?)))(y) \otimes (1_H \otimes o^i \star e^i(\alpha^{-1}(?)))(z) \\
= \ & \sum(\alpha^{-\mathfrak{r}-3}\beta^{-\mathfrak{s}+1}\phi^{-\mathfrak{u}}\psi^{-\mathfrak{v}-2}(z_2)\alpha^{-\mathfrak{r}-2}\beta^{-\mathfrak{s}}\phi^{-\mathfrak{u}}\psi^{-\mathfrak{v}-1}(y) \otimes \varepsilon(x)) \\
&\quad \otimes (\alpha^{-\mathfrak{r}-2}\beta^{-\mathfrak{s}+1}\phi^{-\mathfrak{u}-1}\psi^{-\mathfrak{v}-1}(z_1) \otimes 1_{\Bbbk}) \otimes (1_H \otimes 1_{\Bbbk}),
\end{aligned}$$

and

$$\begin{aligned}
&\sum(\alpha^{-\mathfrak{r}-1}\beta^{-\mathfrak{s}+1}\phi^{-\mathfrak{u}}\psi^{-\mathfrak{v}-1}(e_i) \otimes \varepsilon)(x) \\
&\quad \otimes ((1_H \otimes e^i)\sharp(\alpha^{-\mathfrak{r}-2}\beta^{-\mathfrak{s}+1}\phi^{-\mathfrak{u}}\psi^{-\mathfrak{v}-1}(o_i) \otimes \varepsilon))(y) \otimes (1_H \otimes o^i(\beta^{-1}(?)))(z) \\
= \ & \sum(\alpha^{-\mathfrak{r}-1}\beta^{-\mathfrak{s}+1}\phi^{-\mathfrak{u}}\psi^{-\mathfrak{v}-1}(e_i) \otimes \varepsilon(x)) \\
&\quad \otimes (\alpha^{-\mathfrak{r}-2}\beta^{-\mathfrak{s}+2}\phi^{-\mathfrak{u}-1}\psi^{-\mathfrak{v}-1}(o_{i1}) \otimes e^i(\alpha^{-2}\beta\psi^{-1}(o_{i2})\alpha^{-1}\beta^{-1}(y))) \otimes (1_H \otimes o^i(\beta^{-1}(z))) \\
= \ & \sum(\alpha^{-\mathfrak{r}-3}\beta^{-\mathfrak{s}+1}\phi^{-\mathfrak{u}}\psi^{-\mathfrak{v}-2}(z_2)\alpha^{-\mathfrak{r}-2}\beta^{-\mathfrak{s}}\phi^{-\mathfrak{u}}\psi^{-\mathfrak{v}-1}(y) \otimes \varepsilon(x)) \\
&\quad \otimes (\alpha^{-\mathfrak{r}-2}\beta^{-\mathfrak{s}+1}\phi^{-\mathfrak{u}-1}\psi^{-\mathfrak{v}-1}(z_1) \otimes 1_{\Bbbk}) \otimes (1_H \otimes 1_{\Bbbk}),
\end{aligned}$$

where a_i and a^i and o_i and o^i are both dual bases of H and $H^\star$, respectively. This implies that $\mathcal{R} = \sum(\alpha^{-\mathfrak{r}-2}\beta^{-\mathfrak{s}+1}\ \phi^{-\mathfrak{u}}\psi^{-\mathfrak{v}-1}(e_i) \otimes \varepsilon) \otimes (1_H \otimes e^i)$ is a solution of the BiHom-Hopf equation. □

Corollary 1. $\sum(S\alpha^{-\mathfrak{r}-1}\beta^{-\mathfrak{s}}\phi^{-\mathfrak{u}+1}\psi^{-\mathfrak{v}-2}(e_i)\otimes\varepsilon)\otimes(1_H\otimes e^i)\in\mathfrak{H}_{\mathfrak{r},\mathfrak{s},\mathfrak{u},\mathfrak{v}}(H)\otimes\mathfrak{H}_{\mathfrak{r},\mathfrak{s},\mathfrak{u},\mathfrak{v}}(H)$ *is a solution of the BiHom-pentagon equation.*

Proof. It is easy to check (by using Equation (3)) that

$$\sum(S\alpha^{-\mathfrak{r}-1}\beta^{-\mathfrak{s}}\phi^{-\mathfrak{u}+1}\psi^{-\mathfrak{v}-2}(e_i)\otimes\varepsilon)\otimes(1_H\otimes e^i)$$

is the inverse of $\mathcal{R}=\sum(\alpha^{-\mathfrak{r}-2}\beta^{-\mathfrak{s}+1}\phi^{-\mathfrak{u}}\psi^{-\mathfrak{v}-1}(e_i)\otimes\varepsilon)\otimes(1_H\otimes e^i)$. Hence, the conclusion holds because of Proposition 2. □

4. The BiHom-Long Dimodules and the BiHom-$\mathcal{D}$ Equation

In this section, we will describe the algebraic solutions of the BiHom–Yang-Baxter equation and BiHom-$\mathcal{D}$ equation.

4.1. The Parametric Generalized BiHom-Long Dimodules

In this subsection, we will introduce the generalized BiHom-Long dimodules which play an important role in the BiHom–Yang-Baxter equation and BiHom-$\mathcal{D}$ equation. Assume that $(H,S_H,\alpha_H,\beta_H,\phi_H,\psi_H)$ and $(B,S_B,\alpha_B,\beta_B,\phi_B,\psi_B)$ are two BiHom-Hopf algebras.

Definition 3. *A $\Bbbk$-space U is called* a left-right generalized BiHom-Long dimodule *of H and B, if there exist morphisms $\alpha_U,\beta_U,\phi_U,\psi_U\in Aut(U)$ such that $(U,\alpha_U,\beta_U,\phi_U,\psi_U)$ is both a left H-BiHom-module and a right B-BiHom-comodule, and the following compatibility condition is satisfied:*

$$\underline{h\cdot u_0}\otimes\underline{h\cdot u}_1=\phi_H(h)\cdot u_0\otimes\beta_B(u_1), \tag{11}$$

for all $u\in U$ and $h\in H$. We denote by ${}_H\mathcal{L}^B$ the category of generalized BiHom-Long dimodules, with morphisms being H-linear and B-colinear.

Example 6.

(1) *For any $\mathfrak{r},\mathfrak{s},\mathfrak{u},\mathfrak{v},\mathfrak{t},\mathfrak{w},\mathfrak{x},\mathfrak{y}\in\mathbb{Z}$, define the left H-action $\rightharpoonup$ on $H\otimes B$ by*

$$x\rightharpoonup(h\otimes a)=\alpha_H^{\mathfrak{r}}\beta_H^{\mathfrak{s}}\phi_H^{\mathfrak{u}}\psi_H^{\mathfrak{v}}(x)h\otimes\beta_B(a),$$

and define the right B-coaction on $H\otimes B$ by

$$\rho(h\otimes a)=\phi_H(h)\otimes a_1\otimes\alpha_B^{\mathfrak{t}}\beta_B^{\mathfrak{w}}\phi_B^{\mathfrak{x}}\psi_B^{\mathfrak{y}}(a_2),$$

then it is straightforward to check that $(H\otimes B,\rightharpoonup,\rho,\alpha_H\otimes\alpha_B,\beta_H\otimes\beta_B,\phi_H\otimes\phi_B,\psi_H\otimes\psi_B)$ is a generalized BiHom-Long dimodule.

(2) *Similarly, for $\mathfrak{r},\mathfrak{s},\mathfrak{u},\mathfrak{v},\mathfrak{t},\mathfrak{w},\mathfrak{x},\mathfrak{y}\in\mathbb{Z}$, if we define the left H-action $\rightharpoondown$ on $B\otimes H$ by*

$$x\rightharpoondown(a\otimes h)=\beta_B(a)\otimes\alpha_H^{\mathfrak{r}}\beta_H^{\mathfrak{s}}\phi_H^{\mathfrak{u}}\psi_H^{\mathfrak{v}}(x)h,$$

and define the right B-coaction on $B\otimes H$ by

$$\varrho(a\otimes h)=a_1\otimes\phi_H(h)\otimes\alpha_B^{\mathfrak{t}}\beta_B^{\mathfrak{w}}\phi_B^{\mathfrak{x}}\psi_B^{\mathfrak{y}}(a_2),$$

then, it is straightforward to check that $(B\otimes H,\rightharpoondown,\varrho,\alpha_H\otimes\alpha_B,\beta_H\otimes\beta_B,\phi_H\otimes\phi_B,\psi_H\otimes\psi_B)$ is also an object in ${}_H\mathcal{L}^B$.

For any $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d},\mathfrak{e},\mathfrak{f},\mathfrak{g},\mathfrak{h}\in\mathbb{Z}$, we can define the monoidal structures in ${}_H\mathcal{L}^B$ as follows:

- the monoidal product of $(U,\alpha_U,\beta_U,\phi_U,\psi_U)$ of $(V,\alpha_V,\beta_V,\phi_V,\psi_V)$ is $(U\otimes V,\alpha_U\otimes\alpha_V,\beta_U\otimes\beta_V,\phi_U\otimes\phi_V,\psi_U\otimes\psi_V)$, where the BiHom-module and BiHom-comodule structures are given by

$$h\cdot(u\otimes v)=\alpha_H^{\mathfrak{a}}\beta_H^{\mathfrak{b}}\phi_H^{\mathfrak{c}}\psi_H^{\mathfrak{d}}(h_1)\cdot u\otimes\alpha_H^{\mathfrak{e}}\beta_H^{\mathfrak{f}}\phi_H^{\mathfrak{g}}\psi_H^{\mathfrak{h}}(h_2)\cdot v,$$
$$\rho^{U\otimes V}(u\otimes v)=u_0\otimes v_0\otimes\alpha_B^{-\mathfrak{a}-1}\beta_B^{-\mathfrak{b}}\phi_B^{-\mathfrak{c}-1}\psi_B^{-\mathfrak{d}}(u_1)\alpha_B^{-\mathfrak{e}}\beta_B^{-\mathfrak{f}-1}\phi_B^{-\mathfrak{g}}\psi_B^{-\mathfrak{h}-1}(v_1);$$

- the unit object is $(\Bbbk, id, id, id, id)$ with the trivial H-action and trivial B-coaction.

Theorem 2. *For any $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d},\mathfrak{e},\mathfrak{f},\mathfrak{g},\mathfrak{h}\in\mathbb{Z}$, ${}_H\mathcal{L}^B$ forms a monoidal category under the above structures.*

Proof. First, for any $u\in U$, $v\in V$, we have

$$\begin{aligned}\rho(h\cdot(u\otimes v))&=\underline{\alpha_H^{\mathfrak{a}}\beta_H^{\mathfrak{b}}\phi_H^{\mathfrak{c}}\psi_H^{\mathfrak{d}}(h_1)\cdot u}_0\otimes\underline{\alpha_H^{\mathfrak{e}}\beta_H^{\mathfrak{f}}\phi_H^{\mathfrak{g}}\psi_H^{\mathfrak{h}}(h_2)\cdot v}_0\\&\quad\otimes\alpha_B^{-\mathfrak{a}-1}\beta_B^{-\mathfrak{b}}\phi_B^{-\mathfrak{c}-1}\psi_B^{-\mathfrak{d}}(\underline{\alpha_H^{\mathfrak{a}}\beta_H^{\mathfrak{b}}\phi_H^{\mathfrak{c}}\psi_H^{\mathfrak{d}}(h_1)\cdot u}_1)\alpha_B^{-\mathfrak{e}}\beta_B^{-\mathfrak{f}-1}\phi_B^{-\mathfrak{g}}\psi_B^{-\mathfrak{h}-1}(\underline{\alpha_H^{\mathfrak{e}}\beta_H^{\mathfrak{f}}\phi_H^{\mathfrak{g}}\psi_H^{\mathfrak{h}}(h_2)\cdot v}_1)\\&=\alpha_H^{\mathfrak{a}}\beta_H^{\mathfrak{b}}\phi_H^{\mathfrak{c}+1}\psi_H^{\mathfrak{d}}(h_1)\cdot u_0\otimes\alpha_H^{\mathfrak{e}}\beta_H^{\mathfrak{f}}\phi_H^{\mathfrak{g}+1}\psi_H^{\mathfrak{h}}(h_2)\cdot v_0\otimes\alpha_B^{-\mathfrak{a}-1}\beta_B^{-\mathfrak{b}+1}\phi_B^{-\mathfrak{c}-1}\psi_B^{-\mathfrak{d}}(u_1)\alpha_B^{-\mathfrak{e}}\beta_B^{-\mathfrak{f}}\phi_B^{-\mathfrak{g}}\psi_B^{-\mathfrak{h}-1}(v_1)\\&=\phi_H(h)\cdot(\underline{u\otimes v}_0)\otimes\phi_B(\underline{u\otimes v}_1),\end{aligned}$$

which implies Equation (11). Hence, $U\otimes V\in{}_H\mathcal{L}^B$.

Second, define the the the associativity $\mathbf{a}$ and the unit constraints $\mathbf{l},\mathbf{r}$ by

$$\mathbf{a}_{U,V,W}((u\otimes v)\otimes w)=\alpha_U^{-\mathfrak{a}}\beta_U^{-\mathfrak{b}}\phi_U^{-\mathfrak{c}-1}\psi_U^{-\mathfrak{d}}(u)\otimes(v\otimes\alpha_W^{\mathfrak{e}}\beta_W^{\mathfrak{f}}\phi_W^{\mathfrak{g}}\psi_W^{\mathfrak{h}+1}(w)),$$
$$\mathbf{l}_U(1_\Bbbk\otimes u)=\alpha_U^{-\mathfrak{e}}\beta_U^{-\mathfrak{f}}\phi_U^{-\mathfrak{g}}\psi_U^{-\mathfrak{h}-1}(u),\quad \mathbf{r}_U(u\otimes 1_\Bbbk)=\alpha_U^{-\mathfrak{a}}\beta_U^{-\mathfrak{b}}\phi_U^{-\mathfrak{c}-1}\psi_U^{-\mathfrak{d}}(u);$$

where $U,V,W\in{}_H\mathcal{L}^B$, then, it is not hard to check that $({}_H\mathcal{L}^B,\otimes,\Bbbk,\mathbf{a},\mathbf{l},\mathbf{r})$ is a monoidal category. □

Remark 5. *We denote $({}_H\mathcal{L}^B,\otimes,\Bbbk,\mathbf{a},\mathbf{l},\mathbf{r})$ (under the monoidal structures given above) by ${}_H\mathcal{L}^{B\,\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\ \ \mathfrak{e},\mathfrak{f},\mathfrak{g},\mathfrak{h}}$.*

Proposition 4. *For any $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d},\mathfrak{e},\mathfrak{f},\mathfrak{g},\mathfrak{h},\mathfrak{a}',\mathfrak{b}',\mathfrak{c}',\mathfrak{d}',\mathfrak{e}',\mathfrak{f}',\mathfrak{g}',\mathfrak{h}'\in\mathbb{Z}$, ${}_H\mathcal{L}^{B\,\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\ \ \mathfrak{e},\mathfrak{f},\mathfrak{g},\mathfrak{h}}$ is monoidal isomorphic to ${}_H\mathcal{L}^{B\,\mathfrak{a}',\mathfrak{b}',\mathfrak{c}',\mathfrak{d}'}_{\ \ \mathfrak{e}',\mathfrak{f}',\mathfrak{g}',\mathfrak{h}'}$.*

Proof. Define functor $\mathscr{S}=(\mathscr{S},\mathscr{S}_2,\mathscr{S}_0):{}_H\mathcal{L}^{B\,\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\ \ \mathfrak{e},\mathfrak{f},\mathfrak{g},\mathfrak{h}}\to{}_H\mathcal{L}^{B\,\mathfrak{a}',\mathfrak{b}',\mathfrak{c}',\mathfrak{d}'}_{\ \ \mathfrak{e}',\mathfrak{f}',\mathfrak{g}',\mathfrak{h}'}$ by

$$\mathscr{F}(U)=U\text{ as BiHom-Long dimodule },\quad\mathscr{F}(f)=f,$$

where $(U,\alpha_U,\beta_U,\phi_U,\psi_U)\in{}_H\mathcal{L}^B$, $f\in Mor({}_H\mathcal{L}^B)$, and $\mathscr{S}_{2U,U}$ is given by

$$\mathscr{S}_{2U,V}(u\otimes v)=\alpha_U^{\mathfrak{a}-\mathfrak{a}'}\beta_U^{\mathfrak{b}-\mathfrak{b}'}\phi_U^{\mathfrak{c}-\mathfrak{c}'}\psi_U^{\mathfrak{d}-\mathfrak{d}'}(u)\otimes\alpha_V^{\mathfrak{e}-\mathfrak{e}'}\beta_V^{\mathfrak{f}-\mathfrak{f}'}\phi_V^{\mathfrak{g}-\mathfrak{g}'}\psi_V^{\mathfrak{h}-\mathfrak{h}'}(v),$$

for any $U,V\in{}_H\mathcal{L}^B$, $u\in U$, $v\in V$. Obviously, $\mathscr{S}=(\mathscr{S},\mathscr{S}_2,\mathscr{S}_0)$ is a monoidal isomorphic functor. □

4.2. BiHom-Type Yang-Baxter Equation

In this subsection, we will show that the generalized BiHom-Long dimodules will provide the algebraic solutions of the BiHom–Yang-Baxter equation.

Definition 4. *Let H be a BiHom-Hopf algebra. Recall from [22] that a* quasitriangular structure of *H is an invertible element $\mathrm{R}\in H\otimes H$, such that the following conditions hold:*

$$\begin{cases}(Q1)\ \ (\alpha\otimes\alpha)\mathrm{R}=(\beta\otimes\beta)\mathrm{R}=(\phi\otimes\phi)\mathrm{R}=(\psi\otimes\psi)\mathrm{R}=\mathrm{R};\\(Q2)\ \ \sum\mathrm{R}^{(1)}\phi^{-1}\psi(h_1)\otimes\mathrm{R}^{(2)}\phi\psi^{-1}(h_2)=\sum\alpha^{-1}\beta(h_2)\mathrm{R}^{(1)}\otimes\alpha^{-1}\beta(h_1)\mathrm{R}^{(2)};\\(Q3)\ \ \sum\mathrm{R}_1^{(1)}\otimes\mathrm{R}_2^{(1)}\otimes\mathrm{R}^{(2)}=\sum\alpha\phi(\dot{\mathrm{R}}^{(1)})\otimes\beta\psi(\mathrm{R}^{(1)})\otimes\dot{\mathrm{R}}^{(2)}\mathrm{R}^{(2)};\\(Q4)\ \ \sum R^{(1)}\otimes R_1^{(2)}\otimes R_2^{(2)}=\sum\dot{\mathrm{R}}^{(1)}\mathrm{R}^{(1)}\otimes\beta\phi(\mathrm{R}^{(2)})\otimes\alpha\psi(\dot{\mathrm{R}}^{(2)}),\end{cases}$$

for any $h \in H$*, where* $\dot{\mathrm{R}} = \mathrm{R} = \sum \mathrm{R}^{(1)} \otimes \mathrm{R}^{(2)} = \sum \dot{\mathrm{R}}^{(1)} \otimes \dot{\mathrm{R}}^{(2)}$.

Remark 6. *Let* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{j}$ *be integers,* R *and* R' *be two elements in* $H \otimes H$*. Recall from ([22], Section 3.2) that, for any* $M, N \in {}_H\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{e},\mathfrak{f},\mathfrak{g},\mathfrak{h}}$*, if we define families of maps* $\mathbf{T} : \otimes \Rightarrow \otimes^{op}$ *and* $\mathbf{T}' : \otimes^{op} \Rightarrow \otimes$ *as follows:*

- $\mathbf{T}_{M,N} : M \otimes N \to N \otimes M$ *is given by*

$$m \otimes n \mapsto \sum \alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\phi^{\mathfrak{c}}\psi^{\mathfrak{d}}(\mathrm{R}^{(2)}) \cdot \alpha_N^{\mathfrak{a}-\mathfrak{e}}\beta_N^{\mathfrak{b}-\mathfrak{f}-1}\phi_N^{\mathfrak{c}-\mathfrak{g}+1}\psi_N^{\mathfrak{d}-\mathfrak{h}-1}(n) \\ \otimes \alpha^{\mathfrak{e}}\beta^{\mathfrak{f}}\phi^{\mathfrak{g}}\psi^{\mathfrak{h}}(\mathrm{R}^{(1)}) \cdot \alpha_M^{-\mathfrak{a}+\mathfrak{e}}\beta_M^{-\mathfrak{b}+\mathfrak{f}-1}\phi_M^{-\mathfrak{c}+\mathfrak{g}-1}\psi_M^{-\mathfrak{d}+\mathfrak{h}+1}(m),$$

- $\mathbf{T}'_{M,N} : N \otimes M \to M \otimes N$ *is given by*

$$n \otimes m \mapsto \sum \alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\phi^{\mathfrak{c}+1}\psi^{\mathfrak{d}-1}(\mathrm{R}'^{(1)}) \cdot \alpha_M^{\mathfrak{a}-\mathfrak{e}}\beta_M^{\mathfrak{b}-\mathfrak{f}-1}\phi_M^{\mathfrak{c}-\mathfrak{g}+1}\psi_M^{\mathfrak{d}-\mathfrak{h}-1}(m) \\ \otimes \alpha^{\mathfrak{e}}\beta^{\mathfrak{f}}\phi^{\mathfrak{g}-1}\psi^{\mathfrak{h}+1}(\mathrm{R}'^{(2)}) \cdot \alpha_N^{-\mathfrak{a}+\mathfrak{e}}\beta_N^{-\mathfrak{b}+\mathfrak{f}-1}\phi_N^{-\mathfrak{c}+\mathfrak{g}-1}\psi_N^{-\mathfrak{d}+\mathfrak{h}+1}(n),$$

then $\mathbf{T}$ *is a braiding in* ${}_H\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{e},\mathfrak{f},\mathfrak{g},\mathfrak{h}}$ *with the inverse* $\mathbf{T}'$ *if and only if* R *is a quasitriangular structure of* H *with the inverse element* R'*.*

Lemma 1. *If* R *is a quasitriangular structure of* H*, then*

$$\sum \varepsilon(\mathrm{R}^{(1)})\mathrm{R}^{(2)} = \sum \mathrm{R}^{(1)}\varepsilon(\mathrm{R}^{(2)}) = 1_H. \tag{12}$$

Proof. Be similar to ([23], Lemma 2.1.2). □

Lemma 2. *If* R *is a quasitriangular structure of* H*, then, for any* $h \in H$*, we have*

$$\sum \phi^{-1}\psi(h_1)\mathrm{r}^{(1)} \otimes \phi\psi^{-1}(h_2)\mathrm{r}^{(2)} = \sum \mathrm{r}^{(1)}\alpha\beta^{-1}(h_2) \otimes \mathrm{r}^{(2)}\alpha\beta^{-1}(h_1), \tag{13}$$

where $\mathrm{r} = \sum \mathrm{r}^{(1)} \otimes \mathrm{r}^{(2)} = \mathrm{R}^{-1}$.

Proof. Equation (13) holds because of Equation (Q2). Actually,

$$\begin{aligned} & \sum \mathrm{R}^{(1)}\phi^{-1}\psi(h_1) \otimes \mathrm{R}^{(2)}\phi\psi^{-1}(h_2) = \sum \alpha^{-1}\beta(h_2)\mathrm{R}^{(1)} \otimes \alpha^{-1}\beta(h_1)\mathrm{R}^{(2)} \\ \Leftrightarrow\ & \sum (\mathrm{r}^{(1)}\alpha(\mathrm{R}^{(1)}))(\beta\phi^{-1}\psi(h_1)\beta^{-1}(\mathrm{s}^{(1)})) \otimes (\mathrm{r}^{(2)}\alpha(\mathrm{R}^{(2)}))(\beta\phi\psi^{-1}(h_2)\beta^{-1}(\mathrm{s}^{(2)})) \\ & \quad = \sum (\mathrm{r}^{(1)}\beta(h_2))(\beta(\mathrm{R}^{(1)})\beta^{-1}(\mathrm{s}^{(1)})) \otimes (\mathrm{r}^{(2)}\beta(h_1))(\beta(\mathrm{R}^{(2)})\beta^{-1}(\mathrm{s}^{(2)})) \\ \overset{(Q1)}{\Leftrightarrow}\ & \sum 1_H(\beta\phi^{-1}\psi(h_1)\mathrm{s}^{(1)}) \otimes 1_H(\beta\phi\psi^{-1}(h_2)\mathrm{s}^{(2)}) = \sum (\mathrm{r}^{(1)}\beta(h_2))1_H \otimes (\mathrm{r}^{(2)}\beta(h_1))1_H \\ \overset{(Q1)}{\Leftrightarrow}\ & \sum \phi^{-1}\psi(h_1)\mathrm{r}^{(1)} \otimes \phi\psi^{-1}(h_2)\mathrm{r}^{(2)} = \sum \mathrm{r}^{(1)}\alpha\beta^{-1}(h_2) \otimes \mathrm{r}^{(2)}\alpha\beta^{-1}(h_1), \end{aligned}$$

which implies the conclusion. □

Lemma 3. *If* H *is a BiHom-Hopf algebra with bijective antipode* S*, and* R *is a quasitriangular structure of* H*, then we have*

$$(S\alpha^{-1}\beta\phi^{-1}\psi \otimes id)\mathrm{R} = (id \otimes S^{-1}\alpha^{-1}\beta\phi^{-1}\psi)\mathrm{R} = \mathrm{R}^{-1}, \tag{14}$$

and hence

$$(S \otimes S)\mathrm{R} = \mathrm{R}. \tag{15}$$

Proof. Firstly, due to Equation (12), we immediately obtain that

$$\sum \mathrm{R}_1^{(1)}S(\mathrm{R}_2^{(1)}) \otimes \mathrm{R}^{(2)} = 1_H \otimes 1_H.$$

thus, from Equation (Q3), we have

$$\sum \dot{\mathrm{R}}^{(1)} S\alpha^{-1}\beta\phi^{-1}\psi(\mathrm{R}^{(1)}) \otimes \dot{\mathrm{R}}^{(2)}\mathrm{R}^{(2)} = 1_H \otimes 1_H,$$

which implies $(S\alpha^{-1}\beta\phi^{-1}\psi \otimes id)\mathrm{R} = \mathrm{R}^{-1}$.

Secondly, we can obtain $(id \otimes S^{-1}\alpha^{-1}\beta\phi^{-1}\psi)\mathrm{R} = \mathrm{R}^{-1}$ in a similar way.

Finally, one can easily obtain that Equation (15) holds because of Equation (3). □

Definition 5. *Recall from ([22], Definition 3.18) that* a coquasitriangular structure on a BiHom-Hopf algebra H *is a bilinear form* $\sigma : H \otimes H \to \Bbbk$, *such that* σ *is invertible under the convolution invertible, and the following formulae are satisfied:*

$$\begin{cases} (CQ1) & \sigma(\alpha(a),\alpha(b)) = \sigma(\beta(a),\beta(b)) = \sigma(\phi(a),\phi(b)) = \sigma(\psi(a),\psi(b)) = \sigma(a,b); \\ (CQ2) & \sigma(a_1,b_1)\phi\psi^{-1}(a_2)\phi\psi^{-1}(b_2) = \alpha^{-1}\beta(b_1)\alpha\beta^{-1}(a_1)\sigma(a_2,b_2); \\ (CQ3) & \sigma(\alpha\beta(a),bc) = \sigma(\alpha(a_1),\phi(c))\sigma(\beta(a_2),\psi(b)); \\ (CQ4) & \sigma(ab,\phi\psi(c)) = \sigma(\alpha(a),\psi(c_1))\sigma(\beta(b),\phi(c_2)). \end{cases}$$

Remark 7. *For any bilinear form* $\sigma \in hom(H \otimes H, \Bbbk)$, $U, V \in ({}_{\mathcal{B}}\mathcal{M}^H)^{\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}}_{\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q}}$ *(where* $\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l},\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q}$ *mean any integers), define the families of maps* $\mathbf{B}_{U,V} : U \otimes V \to V \otimes U$ *by*

$$u \otimes v \mapsto \alpha_V^{-\mathfrak{i}+\mathfrak{m}-1}\beta_V^{-\mathfrak{j}+\mathfrak{n}+1}\phi_V^{-\mathfrak{k}+\mathfrak{p}-1}\psi_V^{-\mathfrak{l}+\mathfrak{q}}(v_0) \otimes \alpha_U^{\mathfrak{i}-\mathfrak{m}+1}\beta_U^{\mathfrak{j}-\mathfrak{n}-1}\phi_U^{\mathfrak{k}-\mathfrak{p}-1}\psi_U^{\mathfrak{l}-\mathfrak{q}}(u_0)$$
$$\sigma(\alpha^{\mathfrak{i}}\beta^{\mathfrak{j}}\phi^{\mathfrak{k}}\psi^{\mathfrak{l}}(u_{(1)})\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}}\phi^{\mathfrak{p}}\psi^{\mathfrak{q}}(v_{(1)})).$$

Then, recall from ([22], Theorem 3.20) that σ *is a coquasitriangular form of* H *if and only if* $({}_{\mathcal{B}}\mathcal{M}^H)^{\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}}_{\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q}}$ *is a braided category with the braiding* $\mathbf{B}$.

Being similar to Lemmas 1 and 3, we have the following property.

Lemma 4.

(1) *If* σ *is a coquasitriangular form of* H, *then for any* $h \in H$, *we have*

$$\sigma(h, 1_H) = \sigma(1_H, h) = \varepsilon(h).$$

(2) *If* H *is a BiHom-Hopf algebra with bijective antipode* S, *and* σ *is a coquasitriangular form of* H, *then, for any* $h, g \in H$, *we have*

$$\sigma(S\alpha\beta^{-1}\phi\psi^{-1}(h_1), g)\sigma(h_2, g_2) = \sigma(h_1, S^{-1}\alpha\beta^{-1}\phi\psi^{-1}(g_2)) = \varepsilon(h)\varepsilon(g). \tag{16}$$

Now, assume that $(H, S_H, \alpha_H, \beta_H, \phi_H, \psi_H)$ and $(B, S_B, \alpha_B, \beta_B, \phi_B, \psi_B)$ are two BiHom-Hopf algebras.

Theorem 3. *If* H *is quasitriangular and* B *is coquasitriangular, then* ${}_H\mathcal{L}^B$ *forms a braided category.*

Proof. Suppose that R is a quasitriangular structure of H, and σ is a coquasitriangular structure on B. For any $U, V \in {}_H\mathcal{L}^B$, $u \in U$, $v \in V$, define $\mathbf{C}_{U,V}(u \otimes v) =:$

$$\sum \sigma(\alpha_B^{-\mathfrak{a}-1}\beta_B^{-\mathfrak{b}}\phi_B^{-\mathfrak{c}-1}\psi_B^{-\mathfrak{d}}(u_1), \alpha_B^{-\mathfrak{e}}\beta_B^{-\mathfrak{f}-1}\phi_B^{-\mathfrak{g}}\psi_B^{-\mathfrak{h}-1}(v_1))\alpha_H^{\mathfrak{a}}\beta_H^{\mathfrak{b}}\phi_H^{\mathfrak{c}}\psi_H^{\mathfrak{d}}(\mathrm{R}^{(2)})$$
$$\cdot \alpha_V^{\mathfrak{a}-\mathfrak{e}}\beta_V^{\mathfrak{b}-\mathfrak{f}-1}\phi_V^{\mathfrak{c}-\mathfrak{g}}\psi_V^{\mathfrak{d}-\mathfrak{h}-1}(v_0) \otimes \alpha_H^{\mathfrak{e}}\beta_H^{\mathfrak{f}}\phi_H^{\mathfrak{g}}\psi_H^{\mathfrak{h}}(\mathrm{R}^{(1)}) \cdot \alpha_U^{\mathfrak{e}-\mathfrak{a}}\beta_U^{\mathfrak{f}-\mathfrak{b}-1}\phi_U^{\mathfrak{g}-\mathfrak{c}-2}\psi_U^{\mathfrak{h}-\mathfrak{d}+1}(u_0).$$

Obviously, $\mathbf{C}$ is compatible with the BiHom-structure maps. Since we have

$$
\begin{aligned}
&\mathbf{C}_{U,V}(h\cdot(u\otimes v))\\
\overset{(CQ1)}{=}\ &\sum\sigma(\alpha_B^{-\mathfrak{a}-1}\beta_B^{-\mathfrak{b}}\phi_B^{-\mathfrak{c}-1}\psi_B^{-\mathfrak{d}}(u_1),\alpha_B^{-\mathfrak{e}}\beta_B^{-\mathfrak{f}-1}\phi_B^{-\mathfrak{g}}\psi_B^{-\mathfrak{h}-1}(v_1))\\
&\ \alpha_H^{\mathfrak{a}}\beta_H^{\mathfrak{b}}\phi_H^{\mathfrak{c}}\psi_H^{\mathfrak{d}}(\mathrm{R}^{(2)})\cdot(\alpha_H^{\mathfrak{a}}\beta_H^{\mathfrak{b}-1}\phi_H^{\mathfrak{c}+1}\psi_H^{\mathfrak{d}-1}(h_2)\cdot\alpha_V^{\mathfrak{a}-\mathfrak{e}}\beta_V^{\mathfrak{b}-\mathfrak{f}-1}\phi_V^{\mathfrak{c}-\mathfrak{g}}\psi_V^{\mathfrak{d}-\mathfrak{h}-1}(v_0))\\
&\ \otimes\alpha_H^{\mathfrak{e}}\beta_H^{\mathfrak{f}}\phi_H^{\mathfrak{g}}\psi_H^{\mathfrak{h}}(\mathrm{R}^{(1)})\cdot(\alpha_H^{\mathfrak{e}}\beta_H^{\mathfrak{f}-1}\phi_H^{\mathfrak{g}-1}\psi_H^{\mathfrak{h}+1}(h_1)\cdot\alpha_V^{\mathfrak{e}-\mathfrak{a}}\beta_V^{\mathfrak{f}-\mathfrak{b}-1}\phi_V^{\mathfrak{g}-\mathfrak{c}-2}\psi_V^{\mathfrak{h}-\mathfrak{d}+1}(u_0))\\
\overset{(Q1)}{=}\ &\sum\sigma(\alpha_B^{-\mathfrak{a}-1}\beta_B^{-\mathfrak{b}}\phi_B^{-\mathfrak{c}-1}\psi_B^{-\mathfrak{d}}(u_1),\alpha_B^{-\mathfrak{e}}\beta_B^{-\mathfrak{f}-1}\phi_B^{-\mathfrak{g}}\psi_B^{-\mathfrak{h}-1}(v_1))\\
&\ \alpha_H^{\mathfrak{a}}\beta_H^{\mathfrak{b}-1}\phi_H^{\mathfrak{c}}\psi_H^{\mathfrak{d}}(\mathrm{R}^{(2)}\phi_H\psi_H^{-1}(h_2))\cdot\alpha_V^{\mathfrak{a}-\mathfrak{e}}\beta_V^{\mathfrak{b}-\mathfrak{f}}\phi_V^{\mathfrak{c}-\mathfrak{g}}\psi_V^{\mathfrak{d}-\mathfrak{h}-1}(v_0)\\
&\ \otimes\alpha_H^{\mathfrak{e}}\beta_H^{\mathfrak{f}-1}\phi_H^{\mathfrak{g}}\psi_H^{\mathfrak{h}}(\mathrm{R}^{(1)}\phi_H^{-1}\psi_H(h_1))\cdot\alpha_V^{\mathfrak{e}-\mathfrak{a}}\beta_V^{\mathfrak{f}-\mathfrak{b}}\phi_V^{\mathfrak{g}-\mathfrak{c}-2}\psi_V^{\mathfrak{h}-\mathfrak{d}+1}(u_0)\\
\overset{(Q2)}{=}\ &\sum\sigma(\alpha_B^{-\mathfrak{a}-1}\beta_B^{-\mathfrak{b}}\phi_B^{-\mathfrak{c}-1}\psi_B^{-\mathfrak{d}}(u_1),\alpha_B^{-\mathfrak{e}}\beta_B^{-\mathfrak{f}-1}\phi_B^{-\mathfrak{g}}\psi_B^{-\mathfrak{h}-1}(v_1))\\
&\ (\alpha_H^{\mathfrak{a}-1}\beta_H^{\mathfrak{b}}\phi_H^{\mathfrak{c}}\psi_H^{\mathfrak{d}}(h_1)\alpha_H^{\mathfrak{a}}\beta_H^{\mathfrak{b}-1}\phi_H^{\mathfrak{c}}\psi_H^{\mathfrak{d}}(\mathrm{R}^{(2)}))\cdot\alpha_V^{\mathfrak{a}-\mathfrak{e}}\beta_V^{\mathfrak{b}-\mathfrak{f}}\phi_V^{\mathfrak{c}-\mathfrak{g}}\psi_V^{\mathfrak{d}-\mathfrak{h}-1}(v_0)\\
&\ \otimes(\alpha_H^{\mathfrak{e}-1}\beta_H^{\mathfrak{f}}\phi_H^{\mathfrak{g}}\psi_H^{\mathfrak{h}}(h_2)\alpha_H^{\mathfrak{e}}\beta_H^{\mathfrak{f}-1}\phi_H^{\mathfrak{g}}\psi_H^{\mathfrak{h}}(\mathrm{R}^{(1)}))\cdot\alpha_V^{\mathfrak{e}-\mathfrak{a}}\beta_V^{\mathfrak{f}-\mathfrak{b}}\phi_V^{\mathfrak{g}-\mathfrak{c}-2}\psi_V^{\mathfrak{h}-\mathfrak{d}+1}(u_0)\\
\overset{(Q1)}{=}\ &\sum\sigma(\alpha_B^{-\mathfrak{a}-1}\beta_B^{-\mathfrak{b}}\phi_B^{-\mathfrak{c}-1}\psi_B^{-\mathfrak{d}}(u_1),\alpha_B^{-\mathfrak{e}}\beta_B^{-\mathfrak{f}-1}\phi_B^{-\mathfrak{g}}\psi_B^{-\mathfrak{h}-1}(v_1))\\
&\ \alpha_H^{\mathfrak{a}}\beta_H^{\mathfrak{b}}\phi_H^{\mathfrak{c}}\psi_H^{\mathfrak{d}}(h_1)\cdot(\alpha_H^{\mathfrak{a}}\beta_H^{\mathfrak{b}}\phi_H^{\mathfrak{c}}\psi_H^{\mathfrak{d}}(\mathrm{R}^{(2)})\cdot\alpha_V^{\mathfrak{a}-\mathfrak{e}}\beta_V^{\mathfrak{b}-\mathfrak{f}-1}\phi_V^{\mathfrak{c}-\mathfrak{g}}\psi_V^{\mathfrak{d}-\mathfrak{h}-1}(v_0))\\
&\ \otimes\alpha_H^{\mathfrak{e}-1}\beta_H^{\mathfrak{f}}\phi_H^{\mathfrak{g}}\psi_H^{\mathfrak{h}}(h_2)\cdot(\alpha_H^{\mathfrak{e}}\beta_H^{\mathfrak{f}}\phi_H^{\mathfrak{g}}\psi_H^{\mathfrak{h}}(\mathrm{R}^{(1)})\cdot\alpha_V^{\mathfrak{e}-\mathfrak{a}}\beta_V^{\mathfrak{f}-\mathfrak{b}-1}\phi_V^{\mathfrak{g}-\mathfrak{c}-2}\psi_V^{\mathfrak{h}-\mathfrak{d}+1}(u_0))\\
=\ \ &h\cdot\mathbf{C}_{U,V}((u\otimes v)),
\end{aligned}
$$

$\mathbf{C}$ is H-linear. Similarly, we have

$$(\mathbf{C}_{U,V}\otimes id_B)\circ\rho^{U\otimes V}=\rho^{V\otimes U}\circ\mathbf{C}_{U,V},$$

which implies $\mathbf{C}$ is B-colinear. Moreover, we also have

$$
\begin{aligned}
&((id_V\otimes\mathbf{C}_{U,W})\circ\mathbf{a}_{V,U,W}\circ(\mathbf{C}_{U,V}\otimes id_W))((u\otimes v)\otimes w)\\
=\ \ &(id_V\otimes\mathbf{C}_{U,W})(\sum\sigma(\alpha_B^{-\mathfrak{a}-1}\beta_B^{-\mathfrak{b}}\phi_B^{-\mathfrak{c}-1}\psi_B^{-\mathfrak{d}}(u_1),\alpha_B^{-\mathfrak{e}}\beta_B^{-\mathfrak{f}-1}\phi_B^{-\mathfrak{g}}\psi_B^{-\mathfrak{h}-1}(v_1))\phi_H^{-1}(\mathrm{R}^{(2)})\\
&\ \cdot\alpha_V^{-\mathfrak{e}}\beta_V^{-\mathfrak{f}-1}\phi_V^{-\mathfrak{g}-1}\psi_V^{-\mathfrak{h}-1}(v_0)\otimes(\alpha_H^{\mathfrak{e}}\beta_H^{\mathfrak{f}}\phi_H^{\mathfrak{g}}\psi_H^{\mathfrak{h}}(\mathrm{R}^{(1)})\cdot\alpha_U^{\mathfrak{e}-\mathfrak{a}}\beta_U^{\mathfrak{f}-\mathfrak{b}-1}\phi_U^{\mathfrak{g}-\mathfrak{c}-2}\psi_U^{\mathfrak{h}-\mathfrak{d}+1}(u_0)\\
&\ \otimes\alpha_W^{\mathfrak{e}}\beta_W^{\mathfrak{f}}\phi_W^{\mathfrak{g}}\psi_W^{\mathfrak{h}+1}(w)))\\
\overset{(CQ1),(Q1)}{=}\ &\sum\sigma(\alpha_B(u_{11}),\alpha_B^{2\mathfrak{a}-\mathfrak{e}+2}\beta_B^{2\mathfrak{b}-\mathfrak{f}-1}\phi_B^{2\mathfrak{c}-\mathfrak{g}+3}\psi_B^{2\mathfrak{d}-\mathfrak{h}-1}(w_1))\\
&\ \sigma(\beta_B(u_{12}),\alpha_B^{\mathfrak{a}-\mathfrak{e}+1}\beta_B^{\mathfrak{b}-\mathfrak{f}}\phi_B^{\mathfrak{c}-\mathfrak{g}+1}\psi_B^{\mathfrak{d}-\mathfrak{h}}(v_1))\\
&\ \alpha_H^{\mathfrak{a}-2\mathfrak{e}}\beta_H^{\mathfrak{b}-2\mathfrak{f}+1}\phi_H^{\mathfrak{c}-2\mathfrak{g}}\psi_H^{\mathfrak{d}-2\mathfrak{h}-1}(\mathrm{R}^{(2)})\cdot\alpha_V^{-\mathfrak{e}}\beta_V^{-\mathfrak{f}-1}\phi_V^{-\mathfrak{g}-1}\psi_V^{-\mathfrak{h}-1}(v_0)\\
&\ \otimes(\alpha_H^{\mathfrak{a}-\mathfrak{e}+1}\beta_H^{\mathfrak{b}-\mathfrak{f}}\phi_H^{\mathfrak{c}-\mathfrak{g}}\psi_H^{\mathfrak{d}-\mathfrak{h}}(\dot{\mathrm{R}}^{(2)})\cdot\alpha_W^{\mathfrak{a}}\beta_W^{\mathfrak{b}-1}\phi_W^{\mathfrak{c}}\psi_W^{\mathfrak{d}}(w_0)\\
&\ \otimes(\dot{\mathrm{R}}^{(1)}\mathrm{R}^{(1)})\cdot\alpha_U^{2\mathfrak{e}-2\mathfrak{a}}\beta_U^{2\mathfrak{f}-2\mathfrak{b}-1}\phi_U^{2\mathfrak{g}-2\mathfrak{c}-3}\psi_U^{2\mathfrak{h}-2\mathfrak{d}+2}(u_0))\\
\overset{(CQ3),(Q4)}{=}\ &\sum\sigma(\alpha_B\beta_B(u_1),\alpha_B^{\mathfrak{a}-\mathfrak{e}+1}\beta_B^{\mathfrak{b}-\mathfrak{f}}\phi_B^{\mathfrak{c}-\mathfrak{g}+1}\psi_B^{\mathfrak{d}-\mathfrak{h}-1}(v_1)\alpha_B^{2\mathfrak{a}-\mathfrak{e}+2}\beta_B^{2\mathfrak{b}-\mathfrak{f}-1}\phi_B^{2\mathfrak{c}-\mathfrak{g}+2}\psi_B^{2\mathfrak{d}-\mathfrak{h}-1}(w_1))\\
&\ \alpha_H^{\mathfrak{a}-2\mathfrak{e}}\beta_H^{\mathfrak{b}-2\mathfrak{f}}\phi_H^{\mathfrak{c}-2\mathfrak{g}-1}\psi_H^{\mathfrak{d}-2\mathfrak{h}-1}(\mathrm{R}_1^{(2)})\cdot\alpha_V^{-\mathfrak{e}}\beta_V^{-\mathfrak{f}-1}\phi_V^{-\mathfrak{g}-1}\psi_V^{-\mathfrak{h}-1}(v_0)\\
&\ \otimes(\alpha_H^{\mathfrak{a}-\mathfrak{e}}\beta_H^{\mathfrak{b}-\mathfrak{f}}\phi_H^{\mathfrak{c}-\mathfrak{g}}\psi_H^{\mathfrak{d}-\mathfrak{h}-1}(\mathrm{R}_2^{(2)})\cdot\alpha_W^{\mathfrak{a}}\beta_W^{\mathfrak{b}-1}\phi_W^{\mathfrak{c}}\psi_W^{\mathfrak{d}}(w_0)\\
&\ \otimes\mathrm{R}^{(1)}\cdot\alpha_U^{2\mathfrak{e}-2\mathfrak{a}}\beta_U^{2\mathfrak{f}-2\mathfrak{b}-1}\phi_U^{2\mathfrak{g}-2\mathfrak{c}-3}\psi_U^{2\mathfrak{h}-2\mathfrak{d}+2}(u_0))\\
\overset{(CQ1),(Q1)}{=}\ &\sum\sigma(\alpha_B^{-2\mathfrak{a}-1}\beta_B^{-2\mathfrak{b}}\phi_B^{-2\mathfrak{c}-2}\psi_B^{-2\mathfrak{d}}(u_1),\alpha_B^{-\mathfrak{a}-\mathfrak{e}-1}\beta_B^{-\mathfrak{b}-\mathfrak{f}-1}\phi_B^{-\mathfrak{c}-\mathfrak{g}-1}\psi_B^{-\mathfrak{d}-\mathfrak{h}-1}(v_1)\\
&\ \alpha_B^{-\mathfrak{e}}\beta_B^{-\mathfrak{f}-2}\phi_B^{-\mathfrak{g}}\psi_B^{-\mathfrak{h}-1}(w_1))\alpha_H^{\mathfrak{a}}\beta_H^{\mathfrak{b}}\phi_H^{\mathfrak{c}-1}\psi_H^{\mathfrak{d}}(\mathrm{R}_1^{(2)})\cdot\alpha_V^{-\mathfrak{e}}\beta_V^{-\mathfrak{f}-1}\phi_V^{-\mathfrak{g}-1}\psi_V^{-\mathfrak{h}-1}(v_0)\\
&\ \otimes(\alpha_H^{\mathfrak{a}+\mathfrak{e}}\beta_H^{\mathfrak{b}+\mathfrak{f}}\phi_H^{\mathfrak{c}+\mathfrak{g}}\psi_H^{\mathfrak{d}+\mathfrak{h}}(\mathrm{R}_2^{(2)})\cdot\alpha_W^{\mathfrak{a}}\beta_W^{\mathfrak{b}-1}\phi_W^{\mathfrak{c}}\psi_W^{\mathfrak{d}}(w_0)\\
&\ \otimes\alpha_H^{2\mathfrak{e}}\beta_H^{2\mathfrak{f}}\phi_H^{2\mathfrak{g}}\psi_H^{2\mathfrak{h}+1}(\mathrm{R}^{(1)})\cdot\alpha_U^{2\mathfrak{e}-2\mathfrak{a}}\beta_U^{2\mathfrak{f}-2\mathfrak{b}-1}\phi_U^{2\mathfrak{g}-2\mathfrak{c}-3}\psi_U^{2\mathfrak{h}-2\mathfrak{d}+2}(u_0))
\end{aligned}
$$

$$
\begin{aligned}
= \; & \mathbf{a}_{V,W,U}\Big(\sum \sigma(\alpha_B^{-2\mathfrak{a}-1}\beta_B^{-2\mathfrak{b}}\phi_B^{-2\mathfrak{c}-2}\psi_B^{-2\mathfrak{d}}(u_1), \alpha_B^{-\mathfrak{e}-\mathfrak{a}-1}\beta_B^{-\mathfrak{f}-\mathfrak{b}-1}\phi_B^{-\mathfrak{g}-\mathfrak{c}-1}\psi_B^{-\mathfrak{h}-\mathfrak{d}-1}(v_1) \\
& \quad \alpha_B^{-\mathfrak{e}}\beta_B^{-\mathfrak{f}-2}\phi_B^{-\mathfrak{g}}\psi_B^{-\mathfrak{h}-1}(w_1))(\alpha_H^{2\mathfrak{a}}\beta_H^{2\mathfrak{b}}\phi_H^{2\mathfrak{c}}\psi_H^{2\mathfrak{d}}(\mathrm{R}_1^{(2)}) \cdot \alpha_V^{\mathfrak{a}-\mathfrak{e}}\beta_V^{\mathfrak{b}-\mathfrak{f}-1}\phi_V^{\mathfrak{c}-\mathfrak{g}}\psi_V^{\mathfrak{d}-\mathfrak{h}-1}(v_0) \\
& \quad \otimes \alpha_H^{\mathfrak{a}+\mathfrak{e}}\beta_H^{\mathfrak{b}+\mathfrak{f}}\phi_H^{\mathfrak{c}+\mathfrak{g}}\psi_H^{\mathfrak{d}+\mathfrak{h}}(\mathrm{R}_2^{(2)}) \cdot \alpha_W^{\mathfrak{a}}\beta_W^{\mathfrak{b}-1}\phi_W^{\mathfrak{c}}\psi_W^{\mathfrak{d}}(w_0)) \\
& \quad \otimes \alpha_H^{\mathfrak{e}}\beta_H^{\mathfrak{f}}\phi_H^{\mathfrak{g}}\psi_H^{\mathfrak{h}}(\mathrm{R}^{(1)}) \cdot \alpha_U^{\mathfrak{e}-2\mathfrak{a}}\beta_U^{\mathfrak{f}-2\mathfrak{b}-1}\phi_U^{\mathfrak{g}-2\mathfrak{c}-3}\psi_U^{\mathfrak{h}-2\mathfrak{d}+1}(u_0)\Big) \\
= \; & (\mathbf{a}_{V,W,U} \circ \mathbf{C}_{U,V\otimes W} \circ \mathbf{a}_{U,V,W})((u \otimes v) \otimes w),
\end{aligned}
$$

and similarly we can obtain $\mathbf{a}^{-1}_{W,U,V} \circ \mathbf{C}_{U\otimes V,W} \circ \mathbf{a}^{-1}_{U,V,W} = (\mathbf{C}_{U,W} \otimes id_V) \circ \mathbf{a}^{-1}_{U,W,V} \circ (id_U \otimes \mathbf{C}_{V,W})$.

Now, for any $U, V \in {}_H\mathcal{L}^B$, consider $\mathbf{C}' : \otimes^{op} \Rightarrow \otimes$, where $\mathbf{C}'_{U,V}(v \otimes u) :=$

$$
\begin{aligned}
& \sum \sigma(S\alpha_B^{\mathfrak{a}}\beta_B^{\mathfrak{b}}\phi_B^{\mathfrak{c}+2}\psi_B^{\mathfrak{d}-2}(u_1), \alpha_B^{\mathfrak{e}}\beta_B^{\mathfrak{f}}\phi_B^{\mathfrak{g}}\psi_B^{\mathfrak{h}}(v_1))S\alpha_H^{\mathfrak{a}-1}\beta_H^{\mathfrak{b}+1}\phi_H^{\mathfrak{c}+1}\psi_H^{\mathfrak{d}-1}(\mathrm{R}^{(1)}) \\
& \quad \cdot \alpha_U^{\mathfrak{a}-\mathfrak{e}}\beta_U^{\mathfrak{b}-\mathfrak{f}-1}\phi_U^{\mathfrak{c}-\mathfrak{g}}\psi_U^{\mathfrak{d}-\mathfrak{h}-1}(u_0) \otimes \alpha_H^{\mathfrak{e}}\beta_H^{\mathfrak{f}}\phi_H^{\mathfrak{g}}\psi_H^{\mathfrak{h}}(\mathrm{R}^{(2)}) \cdot \alpha_V^{\mathfrak{e}-\mathfrak{a}}\beta_V^{\mathfrak{f}-\mathfrak{b}-1}\phi_V^{\mathfrak{g}-\mathfrak{c}-2}\psi_V^{\mathfrak{h}-\mathfrak{d}+1}(v_0).
\end{aligned}
$$

Next, we will show $\mathbf{C}'$ is the inverse of $\mathbf{C}$. Indeed, we have

$$
\begin{aligned}
& (\mathbf{C}'_{U,V} \circ \mathbf{C}_{U,V})(u \otimes v) \\
= \; & \mathbf{C}'_{U,V}\Big(\sum \sigma(\alpha_B^{-\mathfrak{a}-1}\beta_B^{-\mathfrak{b}}\phi_B^{-\mathfrak{c}-1}\psi_B^{-\mathfrak{d}}(u_1), \alpha_B^{-\mathfrak{e}}\beta_B^{-\mathfrak{f}-1}\phi_B^{-\mathfrak{g}}\psi_B^{-\mathfrak{h}-1}(v_1))\alpha_H^{\mathfrak{a}}\beta_H^{\mathfrak{b}}\phi_H^{\mathfrak{c}}\psi_H^{\mathfrak{d}}(\mathrm{R}^{(2)}) \\
& \quad \cdot \alpha_V^{\mathfrak{a}-\mathfrak{e}}\beta_V^{\mathfrak{b}-\mathfrak{f}-1}\phi_V^{\mathfrak{c}-\mathfrak{g}}\psi_V^{\mathfrak{d}-\mathfrak{h}-1}(v_0) \otimes \alpha_H^{\mathfrak{e}}\beta_H^{\mathfrak{f}}\phi_H^{\mathfrak{g}}\psi_H^{\mathfrak{h}}(\mathrm{R}^{(1)}) \cdot \alpha_U^{\mathfrak{e}-\mathfrak{a}}\beta_U^{\mathfrak{f}-\mathfrak{b}-1}\phi_U^{\mathfrak{g}-\mathfrak{c}-2}\psi_U^{\mathfrak{h}-\mathfrak{d}+1}(u_0)\Big) \\
\overset{(CQ1),(Q1)}{=} \; & \sum \sigma(\alpha_B^{-\mathfrak{a}-1}\beta_B^{-\mathfrak{b}}\phi_B^{-\mathfrak{c}-1}\psi_B^{-\mathfrak{d}}(u_{12}), \alpha_B^{-\mathfrak{e}}\beta_B^{-\mathfrak{f}-1}\phi_B^{-\mathfrak{g}}\psi_B^{-\mathfrak{h}-1}(v_{12})) \\
& \quad \sigma(S\alpha_B^{\mathfrak{e}}\beta_B^{\mathfrak{f}}\phi_B^{\mathfrak{g}}\psi_B^{\mathfrak{h}}(u_{11}), \alpha_B^{\mathfrak{a}}\beta_B^{\mathfrak{b}}\phi_B^{\mathfrak{c}}\psi_B^{\mathfrak{d}}(v_1)) \\
& \quad (S\alpha_H^{\mathfrak{a}-1}\beta_H^{\mathfrak{b}+1}\phi_H^{\mathfrak{c}+1}\psi_H^{\mathfrak{d}-1}(\dot{\mathrm{R}}^{(1)})\alpha_H^{\mathfrak{a}}\beta_H^{\mathfrak{b}}\phi_H^{\mathfrak{c}+2}\psi_H^{\mathfrak{d}-2}(\mathrm{R}^{(1)})) \cdot \beta_U^{-1}\phi_U^{-1}(u_0) \\
& \quad \otimes (\alpha_H^{\mathfrak{e}}\beta_H^{\mathfrak{f}}\phi_H^{\mathfrak{g}}\psi_H^{\mathfrak{h}}(\dot{\mathrm{R}}^{(2)})\alpha_H^{\mathfrak{e}}\beta_H^{\mathfrak{f}}\phi_H^{\mathfrak{g}}\psi_H^{\mathfrak{h}}(\mathrm{R}^{(2)})) \cdot \beta_V^{-1}\phi_V^{-1}(v_0) \\
\overset{(CQ1),(4.4)}{=} \; & \sigma(\alpha_B^{\mathfrak{e}-1}\beta_B^{\mathfrak{f}+1}\phi_B^{\mathfrak{g}-1}\psi_B^{\mathfrak{h}+1}(u_{12}), \alpha_B^{\mathfrak{a}}\beta_B^{\mathfrak{b}}\phi_B^{\mathfrak{c}}\psi_B^{\mathfrak{d}}(v_{12})) \\
& \quad \sigma(S\alpha_B^{\mathfrak{e}}\beta_B^{\mathfrak{f}}\phi_B^{\mathfrak{g}}\psi_B^{\mathfrak{h}}(u_{11}), \alpha_B^{\mathfrak{a}}\beta_B^{\mathfrak{b}}\phi_B^{\mathfrak{c}}\psi_B^{\mathfrak{d}}(v_1))\phi_U^{-1}(u_0) \otimes \phi_V^{-1}(v_0) \\
\overset{(CQ1),(4.6)}{=} \; & u \otimes v,
\end{aligned}
$$

and similarly we can obtain $\mathbf{C}_{U,V} \circ \mathbf{C}'_{U,V} = id$. This means that $({}_H\mathcal{L}^{B\,\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\;\;\mathfrak{e},\mathfrak{f},\mathfrak{g},\mathfrak{h}}, \otimes, \mathbb{k}, \mathbf{a}, \mathbf{l}, \mathbf{r}, \mathbf{C})$ is a braided category. □

Under the consideration above, we obtain the following result.

Corollary 2. *The family of maps $\mathbf{C}_{U,V}$ for any $U, V \in {}_H\mathcal{L}^{B\,\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\;\;\mathfrak{e},\mathfrak{f},\mathfrak{g},\mathfrak{h}}$ is a solution of the following BiHom-type Yang-Baxter Equation:*

$$
\begin{aligned}
& (id_W \otimes \mathbf{C}_{U,V}) \circ \mathbf{a}_{W,U,V} \circ (\mathbf{C}_{U,W} \otimes id_V) \circ \mathbf{a}^{-1}_{U,W,V} \circ (id_U \otimes \mathbf{C}_{V,W}) \circ \mathbf{a}_{U,V,W} \\
& = \mathbf{a}_{W,V,U} \circ (\mathbf{C}_{W,V} \otimes id_U) \circ \mathbf{a}^{-1}_{W,V,U} \circ (id_V \otimes \mathbf{C}_{U,W}) \circ \mathbf{a}_{V,U,W} \circ (\mathbf{C}_{U,V} \otimes id_W).
\end{aligned}
$$

Proof. Straightforward. □

4.3. Generalized $\mathcal{D}$-Equation

In this section, we will show that the generalized BiHom-Long dimodules will provide the algebraic solutions of BiHom-type $\mathcal{D}$-equation. From now on, we always assume that $H = (H, \mu, 1_H, \Delta, \varepsilon, S, \alpha, \beta, \phi, \psi)$ is a BiHom-Hopf algebra over $\mathbb{k}$.

Definition 6. *Let $\xi : \otimes \Rightarrow \otimes$ be a natural transformation in $Vec_{\Bbbk}$. If the following diagram is commutative*

$$\begin{array}{ccc} U \otimes V \otimes W & \xrightarrow{id_U \otimes \xi_{V,W}} & U \otimes V \otimes W \\ \downarrow{\scriptstyle \xi_{U,V} \otimes id_W} & & \downarrow{\scriptstyle \xi_{U,V} \otimes id_W} \\ U \otimes V \otimes W & \xrightarrow[id_U \otimes \xi_{V,W}]{} & U \otimes (V \otimes W) \end{array}$$

in $Vec_{\Bbbk}$, then we say ξ is a solution of the $\mathcal{D}$-equation.

Theorem 4. *For any integer $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}, \mathfrak{f}, \mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l} \in \mathbb{Z}$, the following $\Bbbk$-linear maps*

$$\begin{aligned} \xi^{\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}}_{U,V} : U \otimes V &\longrightarrow U \otimes V \\ u \otimes v &\longmapsto \alpha^{\mathfrak{i}}\beta^{\mathfrak{j}}\phi^{\mathfrak{k}}\psi^{\mathfrak{l}}(v_1) \cdot \beta^{-1}_U(u) \otimes \phi^{-1}_V(v_0), \end{aligned}$$

where $U, V \in {}_H\mathcal{L}^H$ satisfies the following generalized BiHom-type $\mathcal{D}$-equation in ${}_H\mathcal{L}^{H}{}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{e},\mathfrak{f},\mathfrak{g},\mathfrak{h}}$:

$$\begin{array}{ccccccc} (U \otimes V) \otimes W & \xrightarrow{\mathbf{a}_{U,V,W}} & U \otimes (V \otimes W) & \xrightarrow{id_U \otimes \xi^{\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}}_{V,W}} & U \otimes (V \otimes W) & \xrightarrow{\mathbf{a}^{-1}_{U,V,W}} & (U \otimes V) \otimes W \\ \downarrow{\scriptstyle \xi^{\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}}_{U,V} \otimes id_W} & & & & & & \downarrow{\scriptstyle \xi^{\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}}_{U,V} \otimes id_W} \\ (U \otimes V) \otimes W & \xrightarrow[\mathbf{a}_{U,W,V}]{} & U \otimes (V \otimes W) & \xrightarrow[id_U \otimes \xi^{\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}}_{V,W}]{} & U \otimes (V \otimes W) & \xrightarrow[\mathbf{a}^{-1}_{U,V,W}]{} & (U \otimes V) \otimes W. \end{array}$$

Proof. For any $u \in U, v \in V, w \in W$, since the following identities

$$\begin{aligned} & ((\xi^{\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}}_{U,V} \otimes id_W) \circ \mathbf{a}^{-1}_{U,W,V} \circ (id_U \otimes \xi^{\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}}_{V,W}) \circ \mathbf{a}_{U,V,W})((u \otimes v) \otimes w) \\ = \; & ((\xi^{\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}}_{U,V} \otimes id_W) \circ \mathbf{a}^{-1}_{U,W,V})(\alpha^{\mathfrak{a}}_U\beta^{\mathfrak{b}}_U\pi^{\mathfrak{c}}_U\psi^{\mathfrak{d}}_U(u) \\ & \quad \otimes (\alpha^{\mathfrak{e}+\mathfrak{i}}\beta^{\mathfrak{f}+\mathfrak{j}}\phi^{\mathfrak{g}+\mathfrak{k}}\psi^{\mathfrak{h}+\mathfrak{l}}(w_1) \cdot \beta^{-1}_V(v) \otimes \alpha^{\mathfrak{e}}_W\beta^{\mathfrak{f}}_W\phi^{\mathfrak{g}-1}_W\psi^{\mathfrak{h}+1}_W(w_0))) \\ = \; & \alpha^{\mathfrak{i}}\beta^{\mathfrak{j}}\phi^{\mathfrak{k}}\psi^{\mathfrak{l}}(v_1) \cdot \beta^{-1}_U(u) \otimes (\alpha^{\mathfrak{e}+\mathfrak{i}}\beta^{\mathfrak{f}+\mathfrak{j}}\phi^{\mathfrak{g}+\mathfrak{k}}\psi^{\mathfrak{h}+\mathfrak{l}+1}(w_1) \cdot \beta^{-1}_V\phi^{-1}_V(v_0) \otimes \phi^{-1}_W(w_0)) \\ = \; & (\mathbf{a}^{-1}_{U,V,W} \circ (id_U \otimes \xi^{\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}}_{V,W}))(\alpha^{-\mathfrak{a}+\mathfrak{i}}\beta^{-\mathfrak{b}+\mathfrak{j}}\phi^{-\mathfrak{c}+\mathfrak{k}-1}\psi^{-\mathfrak{d}+\mathfrak{l}}(v_1) \cdot \alpha^{-\mathfrak{a}}_U\beta^{-\mathfrak{b}-1}_U\phi^{-\mathfrak{c}-1}_U\psi^{-\mathfrak{d}}_U(u) \\ & \quad \otimes (\phi^{-1}_V(v_0) \otimes \alpha^{\mathfrak{e}}_W\beta^{\mathfrak{f}}_W\phi^{\mathfrak{g}}_W\psi^{\mathfrak{h}+1}_W(w))) \\ = \; & (\mathbf{a}^{-1}_{U,V,W} \circ (id_U \otimes \xi^{\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}}_{V,W}) \circ \mathbf{a}_{U,W,V} \circ (\xi^{\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}}_{U,V} \otimes id_W))((u \otimes v) \otimes w), \end{aligned}$$

the conclusion holds. □

Remark 8. *As a special case of Theorem 4, if $\alpha = \beta = \phi = \psi$, then H is a Hom–Hopf algebra, and we immediately obtain ([8], Proposition 5.11).*

5. Conclusions

For a BiHom-Hopf algebra H, we first introduced the parametric Heisenberg doubles of H, and show that they can provide the solutions of the BiHom-Hopf equation and BiHom-pentagon equation. Then, we investigated the generalized BiHom-type Long-dimodules, and the solution of BiHom-$\mathcal{D}$-equation derived from them. Moreover, if H is both quasitriangular and coquasitriangular, then BiHom-type Long-dimodules also provide the solutions of BiHom-Yang-Baxter equation.

Author Contributions: Conceptualization, X.Z. and H.W.; methodology, H.W.; validation, X.Z. and H.W.; formal analysis, X.Z.; investigation, H.W.; resources, H.W.; writing original draft preparation, H.W.; writing review and editing, X.Z.; visualization, H.W.; supervision, X.Z.; project administration, H.W.; funding acquisition, X.Z. and H.W. All authors have read and agreed to the published version of the manuscript.

Funding: The work was partially supported by the Project Funded by the China Postdoctoral Science Foundation (No. 2020M672023) and the National Natural Science Foundation of China (Nos. 11871301, 12271292).

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to express their gratitude to the anonymous referee for their very helpful suggestions and comments which led to the improvement of our original manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Hartwig, J.; Larsson, D.; Silvestrov, S. Deformation of Lie algebras using σ-derivations. *J. Algebra* **2006**, *295*, 314–361. [CrossRef]
2. Hu, N.H. q-Witt algebras, q-Lie algebras, q-holomorph structure and representations. *Algebra Colloq.* **1999**, *6*, 51–70.
3. Makhlouf, A.; Panaite, F. Yetter-Drinfeld modules for Hom-bialgebras. *J. Math. Phys.* **2014**, *55*, 013501. [CrossRef]
4. Makhlouf, A.; Silvestrov, S. Hom-algebras structures. *J. Gen. Lie Theory Appl.* **2008**, *2*, 51–64. [CrossRef]
5. Makhlouf, A.; Silvestrov, S. Hom-Lie admissible Hom-coalgebras and Hom–Hopf algebras. In *Generalized Lie theory in Mathematics, Physics and Beyond*; Springer: Berlin, Germany, 2008; Chapter 17, pp. 189–206.
6. Yau, D. Hom-quantum groups I: Quasitriangular Hom-bialgebras. *J. Phys. A* **2012**, *45*, 065203. [CrossRef]
7. Yau, D. Hom-Yang-Baxter equation, Hom-Lie algebras and quasitriangular bialgebras. *J. Phys. A* **2009**, *42*, 165202. [CrossRef]
8. Zhang, X.H.; Dong, L.H. Braided mixed datums and their applications on Hom-quantum groups. *Glasgow Math. J.* **2018**, *60*, 231–251. [CrossRef]
9. Caenepeel, S.; Goyvaerts, I. Monoidal Hom–Hopf algebras. *Comm. Algebra* **2011**, *39*, 2216–2240. [CrossRef]
10. Graziani, G.; Makhlouf, A.; Menini, C.; Panaite, F. BiHom-associative algebras, BiHom-Lie algebras and BiHom-bialgebras. *SIGMA* **2015**, *11*, 086. [CrossRef]
11. Fang, X.L.; Liu, W. Solutions of the BiHom–Yang-Baxter equations. *Mat. Sb.* **2018**, *209*, 128–145; translation in *Sb. Math.* **2018**, *209*, 901–918. [CrossRef]
12. Guo, S.J.; Zhang, X.H.; Wang, S.X. The construction and deformation of BiHom-Novikov algebras. *J. Geom. Phys.* **2018**, *132*, 460–472. [CrossRef]
13. Li, J.; Chen, L.Y.; Sun, B. Bihom-Nijienhuis operators and T*-extensions of Bihom-Lie superalgebras. *Hacet. J. Math. Stat.* **2019**, *48*, 785–799. [CrossRef]
14. Liu, L.; Makhlouf, A.; Menini, C.; Panaite, F. σ,τ-Rota-Baxter operators, infinitesimal Hom-bialgebras and the associative (Bi)Hom-Yang-Baxter equation. *Canad. Math. Bull.* **2019**, *62*, 355–372. [CrossRef]
15. Zhang, X.H.; Wang, D.G. Cotwists of bicomonads and BiHom-bialgebras. *Algebra Represent. Theor.* **2020**, *23*, 1355–1385. [CrossRef]
16. Caenepeel, S.; Militaru, G.; Zhu, S.L. Frobenius and separable functors for generalized module categories and nonlinear equations. In *Lecture Notes in Mathematics (1787)*; Springer: Berlin/Heidelberg, Germany, 2002.
17. Militaru, G. The Long dimodules category and nonlinear equations. *Algebra Represet. Theor.* **1999**, *2*, 177–200. [CrossRef]
18. Long, F. The Brauer group of dimodule algebras. *J. Algebra* **1974**, *30*, 559–601. [CrossRef]
19. Ma, T.S.; Li, J. Nonhomogeneous associative Yang-Baxter equations. *Bull. Math. Soc. SCI. Math. Roumanie (N.S.)* **2022**, *65*, 97–118.
20. Wang, S.X.; Zhang, X.H.; Guo, S.J. The Hom-Long dimodule category and nonlinear equations. *Electron. Res. Arch.* **2022**, *30*, 362–381. [CrossRef]
21. Sweedler, M.E. *Hopf Algebras*; Benjamin: New York, NY, USA, 1969.
22. Zhang, X.H.; Wang, W.; Chen, J.Z. Parametric representations of BiHom-Hopf algebras. 2022, *Submitted.*
23. Majid, S. *Foundations of Quantum Group Theory*; Cambridge University Press: London, UK, 1995.