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Infection Eradication Criterion in a General Epidemic Model with Logistic Growth, Quarantine Strategy, Media Intrusion, and Quadratic Perturbation

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Abstract: This article explores and highlights the effect of stochasticity on the extinction behavior of a disease in a general epidemic model. Specifically, we consider a sophisticated dynamical model that combines logistic growth, quarantine strategy, media intrusion, and quadratic noise. The amalgamation of all these hypotheses makes our model more practical and realistic. By adopting new analytical techniques, we provide a sharp criterion for disease eradication. The theoretical results show that the extinction criterion of our general perturbed model is mainly determined by the parameters closely related to the linear and quadratic perturbations as well as other deterministic parameters of the system. In order to clearly show the strength of our new result in a practical way, we perform numerical examples using the case of herpes simplex virus (HSV) in the USA. We conclude that a great amount of quadratic noise minimizes the period of HSV and affects its eradication time.

Keywords: epidemic model; logistic growth; quarantine strategy; media intrusion; quadratic noise

MSC: 65M06; 39A14; 35L53; 92D25

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1. Introduction

Recently, the spread of contagious infections is having an adverse influence on the healthcare systems globally, and has imposed a heavy load on people's lifestyles [1,2]. To counteract the negative effects of epidemics and prevent their rapid transmission, some medical and non-medical strategies such as drugs, vaccination, quarantine, isolation and media intervention are widely used [3,4]. Although we can discover and grasp the biological characteristics of epidemics, overall infection control is not deterministic due to external factors. Human intervention, financial crises, and political decisions clearly influence the spread of epidemics [5]. Climatic alterations lead to changes in global average temperature and sometimes unpredictable rainfall, phenomena that modify the conditions for the development of vectors [6]. These environmental changes can increase the presence of viruses such as influenza in certain areas where average temperatures are cooler [7]. Despite the challenges that extrinsic conditions can pose for controlling vector-borne infectious diseases, researchers strive to mathematically describe this randomness using different processes and methods [8]. The main goal is to provide a sophisticated formulation that simulates the effects of the above strategies and perturbations on disease spread [9,10].

Researchers in the biology and ecology domains investigate the complex characters of natural phenomena, and therefore require adapted methods of description and abstraction [11]. Probabilistic systems, being both powerful in their characterization and pliable in their analytical treatment, are the most appropriate tools for probing such scenarios and situations [12]. The integration of stochastic modeling with complex real-word model analysis provides wisdom in species and disease dynamics under global oscillations [13–17]. Using white noise, various analytical tools have been adopted to establish information and predict the future of the analyzed phenomenon [18]. In its standard form, many researchers have proportionally integrated white noise in order to stimulate the impact of certain continuous variations on the long-term dynamics of biological systems [19–23]. Asymptotic characteristics such as stability [24–27], dynamical bifurcation [28–30], random bifurcation [31], synchronization, chaos theory [32,33], and other properties have been investigated [34–40]. When encountering noisy and complex disturbances, consideration of second-order white noise is an effective approach for depicting strong differences in species and individuals. This view was first proposed by Liu and Jiang [41] in 2017 by analyzing the asymptotic behavior of a stochastic differential system driven by a quadratic polynomial diffusion part. Subsequently, the authors of [42] studied the stationarity and extinction properties of a multi-stage HIV system with quadratic perturbation. In [43], Liu analyzed the stationarity and ergodicity properties of a SICA-HIV system with quadratic perturbation. In [44], the authors treated the existence of a single probability measure and extinction of the illness for a general SIRS epidemic system with additional hypotheses and quadratic perturbation. In [45], the authors investigated the periodicity and stationarity properties of a perturbed epidemic model with relapse and quadratic perturbation. In [46], Liu and Jiang proposed a new AIDS system with enhanced hypotheses. They studied the stationarity and extinction of their model with quadratic perturbation. In [47], the authors introduced a new multi-stage HIV-AIDS system with quadratic perturbation and established the sufficient conditions for the stationarity and extinction of the disease. In [48], the authors considered an epidemic model with relapse hypothesis and media intervention. They obtained the sufficient criteria for ergodicity and extinction in the case of quadratic perturbation. In [49], Lv et al. provided the sufficient conditions of the stationarity and extinction of an impulsive chemostat system with quadratic perturbation. In [50], the authors included regime switching in a predator–prey system. By assuming the quadratic perturbation, they proved the existence of a single stationary distribution. In [51], the authors analyzed a perturbed logistic equation with continuous delay and quadratic perturbation. They established the conditions of the extinction and existence of a steady stationary distribution. In [52], Liu et al. showed the periodicity and ergodicity properties of a standard SIR system with quadratic perturbation. In [53], the authors offered the sufficient conditions of the ergodicity property of a switched ecological Lotka–Volterra system under quadratic perturbation. In [54], the authors integrated the quadratic perturbation in an ecological system with additional food and obtained the sufficient condition of stationarity and extinction. In [55], Liu and Jiang analyzed the propagation of HIV by discussing the extinction and stationarity properties under the hypothesis of quadratic perturbation. In [56], the authors provided a nice generalization of a switched perturbed epidemic model with the hypotheses of media intrusion, isolation, and default immunity and established the sufficient criteria for stationarity and extinction. In [57], the authors studied the extinction and stationary distribution of a stochastic COVID-19 epidemic model with time delay. In this study, we upgrade the model proposed in [56] by incorporating the effect of logistic growth. In terms of phenomenological modeling, the logistic growth function is generally used illustratively or phenomenologically because it correlates well with the ultimate leveling of infection as the population develops herd immunity [51]. The adoption of this function makes epidemiological and ecological modeling more real and significant [50]. By considering logistic growth and quadratic white noise, the general epidemic model with medical and non-medical intervention is formulated as follows:

$$
\begin{cases}
\mathrm{d}X_1 = \overbrace{\left(\frac{\mathfrak{r}}{\mathfrak{A}}X_1\Big(\mathfrak{A}-X_1\Big)-\mathfrak{B}_0X_1X_2-\left(\mathfrak{B}_1-\frac{\mathfrak{B}_1^\star X_4}{\mathfrak{m}_1+X_4}\right)X_1X_4-\left(\mathfrak{B}_2-\frac{\mathfrak{B}_2^\star X_5}{\mathfrak{m}_2+X_5}\right)X_1X_5\right)\mathrm{d}t}^{\text{Deterministic part}}+\overbrace{\mathrm{d}\mathbb{Q}_1(t)}^{\text{Stochastic part}},\\
\mathrm{d}X_2 = \left(\mathfrak{B}_0X_1X_2+\left(\mathfrak{B}_1-\frac{\mathfrak{B}_1^\star X_4}{\mathfrak{m}_1+X_4}\right)X_1X_4+\left(\mathfrak{B}_2-\frac{\mathfrak{B}_2^\star X_5}{\mathfrak{m}_2+X_5}\right)X_1X_5-\big(\mathfrak{D}+\mathfrak{E}_1+\mathfrak{E}_2+\mathfrak{F}_1\big)X_2\right)\mathrm{d}t+\mathrm{d}\mathbb{Q}_2(t),\\
\mathrm{d}X_3 = \big(\mathfrak{E}_1X_2-(\mathfrak{D}+\mathfrak{M}+\mathfrak{F}_2)X_3\big)\mathrm{d}t+\mathrm{d}\mathbb{Q}_3(t),\\
\mathrm{d}X_4 = \big(\mathfrak{E}_2X_2-(\mathfrak{D}+\mathfrak{N}+\mathfrak{K}_1+\mathfrak{L}_1)X_4\big)\mathrm{d}t+\mathrm{d}\mathbb{Q}_4(t),\\
\mathrm{d}X_5 = \big(\mathfrak{M}X_3+\mathfrak{N}X_4-(\mathfrak{D}+\mathfrak{K}_2+\mathfrak{L}_2)X_5\big)\mathrm{d}t+\mathrm{d}\mathbb{Q}_5(t),\\
\mathrm{d}X_6 = \big(\mathfrak{F}_1X_2+\mathfrak{F}_2X_3+\mathfrak{K}_1X_4+\mathfrak{K}_2X_5-\mathfrak{D}X_6\big)\mathrm{d}t+\mathrm{d}\mathbb{Q}_6(t),\\
X_k(0)>0, k=1,\dots,6,
\end{cases}
\tag{1}
$$

where the processes X_k, $k = 1, \dots, 6$, are defined in Table 1.

Table 1. Classification of different types of individuals.

Symbol	Epidemiological Classification
X_1	Susceptible persons
X_2	Exposed persons
X_3	Isolated persons
X_4	Infected persons
X_5	Hospitalized persons
X_6	Recovered persons

In the following, we explain the components of the above model.

- Deterministic Part:
  The first part contains the transfer rates between the classes, with the positive parameters defined in Table 2.

Table 2. Epidemiological meaning of the deterministic parameters appearing in (1).

Parameter	Epidemiological Meaning	Unit
$\mathfrak{r}$	*The natural intrinsic growing rate of* X_1	days^{-1}
$\mathfrak{A}$	*The carrying amplitude of* X_1	1 million
$\mathfrak{B}_0$	*The propagation ratio between* X_1 *and* X_2	days^{-1}
$\mathfrak{B}_1$	*The maximal efficient contact rate between* X_1 *and* X_4	days^{-1}
$\mathfrak{B}_2$	*The maximal efficient contact rate between* X_1 *and* X_5	days^{-1}
$\mathfrak{B}_1^\star$	*The reduced active contact rate due to media intrusion associated with* X_4, $(\mathfrak{B}_1 \geq \mathfrak{B}_1^\star)$	–
$\mathfrak{B}_2^\star$	*The reduced active contact rate due to media intrusion associated with* X_5, $(\mathfrak{B}_2 \geq \mathfrak{B}_2^\star)$	–
$\mathfrak{E}_1$	*The isolation rate of* X_2	days^{-1}
$\mathfrak{E}_2$	*The transition rate from* X_2 *to* X_4	days^{-1}
$\mathfrak{D}$	*The normal death rate of* X_k, $k = 2, 3, 4, 5, 6$	days^{-1}
$\mathfrak{F}_1$	*The cure rate of* X_2	days^{-1}
$\mathfrak{F}_2$	*The cure rate of* X_3	days^{-1}
$\mathfrak{K}_1$	*The cure rate of* X_4	days^{-1}
$\mathfrak{K}_2$	*The cure rate of* X_5	days^{-1}
$\mathfrak{L}_1$	*The disease-related mortality rate of* X_4	days^{-1}
$\mathfrak{L}_2$	*The disease-related mortality rate of* X_5	days^{-1}
$\mathfrak{m}_1$	*The coefficient of media intrusion associated with* X_4	–
$\mathfrak{m}_2$	*The coefficient of media intrusion associated with* X_5	–
$\mathfrak{M}$	*The hospitalization ratio of* X_3	days^{-1}
$\mathfrak{N}$	*The hospitalization ratio of* X_4	days^{-1}

If we only consider this part without adding random fluctuations, we obtain a deterministic model that simulates the spread of a given disease under an isolation strategy and media intrusion. To classify and sort the long-term behavior of this disease, we can use the basic reproductive ratio $\mathcal{R}_\circ$ [58]. According to the calculus presented in Section 3 of [59], $\mathcal{R}_\circ$ is expressed as follows:

$$\mathcal{R}_\circ = \frac{\mathfrak{A}}{\mathfrak{U}_2}\left(\mathfrak{B}_0 + \frac{\mathfrak{B}_1\mathfrak{E}_2}{\mathfrak{U}_4} + \frac{\mathfrak{B}_2\mathfrak{M}\mathfrak{E}_1}{\mathfrak{U}_3\mathfrak{U}_5} + \frac{\mathfrak{B}_2\mathfrak{N}\mathfrak{E}_2}{\mathfrak{U}_4\mathfrak{U}_5}\right),$$

where

$$\mathfrak{U}_2 = \mathfrak{D} + \mathfrak{E}_1 + \mathfrak{E}_2 + \mathfrak{F}_1, \quad \mathfrak{U}_3 = \mathfrak{D} + \mathfrak{M} + \mathfrak{F}_2, \quad \mathfrak{U}_4 = \mathfrak{D} + \mathfrak{N} + \mathfrak{K}_1 + \mathfrak{L}_1, \quad \mathfrak{U}_5 = \mathfrak{D} + \mathfrak{K}_2 + \mathfrak{L}_2.$$

- Stochastic Part:
  This part characterizes and describes the effects of complex environmental fluctuations, where

$$\begin{cases} d\mathbb{Q}_1(t) &= \overbrace{\Xi_{1\ell}X_1(t)d\mathbb{W}_1(t)}^{\text{Linear part}} + \overbrace{\Xi_{1q}X_1^2(t)d\mathbb{W}_1(t)}^{\text{Quadratic part}}, \\ d\mathbb{Q}_2(t) &= \Xi_{2\ell}X_2(t)d\mathbb{W}_2(t) + \Xi_{2q}X_2^2(t)d\mathbb{W}_2(t), \\ d\mathbb{Q}_3(t) &= \Xi_{3\ell}X_3(t)d\mathbb{W}_3(t) + \Xi_{3q}X_3^2(t)d\mathbb{W}_3(t), \\ d\mathbb{Q}_4(t) &= \Xi_{4\ell}X_4(t)d\mathbb{W}_4(t) + \Xi_{4q}X_4^2(t)d\mathbb{W}_4(t), \\ d\mathbb{Q}_5(t) &= \Xi_{5\ell}X_5(t)d\mathbb{W}_5(t) + \Xi_{5q}X_5^2(t)d\mathbb{W}_5(t), \\ d\mathbb{Q}_6(t) &= \Xi_{6\ell}X_6(t)d\mathbb{W}_6(t) + \Xi_{6q}X_6^2(t)d\mathbb{W}_6(t). \end{cases}$$

  We consider a probability triple $(\Omega, \mathcal{E}, \mathbb{P})$ and an increasing right-continuous filtration $\{\mathcal{E}_t\}_{t\geq 0}$ along with the fact that $\mathcal{E}_0$ includes all $\mathbb{P}$-null sets. The six Wiener processes $\mathbb{W}_k(t)$ $(k = 1,2,3,4,5,6)$ are all mutually independent and defined on $(\Omega, \mathcal{E}, \{\mathcal{E}_t\}_{t\geq 0}, \mathbb{P})$; $\Xi_{k\ell} > 0$ $(k = 1,2,3,4,5,6)$ are the intensities of white noises in the linear part, while $\Xi_{kq} > 0$ $(k = 1,2,3,4,5,6)$ are the intensities of white noises in the quadratic part.

In the present study, our main intention is to provide the precise condition for the eradication of the infection in probabilistic model (1). This asymptotic property is considered sufficient to control the future pandemic situation under medical and non-medical strategies. Although model (1) is based on the model presented in [56], this article generalizes the previous results and develops the extinction criteria in the case of logistic growth. Furthermore, we numerically probe the effect of quadratic noise on the temporal extinction of the infection. We show that human intervention and the high amount of stochastic components influence the eradication time of the epidemic, which is an important way to control the future of the epidemic and provide good forecasts.

The remainder of this article is structured as follows. In Section 2, we exhibit the main theoretical results of our paper by providing the sharp criterion of disease eradication. In Section 3, we support our findings with computer simulations using real data on the herpes simplex virus (HSV) in the USA. Furthermore, we explore the effect of quadratic noise on the eradication time of the epidemic. In Section 4, we derive the main conclusions of this article.

2. Theoretical Results

Before manipulating system (1), it is necessary to check that it is mathematically and biologically well-posed. In line with the proof of Theorem 3.1 in [47], we conclude that for any positive initial data

$$\mathcal{X}_0 = (X_1(0), X_2(0), X_3(0), X_4(0), X_5(0), X_6(0)) \in \mathbb{R}^6_+,$$

we have the existence of a unique, global and positive solution

$$\mathcal{X} = (X_1(t), X_2(t), X_3(t), X_4(t), X_5(t), X_6(t)) \in \mathbb{R}^6_+$$

with near certainty (henceforth abbreviated as a.s.). By summing all classes of (1), we define the total class $T_{\text{tot}}(t) = \sum_{k=1}^{6} X_k(t)$. According to the equation of X_1, we consider the following auxiliary system:

$$\begin{cases} \mathrm{d}Y(t) = \frac{\mathfrak{r}}{\mathfrak{A}} Y(t)\Big\{\mathfrak{A} - Y(t)\Big\}\mathrm{d}t + \Xi_{1\ell} Y(t)\mathrm{d}\mathbb{W}_1(t) + \Xi_{1q} Y^2(t)\mathrm{d}\mathbb{W}_1(t), \\ Y(0) = X_1(0) \in \mathbb{R}_+. \end{cases} \tag{2}$$

From the theory presented in [56], Equation (2) is well-posed, and if $\mathfrak{r} - 0.5\Xi^2_{1\ell} > 0$, then (2) admits the following single invariant probability measure Θ^Y:

$$\Theta^Y(y) = \mathbf{M} y^{\frac{2\mathfrak{r}}{\Xi^2_{1\ell}} - 2} (\Xi_{1q} + \Xi_{1\ell} y)^{-\frac{2\mathfrak{r}}{\Xi^2_{1\ell}} - 2} e^{\frac{2\mathfrak{r}(\Xi_{1\ell} + \mathfrak{A}\Xi_{1q})}{\mathfrak{A}\Xi_{1\ell}\Xi_{1q}(\Xi_{1\ell} + \Xi_{1q} y)}}, \quad \forall y > 0,$$

where $\mathbf{M}$ is a constant that satisfies $\int_{\mathbb{R}_+} y\Theta^Y(\mathrm{d}y) = 1$ a.s. Via the ergodic property [60], we can see that

$$\lim_{t\to\infty} t^{-1} \int_0^t Y(s)\mathrm{d}s = \int_{\mathbb{R}_+} y\Theta^Y(\mathrm{d}y), \quad a.s. \tag{3}$$

In the next lemma, we provide an estimation of the time average of $Y(t)$ defined in (3).

Lemma 1. *Presume that $\mathfrak{r} - 0.5\Xi^2_{1\ell} > 0$; then, the time average of Y is estimated as follows:*

$$\lim_{t\to\infty} \int_0^t Y(s)\mathrm{d}s \le \frac{\left(\mathfrak{r} - 0.5\Xi^2_{1\ell}\right)}{\left(\frac{\mathfrak{r}}{\mathfrak{A}} + \Xi_{1\ell}\Xi_{1q}\right)} \quad a.s.$$

Proof. Employing Itô's lemma for drift–diffusion processes, we have

$$\mathrm{d}\ln Y(t) = \left(\frac{\mathfrak{r}}{\mathfrak{A}}\Big(\mathfrak{A} - Y(t)\Big) - 0.5\Big(\Xi_{1\ell} + \Xi_{1q} Y(t)\Big)^2\right)\mathrm{d}t + \Big(\Xi_{1\ell} + \Xi_{1q} Y(t)\Big)\mathrm{d}\mathbb{W}_1(t).$$

Integrating from 0 to t for both sides of the last equality, we obtain

$$\begin{aligned} \ln Y(t) - \ln Y(0) = \mathfrak{r}t - \frac{\mathfrak{r}}{\mathfrak{A}} \int_0^t Y(s)\mathrm{d}s - 0.5\Xi^2_{1\ell} t - \Xi_{1\ell}\Xi_{1q} \int_0^t Y(s)\mathrm{d}s - 0.5\Xi^2_{1q} \int_0^t Y^2(s)\mathrm{d}s \\ + \Xi_{1\ell}\mathbb{W}_1(t) + \Xi_{1q} \int_0^t Y(s)\mathbb{W}_1(s). \end{aligned}$$

Then, we have

$$\int_0^t Y(s)\mathrm{d}s = \frac{\left(\mathfrak{r} - 0.5\Xi^2_{1\ell}\right)t}{\left(\frac{\mathfrak{r}}{\mathfrak{A}} + \Xi_{1\ell}\Xi_{1q}\right)} - 0.5\Xi^2_{1q} \int_0^t Y^2(s)\mathrm{d}s + \Xi_{1q} \int_0^t Y(s)\mathbb{W}_1(s) + \Xi_{1\ell}\mathbb{W}_1(t) - \ln Y(t) + \ln Y(0).$$

Let $g(t) = \Xi_{1q} \int_0^t Y(s)\mathbb{W}_1(s)$; then, the quadratic variation is $\langle g(t), g(t)\rangle = \Xi^2_{1q} \int_0^t Y^2(s)\mathrm{d}s$. According the exponential Martingales inequality, we have

$$\mathbb{P}\left\{\sup_{0\le t\le h} \Big(g(t) - 0.5\langle g(t), g(t)\rangle \ge 2\ln h\Big)\right\} \le \frac{1}{h}.$$

for all $h > 0$. From the Borel-Cantelli lemma, we can be sure that for all $h-1 < t < h$,

$$g(t) \leq 0.5\Xi_{1q}^2 \int_0^t Y^2(s)\mathrm{d}s + \ln h \quad a.s.$$

Consequently,

$$\frac{1}{t}\int_0^t Y(s)\mathrm{d}s \leq \frac{\left(\mathfrak{r} - 0.5\Xi_{1\ell}^2\right)}{\left(\frac{\mathfrak{r}}{\mathfrak{A}} + \Xi_{1\ell}\Xi_{1q}\right)} + \frac{\ln h}{t} + \frac{\Xi_{1\ell}\mathbb{W}_1(t)}{t} - \frac{\ln Y(t)}{t} + \frac{\ln Y(0)}{t}. \tag{4}$$

Then, taking the limit on both sides of (4),

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t Y(s)\mathrm{d}s \leq \frac{\left(\mathfrak{r} - 0.5\Xi_{1\ell}^2\right)}{\left(\frac{\mathfrak{r}}{\mathfrak{A}} + \Xi_{1\ell}\Xi_{1q}\right)} \quad a.s.$$

□

The next theorem aims to provide a sharp criterion for the eradication of the infection. For simplicity, we consider the following list of notations:

$$\begin{cases} \mathfrak{z}_1 &= \dfrac{\max\limits_{0\leq k\leq 2}\{\mathfrak{B}_k\}}{\min\{\mathfrak{z}_2, \mathfrak{z}_4, \mathfrak{z}_5\}}\dfrac{\left(\mathfrak{r} - 0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r} + \mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}, \\ \mathfrak{z}_2 &= \dfrac{\left(\mathfrak{r} - 0.5\Xi_{1\ell}^2\right)\mathcal{R}_\circ}{\left(\mathfrak{r} + \mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}, \\ \mathfrak{z}_3 &= \dfrac{\mathfrak{A}\mathfrak{B}_2\mathfrak{M}\left(\mathfrak{r} - 0.5\Xi_{1\ell}^2\right)}{\mathfrak{U}_3\mathfrak{U}_5\left(\mathfrak{r} + \mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}, \\ \mathfrak{z}_4 &= \dfrac{\mathfrak{A}\left(\mathfrak{B}_1\mathfrak{U}_5 + \mathfrak{B}_2\mathfrak{N}\right)\left(\mathfrak{r} - 0.5\Xi_{1\ell}^2\right)}{\mathfrak{U}_4\mathfrak{U}_5\left(\mathfrak{r} + \mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}, \\ \mathfrak{z}_5 &= \dfrac{\mathfrak{A}\mathfrak{B}_2\left(\mathfrak{r} - 0.5\Xi_{1\ell}^2\right)}{\mathfrak{U}_5\left(\mathfrak{r} + \mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}. \end{cases}$$

Theorem 1. *The eradication of the disease occurs if*

$$\mathcal{R}_\circ < \min\left\{\frac{\left(\mathfrak{r} + \mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}{\left(\mathfrak{r} - 0.5\Xi_{1\ell}^2\right)}, \frac{0.125\min\limits_{2\leq k\leq 5}\{\Xi_{kq}^2\}}{0.5\mathfrak{z}_1\displaystyle\int_{\mathbb{R}_+}\left|y - \frac{\mathfrak{A}\left(\mathfrak{r} - 0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r} + \mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\right|\Theta^Y(\mathrm{d}y)}\right\} = \mathbf{K}.$$

Explicitly, the solution $\mathcal{X}$ of (1) verifies

$$\limsup_{t\to\infty} t^{-1}\ln\sum_{k=2}^{5}\mathfrak{z}_k X_k(t) \leq 0.5\mathfrak{z}_1\mathcal{R}_\circ\int_{\mathbb{R}_+}\left|y - \frac{\mathfrak{A}\left(\mathfrak{r} - 0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r} + \mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\right|\Theta^Y(\mathrm{d}y) - 0.125\min_{2\leq k\leq 5}\{\Xi_{kq}^2\} < 0 \quad \text{a.s.},$$

which implies that $\lim\limits_{t\to\infty} X_k(t) = 0$ a.s. for all $k = 2, 3, 4, 5$.

Proof. In order to reduce notations and provide clear mathematical writing, we set

$$f(t) = \sum_{k=2}^{5}\mathfrak{z}_k X_k(t).$$

Let $f^{-1} = f^{-1}(t) = \frac{1}{f(t)}$ and $f^{-2} = f^{-2}(t) = \frac{1}{f^2(t)}$. By employing Itô's lemma for drift–diffusion processes, we obtain

$$\mathrm{d}\big(\ln f\big) = \mathcal{L}\big(\ln f\big)\mathrm{d}t + f^{-1} \times \sum_{k=2}^{5} \mathfrak{z}_k\Big(\Xi_{k\ell}X_k + \Xi_{kq}X_k^2\Big)\mathrm{d}\mathbb{W}_k(t), \tag{5}$$

where $\mathcal{L}$ is the Itô differential operator, such that

$$\begin{aligned}\mathcal{L}\big(\ln f\big) = f^{-1} \times \Bigg(&\mathfrak{z}_2\bigg(\mathfrak{B}_0X_1X_2 + \bigg(\mathfrak{B}_1 - \frac{\mathfrak{B}_1^\star X_4}{\mathfrak{m}_1 + X_4}\bigg)X_1X_4 + \bigg(\mathfrak{B}_2 - \frac{\mathfrak{B}_2^\star X_5}{\mathfrak{m}_2 + X_5}\bigg)X_1X_5 - \mathfrak{U}_2X_2\bigg)\\ &+ \mathfrak{z}_3\Big(\mathfrak{C}_1X_2 - \mathfrak{U}_3X_3\Big) + \mathfrak{z}_4\Big(\mathfrak{C}_2X_2 - \mathfrak{U}_4X_4\Big) + \mathfrak{z}_5\Big(\mathfrak{M}X_3 + \mathfrak{N}X_4 - \mathfrak{U}_5X_5\Big)\\ &- 0.5f^{-2} \times \sum_{k=2}^{5}\mathfrak{z}_k^2\Big(\Xi_{k\ell}X_k + \Xi_{kq}X_k^2\Big)^2.\end{aligned}$$

In accordance with the positivity of $\mathcal{X}(t)$, we have

$$\begin{aligned}\mathcal{L}\big(\ln f\big) \le f^{-1} \times \Bigg(&\overbrace{\mathfrak{z}_2\Big(\mathfrak{B}_0X_2 + \mathfrak{B}_1X_4 + \mathfrak{B}_2X_5\Big)X_1 - \Big(\mathfrak{z}_2\mathfrak{U}_2 - \mathfrak{z}_3\mathfrak{C}_1 - \mathfrak{z}_4\mathfrak{C}_2\Big)X_2}^{=\Psi_1}\\ &\overbrace{-\Big(\mathfrak{z}_3\mathfrak{U}_3 - \mathfrak{z}_5\mathfrak{M}\Big)X_3 - \Big(\mathfrak{z}_4\mathfrak{U}_4 - \mathfrak{z}_5\mathfrak{N}\Big)X_4 - \mathfrak{z}_5\mathfrak{U}_5X_5}^{=\Psi_2}\Bigg) - 0.5f^{-2} \times \sum_{k=2}^{5}\mathfrak{z}_k^2\Big(\Xi_{k\ell}X_k + \Xi_{kq}X_k^2\Big)^2.\end{aligned} \tag{6}$$

To deal with expressions Ψ_1 and Ψ_2, we consider the following system:

$$\mathbb{S} = \begin{cases}\mathfrak{U}_2\mathfrak{z}_2 - \mathfrak{C}_1\mathfrak{z}_3 - \mathfrak{C}_2\mathfrak{z}_4 + 0\mathfrak{z}_5 &= \frac{\mathfrak{A}\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)\mathfrak{B}_0}{(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q})},\\ 0\mathfrak{z}_2 + \mathfrak{U}_3\mathfrak{z}_3 + 0\mathfrak{z}_4 - \mathfrak{M}\mathfrak{z}_5 &= 0,\\ 0\mathfrak{z}_2 + 0\mathfrak{z}_3 + \mathfrak{U}_4\mathfrak{z}_4 - \mathfrak{N}\mathfrak{z}_5 &= \frac{\mathfrak{A}\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)\mathfrak{B}_1}{(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q})},\\ 0\mathfrak{z}_2 + 0\mathfrak{z}_3 + 0\mathfrak{z}_4 + \mathfrak{U}_5\mathfrak{z}_5 &= \frac{\mathfrak{A}\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)\mathfrak{B}_2}{(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q})}.\end{cases}$$

Obviously, $(\mathfrak{z}_2, \mathfrak{z}_3, \mathfrak{z}_4, \mathfrak{z}_5) = \left(\frac{\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)\mathcal{R}_\circ}{(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q})}, \frac{\mathfrak{A}\mathfrak{B}_2\mathfrak{M}\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\mathfrak{U}_3\mathfrak{U}_5(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q})}, \frac{\mathfrak{A}\left(\mathfrak{B}_1\mathfrak{U}_5+\mathfrak{B}_2\mathfrak{N}\right)\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\mathfrak{U}_4\mathfrak{U}_5(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q})}, \frac{\mathfrak{A}\mathfrak{B}_2\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\mathfrak{U}_5(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q})}\right)$ is the unique solution of $\mathbb{S}$. Using this result, we obtain

$$\begin{aligned}\Psi_1 + \Psi_2 &= \overbrace{\Big(\mathfrak{B}_0X_2 + \mathfrak{B}_1X_4 + \mathfrak{B}_2X_5\Big)}^{=\Psi_3}\left(\frac{\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\mathcal{R}_\circ X_1 - \frac{\mathfrak{A}\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\right)\\ &= \Psi_3\left(\frac{\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\mathcal{R}_\circ X_1 - \frac{\mathfrak{A}\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)} - \mathcal{R}_\circ\frac{\mathfrak{A}\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)^2}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)^2} + \mathcal{R}_\circ\frac{\mathfrak{A}\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)^2}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)^2}\right)\\ &= \frac{\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\mathcal{R}_\circ\Psi_3\left(X_1 - \frac{\mathfrak{A}\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\right) + \frac{\mathfrak{A}\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\Psi_3\underbrace{\left(\frac{\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\mathcal{R}_\circ - 1\right)}_{<0}.\end{aligned}$$

In line with the probabilistic comparison lemma [61], we can conclude that

$$\Psi_1+\Psi_2 \le \frac{\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\mathcal{R}_\circ\Psi_3\left(X_1-\frac{\mathfrak{A}\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\right) \le \frac{\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\mathcal{R}_\circ\Psi_3\left(Y-\frac{\mathfrak{A}\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\right)$$
$$\le \frac{\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\mathcal{R}_\circ\Psi_3\left\{Y-\frac{\mathfrak{A}\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\right\}^+, \tag{7}$$

where $\{\cdot\}^+$ is the the ramp function defined by $\{z\}^+ = \max\{0,z\} = 0.5(z+|z|)$ for all $z \in \mathbb{R}$. Now, we return to the inequality (6) and utilize the result (7); thus,

$$\mathcal{L}(\ln f) \le f^{-1}\times\Psi_3\times\frac{\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\mathcal{R}_\circ\left\{Y-\frac{\mathfrak{A}\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\right\}^+ - 0.5f^{-2}\times\sum_{k=2}^{5}\mathfrak{Z}_k^2\left(\Xi_{k\ell}X_k+\Xi_{kq}X_k^2\right)^2.$$

From the definition of $\mathfrak{Z}_1$, the above inequality implies that

$$\mathcal{L}(\ln f) \le \mathfrak{Z}_1\mathcal{R}_\circ\left\{Y-\frac{\mathfrak{A}\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\right\}^+ - 0.5f^{-2}\times\sum_{k=2}^{5}\mathfrak{Z}_k^2\left(\Xi_{k\ell}X_k+\Xi_{kq}X_k^2\right)^2.$$

Consequently,

$$\mathcal{L}(\ln f) \le 0.5\mathfrak{Z}_1\mathcal{R}_\circ\left(Y-\frac{\mathfrak{A}\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\right)+0.5\mathfrak{Z}_1\mathcal{R}_\circ\left|Y-\frac{\mathfrak{A}\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\right| - 0.5f^{-2}\times\sum_{k=2}^{5}\mathfrak{Z}_k^2\left(\Xi_{k\ell}X_k+\Xi_{kq}X_k^2\right)^2.$$

After which, we make two operations on both sides of (5), that is, integration from 0 to t and division by t; then, the result is

$$t^{-1}\ln f(t) - t^{-1}\ln f(0) \le 0.5\mathfrak{Z}_1\mathcal{R}_\circ\left(t^{-1}\int_0^t Y(s)\mathrm{d}s-\frac{\mathfrak{A}\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\right)+0.5\mathfrak{Z}_1\mathcal{R}_\circ t^{-1}\int_0^t\left|Y(s)-\frac{\mathfrak{A}\left(\mathfrak{r}-0.5\Xi_{1\ell}^2\right)}{\left(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q}\right)}\right|\mathrm{d}s$$
$$-0.5t^{-1}\sum_{k=2}^{5}\int_0^t \mathfrak{Z}_k^2 f^{-2}(s)\left(\Xi_{k\ell}X_k(s)+\Xi_{kq}X_k^2(s)\right)^2\mathrm{d}s$$
$$+t^{-1}\sum_{k=2}^{5}\int_0^t \mathfrak{Z}_k f^{-1}(s)\left(\Xi_{k\ell}X_k(s)+\Xi_{kq}X_k^2(s)\right)\mathrm{d}\mathbb{W}_k(s). \tag{8}$$

The next step is based on the use of the exponential Martingales inequality [62], which leads to

$$\mathbb{P}\left(\sup_{0\le t\le h}\left\{\int_0^t \mathfrak{Z}_k f^{-1}(s)\left(\Xi_{k\ell}X_k(s)+\Xi_{kq}X_k^2(s)\right)\mathrm{d}\mathbb{W}_k(s)-0.5\mathfrak{n}\int_0^t \mathfrak{Z}_k^2 f^{-2}(s)\left(\Xi_{k\ell}X_k(s)+\Xi_{kq}X_k^2(s)\right)^2\mathrm{d}s\right\} > \frac{2\ln h}{\mathfrak{n}}\right) \le \frac{1}{h^2},$$

for all $k = 2,3,4,5$; $0 < \mathfrak{n} < 1$ and $h > 0$. From the Borel-Cantelli result [62], we can be assured of the existence of $h_\omega = h(\omega)$ for all ω in Ω such that the inequality

$$\int_0^t \mathfrak{Z}_k f^{-1}(s)\left(\Xi_{k\ell}X_k(s)+\Xi_{kq}X_k^2(s)\right)\mathrm{d}\mathbb{W}_k(s) \le 0.5\mathfrak{n}\int_0^t \mathfrak{Z}_k^2 f^{-2}(s)\left(\Xi_{k\ell}X_k(s)+\Xi_{kq}X_k^2(s)\right)^2\mathrm{d}s+\frac{2\ln h}{\mathfrak{n}}, \quad \forall k = 2,3,4,5,$$

holds for all $h-1 < t \le h$ and $h \ge h_\omega$ a.s. Under this setup, we can show that

$$
\begin{aligned}
&-0.5t^{-1}\sum_{k=2}^{5}\int_0^t \mathfrak{z}_k^2 f^{-2}(s)\Big(\Xi_{k\ell}X_k(s)+\Xi_{kq}X_k^2(s)\Big)^2\mathrm{d}s+t^{-1}\sum_{k=2}^{5}\int_0^t \mathfrak{z}_k f^{-1}(s)\Big(\Xi_{k\ell}X_k(s)+\Xi_{kq}X_k^2(s)\Big)\mathrm{d}\mathbb{W}_k(s)\\
&\leq 0.5(\mathfrak{n}-1)t^{-1}\int_0^t f^{-2}(s)\sum_{k=2}^{5}\mathfrak{z}_k^2\Big(\Xi_{k\ell}X_k(s)+\Xi_{kq}X_k^2(s)\Big)^2\mathrm{d}s+\frac{8\ln h}{\mathfrak{n}t}\\
&\leq 0.5(\mathfrak{n}-1)t^{-1}\int_0^t\left(4\sum_{k=2}^{5}\mathfrak{z}_k^2X_k^2(t)\right)^{-1}\left(\sum_{k=2}^{5}\mathfrak{z}_k^2\Xi_{k\ell}^2X_k^2(s)\right)\mathrm{d}s+\frac{8\ln h}{\mathfrak{n}(h-1)}\\
&\leq 0.125(\mathfrak{n}-1)\min_{2\leq k\leq 5}\{\Xi_{kq}^2\}+\frac{8\ln h}{\mathfrak{n}(h-1)}.
\end{aligned}
$$

By taking the sup limit on two sides of (8), h tends to ∞ and

$$
\begin{aligned}
\limsup_{t\to\infty} t^{-1}\ln f(t) &\leq 0.5\mathfrak{z}_1\mathcal{R}_\circ\limsup_{t\to\infty} t^{-1}\int_0^t\left|Y(s)-\frac{\mathfrak{A}\big(\mathfrak{r}-0.5\Xi_{1\ell}^2\big)}{(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q})}\right|\mathrm{d}s-0.125(1-\mathfrak{n})\min_{2\leq k\leq 5}\{\Xi_{kq}^2\}+\underbrace{\lim_{h\to\infty}\frac{8\ln h}{\mathfrak{n}(h-1)}}_{=0}\\
&=0.5\mathfrak{z}_1\mathcal{R}_\circ\int_{\mathbb{R}_+}\left|y-\frac{\mathfrak{A}\big(\mathfrak{r}-0.5\Xi_{1\ell}^2\big)}{(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q})}\right|\Theta^Y(\mathrm{d}y)-0.125\min_{2\leq k\leq 5}\{\Xi_{kq}^2\}+0.125\mathfrak{n}\min_{2\leq k\leq 5}\{\Xi_{kq}^2\}\quad \text{a.s.}
\end{aligned}
$$

We let $\mathfrak{n}$ tend to 0^+; then, the obtained result is

$$
\limsup_{t\to\infty} t^{-1}\ln f(t)\leq 0.5\mathfrak{z}_1\mathcal{R}_\circ\int_{\mathbb{R}_+}\left|y-\frac{\mathfrak{A}\big(\mathfrak{r}-0.5\Xi_{1\ell}^2\big)}{(\mathfrak{r}+\mathfrak{A}\Xi_{1\ell}\Xi_{1q})}\right|\Theta^Y(\mathrm{d}y)-0.125\min_{2\leq k\leq 5}\{\Xi_{kq}^2\}<0,\quad \text{a.s.}
$$

Because the probabilistic eradication of the infection implies its extinction with near certainty, $\lim_{t\to\infty} f(t)=0$ a.s., and the positivity of $\mathcal{X}$ affirms that $\lim_{t\to\infty} X_k(t)=0$ a.s. for all $k=2,3,4,5$. In other words, the classes of individuals carrying the virus disappear despite their different characteristics. □

3. Numerical Application: Herpes Simplex Virus (HSV)

Herpes is a viral, contagious, and recurrent illness. Herpes is characterized by the appearance of blisters (vesicles) grouped in clumps. It is a highly contagious viral disease. Its best known manifestation is the classic labial cold sore, and genital localizations exist as well. Herpes simplex (herpes virus) belongs to the herpesviridae family, which includes the varicella zoster virus, infectious mononucleosis virus (Epstein–Barr), and cytomegalovirus [63].

In this section, and by choosing the parameter values from the real data on HSV (US) depicted in Table 3, we set forth a number of simulations to shed light on the strength of our sharp extinction theorem associated with the general epidemic model (1). The solution of this latter is simulated in our case with the initial data

$$
\mathcal{X}_0=(X_1(0),X_2(0),X_3(0),X_4(0),X_5(0),X_6(0))=(0.4,0.2,0.2,0.2,0.18,0.17).
$$

The linear and quadratic random intensities are selected as follows: $\Xi_{1\ell}=0.07$, $\Xi_{2\ell}=0.17$, $\Xi_{3\ell}=0.1835$, $\Xi_{4\ell}=0.764$, $\Xi_{5\ell}=0.148$, $\Xi_{6\ell}=0.141$, $\Xi_{1q}=0.012$, $\Xi_{2q}=0.027$, $\Xi_{3q}=0.0135$, $\Xi_{4q}=0.024$, $\Xi_{5q}=0.028$, $\Xi_{6q}=0.011$. Henceforth, the units adopted for time and number of individuals are respectively one day and one million population.

Table 3. Numerical values of the system parameters used in the simulations.

Parameter	Case 1	Case 2	Source
$\mathfrak{r}$	*0.1*	*0.1*	Estimated
$\mathfrak{A}$	*3.8*	*4.4*	[63]
$\mathfrak{B}_0$	*0.02*	*0.02*	Estimated
$\mathfrak{B}_1$	*0.2*	*0.2*	Estimated
$\mathfrak{B}_2$	*0.02*	*0.02*	Estimated
$\mathfrak{B}_1^\star$	*0.13*	*0.13*	Estimated
$\mathfrak{B}_2^\star$	*0.16*	*0.16*	Estimated
$\mathfrak{C}_1$	*0.01*	*0.01*	[63]
$\mathfrak{C}_2$	*0.1*	*0.1*	[63]
$\mathfrak{D}$	*0.05*	*0.05*	Estimated
$\mathfrak{F}_1$	*0.2857*	*0.2857*	Estimated
$\mathfrak{F}_2$	*0.3*	*0.3*	Estimated
$\mathfrak{K}_1$	*0.08*	*0.08*	Estimated
$\mathfrak{K}_2$	*0.1*	*0.1*	Estimated
$\mathfrak{L}_1$	*0.042*	*0.042*	[63]
$\mathfrak{L}_2$	*0.028*	*0.028*	[63]
$\mathfrak{m}_1$	*1*	*1*	Supposed
$\mathfrak{m}_2$	*1.5*	*1.5*	Supposed
$\mathfrak{M}$	*0.057*	*0.057*	[63]
$\mathfrak{N}$	*0.051*	*0.051*	[63]

3.1. Case 1: When $\mathcal{R}_\circ < 1 < \mathbf{K}$

In this example, we choose the parameter values presented in the second column of Table 2 in order to examine the sharpness of the condition of Theorem 1. Using numerical approximations and calculus, we obtain $\mathfrak{r} - 0.5\Xi_{1\ell}^2 = 0.0976 > 0$, $\mathcal{R}_\circ = 0.9359 < 1$ and $\mathbf{K} = 1.1615 > 1$. As can be seen from Figure 1, the HSV is almost sure to die out exponentially in the classes X_k, $k = 2, 3, 4, 5, 6$.

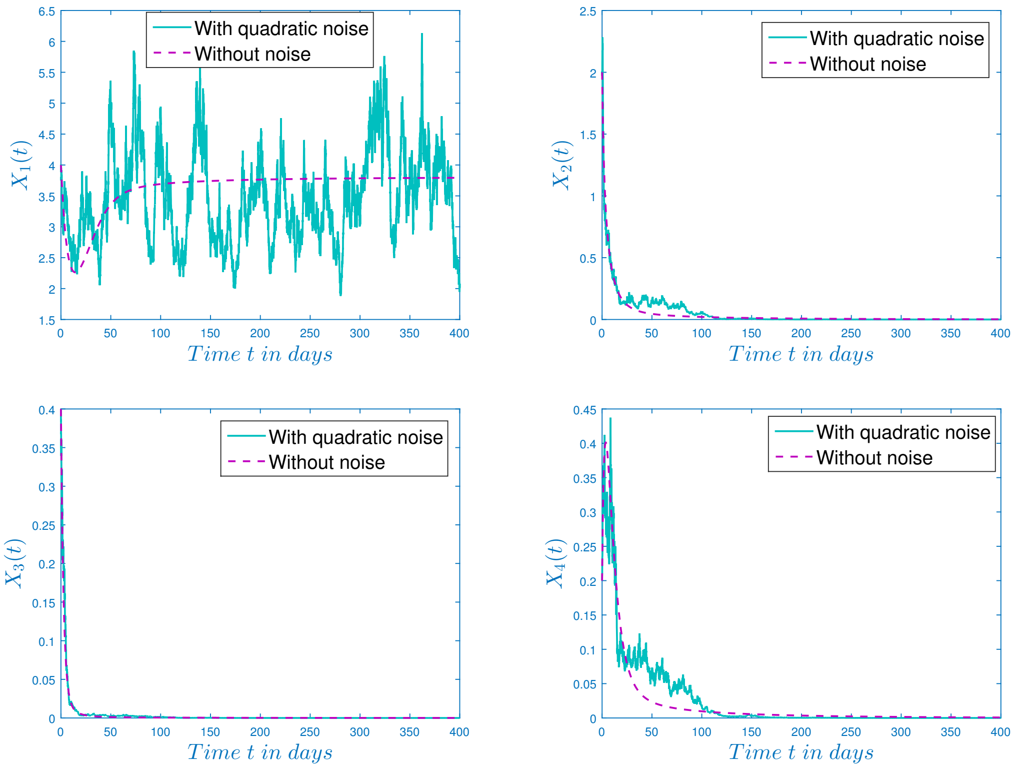

Figure 1. *Cont.*

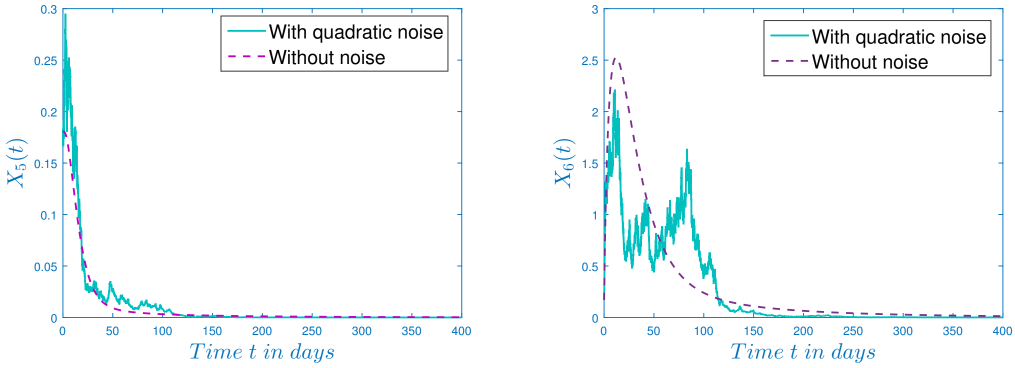

Figure 1. Numerical plot of the stochastic paths X_k, $k = 1, 2, 3, 4, 5, 6$ and their associated deterministic trajectories. The estimated deterministic values are selected from Table 2 and the stochastic amplitudes are chosen as follows: $\Xi_{1\ell} = 0.07$, $\Xi_{2\ell} = 0.17$, $\Xi_{3\ell} = 0.1835$, $\Xi_{4\ell} = 0.764$, $\Xi_{5\ell} = 0.148$, $\Xi_{6\ell} = 0.141$, $\Xi_{1q} = 0.012$, $\Xi_{2q} = 0.027$, $\Xi_{3q} = 0.0135$, $\Xi_{4q} = 0.024$, $\Xi_{5q} = 0.028$, $\Xi_{6q} = 0.011$.

3.2. Case 2: When $1 < \mathcal{R}_\circ < \mathbf{K}$

Now, we select the values of the parameters from the third column of Table 2. Numerically, we obtain $\mathfrak{r} - 0.5\Xi_{1\ell}^2 = 0.0976 > 0$, $\mathcal{R}_\circ = 1.0837 > 1$ and $\mathbf{K} = 1.1830 >$1. In this particular case, we have a different behavior of the deterministic model and the corresponding stochastic paths. From Figure 2, it can be clearly seen that the non-probabilistic solution in classes X_k, $k = 2, 3, 4, 5, 6$ persists, while the stochastic paths die out. This different attitude shows the effect of fluctuations on the long-term behavior of HSV. In other words, stochastic perturbations favor the eradication of HSV.

3.3. Impact of Quadratic Noise on Eradication Time of HSV

In this part, we probe the impact of the growth of the quadratic noise amplitude on the eradication time of HSV. From Figures 3 and 4, we show that the X_2 and X_4 processes reach zero differently depending on the strength of the quadratic noise. For example, in the case of the X_2 compartment, when the quadratic noise is equal to 0.01, the eradication time of HSV is approximately 170 days, whereas for quadratic noise equal to 0.09 , the HSV only becomes extinct after 15 days. That is, a large increasing value of the quadratic intensity minimizes the period of the HSV. Therefore, we can infer that quadratic noise positively changes the epidemic situation.

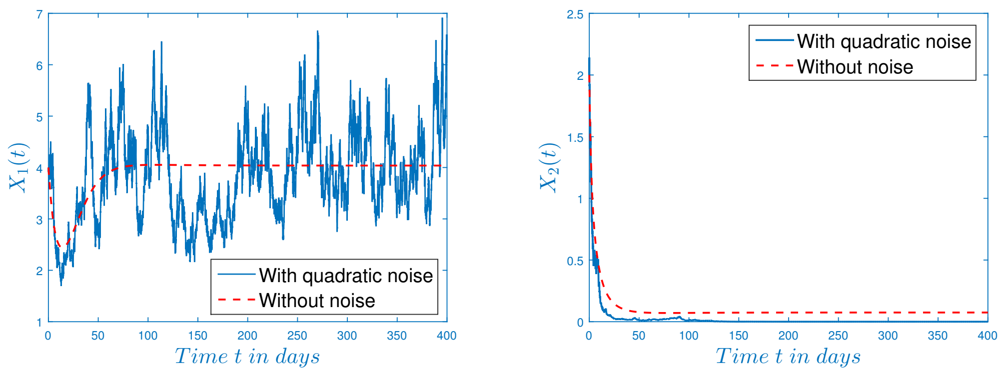

Figure 2. *Cont.*

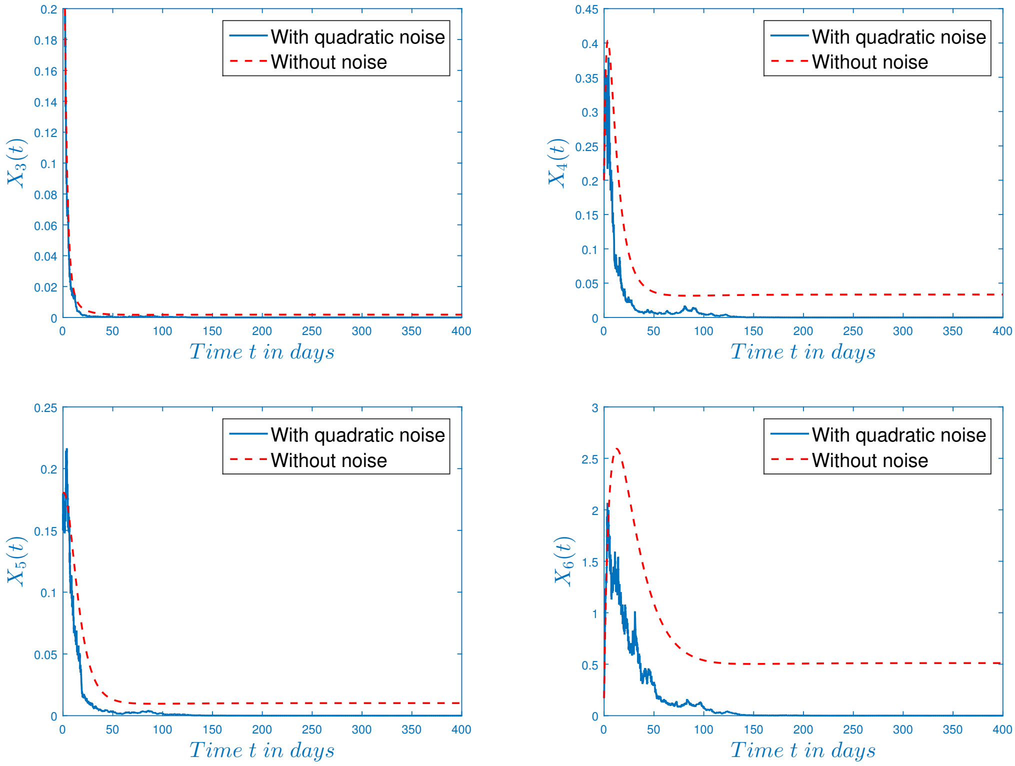

Figure 2. Numerical plot of the stochastic paths X_k, $k = 1, 2, 3, 4, 5, 6$ and their associated deterministic trajectories. The estimated deterministic values are selected from Table 2 and the stochastic amplitudes are chosen as follows: $\Xi_{1\ell} = 0.07$, $\Xi_{2\ell} = 0.17$, $\Xi_{3\ell} = 0.1835$, $\Xi_{4\ell} = 0.764$, $\Xi_{5\ell} = 0.148$, $\Xi_{6\ell} = 0.141$, $\Xi_{1q} = 0.012$, $\Xi_{2q} = 0.027$, $\Xi_{3q} = 0.0135$, $\Xi_{4q} = 0.024$, $\Xi_{5q} = 0.028$, $\Xi_{6q} = 0.011$.

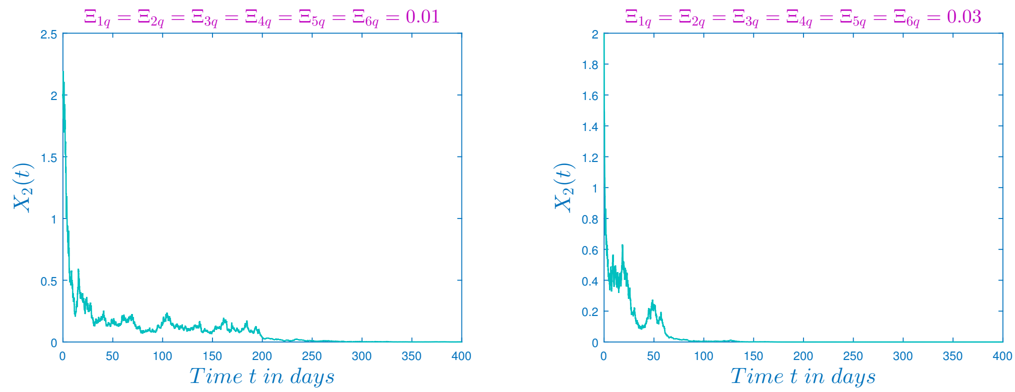

Figure 3. *Cont.*

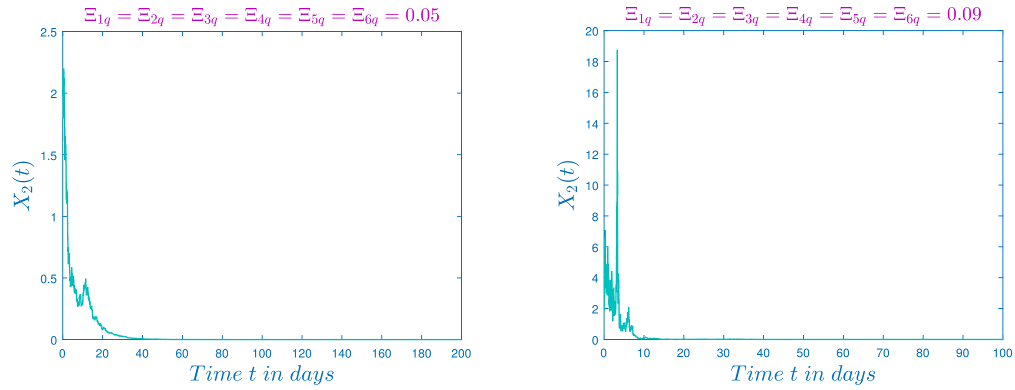

Figure 3. Numerical plot of the stochastic solution X_k, $k = 1,2,3,4,5,6$ with different quadratic noise. The estimated deterministic values are selected from Table 2 and the linear stochastic amplitudes are chosen as follows: $\Xi_{1\ell} = 0.07$, $\Xi_{2\ell} = 0.17$, $\Xi_{3\ell} = 0.1835$, $\Xi_{4\ell} = 0.764$, $\Xi_{5\ell} = 0.148$, $\Xi_{6\ell} = 0.141$.

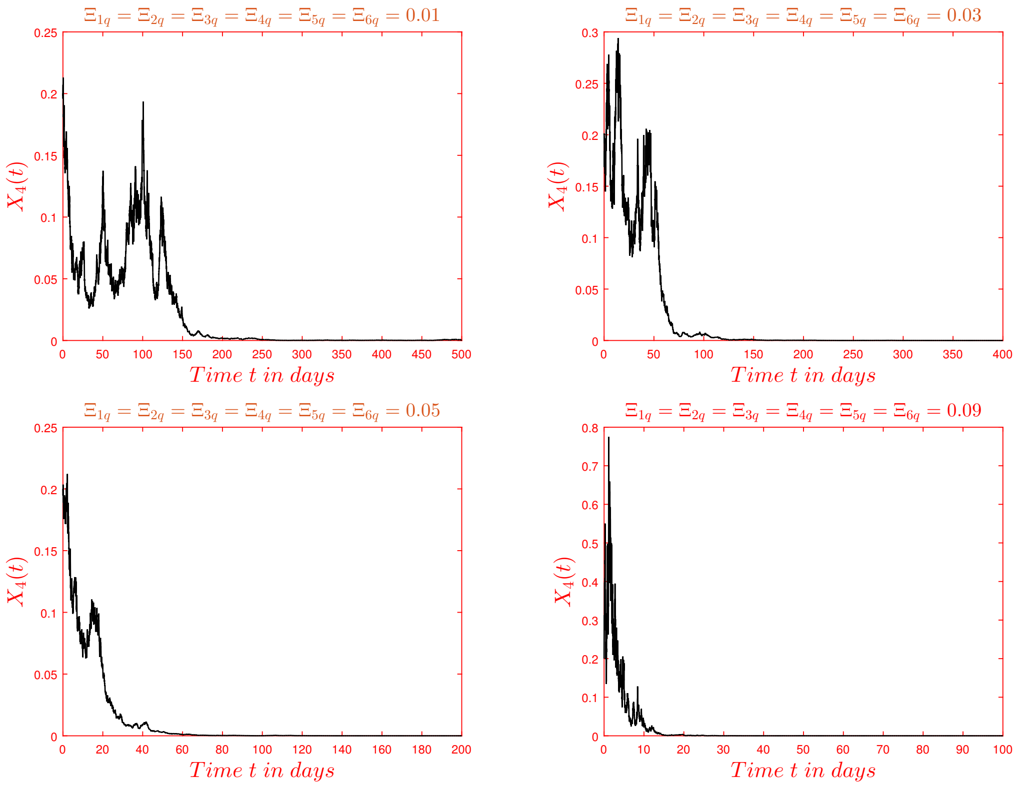

Figure 4. Numerical plot of the stochastic solution X_k, $k = 1,2,3,4,5,6$ with different quadratic noise. The estimated deterministic values are selected from Table 2 and the linear stochastic amplitudes are chosen as follows: $\Xi_{1\ell} = 0.07$, $\Xi_{2\ell} = 0.17$, $\Xi_{3\ell} = 0.1835$, $\Xi_{4\ell} = 0.764$, $\Xi_{5\ell} = 0.148$, $\Xi_{6\ell} = 0.141$.

4. Conclusions

Stochastic modeling characterizes the random nature of certain natural fluctuations. The amount of randomness affects outbreak situation scenarios. In this study, we have considered a general epidemic model that integrates medical and non-medical strategies such as quarantine, different periods of immunity, and media intervention. In addition, our model incorporates logistic growth and the effect of stochasticity. More precisely, we have treated a multidimensional system perturbed by a quadratic white noise. Analytically, we have proposed an acute condition for the eradication of the disease. This result provides insight into the future of epidemic situations under huge environmental fluctuations. Numerically, we have presented an example to confirm the accuracy of our condition.

Additionally, we have discussed the impact of quadratic noise on HSV eradication time. We conclude that increasing the loudness of the noise leads to rapid extinction. This means that the noise has a great influence on the outbreak period.

In general, we emphasize that our study generalizes previous works by considering the logistic growth function and second-order perturbation. Moreover, it provides new knowledge for a better understanding of the extinction scenario of an epidemic in complex real-world conditions. Due to analytical complexity, we derived only the critical condition for extinction, which is less than ideal with respect to epidemiological models. We intend to deal with the conditions of persistence and stationarity in our future work.

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