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Review

Multiple Derivative Inversions and Lagrange-Good Expansion Formulae

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Abstract: By establishing new multiple inverse series relations (with their connection coefficients being given by higher derivatives of fixed multivariate analytic functions), we illustrate a general framework to provide new proofs for MacMahon's master theorem and the multivariate expansion formula due to Good (1960). Further multivariate extensions of the derivative identities due to Pfaff (1795) and Cauchy (1826) will be derived and the generalized multifold convolution identities due to Carlitz (1977) will be reviewed.

Keywords: leibniz rule; Pfaff and Cauchy derivative identities; inverse series relations; MacMahon's master theorem; good's formula of multivariate Lagrange expansion

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1. Introduction and Motivation

Let D be the derivative operator with respect to x. Given two n-times differentiable functions $u := u(x)$ and $v := v(x)$, the higher derivatives of their product uv can usually be computed by the well-known Leibniz formula. This formula plays a fundamental role in calculus, which is recorded as follows:

$$D^n(uv) = \sum_{k=0}^{n}\binom{n}{k} D^k u\, D^{n-k} v.$$

Pfaff ([1], 1795; cf. Johnson [2], Equation (1.1)) first discovered the following generalization of Leibniz' rule with an additional h-function. Suppose that $u := u(x)$, $v := v(x)$ and $h := h(x)$ are three n-times differentiable functions with respect to x. Then the following formula holds:

$$D^n(uv) = \sum_{k=0}^{n}\binom{n}{k} D^{k-1}(h^k u')\, D^{n-k}(h^{-k} v). \tag{1}$$

In fact, this identity reduces to the Leibniz formula when $h \equiv 1$. Making the replacement v by $h^n v$ in (1), we can easily recover the following formula of Cauchy [3] (cf. Johnson [2], Equation (1.3)):

$$D^n\{h^n uv\} = \sum_{k=0}^{n}\binom{n}{k} D^{k-1}(h^k u')\, D^{n-k}(h^{n-k} v). \tag{2}$$

There exists a symmetric formula also due to Cauchy ([3], 1826; cf. [2], Equation (1.4)), where one can also find a special case of it due to Olver [4]), which reads as

$$D^{n-1}\{h^n D(uv)\} = \sum_{k=0}^{n}\binom{n}{k} D^{k-1}(h^k u')\, D^{n-k-1}(h^{n-k} v'). \tag{3}$$

In the same paper, Pfaff ([1], 1795) also got the derivative identity, nowadays called the "Jensen type" (cf. [2], Equation (1.2)):

$$\sum_{k=0}^{n}\binom{n}{k}D^k\big(h^k u\big)D^{n-k}\big(h^{-k}v\big) = \sum_{k=0}^{n}\frac{n!}{k!}D^k\Big\{\big(h'/h\big)^{n-k}uv\Big\}. \tag{4}$$

Among these derivative identities, Pfaff [1] discovered (1) and (4) more than 200 years ago. Thirty years later, Cauchy [3] found two similar ones (2) and (3). Since then, these findings remained almost unnoticed. In 2007, Johnson [2] unearthed these important generalizations and investigated their applications to Lagrange series, Laguerre and Jacobi polynomials, as well as Hurwitz' extended binomial theorems [5], including the binomial convolution identities of Abel–Rothe type.

By making use of these derivative identities, the author discovered a useful pair of derivative inverse series relations (cf. [6]), which can be reproduced as follows: Let $\phi := \phi(x)$ and $w := w(x)$ be two infinitely differentiable functions with respect to x. Define the θ-function by

$$\theta(\lambda) = 1 - \frac{\lambda}{D}\times\frac{\phi'}{\phi}. \tag{5}$$

Then the system of equations

$$f(n) = \sum_{k=0}^{n}\binom{n}{k}D^{n-k}\{\theta(n-k)\phi^n w\}g(k), \quad n = 0,1,2,\cdots \tag{6}$$

is equivalent to the system

$$g(n) = \sum_{k=0}^{n}\binom{n}{k}D^{n-k}\{\phi^{-k}/w\}\, f(k), \qquad n = 0,1,2,\cdots. \tag{7}$$

By switching the θ-factor to the dual relation, we get another inverse pair

$$f(n) = \sum_{k=0}^{n}\binom{n}{k}D^{n-k}\{\phi^n w\}\, g(k), \tag{8}$$

$$g(n) = \sum_{k=0}^{n}\binom{n}{k}D^{n-k}\{\theta(n-k)\phi^{-k}/w\}\, f(k). \tag{9}$$

These derivative inversions are closely connected to the important expansion formula discovered by Lagrange [7] (cf. [6,8], Section 3.8 also) For details, the reader can consult the papers by Jacobi [9], Laplace [10] and Johnson ([2], Section 4). The related q-series counterparts exist too, which can be found in [11–13] and the references therein.

The purpose of this paper is to establish new multiple inversion theorems, which may serve as multifold analogues for the above derivative inverse series relations and a general framework to review their connections with other related topics. The main results are given in the second section, where we shall prove two new pairs of multiple inverse series relations with their connection coefficients being given by higher derivatives of fixed multivariate analytic functions. Then the rest of the paper is dedicated to reviewing several important applications. In Section 3, the rotated forms of the new multiple inverse pairs is highlighted and then utilized to present a new proof for MacMahon's master theorem [14] (§64). In Section 4, we shall prove that our multiple inverse series relations imply the multivariate Lagrange expansion formula due to Good [15]. Section 5 is devoted to deriving multivariate extensions of derivative identities (1–3). New proofs of the generalized convolution formulae due to Carlitz [16,17] are provided in Section 6. Comparisons are made in Section 7 between the inversion theorems established in this paper and those obtained earlier by the author [18,19]. Finally, the paper is concluded in

Section 8, where a brief comment on the contribution of this paper is made and further prospective topics are proposed.

2. Multivariate Inverse Series Relations

In this section, we shall present the main results by demonstrating two Theorems 1 and 2. In particular, two different proofs (one is algebraic and another is by the multiple Cauchy contour integration) are provided for the crucial algebraic identity (15).

Denote by $\mathbb{N}_0$ and $\mathbb{C}$ the sets of nonnegative integers and complex numbers, respectively. Their ℓ-fold tensor products are given by $\mathbb{N}_0^\ell$ and $\mathbb{C}^\ell$, with the zero vector being indicated by $\mathbf{0}$. Then we write for brevity the usual scalar product and binomial coefficient product by

$$\langle \mathbf{a}, \mathbf{c} \rangle = \sum_{i=1}^{\ell} a_i c_i \qquad \text{and} \qquad \binom{\mathbf{n}}{\mathbf{k}} = \prod_{i=1}^{\ell} \binom{n_i}{k_i},$$

where $\mathbf{a}, \mathbf{c} \in \mathbb{C}^\ell$ and $\mathbf{k}, \mathbf{n} \in \mathbb{N}_0^\ell$ with $\mathbf{0} \le \mathbf{k} \le \mathbf{n}$, ordered in component-wise. We shall use the usual Kronecker symbol $\delta(\mathbf{k}, \mathbf{n})$, which is equal to one for $\mathbf{k} = \mathbf{n}$ and zero otherwise. When $k, n \in \mathbb{N}_0$, we fix similarly $\delta_{k,n} := \delta(k, n)$ for brevity.

Denote by $[\ell]$ the set of the first ℓ natural numbers. For the derivative operator D_{x_i} with respect to the variable x_i, we define the corresponding multivariate form by

$$\mathbf{D}^{\mathbf{n}} = \prod_{i=1}^{\ell} D_{x_i}^{n_i} \qquad \text{where} \qquad \mathbf{n} \in \mathbb{N}_0^\ell.$$

In addition, let $h_i := h_i(\mathbf{x}) = h_i(x_1, x_2, \cdots, x_\ell)$ with $i \in [\ell]$. Then for $\mathbf{n} \in \mathbb{N}_0^\ell$, we fix further the following two notations:

$$\mathbf{h}^{\mathbf{n}} = \prod_{i=1}^{\ell} h_i^{n_i} \qquad \text{and} \qquad \vartheta(\mathbf{n}) = \det_{1 \le i,j \le \ell} \left[\delta_{i,j} - \frac{n_i}{D_{x_i}} \frac{h'_{ij}}{h_i} \right], \tag{10}$$

where $h'_{ij} := D_{x_j} h_i$ for $i, j = 1, 2, \cdots, \ell$. Here and forth, $\vartheta(\mathbf{n})$ will be coupled with $\mathbf{D}^{\mathbf{n}}$ in the product $\mathbf{D}^{\mathbf{n}} \vartheta(\mathbf{n})$. When the nonzero components of $\mathbf{n}$ are specified through their indices $\sigma \subseteq [\ell]$

$$\mathbf{n} = (n_1, n_2, \cdots, n_\ell) : \begin{cases} n_i > 0, & i \in \sigma; \\ n_i = 0, & i \notin \sigma; \end{cases}$$

the precedent product is explicitly given as follows:

$$\mathbf{D}^{\mathbf{n}} \vartheta(\mathbf{n}) = \prod_{i \in \sigma} D_{x_i}^{n_i - 1} \times \det_{1 \le i,j \le \ell} \begin{bmatrix} \delta_{i,j}, & \text{if } i \notin \sigma \\ \delta_{i,j} D_{x_i} - n_i \frac{h'_{ij}}{h_i}, & \text{if } i \in \sigma \end{bmatrix}.$$

Now, we are ready to state the main theorems about multivariate inverse series relations whose connection coefficients are expressed in terms of higher derivatives of products of the fixed multivariate analytic functions.

Theorem 1. *Suppose that $w := w(\mathbf{x}) = w(x_1, x_2, \cdots, x_\ell)$ and $h_i := h_i(\mathbf{x}) = h_i(x_1, x_2, \cdots, x_\ell)$ are infinitely differentiable multivariate functions with $i = 1, 2, \cdots, \ell$. Then the system of equations*

$$f(\mathbf{n}) \ = \sum_{\mathbf{0} \le \mathbf{k} \le \mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} \mathbf{D}^{\mathbf{n}-\mathbf{k}} \Big\{ \vartheta(\mathbf{n} - \mathbf{k}) \mathbf{h}^{\mathbf{n}} w \Big\}\, g(\mathbf{k}), \qquad (\mathbf{n} \in \mathbb{N}_0^\ell) \tag{11a}$$

is equivalent to the system

$$g(\mathbf{n}) \ = \sum_{\mathbf{0} \le \mathbf{k} \le \mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} \mathbf{D}^{\mathbf{n}-\mathbf{k}} \Big\{ \mathbf{h}^{-\mathbf{k}} / w \Big\}\, f(\mathbf{k}), \qquad (\mathbf{n} \in \mathbb{N}_0^\ell). \tag{11b}$$

It is curious to notice that the ϑ-factor in the inversions can be switched to the dual relation. In another phrase, the following "dual" inverse series relations hold either.

Theorem 2. *Under the same condition of Thoerem 1, we have the multifold inverse pair:*

$$f(\mathbf{n}) = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} \mathbf{D}^{\mathbf{n}-\mathbf{k}}\Big\{\mathbf{h}^{\mathbf{n}} w\Big\}\, g(\mathbf{k}), \tag{12a}$$

$$g(\mathbf{n}) = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} \mathbf{D}^{\mathbf{n}-\mathbf{k}}\Big\{\frac{\vartheta(\mathbf{n}-\mathbf{k})}{\mathbf{h}^{\mathbf{k}} w}\Big\}\, f(\mathbf{k}). \tag{12b}$$

Proof. The inverse pair in the above theorem implies that for every identity of the form (11a) or (11b), there is a companion identity from the dual form. To prove each is equivalent to prove both. Hence it is enough to show that the latter implies the former. Now, by substituting (11b) into (11a) and then applying the cross product equality of binomial coefficients

$$\binom{\mathbf{n}}{\mathbf{k}}\binom{\mathbf{k}}{\mathbf{m}} = \binom{\mathbf{n}}{\mathbf{m}}\binom{\mathbf{n}-\mathbf{m}}{\mathbf{k}-\mathbf{m}},$$

we can reformulate the double sum expression as follows:

$$\begin{aligned} &\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} \mathbf{D}^{\mathbf{n}-\mathbf{k}}\Big\{\vartheta(\mathbf{n}-\mathbf{k})\mathbf{h}^{\mathbf{n}} w\Big\} \sum_{\mathbf{0}\le\mathbf{m}\le\mathbf{k}} \binom{\mathbf{k}}{\mathbf{m}} \mathbf{D}^{\mathbf{k}-\mathbf{m}}\Big\{\mathbf{h}^{-\mathbf{m}}/w\Big\}\, f(\mathbf{m}) \\ =&\sum_{\mathbf{0}\le\mathbf{m}\le\mathbf{n}} \binom{\mathbf{n}}{\mathbf{m}} f(\mathbf{m}) \sum_{\mathbf{m}\le\mathbf{k}\le\mathbf{n}} \binom{\mathbf{n}-\mathbf{m}}{\mathbf{k}-\mathbf{m}} \mathbf{D}^{\mathbf{k}-\mathbf{m}}\Big\{\mathbf{h}^{-\mathbf{m}}/w\Big\} \mathbf{D}^{\mathbf{n}-\mathbf{k}}\Big\{\vartheta(\mathbf{n}-\mathbf{k})\mathbf{h}^{\mathbf{n}} w\Big\}. \end{aligned}$$

In order to show that the above double sum reduces to $f(\mathbf{n})$, it is sufficient to prove the following orthogonal relation:

$$\sum_{\mathbf{m}\le\mathbf{k}\le\mathbf{n}} \binom{\mathbf{n}-\mathbf{m}}{\mathbf{k}-\mathbf{m}} \mathbf{D}^{\mathbf{k}-\mathbf{m}}\Big\{\mathbf{h}^{-\mathbf{m}}/w\Big\} \mathbf{D}^{\mathbf{n}-\mathbf{k}}\Big\{\vartheta(\mathbf{n}-\mathbf{k})\mathbf{h}^{\mathbf{n}} w\Big\} = \delta(\mathbf{m},\mathbf{n}). \tag{13}$$

Observe that there holds the following expansion for the ϑ-factor:

$$\vartheta(\mathbf{n}) = \sum_{\sigma\subseteq[\ell]} (-1)^{|\sigma|} \prod_{i\in\sigma} \frac{n_i}{D_{x_i}} \det_{i,j\in\sigma}\Big[\frac{h'_{ij}}{h_i}\Big], \tag{14}$$

where $|\sigma|$ stands for the cardinality of $\sigma\subseteq[\ell]$. The left member of (13) can consequently be manipulated, through the Leibniz rule, as follows:

$$\begin{aligned} &\sum_{\mathbf{m}\le\mathbf{k}\le\mathbf{n}} \binom{\mathbf{n}-\mathbf{m}}{\mathbf{k}-\mathbf{m}} \mathbf{D}^{\mathbf{k}-\mathbf{m}}\Big\{\mathbf{h}^{-\mathbf{m}}/w\Big\} \mathbf{D}^{\mathbf{n}-\mathbf{k}}\Big\{\vartheta(\mathbf{n}-\mathbf{k})\mathbf{h}^{\mathbf{n}} w\Big\} \\ =& \sum_{\sigma\subseteq[\ell]} (-1)^{|\sigma|} \sum_{\mathbf{m}\le\mathbf{k}\le\mathbf{n}} \binom{\mathbf{n}-\mathbf{m}}{\mathbf{k}-\mathbf{m}} \prod_{i\in\sigma}(n_i-k_i) \\ &\qquad \times \mathbf{D}^{\mathbf{k}-\mathbf{m}}\Big\{\mathbf{h}^{-\mathbf{m}}/w\Big\} \frac{\mathbf{D}^{\mathbf{n}-\mathbf{k}}}{\prod_{i\in\sigma} D_{x_i}} \Big\{\mathbf{h}^{\mathbf{n}} w \det_{i,j\in\sigma}\Big[\frac{h'_{ij}}{h_i}\Big]\Big\} \\ =& \sum_{\sigma\subseteq[\ell]} (-1)^{|\sigma|} \prod_{i\in\sigma}(n_i-m_i) \frac{\mathbf{D}^{\mathbf{n}-\mathbf{m}}}{\prod_{i\in\sigma} D_{x_i}} \Big\{\mathbf{h}^{\mathbf{n}-\mathbf{m}} \det_{i,j\in\sigma}\Big[\frac{h'_{ij}}{h_i}\Big]\Big\}. \end{aligned}$$

Therefore, we may restate (13) as the following equivalent algebraic identity:

$$\sum_{\sigma\subseteq[\ell]} (-1)^{|\sigma|} \prod_{i\in\sigma}(n_i-m_i) \frac{\mathbf{D}^{\mathbf{n}-\mathbf{m}}}{\prod_{i\in\sigma} D_{x_i}} \Big\{\mathbf{h}^{\mathbf{n}-\mathbf{m}} \det_{i,j\in\sigma}\Big[\frac{h'_{ij}}{h_i}\Big]\Big\} = \delta(\mathbf{m},\mathbf{n}). \tag{15}$$

This identity is clearly true for $\mathbf{m} = \mathbf{n}$. Without loss of generality, it suffices to show that it is also valid for $\mathbf{n} - \mathbf{m} \geq \mathbf{e}$ where $\mathbf{e} \in \mathbb{N}_0^\ell$ with all the components of $\mathbf{e}$ equal to one. The corresponding identity may be reformulated as

$$\mathbf{D}^{\mathbf{n}-\mathbf{m}-\mathbf{e}} \sum_{\sigma \subseteq [\ell]} (-1)^{|\sigma|} \prod_{i \in \sigma} (n_i - m_i) \prod_{j \notin \sigma} D_{x_j} \left\{ \mathbf{h}^{\mathbf{n}-\mathbf{m}} \det_{i,j \in \sigma} \left[\frac{h'_{ij}}{h_i} \right] \right\} = 0.$$

However, there holds the following even stronger relation that we shall confirm:

$$\sum_{\sigma \subseteq [\ell]} (-1)^{|\sigma|} \prod_{i \in \sigma} (n_i - m_i) \prod_{j \notin \sigma} D_{x_j} \left\{ \mathbf{h}^{\mathbf{n}-\mathbf{m}} \det_{i,j \in \sigma} \left[\frac{h'_{ij}}{h_i} \right] \right\} = 0. \tag{16}$$

For $\mathbf{n} \in \mathbb{N}_0^\ell$, let $[(\mathbf{y} - \mathbf{x})^{\mathbf{n}}]\phi(\mathbf{y})$ be the coefficient of $\prod_{i=1}^{\ell} (y_i - x_i)^{n_i}$ in the multivariate formal power series $\phi(\mathbf{y}) := \phi(y_1, y_2, \cdots, y_\ell)$ expanded at $\mathbf{y} = \mathbf{x}$. Then we have the following relation:

$$[(\mathbf{y} - \mathbf{x})^{\mathbf{n}}]\phi(\mathbf{y}) = \frac{\mathbf{D}^{\mathbf{n}}}{\mathbf{n}!}\phi(\mathbf{x}) \quad \text{where} \quad \mathbf{n}! = \prod_{i=1}^{n} n_i!.$$

From this relation, we can further reformulate (16) as follows:

$$[(\mathbf{y} - \mathbf{x})^{\mathbf{e}}] \sum_{\sigma \subseteq [\ell]} (-1)^{|\sigma|} \prod_{i \in \sigma} (n_i - m_i)(y_i - x_i) \left\{ \mathbf{h}^{\mathbf{n}-\mathbf{m}}(\mathbf{y}) \det_{i,j \in \sigma} \left[\frac{h'_{ij}(\mathbf{y})}{h_i(\mathbf{y})} \right] \right\} = 0.$$

For the sake of brevity, let $\phi_i(\mathbf{y}) := h_i^{n_i - m_i}(\mathbf{y})$ with $i = 1, 2, \cdots, \ell$. Applying (14) to the last equation, we get the following equivalent identity of (16):

$$[(\mathbf{y} - \mathbf{x})^{\mathbf{e}}] \det_{1 \leq i,j \leq \ell} \left[\phi_i(\mathbf{y})\delta_{i,j} - (y_j - x_j) \frac{\partial \phi_i(\mathbf{y})}{\partial y_j} \right] = 0. \tag{17}$$

Making further the following substitutions in the last relation:

$$y_k := x_k + 1/z_k \quad \text{and} \quad \psi_k(\mathbf{z}) := z_k \phi_k(\mathbf{y}) \quad \text{for} \quad k = 1, 2, \cdots, \ell,$$

we can express it as the multiple residue of the Jacobian determinant

$$[z_1^{-1} z_2^{-1} \cdots z_\ell^{-1}] \det_{1 \leq i,j \leq \ell} \left[\frac{\partial \psi_i(\mathbf{z})}{\partial z_j} \right] = 0. \tag{18}$$

Denote by A_k the cofactor of $\frac{\partial \psi_1(\mathbf{z})}{\partial z_k}$ in $\left[\frac{\partial \psi_i(\mathbf{z})}{\partial z_j} \right]_{1 \leq i,j \leq \ell}$. According to Jacobi's lemma (see Turnbull ([20], p. 125) for example)

$$\sum_{k=1}^{\ell} \frac{\partial A_k}{\partial z_k} = 0,$$

we can expand the determinant in (18) through the Laplace formula

$$\det_{1 \leq i,j \leq \ell} \left[\frac{\partial \psi_i(\mathbf{z})}{\partial z_j} \right] = \sum_{k=1}^{\ell} A_k \frac{\partial \psi_1(\mathbf{z})}{\partial z_k} = \sum_{k=1}^{\ell} \frac{\partial \{\psi_1(\mathbf{z}) A_k\}}{\partial z_k}.$$

Then the multiple residue identity on the Jacobian determinant in (18) follows from the fact that $\frac{\partial \psi(\mathbf{z})}{\partial z_k}$ contains no terms in z_k^{-1} for any Laurent series $\psi(\mathbf{z})$, as observed by Gessel [21].

This proves orthogonal relation (13) and Theorem 1 consequently. □

Similarly, substituting (12b) into (12a) results in the following orthogonal relation:

$$\sum_{\mathbf{m}\leq\mathbf{k}\leq\mathbf{n}}\binom{\mathbf{n}-\mathbf{m}}{\mathbf{k}-\mathbf{m}}\mathbf{D}^{\mathbf{k}-\mathbf{m}}\left\{\frac{\vartheta(\mathbf{k}-\mathbf{m})}{\mathbf{h}^{\mathbf{m}}w}\right\}\mathbf{D}^{\mathbf{n}-\mathbf{k}}\left\{\mathbf{h}^{\mathbf{n}}w\right\} \;=\; \delta(\mathbf{m},\mathbf{n}).$$

This is substantially equivalent to (13), which can be justified by making the replacements $\mathbf{k}\to\mathbf{m}+\mathbf{n}-\mathbf{k}$ and $w\to\mathbf{h}^{-\mathbf{m}-\mathbf{n}}/w$.

Instead, if we substitute (11a) into (11b), we would come across another orthogonal relation

$$\sum_{\mathbf{m}\leq\mathbf{k}\leq\mathbf{n}}\binom{\mathbf{n}-\mathbf{m}}{\mathbf{k}-\mathbf{m}}\mathbf{D}^{\mathbf{n}-\mathbf{k}}\left\{\mathbf{h}^{-\mathbf{k}}/w\right\}\mathbf{D}^{\mathbf{k}-\mathbf{m}}\left\{\vartheta(\mathbf{k}-\mathbf{m})\mathbf{h}^{\mathbf{k}}w\right\} \;=\; \delta(\mathbf{m},\mathbf{n}). \tag{19}$$

However, it would become much more tedious to prove this orthogonal relation.

Considering the elegance of the algebraic identity (15) and its importance for validating Theorem 1, we offer, in addition, the following alternative proof for it through multiple Cauchy contour integrals. Now that the identity displayed in (15) is obviously true for $\mathbf{m}=\mathbf{n}$, we have to show that it is also valid for $\mathbf{n}-\mathbf{m}=\mathbf{k}$ with $\mathbf{k}\in\mathbb{N}_0^\ell$ and $\mathbf{k}\neq\mathbf{0}$. The corresponding identity may be restated as

$$\sum_{\sigma\subseteq[\ell]}(-1)^{|\sigma|}\mathbf{D}^{\mathbf{k}}\prod_{i\in\sigma}\frac{k_i}{D_{x_i}}\left\{\mathbf{h}^{\mathbf{k}}\det_{i,j\in\sigma}\left[\frac{h'_{ij}}{h_i}\right]\right\}=0. \tag{20}$$

Let $\mathbf{e}\in\mathbb{N}_0^\ell$ with all the components equal to one. According to the Cauchy residue theorem, for a multivariate holomorphic function $\phi(\mathbf{y})$, the following expressions are held:

$$\begin{aligned}\phi(\mathbf{x}) &= \frac{1}{(2\pi i)^\ell}\oint\!\!\oint\cdots\oint_{|\mathbf{X}-\mathbf{x}|=\epsilon}\frac{\phi(\mathbf{X})}{(\mathbf{X}-\mathbf{x})^{\mathbf{e}}}d\mathbf{X},\\ \mathbf{D}^{\mathbf{k}}\phi(\mathbf{x}) &= \frac{\mathbf{k}!}{(2\pi i)^\ell}\oint\!\!\oint\cdots\oint_{|\mathbf{X}-\mathbf{x}|=\epsilon}\frac{\phi(\mathbf{X})}{(\mathbf{X}-\mathbf{x})^{\mathbf{k}+\mathbf{e}}}d\mathbf{X};\end{aligned}$$

where $d\mathbf{X}=dx_1dx_2\cdots dx_\ell$ and the integrals run over the polydisk $|\mathbf{X}-\mathbf{x}|=\epsilon$ defined by $\{X\in\mathbb{C}^\ell \mid |X_i-x_i|=\epsilon_i\}$ for $\epsilon=(\epsilon_1,\epsilon_2,\cdots,\epsilon_\ell)$ with each $\epsilon_i>0$.

Then we can accordingly express (20) as an integral equation

$$\frac{\mathbf{k}!}{(2\pi i)^\ell}\sum_{\sigma\subseteq[\ell]}(-1)^{|\sigma|}\oint\!\!\oint\cdots\oint_{|\mathbf{X}-\mathbf{x}|=\epsilon}\frac{\mathbf{h}^{\mathbf{k}}}{(\mathbf{X}-\mathbf{x})^{\mathbf{k}+\mathbf{e}}}\det_{i,j\in\sigma}\left[\frac{h'_{ij}}{h_i}\right]\prod_{i\in\sigma}(X_i-x_i)\,d\mathbf{X} \;=\; 0.$$

Interchanging the order between sum and integral and then evaluating the sum, we reduce (20) to the following equivalent equation:

$$\frac{\mathbf{k}!}{(2\pi i)^\ell}\oint\!\!\oint\cdots\oint_{|\mathbf{X}-\mathbf{x}|=\epsilon}\frac{\mathbf{h}^{\mathbf{k}}}{(\mathbf{X}-\mathbf{x})^{\mathbf{k}+\mathbf{e}}}\det_{1\leq i,j\leq\ell}\left[\delta_{i,j}-\frac{h'_{ij}}{h_i}(X_i-x_i)\right]d\mathbf{X} \;=\; 0. \tag{21}$$

For the same $\mathbf{k}\in\mathbb{N}_0^\ell$ with $\mathbf{k}\geq\mathbf{0}$, it is trivial to see from the residue theorem that

$$\frac{\mathbf{k}!}{(2\pi i)^\ell}\oint\!\!\oint\cdots\oint_{|\mathbf{Y}-\mathbf{y}|=\vartheta}\frac{d\mathbf{Y}}{(\mathbf{Y}-\mathbf{y})^{\mathbf{k}+\mathbf{e}}} \;=\; 0,$$

where the contour $|\mathbf{Y}-\mathbf{y}| = \vartheta$ is similarly defined as $|\mathbf{X}-\mathbf{x}| = \epsilon$ before. Making the changes of variables $Y_i - y_i = (X_i - x_i)/h_i(\mathbf{X})$ for $i = 1, 2, \cdots, \ell$, and then computing the corresponding Jacobian determinant

$$\left|\frac{\partial(y_1, y_2, \cdots, y_\ell)}{\partial(x_1, x_2, \cdots, x_\ell)}\right| = \det_{1\le i,j\le \ell}\left[\delta_{i,j} - \frac{h'_{ij}}{h_i}(X_i - x_i)\right] \Big/ \prod_{i=1}^{\ell} h_i,$$

which results in *predominantly diagonal* (see Good [15]), and we find that the last integral becomes exactly the integral in Equation (21).

3. MacMahon's Master Theorem

In this section, the rotated form of Theorem 2 is stated and then utilized to present a new proof for MacMahon's master theorem.

Suppose that the following linear relations are inverse each other.

$$f(\mathbf{n}) = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} A(\mathbf{n},\mathbf{k})\, g(\mathbf{k}) \quad\Longleftrightarrow\quad g(\mathbf{n}) = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} B(\mathbf{n},\mathbf{k})\, f(\mathbf{k}).$$

Then we have the corresponding inverse pair of the lower triangular matrices $A := [A(\mathbf{n},\mathbf{k})]$ and $B := [B(\mathbf{n},\mathbf{k})]$ formed by the connection coefficients. According to Riordan [22] (§2.2), the new inverse pair of the transposes of A and B yields another pair of reciprocal relations:

$$f(\mathbf{n}) = \sum_{\mathbf{k}\ge\mathbf{n}} A(\mathbf{k},\mathbf{n})\, g(\mathbf{k}) \quad\Longleftrightarrow\quad g(\mathbf{n}) = \sum_{\mathbf{k}\ge\mathbf{n}} B(\mathbf{k},\mathbf{n})\, f(\mathbf{k}).$$

Therefore, we obtain the "rotated form" of the inverse series relations displayed in Theorem 2.

Theorem 3 (Multifold inverse series relations)**.**

$$f(\mathbf{n}) = \sum_{\mathbf{k}\ge\mathbf{n}} \binom{\mathbf{k}}{\mathbf{n}} \mathbf{D}^{\mathbf{k}-\mathbf{n}}\{\mathbf{h}^{\mathbf{k}} w\}\, g(\mathbf{k}), \tag{22a}$$

$$g(\mathbf{n}) = \sum_{\mathbf{k}\ge\mathbf{n}} \binom{\mathbf{k}}{\mathbf{n}} \mathbf{D}^{\mathbf{k}-\mathbf{n}}\left\{\frac{\vartheta(\mathbf{k}-\mathbf{n})}{\mathbf{h}^{\mathbf{n}} w}\right\} f(\mathbf{k}). \tag{22b}$$

For $\mathbf{n} \in \mathbb{N}_0^\ell$ and a multivariate formal power series $\phi(\mathbf{y}) := \phi(y_1, y_2, \cdots, y_\ell)$, denote by $[\mathbf{y}^{\mathbf{n}}]\phi(\mathbf{y})$ the coefficient of $\mathbf{y}^{\mathbf{n}}$ in $\phi(\mathbf{y})$. Then we have the following relation:

$$[\mathbf{y}^{\mathbf{n}}]\phi(\mathbf{y}) = \frac{\mathbf{D}_0^{\mathbf{n}}}{\mathbf{n}!}\phi(\mathbf{x}) \quad \text{where} \quad \mathbf{D}_0^{\mathbf{n}}\phi(\mathbf{x}) = \mathbf{D}^{\mathbf{n}}\phi(\mathbf{x})\big|_{\mathbf{x}=\mathbf{0}}.$$

Let $\{x_i, y_i\}_{i=1}^{\ell}$ be the two sets of variables related by the equations

$$x_i = y_i/h_i(\mathbf{y}) \quad \text{with} \quad h_i(\mathbf{y}) := \exp\left(\sum_{j=1}^{\ell} b_{ij} y_j\right) \quad \text{for} \quad i = 1, 2, \cdots, \ell.$$

Consider the formal power series expansion of $\{x_i\}$ in $\{y_j\}$:

$$\begin{aligned}
\mathbf{x^n}\det\left[\delta_{i,j}-y_i\frac{h'_{ij}(\mathbf{y})}{h_i(\mathbf{y})}\right] &= \sum_{\mathbf{k}\ge\mathbf{n}}\mathbf{y^k}\,[\mathbf{y^k}]\,\mathbf{x^n}\det\left[\delta_{i,j}-y_ib_{ij}\right]\\
&= \sum_{\mathbf{k}\ge\mathbf{n}}\mathbf{y^k}\,[\mathbf{y^k}]\,\frac{\mathbf{y^n}}{\mathbf{h^n}(\mathbf{y})}\det\left[\delta_{i,j}-y_ib_{ij}\right]\\
&= \sum_{\mathbf{k}\ge\mathbf{n}}\mathbf{y^k}\,[\mathbf{y^{k-n}}]\,\frac{\det\left[\delta_{i,j}-y_ib_{ij}\right]}{\mathbf{h^n}(\mathbf{y})}\\
&= \sum_{\mathbf{k}\ge\mathbf{n}}\mathbf{y^k}\frac{\mathbf{D_0^{k-n}}}{(\mathbf{k-n})!}\left\{\frac{\det\left[\delta_{i,j}-\frac{k_i-n_i}{D_{x_j}}b_{ij}\right]}{\mathbf{h^n}(\mathbf{x})}\right\}.
\end{aligned}$$

Rewriting this relation in terms of (22b)

$$\frac{\mathbf{x^n}}{\mathbf{n}!} = \sum_{\mathbf{k}\ge\mathbf{n}}\binom{\mathbf{k}}{\mathbf{n}}\mathbf{D_0^{k-n}}\left\{\frac{\vartheta(\mathbf{k-n})}{\mathbf{h^n}(\mathbf{x})}\right\}\frac{\mathbf{y^k}}{\mathbf{k}!}\det{}^{-1}\left[\delta_{i,j}-y_ib_{ij}\right], \tag{23a}$$

we get the dual relation of $\{y_i\}$ in $\{x_j\}$ corresponding to (22a)

$$\frac{\mathbf{y^n}}{\mathbf{n}!}\det{}^{-1}\left[\delta_{i,j}-y_ib_{ij}\right] = \sum_{\mathbf{k}\ge\mathbf{n}}\binom{\mathbf{k}}{\mathbf{n}}\mathbf{D_0^{k-n}}\{\mathbf{h^k}(\mathbf{x})\}\frac{\mathbf{x^k}}{\mathbf{k}!}. \tag{23b}$$

Recalling the exponential functions $h_i(\mathbf{y})$ for $i=1,2,\cdots,\ell$, we can deduce the following relations:

$$\mathbf{h^k}(\mathbf{y}) = \exp\Big(\sum_{j=1}^{\ell}y_j\sum_{i=1}^{\ell}k_ib_{ij}\Big),$$

$$\mathbf{D_0^k}\{\mathbf{h^k}(\mathbf{x})\} = \prod_{j=1}^{\ell}\Big(\sum_{i=1}^{\ell}k_ib_{ij}\Big)^{k_j}.$$

Then we recover from the particular case $\mathbf{n}=\mathbf{0}$ of (23b) the expansion formula due to Carlitz [23]:

$$\det{}^{-1}\left[\delta_{i,j}-y_ib_{ij}\right] = \sum_{\mathbf{k}\ge\mathbf{0}}\frac{\mathbf{y^k}}{\mathbf{k}!}\prod_{j=1}^{\ell}\Big\{h_j^{-1}(\mathbf{y})\sum_{i=1}^{\ell}k_ib_{ij}\Big\}^{k_j}. \tag{24}$$

Note further the expansion

$$\frac{\mathbf{x^k}}{\mathbf{k}!} = \frac{\mathbf{y^k}}{\mathbf{k}!}\Big/\mathbf{h^k}(\mathbf{y}) = \sum_{\mathbf{m}\ge\mathbf{k}}(-1)^{|\mathbf{m-k}|}\binom{\mathbf{m}}{\mathbf{k}}\frac{\mathbf{y^m}}{\mathbf{m}!}\prod_{j=1}^{\ell}\Big(\sum_{i=1}^{\ell}k_ib_{ij}\Big)^{m_j-k_j},$$

where $|\mathbf{k}|$ denotes the coordinate sum for $\mathbf{k}\in\mathbb{N}_0^{\ell}$. By interchanging the order of summation, we can reformulate (24) as follows:

$$\begin{aligned}
\det{}^{-1}\left[\delta_{i,j}-y_ib_{ij}\right] &= \sum_{\mathbf{k}\ge\mathbf{0}}\frac{\mathbf{x^k}}{\mathbf{k}!}\mathbf{D_0^k}\{\mathbf{h^k}(\mathbf{x})\}\\
&= \sum_{\mathbf{m}\ge\mathbf{0}}\frac{\mathbf{y^m}}{\mathbf{m}!}\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{m}}(-1)^{|\mathbf{m-k}|}\binom{\mathbf{m}}{\mathbf{k}}\prod_{j=1}^{\ell}\Big(\sum_{i=1}^{\ell}k_ib_{ij}\Big)^{m_j}.
\end{aligned}$$

Observing that the last multiple sum results in a multivariate difference of order $\mathbf{m}$ in $\mathbf{x}$ for a multivariate polynomial $\prod_{j=1}^{\ell}\big(\sum_{i=1}^{\ell}x_ib_{ij}\big)^{m_j}$ of degree $|\mathbf{m}|$, we assert that it is equal to $\mathbf{m}!$ times the coefficient of $\mathbf{x^m}$ in the same polynomial. This leads us to the following celebrated master theorem.

Theorem 4 (MacMahon's master theorem [14] (§64))**.**

$$\det^{-1}\big[\delta_{i,j}-y_ib_{ij}\big] \;=\; \sum_{\mathbf{m}\geq\mathbf{0}}\mathbf{y}^{\mathbf{m}}\,[\mathbf{x}^{\mathbf{m}}]\prod_{j=1}^{\ell}\Big(\sum_{i=1}^{\ell}x_ib_{ij}\Big)^{m_j}.$$

Different proofs and applications of this theorem may be found, for example, in Carlitz [23], Chu [24,25], Good [26,27], and Goulden–Jackson [28] (§1.2.11–12).

4. Multivariate Lagrange Expansion Formula

In this section, we shall review the celebrated multivariate Lagrange expansion formula discovered by Good [15] by making use of the rotated form of Theorem 1.

Analogous to the consideration of Theorem 2 in the last section, the inverse series relations displayed in Theorem 1 admit the following rotated form.

Theorem 5 (Multifold inverse series relations)**.**

$$f(\mathbf{n}) \;=\; \sum_{\mathbf{k}\geq\mathbf{n}}\binom{\mathbf{k}}{\mathbf{n}}\mathbf{D}^{\mathbf{k}-\mathbf{n}}\big\{\vartheta(\mathbf{k}-\mathbf{n})\mathbf{h}^{\mathbf{k}}w\big\}\,g(\mathbf{k}), \tag{25a}$$

$$g(\mathbf{n}) \;=\; \sum_{\mathbf{k}\geq\mathbf{n}}\binom{\mathbf{k}}{\mathbf{n}}\mathbf{D}^{\mathbf{k}-\mathbf{n}}\big\{\mathbf{h}^{-\mathbf{n}}/w\big\}\,f(\mathbf{k}). \tag{25b}$$

Let $W(\mathbf{y})$ and $h_i(\mathbf{y})$ be formal power series with $W(\mathbf{0})\neq 0$ and $h_i(\mathbf{0})\neq 0$ for $i=1,2,\cdots,\ell$. For the two sets of variables $\{x_i,y_i\}_{i=1}^{\ell}$ related by the equations $x_i=y_i/h_i(\mathbf{y})$, consider the formal power series expansion in $\{y_j\}$:

$$\begin{aligned}
\frac{\mathbf{x}^{\mathbf{n}}}{W(\mathbf{y})} &= \sum_{\mathbf{k}\geq\mathbf{n}}\mathbf{y}^{\mathbf{k}}\,[\mathbf{y}^{\mathbf{k}}]\,\frac{\mathbf{x}^{\mathbf{n}}}{W(\mathbf{y})}\\
&= \sum_{\mathbf{k}\geq\mathbf{n}}\mathbf{y}^{\mathbf{k}}\,[\mathbf{y}^{\mathbf{k}}]\Big\{\frac{\mathbf{y}^{\mathbf{n}}}{W(\mathbf{y})}\mathbf{h}^{-\mathbf{n}}(\mathbf{y})\Big\}\\
&= \sum_{\mathbf{k}\geq\mathbf{n}}\mathbf{y}^{\mathbf{k}}\,[\mathbf{y}^{\mathbf{k}-\mathbf{n}}]\Big\{\mathbf{h}^{-\mathbf{n}}(\mathbf{y})/W(\mathbf{y})\Big\}\\
&= \sum_{\mathbf{k}\geq\mathbf{n}}\mathbf{y}^{\mathbf{k}}\frac{\mathbf{D}_0^{\mathbf{k}-\mathbf{n}}}{(\mathbf{k}-\mathbf{n})!}\Big\{\mathbf{h}^{-\mathbf{n}}(\mathbf{x})/W(\mathbf{x})\Big\}.
\end{aligned}$$

Equating the first and the last expressions, we may state the resulting equation as

$$\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} = \sum_{\mathbf{k}\geq\mathbf{n}}\binom{\mathbf{k}}{\mathbf{n}}\mathbf{D}_0^{\mathbf{k}-\mathbf{n}}\Big\{\mathbf{h}^{-\mathbf{n}}(\mathbf{x})/W(\mathbf{x})\Big\}\frac{\mathbf{y}^{\mathbf{k}}}{\mathbf{k}!}W(\mathbf{y}). \tag{26a}$$

Comparing this relation with (25b), we get the dual relation corresponding to (25a)

$$\frac{\mathbf{y}^{\mathbf{n}}}{\mathbf{n}!}W(\mathbf{y}) = \sum_{\mathbf{k}\geq\mathbf{n}}\binom{\mathbf{k}}{\mathbf{n}}\mathbf{D}_0^{\mathbf{k}-\mathbf{n}}\Big\{\vartheta(\mathbf{k}-\mathbf{n})\mathbf{h}^{\mathbf{k}}(\mathbf{x})W(\mathbf{x})\Big\}\frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!}. \tag{26b}$$

Observing the equation

$$\mathbf{D}_0^{\mathbf{k}-\mathbf{n}}\Big\{\vartheta(\mathbf{k}-\mathbf{n})\mathbf{h}^{\mathbf{k}}(\mathbf{x})W(\mathbf{x})\Big\} \;=\; \mathbf{D}_0^{\mathbf{k}-\mathbf{n}}\Big\{\det\Big[\delta_{i,j}-y_i\frac{h'_{ij}(\mathbf{y})}{h_i(\mathbf{y})}\Big]\mathbf{h}^{\mathbf{k}}(\mathbf{x})W(\mathbf{x})\Big\},$$

we find from (26b) the following elegant formal power series expansion:

$$\frac{\mathbf{y}^{\mathbf{n}}}{\mathbf{n}!}W(\mathbf{y}) = \sum_{\mathbf{k}\geq\mathbf{n}} \binom{\mathbf{k}}{\mathbf{n}} \mathbf{D}_{\mathbf{0}}^{\mathbf{k}-\mathbf{n}} \left\{ \det\left[\delta_{i,j} - y_i \frac{h'_{ij}(\mathbf{y})}{h_i(\mathbf{y})}\right] \mathbf{h}^{\mathbf{k}}(\mathbf{x}) W(\mathbf{x}) \right\} \frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!}, \tag{27}$$

where the condition $W(\mathbf{0}) \neq 0$ imposed initially can be removed now since it is just a formal power series equation.

Letting $\mathbf{n} = \mathbf{0}$ in (27) leads us to the multivariate Lagrange expansion formula.

Theorem 6 (Good [15,29])**.** *Let $\{x_i, y_i\}_{i=1}^{\ell}$ be the two sets of variables related by the equations $x_i = y_i/h_i(\mathbf{y})$ with $h_i(\mathbf{y})$ being formal power series and $h_i(\mathbf{0}) \neq 0$ for $i = 1, 2, \cdots, \ell$. Then these equations determine implicit functions $y_j := y_j(\mathbf{x})$ with $j = 1, 2, \cdots, \ell$. For any formal power series $W(\mathbf{y})$, there holds the following expansion formula:*

$$W(\mathbf{y}) = \sum_{\mathbf{k}\geq\mathbf{0}} \mathbf{D}_{\mathbf{0}}^{\mathbf{k}} \left\{ \det\left[\delta_{i,j} - y_i \frac{h'_{ij}(\mathbf{y})}{h_i(\mathbf{y})}\right] \mathbf{h}^{\mathbf{k}}(\mathbf{x}) W(\mathbf{x}) \right\} \frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!}.$$

This theorem is very important in combinatorial computation and enumerative combinatorics. Gessel [21] has provided a combinatorial proof, where a useful survey has also been given for variants of the multivariate Lagrange inversion formula due to Abhyankar [30], Jacobi [9], Joni [31,32], and Stieltjes [33]. Different proofs and applications can be found in Refs. [29,30,34–41].

5. Multivariate Derivative Identities

For the two derivative identities (1) and (3) discovered by Pfaff ([1], 1795) and Cauchy ([3], 1826), we shall derive their multivariate generalizations in this section. This is accomplished by utilizing the tensor product of the Leibniz rule and the multivariate inverse series relations established in the second section. However for the third identity (4) of Jensen type [42], our effort has unfortunately failed to fulfill analogous multivariate generalization to Theorems 7 and 8 for (1) and (3), respectively.

5.1. Extension of Pfaff's Derivative Identity

For the identities of (1) and (2), their common multivariate extension is given as in the theorem below.

Theorem 7 (Derivative identity)**.** *Suppose that $u := u(\mathbf{x})$, $v := v(\mathbf{x})$ and $h_\kappa := h_\kappa(\mathbf{x})$ are the $\mathbf{n}$-times differentiable functions with respect to $\mathbf{x}$* (hence, $n_\imath$-times differentiable functions with respect to each $x_\imath$ for $\imath, \kappa = 1, 2, \cdots, \ell$)*. Then, the following derivative formula holds:*

$$\mathbf{D}^{\mathbf{n}}(uv) = \sum_{\mathbf{0}\leq\mathbf{k}\leq\mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} \mathbf{D}^{\mathbf{k}}\{\mathbf{h}^{\mathbf{k}}u\} \mathbf{D}^{\mathbf{n}-\mathbf{k}} \left\{ \frac{\vartheta(\mathbf{n}-\mathbf{k})}{\mathbf{h}^{\mathbf{k}}} v \right\}.$$

When $h_\kappa \equiv 1$, this identity turns out to be the tensor product of the Leibniz formula. Its univariate case is recorded as follows:

$$D^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D^k(h^k u) D^{n-k} \left\{ \left(1 - \frac{n-k}{D}\frac{h'}{h}\right) \frac{v}{h^k} \right\}.$$

Replacing v by $h^n v$ and keeping in mind the equality

$$D(h^{n-k}v) = (n-k)h^{n-k-1}h'v + h^{n-k}v',$$

we can rewrite the last identity in the following manner:

$$D^n\left(h^n uv\right) = \sum_{k=0}^{n}\binom{n}{k} D^k\left(h^k u\right) D^{n-k-1}\left\{h^{n-k}v'\right\},$$

which is evidently the same as (2) after having made the involution $k \to n-k$.

Proof. It is not hard to verify that the derivative identity stated in Theorem 7 matches exactly (12b) under the following specifications:

$$w = 1/v, \qquad f(\mathbf{k}) = \mathbf{D}^{\mathbf{k}}\{\mathbf{h}^{\mathbf{k}}u\} \quad \text{and} \quad g(\mathbf{n}) = \mathbf{D}^{\mathbf{n}}(uv).$$

Then to prove the identity enunciated in the theorem, it is enough to validate the following dual relation corresponding to (12a):

$$\mathbf{D}^{\mathbf{n}}\{\mathbf{h}^{\mathbf{n}}u\} = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}}\binom{\mathbf{n}}{\mathbf{k}} \mathbf{D}^{\mathbf{k}}(uv)\mathbf{D}^{\mathbf{n}-\mathbf{k}}\{\mathbf{h}^{\mathbf{n}}/v\}.$$

This follows immediately from the tensor product of the Leibniz rule. □

5.2. Extension of Cauchy's Derivative Identity

Making the substitution v by $\mathbf{h}^{\mathbf{n}}v$ in Theorem 7, we derive the following equivalent expression:

$$\mathbf{D}^{\mathbf{n}}\left(\mathbf{h}^{\mathbf{n}}uv\right) = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}}\binom{\mathbf{n}}{\mathbf{k}} \mathbf{D}^{\mathbf{k}}\{\mathbf{h}^{\mathbf{k}}u\}\mathbf{D}^{\mathbf{n}-\mathbf{k}}\left\{\vartheta(\mathbf{n}-\mathbf{k})\mathbf{h}^{\mathbf{n}-\mathbf{k}}v\right\}. \tag{28}$$

This is further applied, in turn, for calculating the following derivatives:

$$\begin{aligned}\mathbf{D}^{\mathbf{n}}\left\{\vartheta(\mathbf{n})\mathbf{h}^{\mathbf{n}}uv\right\} &= \mathbf{D}^{\mathbf{n}}\left\{\sum_{\sigma\subseteq[\ell]}(-1)^{|\sigma|}\prod_{i\in\sigma}\frac{n_i}{D_{x_i}}\det_{i,j\in\sigma}\left[\frac{h'_{ij}}{h_i}\right]\mathbf{h}^{\mathbf{n}}uv\right\}\\ &= \sum_{\sigma\subseteq[\ell]}(-1)^{|\sigma|}\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}}\binom{\mathbf{n}}{\mathbf{k}}\mathbf{D}^{\mathbf{k}}\left\{\prod_{i\in\sigma}\frac{k_i}{D_{x_i}}\det_{i,j\in\sigma}\left[\frac{h'_{ij}}{h_i}\right]\mathbf{h}^{\mathbf{k}}u\right\}\mathbf{D}^{\mathbf{n}-\mathbf{k}}\left\{\vartheta(\mathbf{n}-\mathbf{k})\mathbf{h}^{\mathbf{n}-\mathbf{k}}v\right\}.\end{aligned}$$

Interchanging the order of summation, we may reformulate the last double sum as

$$\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}}\binom{\mathbf{n}}{\mathbf{k}}\mathbf{D}^{\mathbf{k}}\left\{\sum_{\sigma\subseteq[\ell]}(-1)^{|\sigma|}\prod_{i\in\sigma}\frac{k_i}{D_{x_i}}\det_{i,j\in\sigma}\left[\frac{h'_{ij}}{h_i}\right]\mathbf{h}^{\mathbf{k}}u\right\}\mathbf{D}^{\mathbf{n}-\mathbf{k}}\left\{\vartheta(\mathbf{n}-\mathbf{k})\mathbf{h}^{\mathbf{n}-\mathbf{k}}v\right\}.$$

Keeping in mind the definition of ϑ-function, we derive the following symmetric counterpart of (28), which gives the multivariate extension of Cauchy's identity (3).

Theorem 8 (Derivative identity)**.** *Suppose that $u := u(\mathbf{x})$, $v := v(\mathbf{x})$ and $h_\kappa := h_\kappa(\mathbf{x})$ are the $\mathbf{n}$-times differentiable functions with respect to $\mathbf{x}$* (i.e., $n_\imath$-times differentiable functions with respect to $x_\imath$ for $\imath, \kappa = 1, 2, \cdots, \ell$)*. Then the following formula holds:*

$$\mathbf{D}^{\mathbf{n}}\left\{\vartheta(\mathbf{n})\mathbf{h}^{\mathbf{n}}uv\right\} = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}}\binom{\mathbf{n}}{\mathbf{k}}\mathbf{D}^{\mathbf{k}}\left\{\vartheta(\mathbf{k})\mathbf{h}^{\mathbf{k}}u\right\}\mathbf{D}^{\mathbf{n}-\mathbf{k}}\left\{\vartheta(\mathbf{n}-\mathbf{k})\mathbf{h}^{\mathbf{n}-\mathbf{k}}v\right\}.$$

When $h_\kappa \equiv 1$, this identity becomes the tensor product of the Leibniz formula.

Analogous to the proof of Theorem 7, we can also show Theorem 8 through the inverse pair (12a) and (12b). First, under the replacement v by $v/\mathbf{h}^{\mathbf{n}}$, we obtain, from Theorem 8, the equality:

$$\mathbf{D}^{\mathbf{n}}\left\{\vartheta(\mathbf{n})uv\right\} = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}}\binom{\mathbf{n}}{\mathbf{k}}\mathbf{D}^{\mathbf{k}}\left\{\vartheta(\mathbf{k})\mathbf{h}^{\mathbf{k}}u\right\}\mathbf{D}^{\mathbf{n}-\mathbf{k}}\left\{\frac{\vartheta(\mathbf{n}-\mathbf{k})}{\mathbf{h}^{\mathbf{k}}}v\right\}.$$

Then by making a comparison between the last equation and (12b), we affirm that it suffices to prove the following dual relation corresponding to (12a):

$$\mathbf{D}^{\mathbf{n}}\Big\{\vartheta(\mathbf{n})\mathbf{h}^{\mathbf{n}}u\Big\} = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}}\binom{\mathbf{n}}{\mathbf{k}}\mathbf{D}^{\mathbf{k}}\{\vartheta(\mathbf{k})uv\}\mathbf{D}^{\mathbf{n}-\mathbf{k}}\{\mathbf{h}^{\mathbf{n}}/v\}.$$

According to the Leibniz rule, we can finally confirm it as follows:

$$\begin{aligned}&\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}}\binom{\mathbf{n}}{\mathbf{k}}\mathbf{D}^{\mathbf{k}}\{\vartheta(\mathbf{k})uv\}\mathbf{D}^{\mathbf{n}-\mathbf{k}}\{\mathbf{h}^{\mathbf{n}}/v\}\\ =&\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}}\binom{\mathbf{n}}{\mathbf{k}}\mathbf{D}^{\mathbf{k}}\Big\{\sum_{\sigma\subseteq[\ell]}(-1)^{|\sigma|}\prod_{i\in\sigma}\frac{k_i}{D_{x_i}}\det_{i,j\in\sigma}\Big[\frac{h'_{ij}}{h_i}\Big]uv\Big\}\mathbf{D}^{\mathbf{n}-\mathbf{k}}\{\mathbf{h}^{\mathbf{n}}/v\}\\ =&\ \mathbf{D}^{\mathbf{n}}\Big\{\sum_{\sigma\subseteq[\ell]}(-1)^{|\sigma|}\prod_{i\in\sigma}\frac{n_i}{D_{x_i}}\det_{i,j\in\sigma}\Big[\frac{h'_{ij}}{h_i}\Big]\mathbf{h}^{\mathbf{n}}u\Big\} = \mathbf{D}^{\mathbf{n}}\Big\{\vartheta(\mathbf{n})\mathbf{h}^{\mathbf{n}}u\Big\}.\end{aligned}$$ □

As a byproduct of Theorem 8, we can even establish the multifold inverse series pair with more symmetric expressions, as in the following theorem.

Theorem 9 (Multifold inverse series relations)**.** *Suppose that $w := w(\mathbf{x}) = w(x_1, x_2, \cdots, x_\ell)$ and $h_\kappa := h_\kappa(\mathbf{x}) = h_\kappa(x_1, x_2, \cdots, x_\ell)$ are infinitely differentiable multivariate functions. Then the following inverse series relations hold:*

$$f(\mathbf{n}) = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}}\binom{\mathbf{n}}{\mathbf{k}}\mathbf{D}^{\mathbf{n}-\mathbf{k}}\Big\{\vartheta(\mathbf{n}-\mathbf{k})\mathbf{h}^{\mathbf{n}-\mathbf{k}}w\Big\}\,g(\mathbf{k}), \tag{29a}$$

$$g(\mathbf{n}) = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}}\binom{\mathbf{n}}{\mathbf{k}}\mathbf{D}^{\mathbf{n}-\mathbf{k}}\Big\{\vartheta(\mathbf{n}-\mathbf{k})\mathbf{h}^{\mathbf{n}-\mathbf{k}}/w\Big\}\,f(\mathbf{k}). \tag{29b}$$

Proof. Taking into account the symmetric form, we can indiscriminately assume one of the two relations and prove another. Analogous to the proof of Theorem 1, by first substituting (29a) into (29b), and then exchanging the order of summation, we can readily verify that the inverse pair displayed in the theorem is equivalent to the following orthogonality:

$$\sum_{\mathbf{m}\le\mathbf{k}\le\mathbf{n}}\binom{\mathbf{n}-\mathbf{m}}{\mathbf{k}-\mathbf{m}}\mathbf{D}^{\mathbf{n}-\mathbf{k}}\Big\{\vartheta(\mathbf{n}-\mathbf{k})\mathbf{h}^{\mathbf{n}-\mathbf{k}}/w\Big\}\mathbf{D}^{\mathbf{k}-\mathbf{m}}\Big\{\vartheta(\mathbf{k}-\mathbf{m})\mathbf{h}^{\mathbf{k}-\mathbf{m}}w\Big\} = \delta(\mathbf{m},\mathbf{n}).$$

For $\mathbf{m} = \mathbf{n}$, this is obviously true. Otherwise, the sum on the left hand side can be expressed, by means of Theorem 8, as

$$\mathbf{D}^{\mathbf{n}-\mathbf{m}}\{\vartheta(\mathbf{n}-\mathbf{m})\mathbf{h}^{\mathbf{n}-\mathbf{m}}\} = \mathbf{D}^{\mathbf{n}-\mathbf{m}}\Big\{\sum_{\sigma\subseteq[\ell]}(-1)^{|\sigma|}\prod_{i\in\sigma}\frac{n_i-m_i}{D_{x_i}}\det_{i,j\in\sigma}\Big[\frac{h'_{ij}}{h_i}\Big]\mathbf{h}^{\mathbf{n}-\mathbf{m}}\Big\}.$$

According to (15), the above difference vanishes for $\mathbf{m} \ne \mathbf{n}$ with $\mathbf{m} \le \mathbf{n}$. □

6. Multiple Convolution Formulae Due to Carlitz

Utilizing these derivative identities derived in the last section, this section will review the multiple convolution formulae of Abel and Hagen–Rothe type due to Carlitz [16,17].

Throughout this section, we shall further fix the following notations. Let $\mathbf{a} := (a_1, a_2, \cdots, a_\ell)$, $\mathbf{c} = (c_1, c_2, \cdots, c_\ell) \in \mathbb{C}^\ell$ and $\mathbf{B} := [b_{ij}]_{1\le i,j\le\ell} \in \mathbb{C}^{\ell\times\ell}$. Denote by ${}_i\mathbf{b} := (b_{i1}, b_{i2}, \cdots, b_{i\ell})$ and $\mathbf{b}_j := (b_{1j}, b_{2j}, \cdots, b_{\ell j})$ the i-th row and j-th column of $\mathbf{B}$, respectively.

6.1. Multiple Convolutions of Abel Type

Define the exponential functions

$$u := e^{\langle \mathbf{a},\mathbf{x}\rangle}, \qquad v := e^{\langle \mathbf{c},\mathbf{x}\rangle}, \qquad h_\kappa := e^{\langle \mathbf{b}_\kappa,\mathbf{x}\rangle}.$$

Then it is not hard to compute

$$\mathbf{h}^{\mathbf{k}} = \exp\Big\{\sum_{\imath=1}^{\ell}\langle_\imath\mathbf{b},\mathbf{k}\rangle x_\imath\Big\} \quad \text{and} \quad \mathbf{D}^{\mathbf{k}}\{\mathbf{h}^{\mathbf{k}}u\} = \mathbf{h}^{\mathbf{k}}u\prod_{\imath=1}^{\ell}(a_\imath+\langle_\imath\mathbf{b},\mathbf{k}\rangle)^{k_\imath}$$

as well as

$$\begin{aligned}
\mathbf{D}^{\mathbf{k}}\{\vartheta(\mathbf{k})\mathbf{h}^{\mathbf{k}}u\} &= \mathbf{D}^{\mathbf{k}}\Big\{\sum_{\sigma\subseteq[\ell]}(-1)^{|\sigma|}\prod_{i\in\sigma}\frac{k_i}{D_{x_i}}\det_{i,j\in\sigma}\Big[\frac{h'_{ij}}{h_i}\Big]\mathbf{h}^{\mathbf{k}}u\Big\}\\
=&\mathbf{D}^{\mathbf{k}}\Big\{\sum_{\sigma\subseteq[\ell]}(-1)^{|\sigma|}\prod_{i\in\sigma}\frac{k_i}{D_{x_i}}\det_{i,j\in\sigma}\big[b_{ij}\big]\exp\Big(\langle\mathbf{a},\mathbf{x}\rangle+\sum_{\imath=1}^{\ell}\langle_\imath\mathbf{b},\mathbf{k}\rangle x_\imath\Big)\Big\}\\
=&\mathbf{h}^{\mathbf{k}}u\,\varpi_{\mathbf{a}}(\mathbf{k})\prod_{\imath=1}^{\ell}(a_\imath+\langle_\imath\mathbf{b},\mathbf{k}\rangle)^{k_\imath},
\end{aligned}$$

where

$$\varpi_{\mathbf{a}}(\mathbf{k}) := \det_{1\le i,j\le\ell}\Big[\delta_{i,j}-\frac{b_{ij}k_i}{a_i+\langle_i\mathbf{b},\mathbf{k}\rangle}\Big]. \tag{30}$$

Substituting these relations into Theorems 7 and 8, and then removing the common exponential function factors, we recover the multiple convolution identities below.

Corollary 1 (Carlitz [16,17])**.**

$$\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}}\varpi_{\mathbf{a}}(\mathbf{k})\prod_{\imath=1}^{\ell}\frac{(a_\imath+\langle_\imath\mathbf{b},\mathbf{k}\rangle)^{k_\imath}}{k_\imath!}\prod_{\imath=1}^{\ell}\frac{(c_\imath+\langle_\imath\mathbf{b},\mathbf{n}-\mathbf{k}\rangle)^{n_\imath-k_\imath}}{(n_\imath-k_\imath)!} \tag{31a}$$

$$=\prod_{\imath=1}^{\ell}\frac{(a_\imath+c_\imath+\langle_\imath\mathbf{b},\mathbf{n}\rangle)^{n_\imath}}{n_\imath!}; \tag{31b}$$

$$\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}}\varpi_{\mathbf{a}}(\mathbf{k})\prod_{\imath=1}^{\ell}\frac{(a_\imath+\langle_\imath\mathbf{b},\mathbf{k}\rangle)^{k_\imath}}{k_\imath!}\varpi_{\mathbf{c}}(\mathbf{n}-\mathbf{k})\prod_{\imath=1}^{\ell}\frac{(c_\imath+\langle_\imath\mathbf{b},\mathbf{n}-\mathbf{k}\rangle)^{n_\imath-k_\imath}}{(n_\imath-k_\imath)!} \tag{32a}$$

$$=\varpi_{\mathbf{a}+\mathbf{c}}(\mathbf{n})\prod_{\imath=1}^{\ell}\frac{(a_\imath+c_\imath+\langle_\imath\mathbf{b},\mathbf{n}\rangle)^{n_\imath}}{n_\imath!}. \tag{32b}$$

Different proofs of these identities via the generating function method can be found in Chu [36]. When $\ell=1$, these identities reduce to the following well–known formulae originally due to Abel [43] (cf. [8,36] (§3.1), [22,44] (§1.5) also):

$$\begin{aligned}
\sum_{k=0}^{n}\frac{a(a+bk)^{k-1}}{k!}\,\frac{\{c+b(n-k)\}^{n-k}}{(n-k)!} &= \frac{(a+c+bn)^n}{n!};\\
\sum_{k=0}^{n}\frac{a(a+bk)^{k-1}}{k!}\,\frac{c\{c+b(n-k)\}^{n-k-1}}{(n-k)!} &= \frac{(a+c)(a+c+bn)^{n-1}}{n!}.
\end{aligned}$$

6.2. Multiple Convolutions of Hagen–Rothe Type

Instead, we define the binomial functions

$$u := \prod_{\imath=1}^{\ell}(1+x_\imath)^{a_\imath}, \qquad v := \prod_{\imath=1}^{\ell}(1+x_\imath)^{c_\imath}, \qquad h_\kappa := \prod_{\imath=1}^{\ell}(1+x_\imath)^{b_{\imath\kappa}}.$$

Then it is not hard to compute

$$\mathbf{h}^{\mathbf{k}} = \prod_{\imath=1}^{\ell}(1+x_\imath)^{\langle_\imath \mathbf{b},\mathbf{k}\rangle} \quad \text{and} \quad \mathbf{D}^{\mathbf{k}}\{\mathbf{h}^{\mathbf{k}}u\} = \mathbf{h}^{\mathbf{k}}u\prod_{\imath=1}^{\ell}\frac{k_\imath!}{(1+x_\imath)^{k_\imath}}\binom{a_\imath+\langle_\imath \mathbf{b},\mathbf{k}\rangle}{k_\imath}$$

as well as

$$\begin{aligned}
\mathbf{D}^{\mathbf{k}}\{\vartheta(\mathbf{k})\mathbf{h}^{\mathbf{k}}u\} \;&=\; \mathbf{D}^{\mathbf{k}}\Big\{\sum_{\sigma\subseteq[\ell]}(-1)^{|\sigma|}\prod_{i\in\sigma}\frac{k_i}{D_{x_i}}\det_{i,j\in\sigma}\Big[\frac{h'_{ij}}{h_i}\Big]\mathbf{h}^{\mathbf{k}}u\Big\}\\
&=\mathbf{D}^{\mathbf{k}}\Big\{\sum_{\sigma\subseteq[\ell]}(-1)^{|\sigma|}\prod_{i\in\sigma}\frac{k_i}{D_{x_i}}\det_{i,j\in\sigma}\Big[\frac{b_{ij}}{1+x_i}\Big]\prod_{\imath=1}^{\ell}(1+x_\imath)^{a_\imath+\langle_\imath \mathbf{b},\mathbf{k}\rangle}\Big\}\\
&=\mathbf{h}^{\mathbf{k}}u\sum_{\sigma\subseteq[\ell]}(-1)^{|\sigma|}\det_{i,j\in\sigma}\Big[\frac{k_ib_{ij}}{a_i+\langle_i \mathbf{b},\mathbf{k}\rangle}\Big]\prod_{\imath=1}^{\ell}\frac{k_\imath!}{(1+x_\imath)^{k_\imath}}\binom{a_\imath+\langle_\imath \mathbf{b},\mathbf{k}\rangle}{k_\imath}\\
&=\mathbf{h}^{\mathbf{k}}u\,\varpi_{\mathbf{a}}(\mathbf{k})\prod_{\imath=1}^{\ell}\frac{k_\imath!}{(1+x_\imath)^{k_\imath}}\binom{a_\imath+\langle_\imath \mathbf{b},\mathbf{k}\rangle}{k_\imath}.
\end{aligned}$$

Substituting these relations into Theorems 7 and 8, and then making some routine simplifications, we arrive at the following multiple convolution identities.

Corollary 2 (Carlitz [16,17])**.**

$$\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}}\varpi_{\mathbf{a}}(\mathbf{k})\prod_{\imath=1}^{\ell}\binom{a_\imath+\langle_\imath \mathbf{b},\mathbf{k}\rangle}{k_\imath}\prod_{\imath=1}^{\ell}\binom{c_\imath+\langle_\imath \mathbf{b},\mathbf{n}-\mathbf{k}\rangle}{n_\imath-k_\imath} \tag{33a}$$

$$=\prod_{\imath=1}^{\ell}\binom{a_\imath+c_\imath+\langle_\imath \mathbf{b},\mathbf{n}\rangle}{n_\imath}; \tag{33b}$$

$$\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}}\varpi_{\mathbf{a}}(\mathbf{k})\prod_{\imath=1}^{\ell}\binom{a_\imath+\langle_\imath \mathbf{b},\mathbf{k}\rangle}{k_\imath}\varpi_{\mathbf{c}}(\mathbf{n}-\mathbf{k})\prod_{\imath=1}^{\ell}\binom{c_\imath+\langle_\imath \mathbf{b},\mathbf{n}-\mathbf{k}\rangle}{n_\imath-k_\imath} \tag{34a}$$

$$=\varpi_{\mathbf{a}+\mathbf{c}}(\mathbf{n})\prod_{\imath=1}^{\ell}\binom{a_\imath+c_\imath+\langle_\imath \mathbf{b},\mathbf{n}\rangle}{n_\imath}. \tag{34b}$$

Chu and Hsu [45] gave different analytic proofs of these identities. We remark that the above identities are, in fact, multifold extensions of the Hagen–Rothe formulae (see Chu [36], Gould [46,47], and Rothe [48]):

$$\begin{aligned}
\sum_{k=0}^{n}\frac{a}{a+bk}\binom{a+bk}{k}\binom{c+b(n-k)}{n-k} &= \binom{a+c+bn}{n};\\
\sum_{k=0}^{n}\frac{a}{a+bk}\binom{a+bk}{k}\frac{c}{c+b(n-k)}\binom{c+b(n-k)}{n-k} &= \frac{a+c}{a+c+bn}\binom{a+c+bn}{n}.
\end{aligned}$$

7. Further Multivariate Inverse Series Relations

In this section, we shall review some of the author's work in comparison with those in Theorems 1 and 2. Some related works are briefly commented in the sequel. Egorychev produced numerous formulae exclusively by making use of the Cauchy residue method in his monograph [49]. Three multifold inverse relations [49] (§3.3) were recorded in Ref. [50]. They were further extended in quite a different manner by the author [51] to the revisiting multifold identities of Handa and Mohanty [52,53].

Let $\{\mathbf{a}(k)\}_{k\in\mathbb{N}_0}$ and $\{\mathbf{B}(k)\}_{k\in\mathbb{N}_0}$ be two sequences of the vectors and the matrices with $\mathbf{a}(k) := \big(a_1(k), a_2(k), \cdots, a_\ell(k)\big) \in \mathbb{C}^\ell$ and $\mathbf{B}(k) := \big[b_{ij}(k)\big]_{1\le i,j\le\ell} \in \mathbb{C}^{\ell\times\ell}$. Denote as before the i-th row and j-th column of matrix $\mathbf{B}(k)$, respectively, by ${}_i\mathbf{b}(k) := \big(b_{i1}(k), b_{i2}(k), \cdots, b_{i\ell}(k)\big)$ and $\mathbf{b}_j(k) := \big(b_{1j}(k), b_{2j}(k), \cdots, b_{\ell j}(k)\big)$.

Extending Gould–Hsu [54] inversions, Chu [18,19,55] found the following multivariate inverse series relations. Define $\varphi(\mathbf{x};\mathbf{n})$ and $\omega(\mathbf{x};\mathbf{n})$, respectively, by

$$\varphi(\mathbf{x};\mathbf{n}) \;:=\; \prod_{i=1}^{\ell}\prod_{k=0}^{n_i-1}\big\{a_i(k) + \langle {}_i\mathbf{b}(k), \mathbf{x}\rangle\big\}, \tag{35a}$$

$$\omega(\mathbf{x};\mathbf{n}) \;:=\; \det_{1\le i,j\le\ell}\Big[\delta_{i,j} - \frac{(x_i-n_i)b_{ij}(n)}{a_i(n_i) + \langle {}_i\mathbf{b}(n_i), \mathbf{x}\rangle}\Big]; \tag{35b}$$

with the convention that the empty product is equal to one. Then there hold the following inverse series relations:

$$f(\mathbf{n}) \;=\; \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} (-1)^{|\mathbf{k}|}\binom{\mathbf{n}}{\mathbf{k}}\varphi(\mathbf{k};\mathbf{n})g(\mathbf{k}), \tag{36a}$$

$$g(\mathbf{n}) \;=\; \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} (-1)^{|\mathbf{k}|}\binom{\mathbf{n}}{\mathbf{k}}\frac{\omega(\mathbf{n};\mathbf{k})}{\varphi(\mathbf{n};\mathbf{k})}f(\mathbf{k}). \tag{36b}$$

Applications and q-analogue of this inverse pair can be found in Abderrezzak [56]. The above inverse pair clearly has a very different nature from that stated in Theorem 1. However, it is not hard to verify that the multivariate reciprocal pairs in the sequel are contained as their common particular cases.

7.1. Inverse Series Relations of Abel Type

Letting $a_i(k) \equiv a_i$ and $b_{ij}(k) \equiv b_{ij}$ be constant for $1 \le i,j \le \ell$, then we find from (36a) and (36b) the following inverse series relations.

Corollary 3 (Inverse pair of Abel type)**.**

$$f(\mathbf{n}) \;=\; \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} (-1)^{|\mathbf{k}|}\binom{\mathbf{n}}{\mathbf{k}}\prod_{i=1}^{\ell}(a_i + \langle {}_i\mathbf{b}, \mathbf{k}\rangle)^{n_i}\, g(\mathbf{k}), \tag{37a}$$

$$g(\mathbf{n}) \;=\; \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} (-1)^{|\mathbf{k}|}\binom{\mathbf{n}}{\mathbf{k}}\frac{\rho(\mathbf{n};\mathbf{k})}{\prod_{i=1}^{\ell}(a_i + \langle {}_i\mathbf{b}, \mathbf{n}\rangle)^{k_i}}\, f(\mathbf{k}); \tag{37b}$$

where

$$\rho(\mathbf{n};\mathbf{k}) := \det_{1\le i,j\le\ell}\Big[\delta_{i,j} - \frac{(n_i-k_i)b_{ij}}{a_i + \langle {}_i\mathbf{b}, \mathbf{n}\rangle}\Big].$$

This inverse pair follows by specifying $h_k = e^{\langle \mathbf{b}_k, \mathbf{x}\rangle}$ and $w = e^{\langle \mathbf{a}, \mathbf{x}\rangle}$ in Theorem 1.

7.2. Inverse Series Relations of the Hagen—Rothe Type

Letting $a_i(k) := k + a_i$ and $b_{ij}(k) \equiv b_{ij}$ for $1 \le i,j \le \ell$, then we find from (36a) and (36b) another inverse pair.

Corollary 4 (Inverse pair of Hagen–Rothe type)**.**

$$f(\mathbf{n}) = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} (-1)^{|\mathbf{k}|} \binom{\mathbf{n}}{\mathbf{k}} \prod_{i=1}^{\ell} (a_i + \langle {}_i\mathbf{b}, \mathbf{k}\rangle)_{n_i}\, g(\mathbf{k}), \tag{38a}$$

$$g(\mathbf{n}) = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} (-1)^{|\mathbf{k}|} \binom{\mathbf{n}}{\mathbf{k}} \frac{\varrho(\mathbf{n};\mathbf{k})}{\prod_{i=1}^{\ell} (a_i + \langle {}_i\mathbf{b}, \mathbf{n}\rangle)_{k_i}}\, f(\mathbf{k}); \tag{38b}$$

where

$$\varrho(\mathbf{n};\mathbf{k}) := \det_{1\le i,j\le\ell} \left[\delta_{i,j} - \frac{(n_i - k_i)b_{ij}}{a_i + k_i + \langle {}_i\mathbf{b}, \mathbf{n}\rangle}\right]$$

with the rising factorial being given by

$$(x)_0 = 1 \quad \textit{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \textit{for} \quad n \in \mathbb{N}.$$

It can be verified without much difficulty that the last inverse series relations result from Theorem 1 under $h_k = \prod_{i=1}^{\ell}(1+x_i)^{b_{ik}+\delta_{i,k}}$ and $w = \prod_{i=1}^{\ell}(1+x_i)^{a_i}$.

7.3. Two Further Inverse Series Relations

Under the previous two settings for h and w functions, we can further derive from Theorem 9 the following two pairs of multivariate inverse series relations, where the ϖ-function is defined in (30).

Corollary 5 (Two quasi–symmetric inverse pairs)**.**

$$f(\mathbf{n}) = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} \varpi_{\mathbf{a}}(\mathbf{n}-\mathbf{k}) \prod_{i=1}^{\ell} (a_i + \langle {}_i\mathbf{b}, \mathbf{n}-\mathbf{k}\rangle)^{n_i-k_i}\, g(\mathbf{k}), \tag{39a}$$

$$g(\mathbf{n}) = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} \varpi_{-\mathbf{a}}(\mathbf{n}-\mathbf{k}) \prod_{i=1}^{\ell} (-a_i + \langle {}_i\mathbf{b}, \mathbf{n}-\mathbf{k}\rangle)^{n_i-k_i}\, f(\mathbf{k}); \tag{39b}$$

$$f(\mathbf{n}) = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} \varpi_{\mathbf{a}}(\mathbf{n}-\mathbf{k}) \prod_{i=1}^{\ell} (a_i + \langle {}_i\mathbf{b}, \mathbf{n}-\mathbf{k}\rangle)_{n_i-k_i}\, g(\mathbf{k}), \tag{40a}$$

$$g(\mathbf{n}) = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} \varpi_{-\mathbf{a}}(\mathbf{n}-\mathbf{k}) \prod_{i=1}^{\ell} (-a_i + \langle {}_i\mathbf{b}, \mathbf{n}-\mathbf{k}\rangle)_{n_i-k_i}\, f(\mathbf{k}). \tag{40b}$$

However, both inverse pairs do not seem to be obtainable from (36a) and (36b). This fact shows that the inversion theorems established in the present paper are not completely covered by the previous ones displayed in (36a) and (36b).

8. Concluding Remarks

By employing the algebraic calculus, we have established two multiple derivative inverse pairs (Theorems 1 and 2) in this paper. They are then utilized to review MacMahon's master theorem [14] (§64) and Good's multivariate Lagrange formula [15]. Sample applications include multivariate extensions of derivative identities due to Pfaff [1] and Cauchy 28, as well as multiple convolution formulae due to Carlitz [16,17].

It is widely recognized that Gould–Hsu inversions [54], their variants and multifold analogues (cf. [11,12,18,19,44,50,51,55]) have important applications in binomial identities [16,17,46,47] and hypergeometric series evaluations [45,57–59], as well as special functions and orthogonal polynomials [2,4]. Therefore, it is plausible that the multiple derivative inverse pairs established in this paper can find significant applications in combinatorial enumeration [28,29,41], multinomial convolution formulae [27,38,52,53], special functions, and polynomials of several variables [2,31,37,40], as well as multiple contour integrals (see [49] (Chapter 5) and [60] (Chapter 3)). The interested readers are encouraged to make further explorations.

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