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On the Global Behaviour of Solutions for a Delayed Viscoelastic-Type Petrovesky Wave Equation with p-Laplacian Operator and Logarithmic Source

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Abstract: This research is concerned with a nonlinear p-Laplacian-type wave equation with a strong damping and logarithmic source term under the null Dirichlet boundary condition. We establish the global existence of the solutions by using the potential well method. Moreover, we prove the stability of the solutions by the Nakao technique. An example with illustrative figures is provided as an application.

Keywords: global existence; energy decay; viscoelastic wave equation; strong damping; p-Laplacian; logarithmic nonlinearity

MSC: 35L20; 35B40; 45K05

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1. Introduction

Many dynamical systems in physics and engineering have been designed and proposed by means of semilinear hyperbolic equations. For instance, logarithmic nonlinearity has received a great deal of attention from many scientists and researchers, and it has introduced many issues, including the wave equation. This type of nonlinearity appears naturally in supersymmetric field theory and inflation cosmology and has numerous applications in many branches of physics, such as nuclear physics, optics and geophysics [1–3]. In the past decades, many authors have studied the well-posedness, regularity and blow-up of the solution for wave equations in a bounded domain of $\mathbb{R}^n$. For example, Sattinger in [4] treated the following semilinear wave equation with polynomial nonlinearity

$$\ddot{\varkappa} - \Delta\varkappa(\mathrm{y},\mathfrak{s}) = \varrho(\varkappa), \quad \mathrm{y} \in \Omega, \mathfrak{s} > 0, \tag{1}$$

and showed, by introducing the concept of the potential well, the existence of the local as well as the global solutions for Equation (1). Messaoudi, in [5], proved that the obtained solution of the problem

$$\ddot{\varkappa} - \Delta\varkappa(\mathrm{y},\mathfrak{s}) + \int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\Delta\varkappa(\mathrm{y},\zeta)\,\mathrm{d}\zeta + a\varkappa(\mathrm{y},\mathfrak{s})|\dot{\varkappa}|^{m-2} = b\varkappa(\mathrm{y},\mathfrak{s})|\varkappa(\mathrm{y},\mathfrak{s})|^{p-2},$$

for $\mathrm{y} \in \Omega, \mathfrak{s} > 0$, where p is a real number, in a bounded domain Ω with initial and boundary conditions of the Dirichlet type, blow-up in a finite time if $p > m$ $(m \in \mathbb{N})$ and the initial energy is sufficiently negative. Filippo Gazzola and Marco Squassina, in [6], established a blow-up result with an arbitrarily high initial energy for the semilinear Cauchy problem

$$\ddot{\varkappa} - \Delta\varkappa(\mathrm{y},\mathfrak{s}) - w\Delta\dot{\varkappa} + \mu\dot{\varkappa} = b\varkappa(\mathrm{y},\mathfrak{s})|\varkappa(\mathrm{y},\mathfrak{s})|^{p-2}, \qquad \mathrm{y} \in \Omega, \mathfrak{s} > 0,$$

where $p \in \mathbb{R}$, in an open bounded Lipschitz subset Ω of $\mathbb{R}^n$, provided that $w = 0$ and $\mu \geq 0$. This result was later extended by Wang in [7] to the whole space $\mathbb{R}^n$ for the nonlinear Klein–Gordon equation with polynomial nonlinearity of the form

$$\ddot{\varkappa} - \Delta\varkappa(\mathrm{y},\mathfrak{s}) + m^2\varkappa(\mathrm{y},\mathfrak{s}) = \varrho(\varkappa), \qquad \mathrm{y} \in \mathbb{R}^n, \mathfrak{s} > 0,$$

where $m \in \mathbb{N}$. On the other hand, for the p-Laplacian wave equations, the authors in [8] gave, by the multiplier method, the energy decay of the solution of the following problem

$$\ddot{\varkappa} - \Delta_p\varkappa(\mathrm{y},\mathfrak{s}) - \Delta\dot{\varkappa} + \dot{\varkappa}|\dot{\varkappa}|^{q-2} = \varkappa(\mathrm{y},\mathfrak{s})|\varkappa(\mathrm{y},\mathfrak{s})|^{p-2}, \qquad \mathrm{y} \in \Omega, \mathfrak{s} > 0, \tag{2}$$

where $p, q \in \mathbb{R}$. Later, in [9], the authors proved an exponential and polynomial decay rate of the solutions by using the inequality of Nakao for the Problem (2). In the case of logarithmic nonlinearity, Ma et al. [10] studied the following problem

$$\ddot{\varkappa} - \Delta\varkappa(\mathrm{y},\mathfrak{s}) - \Delta\dot{\varkappa} = \varkappa(\mathrm{y},\mathfrak{s}) \ln|\varkappa(\mathrm{y},\mathfrak{s})|^2, \qquad \mathrm{y} \in \Omega, \mathfrak{s} > 0,$$

and proved that the blow-up of the solution occurs in infinite time. In addition, the existence of global solutions was shown by the use of a family of potential wells which includes the single potential well W as a particular case together with the perturbation energy technique. Di et al. [11] considered the strongly damped nonlinear wave equation

$$\ddot{\varkappa} - \Delta\varkappa(\mathrm{y},\mathfrak{s}) - \Delta\dot{\varkappa} = \varkappa(\mathrm{y},\mathfrak{s})|\varkappa(\mathrm{y},\mathfrak{s})|^{p-2} \ln|\varkappa(\mathrm{y},\mathfrak{s})|, \qquad \mathrm{y} \in \Omega, \mathfrak{s} > 0, \tag{3}$$

where $p \in \mathbb{R}$. They discussed the uniqueness, global existence, energy decay estimates and finite time blow-up phenomena to Problem (3) by modifying the potential well method. Tae Gab Ha et al. [12] introduced the following equation

$$\ddot{\varkappa} - \Delta\varkappa(\mathrm{y},\mathfrak{s}) + \int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\Delta\varkappa(\mathrm{y},\zeta)\,\mathrm{d}\zeta - \Delta\dot{\varkappa} = \varkappa(\mathrm{y},\mathfrak{s})|\varkappa(\mathrm{y},\mathfrak{s})|^{p-2} \ln|\varkappa(\mathrm{y},\mathfrak{s})|, \tag{4}$$

for $\mathrm{y} \in \Omega, \mathfrak{s} > 0$ where $p \in \mathbb{R}$. The existence, uniqueness and regularities of the weak solution to the Problem (4) are proved for all $\varkappa(0,\mathfrak{s}) \in H_0^1(\Omega)$ and $\dot{\varkappa}\big|_{(0,\mathfrak{s})} \in L^2(\Omega)$ by applying the Faedo–Galerkin method and the contraction mapping principle. A finite time blow-up property of the solutions was also derived at three different energy levels, $E(0) = 0$, $E(0) < 0$ and $0 < E(0) < d$. This result was later pursued by Liao [13] for certain solutions with a high initial energy. Meanwhile, explicit upper and lower bound estimates to the blow-up time have been proved. Inspired by the above studies, in this work, we investigate the global behaviour of solutions for a delayed viscoelastic-type Petrovesky wave equation with a p-Laplacian operator and logarithmic source under some appropriate conditions.

We consider the initial-boundary value problem of the following nonlinear wave equation with strong damping and logarithmic source terms

$$\begin{cases} |\dot{\varkappa}(\mathrm{y},\mathfrak{s})|^l \ddot{\varkappa}(\mathrm{y},\mathfrak{s}) - \Delta_p\varkappa(\mathrm{y},\mathfrak{s}) \\ \qquad +\Delta^2\left(\varkappa(\mathfrak{s}) - \displaystyle\int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\varkappa(\zeta)\,\mathrm{d}\zeta\right) \\ \qquad -\mu_1\Delta\dot{\varkappa}(\mathrm{y},\mathfrak{s}) - \mu_2\Delta\dot{\varkappa}(\mathrm{y},\mathfrak{s}-\zeta) - \Delta\ddot{\varkappa}(\mathrm{y},\mathfrak{s}) \\ \quad = \varkappa(\mathrm{y},\mathfrak{s})|\varkappa(\mathrm{y},\mathfrak{s})|^{p-2} \ln|\varkappa(\mathrm{y},\mathfrak{s})|, & \mathrm{y} \in \Omega,\ \mathfrak{s} > 0, \\ \varkappa = \Delta\varkappa = 0, & \mathrm{y} \in \partial\Omega,\ \mathfrak{s} > 0, \\ \varkappa(\mathrm{y},0) = \varkappa_0(\mathrm{y}), \quad \dot{\varkappa}\big|_{(\mathrm{y},0)} = \varkappa_1(\mathrm{y}), & \mathrm{y} \in \Omega, \\ \dot{\varkappa}\big|_{(\mathrm{y},\mathfrak{s}-\tau)} = h_0(\mathrm{y},\mathfrak{s}-\tau) & \mathrm{y} \in \Omega,\ 0 < \mathfrak{s} < \tau \end{cases} \tag{5}$$

where $\dot{\varkappa} = \varkappa_{\mathfrak{s}}(\mathrm{y},\mathfrak{s})$, $\ddot{\varkappa} = \varkappa_{\mathfrak{s}\mathfrak{s}}(\mathrm{y},\mathfrak{s})$, Ω is a bounded domain of $\mathbb{R}^n$, $n \in \mathbb{N}$ and $p \in \mathbb{R}$, with smooth boundary $\partial\Omega$, $\varkappa_0(\mathrm{y})$ and $\varkappa_1(\mathrm{y})$ are given initial data. Problem (5) describes an

evolution problem with an interior logarithmic source and the investigation of this problem answers the question what will happen if one replaces the power source term by other source terms, for example, the logarithmic source term? This paper provides an affirmative reply to such a situation.

The rest of this paper can be sketched out concisely as follows: In Section 2, we introduce some basic concepts and state the important and necessary results. In Section 3.1, we shall use the Galerkin approximation technique combined with the potential well method to prove the global existence of weak solutions. Meanwhile, we will derive the decay rate of the energy functional using Nakao's inequality and some techniques on logarithmic nonlinearity in Section 3.2. The numerical examples and simulation are provided in Section 4. Finally, a conclusion is drawn in Section 5.

2. Preliminaries

In this section, we present some material and assumptions needed for the proof of our results. For a Banach space X, $\|.\|_X$ denotes the norm of X. For simplicity, we denote $\|.\|_{L^q(\Omega)}$ by $\|.\|_q$ and $\|.\|_{L^2(\Omega)}$ by $\|.\|_2$. Here, $\Delta_p u$ is the standard p-Laplace operator defined by

$$\Delta_p \varkappa = \operatorname{div}\Big(|\nabla \varkappa|^{p-2}\nabla \varkappa\Big), \qquad p > \max\left\{1, \frac{2n}{n+2}\right\},$$

for $p \in \mathbb{R}$ and $n \in \mathbb{N}$ where $\nabla = \left(\frac{\partial}{\partial \mathfrak{y}}, \frac{\partial}{\partial \mathfrak{s}}\right)$.

Lemma 1. *Theorem 4.4 in* [14]. *Let Ω be a bounded domain with Lipschitz boundary. In the space $X = H^2(\Omega) \cap H_0^1(\Omega)$, the norm $\|\cdot\|_X$ and $\|\cdot\|_2$ are equivalent norms.*

Lemma 2. *Theorem 5.4 in* [15]. *Let Ω be a domain in $\mathbb{R}^n$ that has the cone property; then, for $n > 2$ and $2 \le q \le \frac{2n}{n-2}$, there exist the following imbeddings*

$$H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow W^{1,q}(\Omega) \hookrightarrow L^q(\Omega).$$

There exists constants $c_1, c_*, c_2 > 0$ depending on Ω and q such that

$$\|\varkappa(\mathfrak{s})\|_q \le c_1\|\Delta\varkappa(\mathfrak{s})\|_2, \quad \|\varkappa(\mathfrak{s})\|_q \le c_*\|\nabla\varkappa(\mathfrak{s})\|_q, \quad \|\nabla\varkappa(\mathfrak{s})\|_q \le c_2\|\Delta u(t)\|_2. \tag{6}$$

Lemma 3. *(Logarithmic Sobolev Inequality)* [16]. *Let $\varkappa \in W_0^{1,p}(\Omega)\backslash\{0\}$. Then, for any $p > 1$ and $\mu > 0$, we can write*

$$p\int_\Omega (\varkappa(\zeta,\mathfrak{s}))^p \ln\left(\frac{|\varkappa(\zeta,\mathfrak{s})|}{\|\varkappa(\zeta,\mathfrak{s})\|_p}\right) \mathrm{d}\zeta \le \mu\|\nabla\varkappa(\mathfrak{s})\|_p^p - \frac{n}{p}\ln\left(\frac{p\mu e}{n\mathcal{L}_p}\right)\|\varkappa(\mathfrak{s})\|_p^p,$$

where

$$\mathcal{L}_p = \frac{p}{n}\left(\frac{p-1}{e}\right)^{p-1}\pi^{-p/2}\left(\frac{\Gamma\left(\left(\frac{n}{2}\right)+1\right)}{\Gamma\left(\frac{n(p-1)}{p}+1\right)}\right)^{p/2}.$$

Lemma 4. *(Generalised Young's Inequality) Suppose that X, Y and σ are positive constants and $\zeta \ge 1$. Then, Young's inequality reads*

$$XY \le \frac{\sigma^\zeta}{\zeta}X^\zeta + \frac{\zeta-1}{\zeta}\sigma^{-\zeta/(\zeta-1)}Y^{\zeta/(\zeta-1)}.$$

We prove the global existence and the blow-up property for solutions of Problem (5) under the following suitable assumptions.

(H1) The relaxation function $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a twice-differentiable and bounded function satisfying $g(\mathfrak{s}) \geq 0$, and

$$1 - \int_0^\infty g(\varrho)\,\mathrm{d}\varrho =: \vartheta > 0.$$

There exist constants $\xi_1, \xi_2, \xi_3 > 0$ depending on Ω such that

$$-\xi_1 \leq g'(\mathfrak{s}) \leq -\xi_3 g(\mathfrak{s}), \qquad 0 \leq g''(\mathfrak{s}) \leq \xi_2 g(\mathfrak{s}),$$

whenever $\mathfrak{s} \geq 0$.

(H2) The exponent p satisfies $2 < p < \infty$ for $n \leq 2$ and $2 < p < \frac{n}{n-2}$ for $n > 2$.

(H3) Assume that l satisfies $0 < l < \frac{2}{n-2}$ whenever $n \geq 3$ and $l > 0$ whenever $n = 1, 2$.

To obtain the main result, we have the lemmas as follows.

Lemma 5. [17] *Suppose that $\phi(\mathfrak{s})$ is a bounded nonnegative function nonincreasing on $[0, \delta]$ where $\delta > 0$ and satisfying*

$$\phi^{1+\alpha}(\mathfrak{s}) \leq \sigma(\phi(\mathfrak{s}) - \phi(\mathfrak{s}+1)), \qquad \forall \mathfrak{s} \in [0, \delta],$$

for some positive constants α and σ. Then, we have

$$\phi(\mathfrak{s}) \leq \begin{cases} \phi(0)\exp\left(-[\mathfrak{s}-1]^+ \ln\left(\frac{\sigma}{\sigma-1}\right)\right), & \alpha = 0, \\ \left(\phi(0)^{-\alpha} + \sigma^{-1}\alpha[\mathfrak{s}-1]^+\right)^{\frac{-1}{\alpha}}, & \alpha > 0, \end{cases}$$

where $[\mathfrak{s}-1]^+ = \max\{\mathfrak{s}-1, 0\}$.

To deal with the time delay term, we introduce the following new variable $\top(\mathrm{y}, \eta, \mathfrak{s})$ to represent the delay term as in [18]

$$\top(\mathrm{y}, \eta, \mathfrak{s}) = \dot{\varkappa}\big|_{(\mathrm{y}, \mathfrak{s}-\eta\tau)}. \tag{7}$$

Therefore, $\top(\mathrm{y}, \eta, \mathfrak{s})$ satisfies

$$\tau\dot{\top}\big|_{(\mathrm{y},\eta,\mathfrak{s})} + \top_\eta(\mathrm{y}, \eta, \mathfrak{s}) = 0, \qquad (\mathrm{y}, \eta, \mathfrak{s}) \in \Omega \times (0,1) \times (0, \infty). \tag{8}$$

Using the above transformation (7), system (5) can be rewritten as equivalent form

$$\begin{cases} |\dot{\varkappa}(\mathrm{y},\mathfrak{s})|^l \ddot{\varkappa}(\mathrm{y},\mathfrak{s}) - \Delta_p \varkappa(\mathrm{y},\mathfrak{s}) \\ \qquad +\Delta^2\left(\varkappa - \int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\varkappa(\zeta)\,\mathrm{d}\zeta\right) \\ \qquad -\mu_1\Delta\dot{\varkappa}(\mathrm{y},\mathfrak{s}) - \mu_2\Delta\top(\mathrm{y},1,\mathfrak{s}) - \Delta\ddot{\varkappa}(\mathrm{y},\mathfrak{s}) \\ \quad = \varkappa(\mathrm{y},\mathfrak{s})|\varkappa(\mathrm{y},\mathfrak{s})|^{p-2}\ln|\varkappa(\mathrm{y},\mathfrak{s})|, & \mathrm{y} \in \Omega, \mathfrak{s} > 0, \\ \tau\dot{\top}\big|_{(\mathrm{y},\eta,\mathfrak{s})} + \top_\eta(\mathrm{y},\eta,\mathfrak{s}) = 0, & \mathrm{y} \in \Omega, \eta \in (0,1), \mathfrak{s} > 0, \\ \varkappa = 0, & \mathrm{y} \in \partial\Omega, \mathfrak{s} > 0, \\ \top(\mathrm{y},0,\mathfrak{s}) = \dot{\varkappa}, & \mathrm{y} \in \partial\Omega, \mathfrak{s} > 0, \\ \varkappa(\mathrm{y},0) = \varkappa_0(\mathrm{y}), \quad \dot{\varkappa}\big|_{(\mathrm{y},0)} = \varkappa_1(\mathrm{y}), & \mathrm{y} \in \Omega. \end{cases} \tag{9}$$

Let ξ be a positive constant satisfying

$$\tau\frac{|\mu_2|}{2} \leq \xi \leq \tau\left(\mu_1 - \frac{|\mu_2|}{2}\right) \tag{10}$$

The energy functional associated to Problem (9) is defined as

$$\begin{aligned} E(\mathfrak{s}) \quad &:= \frac{1}{l+2}\|\dot{\varkappa}(\mathfrak{s})\|_{l+2}^{l+2} + \|\nabla\dot{\varkappa}(\mathfrak{s})\|_2^2 + \frac{1}{2}\left(1 - \int_0^{\mathfrak{s}} g(\zeta)\,\mathrm{d}\zeta\right)\|\Delta\varkappa(\mathfrak{s})\|_2^2 \\ &+\frac{1}{p}\|\nabla\varkappa(\mathfrak{s})\|_p^p + \frac{1}{2}(g\circ\Delta\varkappa)(\mathfrak{s}) \\ &-\frac{1}{p}\int_\Omega |\varkappa(\mathrm{y},\mathfrak{s})|^p \ln|\varkappa(\mathrm{y},\mathfrak{s})|\,\mathrm{d}\zeta + \frac{1}{p^2}\|\varkappa(\mathfrak{s})\|_p^p \\ &+\xi\int_\Omega\int_0^1 |\nabla\top(\mathrm{y},\eta,\mathfrak{s})|^2\,\mathrm{d}\eta\mathrm{d}\mathrm{y}, \end{aligned} \tag{11}$$

where

$$(g\circ\phi)(\mathfrak{s}) := \int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\|\phi(\mathfrak{s})-\phi(\zeta)\|_2^2\,\mathrm{d}\zeta.$$

Multiplying the first equation in (9) by $\dot{\varkappa}(\mathrm{y},\mathfrak{s})$ and integrating over Ω, using integration by part, we see that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\Bigg\{\frac{1}{l+2}\|\dot{\varkappa}(\mathfrak{s})\|_{l+2}^{l+2} \quad &+\|\nabla\dot{\varkappa}(\mathfrak{s})\|_2^2 + \frac{1}{2}\left(1-\int_0^{\mathfrak{s}} g(\zeta)\,\mathrm{d}\zeta\right)\|\Delta\varkappa(\mathfrak{s})\|_2^2 \\ &+\frac{1}{p}\|\nabla\varkappa(\mathfrak{s})\|_p^p + \frac{1}{2}(g\circ\Delta\varkappa)(\mathfrak{s}) \\ &-\frac{1}{p}\int_\Omega |\varkappa(\mathrm{y},\mathfrak{s})|^p\ln|\varkappa(\mathrm{y},\mathfrak{s})|\,\mathrm{d}\mathrm{y} + \frac{1}{p^2}\|\varkappa(\mathfrak{s})\|_p^p\Bigg\} \\ &+\mu_1\|\nabla\dot{\varkappa}(\mathfrak{s})\|_2^2 + \mu_2\int_\Omega \nabla\top(\mathrm{y},1,\mathfrak{s})\nabla\dot{\varkappa}(\mathrm{y},\mathfrak{s})\,\mathrm{d}\mathrm{y} \\ &= \frac{1}{2}(g'\circ\Delta\varkappa)(\mathfrak{s}) - \frac{1}{2}g(\mathfrak{s})\|\Delta\varkappa(\mathfrak{s})\|_2^2. \end{aligned} \tag{12}$$

We multiply the second equation in (9) by $-\xi\Delta\top(\mathrm{y},\eta,\mathfrak{s})$ and integrate the result over $\Omega\times(0,1)$ to obtain

$$\begin{aligned} -\xi\int_\Omega\int_0^1 \dot{\top}(\mathrm{y},\eta,\mathfrak{s})\Delta\top(\mathrm{y},\eta,\mathfrak{s})\,\mathrm{d}\eta\,\mathrm{d}\mathrm{y} &= \frac{\xi}{\tau}\int_\Omega\int_0^1 \top_\eta(\mathrm{y},\eta,\mathfrak{s})\Delta\top(\mathrm{y},\eta,\mathfrak{s})\,\mathrm{d}\eta\,\mathrm{d}\mathrm{y} \\ &= -\frac{\xi}{2\tau}\int_\Omega\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\eta}|\nabla\top(\mathrm{y},\eta,\mathfrak{s})|^2\,\mathrm{d}\eta\,\mathrm{d}\mathrm{y} \\ &= -\frac{\xi}{2\tau}\int_\Omega\Big(|\nabla\top(\mathrm{y},1,\mathfrak{s})|^2 - |\nabla\top(\mathrm{y},0,\mathfrak{s})|^2\Big)\,\mathrm{d}\mathrm{y}, \end{aligned}$$

which gives

$$\begin{aligned} \xi\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\int_\Omega\int_0^1 |\nabla\top(\mathrm{y},\eta,\mathfrak{s})|^2\,\mathrm{d}\eta\,\mathrm{d}\mathrm{y} &= -\frac{\xi}{\tau}\int_\Omega\Big(|\nabla\top(\mathrm{y},1,\mathfrak{s})|^2 - |\nabla\top(\mathrm{y},0,\mathfrak{s})|^2\Big)\,\mathrm{d}\mathrm{y} \\ &= -\frac{\xi}{\tau}\int_\Omega\Big(|\nabla\top(\mathrm{y},1,\mathfrak{s})|^2 - |\nabla\dot{\varkappa}(\mathrm{y},\mathfrak{s})|^2\Big)\,\mathrm{d}\mathrm{y}. \end{aligned} \tag{13}$$

Using Young's inequality, we have

$$\mu_2\int_\Omega \nabla\top(\mathrm{y},1,\mathfrak{s})\nabla\dot{\varkappa}(\mathrm{y},\mathfrak{s})\,\mathrm{d}\mathrm{y} \le \frac{|\mu_2|}{2}\|\nabla\top(\mathrm{y},1,\mathfrak{s})\|_2^2 + \frac{|\mu_2|}{2}\|\nabla\dot{\varkappa}(\mathfrak{s})\|_2^2. \tag{14}$$

Combining (12)–(14), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}E(\mathfrak{s}) \le \frac{1}{2}(g'\circ\Delta\varkappa)(\mathfrak{s}) - \frac{1}{2}g(\mathfrak{s})\|\Delta\varkappa(\mathfrak{s})\|_2^2 - \left(\mu_1 - \frac{\xi}{\tau} - \frac{|\mu_2|}{2}\right)\|\nabla\dot{\varkappa}(\mathfrak{s})\|_2^2 \\ - \left(\frac{\xi}{\tau} - \frac{|\mu_2|}{2}\right)\|\nabla\top(\mathrm{y},1,\mathfrak{s})\|_2^2. \end{aligned} \tag{15}$$

By using condition (10), we obtain $k_0 = \mu_1 - \frac{\xi}{\tau} - \frac{|\mu_2|}{2} > 0$, $k_1 = \frac{\xi}{\tau} - \frac{|\mu_2|}{2} > 0$, which implies

$$\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}E(\mathfrak{s}) \leq \frac{1}{2}(g' \circ \Delta\varkappa)(\mathfrak{s}) - \frac{1}{2}g(\mathfrak{s})\|\nabla\varkappa(\mathfrak{s})\|_2^2 - k_0\|\nabla\dot{\varkappa}(\mathfrak{s})\|_2^2 - k_1\|\nabla\top(\mathrm{y},1,\mathfrak{s})\|_2^2 < 0, \quad (16)$$

for almost $\mathfrak{s} \in [0,\delta)$ and hence

$$E(\mathfrak{s}) + k_0\int_0^{\mathfrak{s}} \|\nabla\dot{\varkappa}(\varrho)\|_2^2\,\mathrm{d}\varrho + k_1\int_0^{\mathfrak{s}} \|\nabla\top(\mathrm{y},1,\varrho)\|_2^2\,\mathrm{d}\varrho \leq E(0). \quad (17)$$

Next, we introduce two potential energy functionals on $H^3(\Omega)\cap H_0^1(\Omega)$ as follows:

$$\begin{aligned} J(\varkappa) = {} & \frac{1}{2}\left(1 - \int_0^{\mathfrak{s}} g(\zeta)\,\mathrm{d}\zeta\right)\|\Delta\varkappa(\mathfrak{s})\|_2^2 + \frac{1}{p}\|\nabla\varkappa(\mathfrak{s})\|_p^p \\ & - \frac{1}{p}\int_\Omega |\varkappa(\mathrm{y},\mathfrak{s})|^p \ln|\varkappa(\mathrm{y},\mathfrak{s})|\,\mathrm{d}\mathrm{y} + \frac{1}{p^2}\|\varkappa(\mathfrak{s})\|_p^p, \end{aligned} \quad (18)$$

$$\begin{aligned} I(\varkappa) = {} & \left(1 - \int_0^{\mathfrak{s}} g(\zeta)\,\mathrm{d}\zeta\right)\|\Delta\varkappa(\mathfrak{s})\|_2^2 + \|\nabla\varkappa(\mathfrak{s})\|_p^p \\ & - \int_\Omega |\varkappa(\mathrm{y},\mathfrak{s})|^p \ln|\varkappa(\mathrm{y},\mathfrak{s})|\,\mathrm{d}\mathrm{y}. \end{aligned} \quad (19)$$

By a direct computation,

$$J(\varkappa) = \left(\frac{1}{2} - \frac{1}{p}\right)\left(1 - \int_0^{\mathfrak{s}} g(\zeta)d\zeta\right)\|\Delta\varkappa(s)\|_2^2 + \frac{1}{p^2}\|\varkappa(s)\|_p^p + \frac{1}{p}I(\varkappa). \quad (20)$$

Clearly,

$$\begin{aligned} E(\mathfrak{s}) \quad = {} & \frac{1}{l+2}\|\dot{\varkappa}(\mathfrak{s})\|_{l+2}^{l+2} + \frac{1}{2}\|\nabla\dot{\varkappa}(\mathfrak{s})\|_2^2 + \frac{1}{2}(g\circ\Delta\varkappa)(\mathfrak{s}) \\ & + \xi\int_\Omega\int_0^1 |\nabla\top(\mathrm{y},\eta,\mathfrak{s})|^2\,\mathrm{d}\eta\,\mathrm{d}\mathrm{y} + J(\varkappa). \end{aligned} \quad (21)$$

Define the Nehari's manifold

$$\mathcal{N} = \left\{\varkappa \in H^3(\Omega)\cap H_0^1(\Omega) | I(\varkappa) = 0, \|\Delta\varkappa\|_2 \neq 0\right\}.$$

Next, let us define the stable set W and the unstable set V as follows:

$$W = \left\{\varkappa \in H^3(\Omega)\cap H_0^1(\Omega) | I(\varkappa) > 0, J(\varkappa) < d\right\} \cup \{0\},$$

$$V = \left\{\varkappa \in H^3(\Omega)\cap H_0^1(\Omega) | I(\varkappa) < 0, J(\varkappa) < d\right\}.$$

The mountain pass level d, also known as potential well depth, is characterized by

$$d = \inf_{0\neq\varkappa\in H^3(\Omega)\cap H_0^1(\Omega)} \sup_{\lambda\geq 0} J(\lambda\varkappa) = \inf_{\varkappa\in\mathcal{N}} J(\varkappa). \quad (22)$$

3. Global Existence and Energy Decay

First, we state and prove a few lemmas.

Lemma 6. *The depth d of the potential well d is positive.*

Proof. Fix $u \in \mathcal{N}$, and using the fact $\varkappa^p \ln|\varkappa| \leq |\varkappa|^{p+\alpha}$ for $|\varkappa| > 1$ and for all $\alpha > 0$, we arrive at

$$\begin{aligned}\left(1 - \int_0^{\mathfrak{s}} g(\zeta)\,\mathrm{d}\zeta\right)\|\Delta\varkappa(\mathfrak{s})\|_2^2 + \|\nabla\varkappa(\mathfrak{s})\|_p^p \\ = \int_\Omega |\varkappa(\mathrm{y},\mathfrak{s})|^p \ln|\varkappa(\mathrm{y},\mathfrak{s})|\,\mathrm{dy} \\ \leq \|\varkappa(\mathfrak{s})\|_{p+\alpha}^{p+\alpha} \leq c_1^{p+\alpha}\|\Delta\varkappa(\mathfrak{s})\|_2^{p+\alpha}.\end{aligned} \tag{23}$$

Taking α such that $p + \alpha \leq 2^* = \dfrac{2n}{n-2}$, we deduce

$$\|\Delta\varkappa(\mathfrak{s})\|_2 \geq \left(\frac{\vartheta}{c_1^{p+\alpha}}\right)^{1/p+\alpha-2},$$

which implies

$$\begin{aligned}J(\varkappa) &= \left(\frac{1}{2} - \frac{1}{p}\right)\left(1 - \int_0^{\mathfrak{s}} g(\zeta)\,\mathrm{d}\zeta\right)\|\Delta\varkappa(\mathfrak{s})\|_2^2 + \frac{1}{p^2}\|\varkappa(\mathfrak{s})\|_p^p \\ &\geq \vartheta\left(\frac{1}{2} - \frac{1}{p}\right)\left(\frac{\vartheta}{c_1^{p+\alpha}}\right)^{2/p+\alpha-2}.\end{aligned}$$

Therefore,

$$d \geq \vartheta\left(\frac{1}{2} - \frac{1}{p}\right)\left(\frac{\vartheta}{c_1^{p+\alpha}}\right)^{2/p+\alpha-2}.$$

The proof is complete. □

Lemma 7. *For any $\varkappa \in H^3(\Omega) \cap H_0^1(\Omega)$, $\|\Delta\varkappa(s)\|_2 \neq 0$ there exists a unique $\lambda_0 = \lambda_0(\varkappa) > 0$ such that the following assertions hold:*

(i) $\lim_{\lambda\to o^+} J(\lambda\varkappa) = 0$, $\lim_{\lambda\to+\infty} J(\lambda u) = -\infty$;

(ii) *$J(\lambda\varkappa)$ is increasing in the interval $0 < \lambda \leq \lambda_0$, decreasing in the interval $\lambda_0 \leq \lambda < +\infty$ and takes its maximum at $\lambda = \lambda_0$ where $\dfrac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda\varkappa)|_{\lambda=\lambda_0} = 0$;*

(iii) *$I(\lambda\varkappa) > 0$ for $0 < \lambda < \lambda_0$, $I(\lambda\varkappa) < 0$ for $\lambda_0 < \lambda < +\infty$ and $I(\lambda_0\varkappa) = 0$.*

Proof. For $\lambda > 0$, we know that

$$\begin{aligned}J(\lambda\varkappa) := \frac{1}{2}\lambda^2\left(1 - \int_0^{\mathfrak{s}} g(\zeta)\,\mathrm{d}\zeta\right)\|\Delta\varkappa(\mathfrak{s})\|_2^2 + \frac{\lambda^p}{p}\|\nabla\varkappa(\mathfrak{s})\|_p^p \\ - \frac{\lambda^p}{p}\int_\Omega |\varkappa(\mathrm{y},\mathfrak{s})|^p \ln|\varkappa(\mathrm{y},\mathfrak{s})|\,\mathrm{dy} - \frac{\lambda^p}{p}\ln|\lambda|\int_\Omega |\varkappa(\mathrm{y},\mathfrak{s})|^p\,\mathrm{dy} \\ + \frac{\lambda^p}{p}\int_\Omega |\varkappa(\mathrm{y},\mathfrak{s})|^p\,\mathrm{dy}.\end{aligned}$$

Taking the derivative of $J(\lambda\varkappa)$, we obtain

$$\begin{aligned}\frac{\partial}{\partial\lambda}J(\lambda\varkappa) = \lambda\left(1 - \int_0^{\mathfrak{s}} g(\zeta)\,\mathrm{d}\zeta\right)\|\Delta\varkappa(\mathfrak{s})\|_2^2 + \lambda^{p-1}\|\nabla\varkappa(\mathfrak{s})\|_p^p \\ - \lambda^{p-1}\int_\Omega |\varkappa(\mathrm{y},\mathfrak{s})|^p \ln|\varkappa(\mathrm{y},\mathfrak{s})|\,\mathrm{dy} - \lambda^{p-1}\ln|\lambda|\int_\Omega |\varkappa(\mathrm{y},\mathfrak{s})|^p\,\mathrm{dy}.\end{aligned} \tag{24}$$

Making (24) equal zero is equivalent to

$$\|\nabla \varkappa(\mathfrak{s})\|_p^p = \int_\Omega |\varkappa(\mathrm{y},\mathfrak{s})|^p \ln|\varkappa(\mathrm{y},\mathfrak{s})|\,\mathrm{d}\mathrm{y} + \ln|\lambda| \int_\Omega |\varkappa(\mathrm{y},\mathfrak{s})|^p\,\mathrm{d}\mathrm{y} - \lambda^{2-p}\left(1 - \int_0^{\mathfrak{s}} g(\zeta)\,\mathrm{d}\zeta\right)\|\Delta\varkappa(\mathfrak{s})\|_2^2. \tag{25}$$

We define

$$\phi_\varkappa(\lambda) := \int_\Omega |\varkappa(\mathrm{y},\mathfrak{s})|^p \ln|\varkappa(\mathrm{y},\mathfrak{s})|\,\mathrm{d}\mathrm{y} + \ln|\lambda| \int_\Omega |\varkappa(\mathrm{y},\mathfrak{s})|^p\,\mathrm{d}\mathrm{y} - \lambda^{2-p}\left(1 - \int_0^{\mathfrak{s}} g(\zeta)\,\mathrm{d}\zeta\right)\|\Delta\varkappa(\mathfrak{s})\|_2^2.$$

The function $\phi_\varkappa(\lambda)$ is increasing on $0 < \lambda < \infty$, and we remark that

$$\lim_{\lambda\to o^+} \phi_\varkappa(\lambda) = -\infty, \qquad \lim_{\lambda\to+\infty} \phi_\varkappa(\lambda) = +\infty.$$

Therefore, there exists a unique λ^* such that $\phi_\varkappa(\lambda^\star) = 0$, $\phi_\varkappa(\lambda) < 0$ on $0 < \lambda < \lambda^\star$ and $\phi_\varkappa(\lambda) > 0$ on $\lambda^\star < \lambda < \infty$. We can clearly perceive that there exists a unique $\lambda_0 > \lambda^\star$ such that $\|\nabla\varkappa(\mathfrak{s})\|_p^p = \phi_\varkappa(\lambda)$. Again, we have

$$\frac{\partial}{\partial\lambda} J(\lambda\varkappa) = \lambda^{p-1}\Big(\|\nabla\varkappa(\mathfrak{s})\|_p^p - \phi_\varkappa(\lambda)\Big) := K_\varkappa(\lambda).$$

We also obtain $\phi_\varkappa(\lambda) \le 0$ for $0 < \lambda < \lambda^\star$, $0 < \phi_\varkappa(\lambda) \le \|\nabla\varkappa(\mathfrak{s})\|_p^p$ for $\lambda^\star < \lambda < \lambda_0$ and $\phi_\varkappa(\lambda) > \|\nabla\varkappa(\mathfrak{s})\|_p^p$ for $\lambda_0 < \lambda < \infty$, and we have (ii). Because $I(\lambda\varkappa) = \lambda\frac{\partial J(\lambda\varkappa)}{\partial\lambda}$ which is verified by a direct computation, then one has (iii). □

Lemma 8. *(Invariant Set W when $E(0) < d$). Let* (H1)–(H4) *hold. Assume that $\varkappa$ is a weak solution of Problem* (5) *with $I(\varkappa_0) > 0$ and $E(0) < d$, then the solution of the Problem* (5) *satisfies $\varkappa(\mathrm{y},\mathfrak{s}) \in W$, $0 \le \mathfrak{s} \le \delta$, where δ is the maximal existence time of the solutions.*

Proof. Arguing by contradiction, we suppose that there exists $\mathfrak{s}_1 \in (0,\delta)$ such that $\varkappa(\delta_1) \in \partial W$, and then we obtain $I(\varkappa(\mathfrak{s}_1)) = 0$ or $J(\varkappa(\mathfrak{s}_1)) = d$. Going back to (21), we have $E(\mathfrak{s}) \ge J(\varkappa)$. Then, $J(\varkappa(\mathfrak{s}_1)) = d$ gives $E(\mathfrak{s}_1) \ge d$ and (17) implies $E(\mathfrak{s}) < d$. This leads to a contradiction. Assuming that $\mathfrak{s}_1$ is the first time such that $I(\varkappa(\mathfrak{s}_1)) = 0$ and $\|\Delta\varkappa\|_2 \ne 0$, then $I(\varkappa) > 0$ in the interval $(0,\mathfrak{s}_1)$ and $\varkappa(\mathfrak{s}_1) \in \mathcal{N}$ which implies $d \le J(\varkappa(\mathfrak{s}_1))$. Using (21), we have $E(\mathfrak{s}_1) \ge d$ which contradicts the nonincreasing property of $E(\mathfrak{s})$. Thus, the proof of Lemma (8) is completed. □

Now, under the hypotheses (H1)–(H4), we shall use the Galerkin approximation technique combined with the potential well method to prove the global existence of weak solutions for Problem (9) as long as $\varkappa_0 \in W$. Meanwhile, we shall obtain the asymptotic stability of the global solutions.

3.1. Global Existence for Low Initial Energy $J(\varkappa_0) < d$

Theorem 1. *Assume that assumptions* (H1)–(H3) *are in place and $h_0 \in H_0^1(\Omega, H^1(0,1))$ satisfying the compatibility condition $h(.,0) = \varkappa_1$. Then, for given initial data $\varkappa_0 \in H^3(\Omega) \cap H_0^1(\Omega)$, $\varkappa_1 \in H_0^1(\Omega)$, the Cauchy problem* (9) *possesses a unique global weak solution*

$$\varkappa \in L^\infty([0,\infty), H^3(\Omega) \cap H_0^1(\Omega)),$$
$$\dot{\varkappa} \in L^\infty([0,\infty), H_0^1(\Omega)),$$
$$\ddot{\varkappa} \in L^2([0,\infty), H_0^1(\Omega)).$$

In addition, said solution satisfies the energy inequality

$$\begin{aligned} E(\mathfrak{s}) - \frac{1}{2}\int_0^{\mathfrak{s}} (g' \circ \Delta\varkappa)(\varrho)\,\mathrm{d}\varrho + \frac{1}{2}\int_0^{\mathfrak{s}} g(\varrho)\|\Delta\varkappa(\varrho)\|_2^2\,\mathrm{d}\varrho \\ + \left(\mu_1 - \frac{\xi}{\tau} - \frac{|\mu_2|}{2}\right)\int_0^{\mathfrak{s}} \|\nabla\dot{\varkappa}(\varrho)\|_2^2\,\mathrm{d}\varrho \\ + \left(\frac{\xi}{\tau} - \frac{|\mu_2|}{2}\right)\int_0^{\mathfrak{s}} \|\nabla\top(\mathrm{y},1,\varrho)\|_2^2\,\mathrm{d}\varrho \leq E(0). \end{aligned} \tag{26}$$

Proof. The main idea is to use the Galerkin method. Let $\delta > 0$ be fixed, and for each $n \in \mathbb{N}$, define $V_n = \mathrm{Span}\{w_1, w_2, \ldots, w_n\}$ the linear space dense in $H^3(\Omega) \cap H_0^1(\Omega)$, where $\{w_1, w_2, \ldots, w_n, \ldots\}$ is an orthonormal basis of $L^2(\Omega)$ generated by the eigenvectors of $\Delta^2 w_i = \lambda_i w_i \in \Omega$, $w_i = \Delta w_i = 0$ on $\partial\Omega$, where $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ are the corresponding eigenvalues. Now, we define the sequence $\{\phi_i(\mathrm{y},\eta)\}_{i=1}^n$ satisfying $\phi_i(\mathrm{y},0) = w_i$. Then, we may extend $\phi_i(\mathrm{y},0)$ by $\phi_i(\mathrm{y},\eta)$ over $L^2(\Omega\times[0,1])$ such that $\{\phi_i\}_{i=1}^n$ forms a basis of $L^2(\Omega, H^2(0,1))$ and denote $\mathfrak{Y}_k$ the space generated by $\{\phi^n\}$. We define

$$\varkappa^n(\mathfrak{s}) = \sum_{i=1}^n \alpha_{in}(\mathfrak{s}) w_i(\mathrm{y}), \qquad \top^n(\mathfrak{s}) = \sum_{i=1}^n \beta_{in}(\mathfrak{s})\phi_i(\mathrm{y},\eta),\ n = 1,2,3,\ldots,$$

as the approximated solutions which satisfy the nonlinear ordinary differential equations for $1 \leq i \leq n$,

$$\begin{cases} \displaystyle\int_\Omega |\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l \ddot{\varkappa}^n(\mathrm{y},\mathfrak{s}) w_i(\mathrm{y})\,\mathrm{d}\mathrm{y} \\ \quad + \displaystyle\int_\Omega |\nabla\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\nabla\varkappa^n(\mathrm{y},\mathfrak{s})\nabla w_i(\mathrm{y})\,\mathrm{d}\mathrm{y} \\ \quad + \displaystyle\int_\Omega \Delta\left(\varkappa^n(\mathrm{y},\mathfrak{s}) - \int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\varkappa^n(\zeta)\,\mathrm{d}\zeta\right)\Delta w_i(\mathrm{y})\,\mathrm{d}\mathrm{y} \\ \quad + \mu_1\displaystyle\int_\Omega \nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\nabla w_i(\mathrm{y})\,\mathrm{d}\mathrm{y} \\ \quad + \mu_2\displaystyle\int_\Omega \nabla\top^n(\mathrm{y},1,\mathfrak{s})\nabla w_i(\mathrm{y})\,\mathrm{d}\mathrm{y} \\ \quad + \displaystyle\int_\Omega \nabla\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\nabla w_i(\mathrm{y})\,\mathrm{d}\mathrm{y} \\ \quad = \displaystyle\int_\Omega \varkappa^n(\mathrm{y},\mathfrak{s})|\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\ln|\varkappa^n(\mathrm{y},\mathfrak{s})| w_i(\mathrm{y})\,\mathrm{d}\mathrm{y}, & \mathrm{y}\in\Omega, \mathfrak{s}>0, \\ \top^n(\mathrm{y},0,\mathfrak{s}) = \dot{\varkappa}^n(\mathrm{y},\mathfrak{s}), & \mathrm{y}\in\partial\Omega, \mathfrak{s}>0, \\ \varkappa^n(\mathrm{y},0) = \varkappa_{0n}(\mathrm{y}), \quad \dot{\varkappa}^n(\mathrm{y},0) = \varkappa_{1n}(\mathrm{y}), & \mathrm{y}\in\Omega, \end{cases} \tag{27}$$

and

$$\begin{cases} \tau\displaystyle\int_0^1\int_\Omega \dot{\top}^n(\mathrm{y},\eta,\mathfrak{s})\phi_i(\mathrm{y},\eta)\,\mathrm{d}\mathrm{y}\,\mathrm{d}\eta \\ \quad + \displaystyle\int_0^1\int_\Omega \top_\eta(\mathrm{y},\eta,\mathfrak{s})\phi_i(\mathrm{y},\eta)\,\mathrm{d}\mathrm{y}\,\mathrm{d}\eta = 0, \\ \top^n(\mathrm{y},\eta,0) = \top_{0n}, \end{cases} \tag{28}$$

where $\varkappa_{0n}$, $\varkappa_{1n}$ are chosen in V_n so that

$$\begin{aligned} &\varkappa_{0n} \to \varkappa_0 \text{ strongly in } H^3(\Omega)\cap H_0^1(\Omega), \\ &u_{1n} \to u_1 \text{ strongly in } H^2(\Omega)\cap H_0^1(\Omega), \end{aligned} \tag{29}$$

and $\top_{0n} \in V_n$ such that

$$\top_{0n} \to h_0(\mathrm{y}, -\eta\tau) \text{ strongly in } H_0^2(\Omega\times[0,1]). \tag{30}$$

As $n \to \infty$. Thanks to the Sobolev embeddings $H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow L^{2(l+1)}(\Omega)$ and using the generalised Hölder inequality with exponents $l/2(l+1), 1/2(l+1)$ and $1/2$, the nonlinear term $(|\dot{\varkappa}^n(\mathfrak{s})|^l \dot{\varkappa}^n(\mathfrak{s}), w_i)_{L^2(\Omega)}$ makes sense

$$\begin{aligned}\int_\Omega |\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l \dot{\varkappa}^n(\mathrm{y},\mathfrak{s}) w_i(\mathrm{y})\,\mathrm{d}\mathrm{y} &\le \|\dot{\varkappa}^n(\mathfrak{s})\|_{2(l+1)}^l \|\dot{\varkappa}^n(\mathfrak{s})\|_2 \|w_i\|_{2(l+1)} \\ &\le c_1^{l+1} \|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^l \|\dot{\varkappa}^n(\mathfrak{s})\|_2 \|\Delta w_i\|_2 \\ &\le C\Big\{\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^{l+1} + \|\dot{\varkappa}^n(\mathfrak{s})\|_2^{l+1}\Big\},\end{aligned}$$

and using the fact $\varkappa^p \ln|\varkappa| \le |\varkappa|^{p+\alpha}$ for $|\varkappa| > 1$ and for all $\alpha > 0$, we can estimate

$$\int_\Omega \varkappa^n(\mathrm{y},\mathfrak{s})|\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2} \ln|\varkappa^n(\mathrm{y},\mathfrak{s})| w_i(\mathrm{y})\,\mathrm{d}\mathrm{y},$$

as follows by choosing $\alpha = 1$,

$$\begin{aligned}&\int_\Omega \varkappa^n(\mathrm{y},\mathfrak{s})|\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2} \ln|\varkappa^n(\mathrm{y},\mathfrak{s})| w_i(\mathrm{y})\,\mathrm{d}\mathrm{y} \\ &\qquad = \int_{\Omega_1} \varkappa^n(\mathrm{y},\mathfrak{s})|\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2} \ln|\varkappa^n(\mathrm{y},\mathfrak{s})| w_i(\mathrm{y})\,\mathrm{d}\mathrm{y} \\ &\qquad\quad + \int_{\Omega_2} \varkappa^n(\mathrm{y},\mathfrak{s})|\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2} \ln|\varkappa^n(\mathrm{y},\mathfrak{s})| w_i(\mathrm{y})\,\mathrm{d}\mathrm{y} \\ &\qquad \le \|w_i\|_2 \int_{\Omega_2} |\varkappa^n(\mathrm{y},\mathfrak{s})|^{2p}\,\mathrm{d}\mathrm{y}, \le C\|\Delta\varkappa^n(\mathfrak{s})\|_2^p,\end{aligned}$$

where

$$\Omega_1 = \{\mathrm{y} \in \Omega : |\varkappa^n(\mathrm{y},\mathfrak{s})| \le 1\}, \qquad \Omega_2 = \{\mathrm{y} \in \Omega : |\varkappa^n(\mathrm{y},\mathfrak{s})| > 1\}.$$

We may show by the standard theory of ordinary differential equation that the initial value Problems (27) and (28) is uniquely solvable. More precisely, we obtain a local solution $\varkappa^n(\mathfrak{s}) \in (H^3[0,\delta_n))^2$ extended to a maximal interval $0 \le \mathfrak{s} \le \delta_n$ with $0 < \delta_n$, owing to Zorn lemma because the unknown functions α_{in}, β_{in} and their time derivatives are continuous, and by using the embedding $H^3[0,\delta_n] \hookrightarrow C^2[0,\delta_n]$, we deduce the solution $\varkappa^n(\mathfrak{s}) \in C^2[0,\delta_n)$. Next, using a standard compactness argument, we shall obtain a priori estimates for the solution of systems (27) and (28), so that it can be extended to a global solution defined for all $\mathfrak{s} > 0$. Multiplying (27) by $\alpha_{in}'(\mathfrak{s})$, summing with respect to i and integrating by parts over the time variable from 0 to $\mathfrak{s}$, we shall obtain from (21) the following estimate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\Big\{&\frac{1}{l+2}\|\dot{\varkappa}^n(\mathfrak{s})\|_{l+2}^{l+2} + \|\nabla\dot{\varkappa}^n(\mathfrak{s})\|_2^2 + \frac{1}{2}\Big(1 - \int_0^{\mathfrak{s}} g(\zeta)\,\mathrm{d}\zeta\Big)\|\Delta\varkappa^n(\mathfrak{s})\|_2^2 \\ &+ \frac{1}{p}\|\nabla\varkappa^n(\mathfrak{s})\|_p^p + \frac{1}{2}(g \circ \Delta\varkappa^n)(\mathfrak{s}) \\ &- \frac{1}{p}\int_\Omega |\varkappa^n(\mathrm{y},\mathfrak{s})|^p \ln|\varkappa(\mathrm{y},\mathfrak{s})|\,\mathrm{d}\mathrm{y} + \frac{1}{p^2}\|\varkappa^n(\mathfrak{s})\|_p^p\Big\} \\ &+ \mu_1\|\nabla\dot{\varkappa}^n(\mathfrak{s})\|_2^2 + \mu_2\int_\Omega \nabla\top^n(\mathrm{y},1,\mathfrak{s})\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{d}\mathrm{y} \\ &= \frac{1}{2}(g' \circ \Delta\varkappa^n)(\mathfrak{s}) - \frac{1}{2}g(\mathfrak{s})\|\Delta\varkappa^n(\mathfrak{s})\|_2^2.\end{aligned} \tag{31}$$

Replacing $\mathfrak{s}$ with ϱ in (31) and integrating over $\varrho \in [0, \mathfrak{s}]$ to arrive at

$$\begin{aligned}\frac{1}{l+2}\|\dot{\varkappa}^n(\mathfrak{s})\|_{l+2}^{l+2} + \|\nabla\dot{\varkappa}^n(\mathfrak{s})\|_2^2 + J(\varkappa^n) + \frac{1}{2}(g\circ\Delta\varkappa^n)(\mathfrak{s}) + \mu_1\int_0^{\mathfrak{s}}\|\nabla\dot{\varkappa}^n(\varrho)\|_2^2\,\mathrm{d}\varrho \\ + \mu_2\int_0^{\mathfrak{s}}\int_\Omega \nabla\top(\mathrm{y},1,\varrho)\nabla\dot{\varkappa}^n(\mathrm{y},\varrho)\,\mathrm{d}\mathrm{y}\,\mathrm{d}\varrho \\ - \frac{1}{2}\int_0^{\mathfrak{s}}(g'\circ\Delta\varkappa^n)(\varrho)\,\mathrm{d}\varrho + \frac{1}{2}\int_0^{\mathfrak{s}} g(\varrho)\|\Delta\varkappa^n(\varrho)\|_2^2\,\mathrm{d}\varrho \\ = J(\varkappa_{0n}) + \frac{1}{l+2}\|\varkappa_{1n}\|_{l+2}^{l+2} + \|\nabla\varkappa_{1n}\|_2^2 \\ + \frac{1}{2}\|\Delta\varkappa_{0n}\|_2^2 + \frac{1}{p}\|\nabla\varkappa_{0n}\|_p^p + \frac{1}{2}(g\circ\Delta\varkappa_{0n}) \\ - \frac{1}{p}\int_\Omega|\varkappa_{0n}|^p\ln|\varkappa_{0n}|\,\mathrm{d}\mathrm{y} + \frac{1}{p^2}\|\varkappa_{0n}\|_p^p.\end{aligned} \tag{32}$$

Replacing $\phi_i(\mathrm{y},\eta)$ in (28) with $-\Delta\phi_i(\mathrm{y},\eta)$, multiplying by $\beta_{in}(\mathfrak{s})/\tau$ and adding those equations with respect to i from 1 to n, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\int_0^1\int_\Omega|\nabla\top^n(\mathrm{y},\eta,\mathfrak{s})|^2\,\mathrm{d}\mathrm{y}\,\mathrm{d}\eta + \frac{1}{2\tau}\int_\Omega|\nabla\top^n(\mathrm{y},1,\mathfrak{s})|^2 - |\nabla\top^n(\mathrm{y},0,\mathfrak{s})|^2\,\mathrm{d}\mathrm{y} = 0.$$

Integrating in terms of $\mathfrak{s}$, we obtain that

$$\begin{aligned}\frac{1}{2}\int_0^1\int_\Omega|\nabla\top^n(\mathrm{y},\eta,\mathfrak{s})|^2\,\mathrm{d}\mathrm{y}\,\mathrm{d}\eta + \frac{1}{2\tau}\int_0^{\mathfrak{s}}\int_\Omega|\nabla\top^n(\mathrm{y},1,\varrho)|^2 - |\nabla\top^n(\mathrm{y},0,\varrho)|^2\,\mathrm{d}\mathrm{y}\,\mathrm{d}\varrho \\ = \frac{1}{2}\int_0^1\int_\Omega|\nabla\top_{0n}|^2\,\mathrm{d}\mathrm{y}\,\mathrm{d}\eta = \frac{1}{2}\|\top_{0n}\|_{H_0^1(\Omega\times(0,1))}^2.\end{aligned} \tag{33}$$

Multiplying (33) by ξ and summing up with (32), we have

$$\begin{aligned}\frac{1}{l+2}\|\dot{\varkappa}^n(\mathfrak{s})\|_{l+2}^{l+2} + \|\nabla\dot{\varkappa}^n(\mathfrak{s})\|_2^2 + J(\varkappa^n) + \frac{1}{2}(g\circ\Delta\varkappa^n)(\mathfrak{s}) \\ + \frac{\xi}{2}\int_0^1\|\nabla\top^n(\mathrm{y},\eta,\mathfrak{s})\|_2^2\,\mathrm{d}\eta \\ + \left(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_0^{\mathfrak{s}}\|\nabla\dot{\varkappa}^n(\varrho)\|_2^2\,\mathrm{d}\varrho \\ + \left(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_0^{\mathfrak{s}}\|\nabla\top^n(\mathrm{y},1,\varrho)\|_2^2\,\mathrm{d}\varrho \\ - \frac{1}{2}\int_0^{\mathfrak{s}}(g'\circ\Delta\varkappa^n)(\varrho)\,\mathrm{d}\varrho + \frac{1}{2}\int_0^{\mathfrak{s}} g(\varrho)\|\Delta\varkappa^n(\varrho)\|_2^2\,\mathrm{d}\varrho \\ \le J(\varkappa_{0n}) + \frac{1}{l+2}\|\varkappa_{1n}\|_{l+2}^{l+2} + \|\nabla\varkappa_{1n}\|_2^2 \\ + \frac{1}{2}\|\Delta\varkappa_{0n}\|_2^2 + \frac{1}{p}\|\nabla\varkappa_{0n}\|_p^p + \frac{1}{2}(g\circ\Delta\varkappa_{0n}) \\ - \frac{1}{p}\int_\Omega|\varkappa_{0n}|^p\ln|\varkappa_{0n}|\,\mathrm{d}\zeta + \frac{1}{p^2}\|\varkappa_{0n}\|_p^p + \frac{\xi}{2}\|\top_{0n}\|_{H_0^1(\Omega\times(0,1))}^2.\end{aligned} \tag{34}$$

On one hand, from the mean value theorem, we assert that

$$\left|\int_\Omega|\varkappa_{0n}|^p\ln|\varkappa_{0n}|\,\mathrm{d}\mathrm{y} - |\varkappa_0|^p\ln|\varkappa_0|\,\mathrm{d}\mathrm{y}\right| \le \|\phi'(\varkappa_0(1-\varepsilon)+\varepsilon\varkappa_{0n})\|_2\cdot\|\varkappa_{0n}-\varkappa_0\|_2,$$

where $\phi(v) = |v|^p \ln|v|$, $\phi'(v) = p|v|^{p-1}\ln|v| + |v|^{p-1}$ and $0 < \varepsilon < 1$. Moreover, by the Minkowski inequality and the fact [19]

$$\varkappa^{\alpha} \ln|u| \leq \begin{cases} |\varkappa|^{\alpha+\beta}, & |\varkappa| > 1, \\ \dfrac{1}{e\alpha}, & |\varkappa| \leq 1, \end{cases} \tag{35}$$

for any $\beta, \alpha > 0$. We obtain by applying Hölder's inequality with $\frac{p}{p-1}$ and p

$$\begin{aligned} \|\phi'(v)\|_2 &\leq p\left(\int_{\Omega_1} \left||v|^{p-1}|\ln|v||\right|^2 \mathrm{d}y + \int_{\Omega_2} |v|^{p-1}|\ln|v||^2 \mathrm{d}y\right)^{\frac{1}{2}} + \left(\int_{\Omega} |v|^{2(p-1)}\mathrm{d}y\right)^{\frac{1}{2}} \\ &\leq p\frac{\sqrt{|\Omega_1|}}{e(p-1)} + (p+1)\left(\int_{\Omega} |v|^{2(p-1)}\mathrm{d}y\right)^{\frac{1}{2}} \\ &\leq p\frac{\sqrt{|\Omega_1|}}{e(p-1)} + (p+1)|\Omega|^{\frac{1}{2p}}\|v\|_{2p}^{p-1} \\ &\leq A + (p+1)c_1^{p-1}|\Omega|^{\frac{1}{2p}}\|\Delta v\|_2^{p-1}. \end{aligned}$$

Therefore, as $p < \frac{n}{n-2}$, we obtain $2p < \frac{2n}{n-2} = 2^*$ and

$$\begin{aligned} &\left|\int_{\Omega} |\varkappa_{0n}|^p \ln|\varkappa_{0n}|\,\mathrm{d}y - |\varkappa_0|^p \ln|\varkappa_0|\,\mathrm{d}y\right| \\ &\qquad \leq c_1\Big\{A + (p+1)c_1^{p-1}|\Omega|^{\frac{1}{2p}}(\|\Delta u_0\|_2^{p-1} \\ &\qquad\quad + \|\Delta\varkappa_{0n}\|_2^{p-1})\Big\} \cdot \|\Delta\varkappa_{0n} - \Delta\varkappa_0\|_2. \end{aligned}$$

So, by (29) and hypothesis (H3), we obtain for sufficiently large n

$$\left|\int_{\Omega} |\varkappa_{0n}|^p \ln|\varkappa_{0n}|\,\mathrm{d}y - |\varkappa_0|^p \ln|\varkappa_0|\,\mathrm{d}y\right| \leq A_p\|\Delta\varkappa_0\|_2^{p-1} \cdot \|\Delta\varkappa_{0n} - \Delta\varkappa_0\|_2.$$

Together with $\varkappa_{0n} \to \varkappa_0$ strongly in $H^3(\Omega) \cap H_0^1(\Omega)$ as $n \to \infty$, we derive

$$\int_{\Omega} |\varkappa_{0n}|^p \ln|\varkappa_{0n}|\,\mathrm{d}y \to \int_{\Omega} |\varkappa_0|^p \ln|\varkappa_0|\,\mathrm{d}y, \qquad as\ n \to \infty. \tag{36}$$

By taking $\phi(v) = |v|^p$, we obtain for sufficiently large n

$$\begin{aligned} &\left|\int_{\Omega} |\nabla\varkappa_{0n}|^p \,\mathrm{d}y - \int_{\Omega} |\nabla\varkappa_0|^p \,\mathrm{d}y\right| \\ &\qquad \leq p\int_{\Omega} |\nabla\varkappa_0(1-\varepsilon) + \varepsilon\nabla\varkappa_{0n}|^{p-1}|\nabla\varkappa_{0n} - \nabla\varkappa_0|\,\mathrm{d}y \\ &\qquad \leq c_2 2^{p-1} p\Big\{\|\Delta\varkappa_{0n}\|_{2(p-1)}^{p-1} + \|\Delta\varkappa_0\|_{2(p-1)}^{p-1}\Big\}.\|\Delta\varkappa_{0n} - \Delta\varkappa_0\|_2 \\ &\qquad \leq A_p\|\Delta\varkappa_0\|_2^{p-1} \cdot \|\Delta\varkappa_{0n} - \Delta\varkappa_0\|_2. \end{aligned} \tag{37}$$

Together with $\varkappa_{0n} \to \varkappa_0$ strongly in $H^3(\Omega) \cap H_0^1(\Omega)$ as $n \to \infty$, we deduce that

$$\int_{\Omega} |\nabla\varkappa_{0n}|^p \,\mathrm{d}\zeta \to \int_{\Omega} |\nabla\varkappa_0|^p \,\mathrm{d}\zeta, \qquad as\ n \to \infty, \tag{38}$$

By (29), we achieve $\varkappa_{0n} \to \varkappa_0$ almost everywhere in $\Omega \times (0, \delta)$, which implies by the continuity of the function $v \mapsto |v|^p v$ that

$$|\varkappa_{0n}|^p\varkappa_{0n} \to |\varkappa_0|^p\varkappa_0, \text{ almost everywhere in } \Omega \times (0, \delta). \tag{39}$$

Using (29)–(39) and from Aubin–Lions Lemma, we derive

$$|\varkappa_{0n}|^p \varkappa_{0n} \to |\varkappa_0|^p \varkappa_0 \text{ weakly in } L^2(0,\delta,L^2(\Omega)). \tag{40}$$

So, by (40), we obtain as $n \to \infty$ that

$$\int_\Omega v(|\varkappa_{0n}|^p \varkappa_{0n} - |\varkappa_0|^p \varkappa_0)\,\mathrm{dy} \to 0 \text{ for any } v \in L^2(\Omega).$$

By taking $v = \varkappa_0$ and Hölder's inequality, we deduce for sufficiently large n that

$$\begin{aligned}
&\left|\int_\Omega |\varkappa_{0n}|^p \,\mathrm{d}\zeta - \int_\Omega |\varkappa_0|^p \,\mathrm{dy}\right| \\
&\qquad \le \left|\int_\Omega (\varkappa_{0n}-\varkappa_0)\varkappa_{0n}|\varkappa_{0n}|^{p-2}\,\mathrm{dy}\right| \\
&\qquad\quad + \left|\int_\Omega \varkappa_0(\varkappa_{0n}|\varkappa_{0n}|^{p-2} - \varkappa_0|\varkappa_0|^{p-2})\,\mathrm{dy}\right| \\
&\qquad \le \|\varkappa_{0n}\|_p^{p-1}\|\varkappa_{0n}-\varkappa_0\|_p \\
&\qquad\quad + \left|\int_\Omega \varkappa_0(\varkappa_{0n}|\varkappa_{0n}|^{p-2} - \varkappa_0|\varkappa_0|^{p-2})\,\mathrm{dy}\right| \\
&\qquad \le c_1\|\varkappa_0\|_p^{p-1}\|\Delta\varkappa_{0n}-\Delta\varkappa_0\|_2 \\
&\qquad\quad + \left|\int_\Omega \varkappa_0(\varkappa_{0n}|\varkappa_{0n}|^{p-2} - \varkappa_0|\varkappa_0|^{p-2})\,\mathrm{dy}\right| \to 0,
\end{aligned}$$

and consequently, for almost everywhere $\mathfrak{s}$ in $(0,\infty)$,

$$\int_\Omega |\varkappa_{0n}|^p \,\mathrm{dy} \to \int_\Omega |\varkappa_0|^p \,\mathrm{dy}, \qquad \text{as } n \to \infty. \tag{41}$$

According to (36), (38) and (41), we assert that $J(\varkappa_{0n}) \to J(\varkappa_0)$ an $n \to \infty$, which together with (34) implies for sufficiently large n that

$$\begin{aligned}
&\frac{1}{l+2}\|\dot{\varkappa}^n(\mathfrak{s})\|_{l+2}^{l+2} + \|\nabla\dot{\varkappa}^n(\mathfrak{s})\|_2^2 + J(\varkappa^n) + \frac{1}{2}(g\circ\Delta\varkappa^n)(\mathfrak{s}) \\
&\qquad + \frac{\xi}{2}\int_0^1 \|\nabla\top^n(\mathrm{y},\eta,\mathfrak{s})\|_2^2\,\mathrm{d}\eta \\
&\qquad + \left(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_0^{\mathfrak{s}} \|\nabla\dot{\varkappa}^n(\varrho)\|_2^2\,\mathrm{d}\varrho \\
&\qquad + \left(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_0^{\mathfrak{s}} \|\nabla\top^n(\mathrm{y},1,\varrho)\|_2^2\,\mathrm{d}\varrho \\
&\qquad - \frac{1}{2}\int_0^{\mathfrak{s}} (g'\circ\Delta\varkappa^n)(\varrho)\,\mathrm{d}\varrho + \frac{1}{2}\int_0^{\mathfrak{s}} g(\varrho)\|\Delta\varkappa^n(\varrho)\|_2^2\,\mathrm{d}\varrho \\
&\quad \le J(\varkappa_0) + M(\|\Delta\varkappa_0\|_2, \|\Delta\varkappa_1\|_2) + \frac{\xi}{2}\|h_0\|^2_{H_0^2(\Omega\times(0,1))} \\
&\quad \le d + M(\|\Delta\varkappa_0\|_2, \|\Delta\varkappa_1\|_2) + \frac{\xi}{2}\|h_0(\mathrm{y},-\eta\tau)\|^2_{H_0^2(\Omega\times(0,1))}.
\end{aligned} \tag{42}$$

Taking into account (20) in (42), we obtain

$$\begin{aligned}\frac{1}{l+2}\|\dot{\varkappa}^n(\mathfrak{s})\|_{l+2}^{l+2}+\|\nabla\dot{\varkappa}^n(\mathfrak{s})\|_2^2+\left(\frac{1}{2}-\frac{1}{p}\right)\left(1-\int_0^{\mathfrak{s}}g(\zeta)\,\mathrm{d}\zeta\right)\|\Delta\varkappa^n(\mathfrak{s})\|_2^2\\+\frac{1}{p^2}\|\varkappa^n(\mathfrak{s})\|_p^p+\frac{1}{p}I(\varkappa^n)+\frac{1}{2}(g\circ\Delta\varkappa^n)(\mathfrak{s})\\+\frac{\xi}{2}\int_0^1\|\nabla\top^n(\mathrm{y},\eta,\mathfrak{s})\|_2^2\,\mathrm{d}\eta\\+\left(\mu_1-\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_0^{\mathfrak{s}}\|\nabla\varkappa_{\mathfrak{s}}^n(\varrho)\|_2^2\,\mathrm{d}\varrho\\+\left(\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_0^{\mathfrak{s}}\|\nabla\top^n(\mathrm{y},1,\varrho)\|_2^2\,\mathrm{d}\varrho\\-\frac{1}{2}\int_0^{\mathfrak{s}}(g'\circ\Delta\varkappa^n)(\varrho)\,\mathrm{d}\varrho+\frac{1}{2}\int_0^{\mathfrak{s}}g(\varrho)\|\Delta\varkappa^n(\varrho)\|_2^2\,\mathrm{d}\varrho\\\leq d+M(\|\Delta\varkappa_0\|_2,\|\Delta\varkappa_1\|_2)+\frac{\xi}{2}\|h_0(\mathrm{y},-\eta\tau)\|_{H_0^2(\Omega\times(0,1))}^2.\end{aligned}\tag{43}$$

Noting that $\varkappa_{0n}\to u_0$ as $n\to\infty$, we see that $\varkappa_0\in W$ implies $\varkappa_{0n}\in W$ for sufficiently large n, and using Lemma (8), we prove that $\varkappa^n(\mathfrak{s})\in W$ for sufficiently large n. Therefore, we conclude that

$$\begin{aligned}\frac{1}{l+2}\|\dot{\varkappa}^n(\mathfrak{s})\|_{l+2}^{l+2}+\|\nabla\dot{\varkappa}^n(\mathfrak{s})\|_2^2+\left(\frac{1}{2}-\frac{1}{p}\right)\left(1-\int_0^{\mathfrak{s}}g(\zeta)\,\mathrm{d}\zeta\right)\|\Delta\varkappa^n(\mathfrak{s})\|_2^2\\+\frac{1}{p^2}\|\varkappa^n(\mathfrak{s})\|_p^p+\frac{1}{2}(g\circ\Delta\varkappa^n)(\mathfrak{s})+\frac{\xi}{2}\int_0^1\|\nabla\top^n(\mathrm{y},\eta,\mathfrak{s})\|_2^2\,\mathrm{d}\eta\\+\left(\mu_1-\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_0^{\mathfrak{s}}\|\nabla\dot{\varkappa}^n(\varrho)\|_2^2\,\mathrm{d}\varrho\\+\left(\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_0^{\mathfrak{s}}\|\nabla\top^n(\mathrm{y},1,\varrho)\|_2^2\,\mathrm{d}\varrho\\-\frac{1}{2}\int_0^{\mathfrak{s}}(g'\circ\Delta\varkappa^n)(\varrho)\,\mathrm{d}\varrho+\frac{1}{2}\int_0^{\mathfrak{s}}g(\varrho)\|\Delta\varkappa^n(\varrho)\|_2^2\,\mathrm{d}\varrho\\\leq d+M(\|\Delta\varkappa_0\|_2,\|\Delta\varkappa_1\|_2)+\frac{\xi}{2}\|h_0(\mathrm{y},-\eta\tau)\|_{H_0^2(\Omega\times(0,1))}^2.\end{aligned}\tag{44}$$

Setting

$$\mathring{d}:=d+M(\|\Delta\varkappa_0\|_2,\|\Delta\varkappa_1\|_2)+\frac{\xi}{2}\|h_0(\mathrm{y},-\eta\tau)\|_{H_0^2(\Omega\times(0,1))}^2.$$

By (44), we know that δ_n can be extended to δ, and we obtain

$$\{\varkappa^n\}_{n=1}^{\infty}\ \text{is bounded in}\ L^\infty(0,\delta;H^2(\Omega)\cap H_0^1(\Omega))\cap L^\infty(0,\delta;L^p(\Omega)),\tag{45}$$

$$\{\dot{\varkappa}^n\}_{n=1}^{\infty}\ \text{is bounded in}\ L^\infty(0,\delta;H_0^1(\Omega))\cap L^2(0,\delta;H_0^1(\Omega)),\tag{46}$$

$$\{\top^n\}_{n=1}^{\infty}\ \text{is bounded in}\ L^\infty(0,\delta;H_0^1(\Omega\times[0,1]))\cap L^2(0,\delta;H_0^1(\Omega)).\tag{47}$$

On the other hand, for $\varkappa_0\in W$ and applying Lemma 3, we obtain

$$\begin{aligned}\frac{1}{p}\|\nabla\varkappa^n(\mathfrak{s})\|_p^p=J(\varkappa^n)-\frac{1}{2}\left(1-\int_0^{\mathfrak{s}}g(\zeta)\,\mathrm{d}\zeta\right)\|\Delta\varkappa^n(\mathfrak{s})\|_2^2\\+\frac{1}{p}\int_\Omega|\varkappa^n(\mathrm{y},\mathfrak{s})|^p\ln|\varkappa^n(\mathrm{y},\mathfrak{s})|\,\mathrm{d}\mathrm{y}-\frac{1}{p^2}\|\varkappa(\mathfrak{s})\|_p^p\\\leq d+\frac{1}{p^2}\left\{p\int_\Omega|\varkappa^n(\mathrm{y},\mathfrak{s})|^p\ln\left(\frac{|\varkappa^n(\mathrm{y},\mathfrak{s})|}{\|\varkappa^n(\mathfrak{s})\|_p}\right)\mathrm{d}\mathrm{y}+p\|\varkappa(\mathfrak{s})\|_p^p\ln\|\varkappa^n(\mathfrak{s})\|_p\right\}\end{aligned}$$

$$\leq d + \frac{1}{p^2}\left\{\mu\|\nabla \varkappa^n(\mathfrak{s})\|_p^p - \frac{n}{p}\ln\left(\frac{p\mu e}{n\mathcal{L}_p}\right)\|\varkappa^n(\mathfrak{s})\|_p^p + p\|\varkappa(\mathfrak{s})\|_p^p \ln\|\varkappa^n\|_p\right\}.$$

Choosing $\mu < p$ and using (45), we deduce that $\|\nabla \varkappa^n(\mathfrak{s})\|_p^p \leq c_d$ which implies that $(\nabla \varkappa^n)$ is bounded in $L^p(\Omega \times (0,\delta))$ and hence $(\nabla \varkappa^n|^{p-2}\nabla \varkappa^n)$ is bounded in $L^{p/p-1}(\Omega \times (0,\delta))$. Applying Dunford–Pettis' Theorem, we observe from (45)–(47) that there exist subsequences $(\dot{\varkappa}^n)$ and $(\dot{\top}^n)$ such that

$$\varkappa^n \rightharpoonup \varkappa \text{ weakly star in } L^\infty(0,\delta; H^2(\Omega)\cap H_0^1(\Omega))\cap L^\infty(0,\delta; L^p(\Omega)), \tag{48}$$

$$\dot{\varkappa}^n \rightharpoonup \dot{\varkappa} \text{ weakly star in } L^\infty(0,\delta; H_0^1(\Omega))\cap L^2(0,\delta; H_0^1(\Omega)), \tag{49}$$

$$\top^n \to \top \text{ weakly star in } L^\infty(0,\delta; H_0^1(\Omega\times[0,1]))\cap L^2(0,\delta; H_0^1(\Omega)), \tag{50}$$

$$|\nabla \varkappa^n|^{p-2}\nabla \varkappa^n \to \psi \text{ weakly in } L^{\frac{p}{p-1}}(\Omega\times(0,\delta)). \tag{51}$$

Using (48), (49) and [20], we deduce that

$$\varkappa^n \to \varkappa \text{ in } C([0,\delta], L^2(\Omega)), \qquad \forall \mathrm{y} \in \bar{\Omega}, \tag{52}$$

and $\varkappa^n \to \varkappa$ a.e $\mathrm{y} \in \Omega$, for all $\mathfrak{s} \geq 0$. Clearly, (52) implies

$$\varkappa^n(\mathrm{y},\mathfrak{s})|\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\ln\varkappa^n(\mathrm{y},\mathfrak{s}) \to \varkappa(\mathrm{y},\mathfrak{s})|\varkappa(\mathrm{y},\mathfrak{s})|^{p-2}\ln\varkappa(\mathrm{y},\mathfrak{s}), \tag{53}$$

a.e $\mathrm{y} \in \Omega$, $\mathfrak{s} \in (0,\delta)$. On the other hand, a direct calculation using (35), (45) and Lemma 2 with $\beta = 1$ gives

$$\begin{aligned}
&\int_\Omega \left|\varkappa^n(\mathrm{y},\mathfrak{s})|\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\ln|\varkappa^n(\mathrm{y},\mathfrak{s})|\right|^2 \mathrm{dy}\\
&\qquad = \int_{\Omega_1}\left|\varkappa^n(\mathrm{y},\mathfrak{s})|\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\ln|\varkappa^n(\mathrm{y},\mathfrak{s})|\right|^2 \mathrm{dy}\\
&\qquad\quad + \int_{\Omega_2}\left|\varkappa^n(\mathrm{y},\mathfrak{s})|\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\ln|\varkappa^n(\mathrm{y},\mathfrak{s})|\right|^2 \mathrm{dy}\\
&\qquad \leq \frac{|\Omega|}{e^2(p-1)^2} + \int_\Omega |\varkappa^n(\mathrm{y},\mathfrak{s})|^{2p}\,\mathrm{dy}\\
&\qquad \leq \frac{|\Omega|}{e^2(p-1)^2} + c_1^{2p}\|\Delta\varkappa^n(\mathfrak{s})\|_2^{2p} \leq C_p(\Omega).
\end{aligned} \tag{54}$$

Recalling Lion's ([21], Lemma 1.3), we arrive at

$$\varkappa^n(\mathrm{y},\mathfrak{s})|\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\ln|\varkappa^n(\mathrm{y},\mathfrak{s})| \rightharpoonup \varkappa(\mathrm{y},\mathfrak{s})|\varkappa(\mathrm{y},\mathfrak{s})|^{p-2}\ln|\varkappa(\mathrm{y},\mathfrak{s})|,$$

weakly star in $L^{\infty}(0,\delta;L^2(\Omega))$. Differentiating (27) with respect to $\mathfrak{s}$ and after multiplying by $\alpha_{in}''(\mathfrak{s})$ and summing over i from 1 to n, we have

$$\begin{aligned}
&l\int_{\Omega}|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^{l-2}\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\\
&\quad+\int_{\Omega}|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l\dddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\\
&\quad+(p-2)\int_{\Omega}|\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^{p-4}(\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\nabla\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s}))(\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\nabla\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s}))\,\mathrm{dy}\\
&\quad+\int_{\Omega}|\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^{p-2}\nabla\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\nabla\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\\
&\quad+\int_{\Omega}\Delta\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\Delta\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}-g(0)\int_{\Omega}\Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\Delta\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\\
&\quad-\int_0^{\mathfrak{s}}\int_{\Omega}g'(\mathfrak{s}-\zeta)\Delta\dot{\varkappa}^n(\zeta)\Delta\ddot{\varkappa}^n(\mathfrak{s})\,\mathrm{d}\zeta\,\mathrm{dy}\\
&\quad+\mu_1\|\nabla\ddot{\varkappa}^n(\mathfrak{s})\|_2^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\|\nabla\ddot{\varkappa}^n(\mathfrak{s})\|_2^2+\mu_2\int_{\Omega}\nabla\top_{\mathfrak{s}}^n(\mathrm{y},1,\mathfrak{s})\nabla\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\\
&=(p-1)\int_{\Omega}\dot{\varkappa}(\mathrm{y},\mathfrak{s})|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^{p-2}\ln|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\mathrm{dy}\\
&\quad+\int_{\Omega}\dot{\varkappa}(\mathrm{y},\mathfrak{s})|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^{p-2}\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\mathrm{dy}.
\end{aligned}\tag{55}$$

On one hand, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\left\{\int_{\Omega}|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l|\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\mathrm{dy}\right\}&=\int_{\Omega}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\left\{|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l\right\}|\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\mathrm{dy}\\
&\quad+\int_{\Omega}|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\left\{|\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\right\}\mathrm{dy},
\end{aligned}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\left\{\int_{\Omega}|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l\right\}\mathrm{dy}=l\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^{l-2}\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s}).$$

So,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\left\{\int_{\Omega}|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l|\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\mathrm{dy}\right\}&=l\int_{\Omega}|\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^{l-2}\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\\
&\quad+2\int_{\Omega}|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l\dddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_{\Omega}|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l\dddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\left\{\int_{\Omega}|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l|\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\mathrm{dy}\right\}\\
&\quad-\frac{l}{2}\int_{\Omega}|\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^{l-2}\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}.
\end{aligned}$$

On the other hand, it is convenient to observe that

$$\begin{aligned}
-g(0)\int_\Omega |\Delta \dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\,\mathrm{d}\mathrm{y} + g(0)\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\int_\Omega \Delta\varkappa^n(\mathrm{y},\mathfrak{s})\Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{d}\mathrm{y} \\
+\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\int_0^{\mathfrak{s}}\int_\Omega g'(\mathfrak{s}-\zeta)\Delta\varkappa^n(\zeta)\Delta\dot{\varkappa}^n(\mathfrak{s})\,\mathrm{d}\mathrm{y}\,\mathrm{d}\zeta \\
-g'(0)\int_\Omega \Delta\varkappa^n(\mathrm{y},\mathfrak{s})\Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{d}\mathrm{y} \\
-\int_0^{\mathfrak{s}}\int_\Omega g''(\mathfrak{s}-\zeta)\Delta\varkappa^n(\zeta)\Delta\dot{\varkappa}^n(\mathfrak{s})\,\mathrm{d}\mathrm{y}\,\mathrm{d}\zeta \\
= g(0)\int_\Omega \Delta\varkappa^n(\mathrm{y},\mathfrak{s})\Delta\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{d}\mathrm{y} \\
+\int_0^{\mathfrak{s}}\int_\Omega g'(\mathfrak{s}-\zeta)\Delta\varkappa^n(\zeta)\Delta\ddot{\varkappa}^n(\mathfrak{s})\,\mathrm{d}\mathrm{y}\,\mathrm{d}\zeta.
\end{aligned} \tag{56}$$

Taking into account (56) and integrating (55) over $(0,\mathfrak{s})$ yields

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\,\mathrm{d}\mathrm{y} + \frac{1}{2}\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^2 + \mu_1\int_0^{\mathfrak{s}}\|\nabla\ddot{\varkappa}^n(\varrho)\|_2^2\,\mathrm{d}\varrho \\
&\quad+\frac{1}{2}\|\nabla\ddot{\varkappa}^n(\mathfrak{s})\|_2^2 + g(0)\int_0^{\mathfrak{s}}\|\Delta\dot{\varkappa}^n(\varrho)\|_2^2\,\mathrm{d}\varrho \\
&\quad= -\frac{l}{2}\int_0^{\mathfrak{s}}\int_\Omega |\ddot{\varkappa}^n(\mathrm{y},\varrho)|^2\dot{\varkappa}^n(\mathrm{y},\varrho)|\dot{\varkappa}^n(\mathrm{y},\varrho)|^{l-2}\ddot{\varkappa}^n(\mathrm{y},\varrho)\,\mathrm{d}\mathrm{y}\,\mathrm{d}\varrho \\
&\quad+\frac{1}{2}\int_\Omega |\varkappa_{1n}(\mathrm{y})|^l|\dot{\varkappa}^n(\mathrm{y},0)|^2\,\mathrm{d}\mathrm{y} + \frac{1}{2}\|\Delta\varkappa_{1n}\|_2^2 + \frac{1}{2}\|\nabla\dot{\varkappa}^n(0)\|_2^2 \\
&\quad-(p-2)\int_\Omega |\nabla\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-4}(\nabla\varkappa^n(\mathrm{y},\mathfrak{s})\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s}))(\nabla\varkappa^n(\mathrm{y},\mathfrak{s})\nabla\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s}))\,\mathrm{d}\mathrm{y} \\
&\quad-\int_0^{\mathfrak{s}}\int_\Omega |\nabla\varkappa^n(\mathrm{y},\varrho)|^{p-2}\nabla\dot{\varkappa}^n_{\mathfrak{s}}(\mathrm{y},\varrho)\nabla\ddot{\varkappa}^n(\mathrm{y},\varrho)\,\mathrm{d}\mathrm{y}\,\mathrm{d}\varrho \\
&\quad+g(0)\left\{\int_\Omega \Delta\varkappa^n(\mathrm{y},\mathfrak{s})\Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{d}\mathrm{y} - \int_\Omega \Delta\varkappa_{0n}\Delta\varkappa_{1n}\,\mathrm{d}\mathrm{y}\right\} \\
&\quad+\int_0^{\mathfrak{s}}\int_\Omega g'(\mathfrak{s}-\zeta)\Delta\varkappa^n(\zeta)\Delta\dot{\varkappa}^n(\mathfrak{s})\,\mathrm{d}\mathrm{y}\,\mathrm{d}\zeta \\
&\quad-g'(0)\int_0^{\mathfrak{s}}\int_\Omega \Delta\varkappa^n(\mathrm{y},\varrho)\Delta\dot{\varkappa}^n(\mathrm{y},\varrho)\,\mathrm{d}\mathrm{y}\,\mathrm{d}\varrho \\
&\quad-\int_0^{\mathfrak{s}}\int_0^{\varrho}\int_\Omega g''(\varrho-\zeta)\Delta\varkappa^n(\zeta)\Delta\dot{\varkappa}^n(\varrho)\,\mathrm{d}\mathrm{y}\,\mathrm{d}\zeta\,\mathrm{d}\varrho \\
&\quad-\mu_2\int_0^{\mathfrak{s}}\int_\Omega \nabla\dagger^n(\mathrm{y},1,\varrho)\nabla\ddot{\varkappa}^n(\mathrm{y},\varrho)\,\mathrm{d}\mathrm{y}\,\mathrm{d}\varrho \\
&\quad+(p-1)\int_0^{\mathfrak{s}}\int_\Omega \dot{\varkappa}(\mathrm{y},\varrho)|\varkappa^n(\mathrm{y},\varrho)|^{p-2}\ln|\varkappa^n(\mathrm{y},\varrho)|\dot{\varkappa}^n(\mathrm{y},\varrho)\mathrm{d}\mathrm{y}\,\mathrm{d}\varrho \\
&\quad+\int_0^{\mathfrak{s}}\int_\Omega \dot{\varkappa}(\mathrm{y},\varrho)|\varkappa^n(\mathrm{y},\varrho)|^{p-2}\ddot{\varkappa}^n(\mathrm{y},\varrho)\mathrm{d}\mathrm{y}\,\mathrm{d}\varrho.
\end{aligned} \tag{57}$$

Differentiating (8) with respect to $\mathfrak{s}$ gives

$$\int_0^1\int_\Omega (\tau\ddot{\dagger}^n(\mathrm{y},\eta,\mathfrak{s}) + \dagger^n_\eta(\mathrm{y},\eta,\mathfrak{s}))\phi_i(\mathrm{y},\eta)\,\mathrm{d}\mathrm{y}\,\mathrm{d}\eta = 0. \tag{58}$$

Replacing $\phi_i(\mathrm{y},\eta)$ with $-\Delta\phi_i(\mathrm{y},\eta)$, multiplying by $\beta'_{in}(\mathfrak{s})$ and summing over i from 1 to n, we obtain

$$\frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\int_0^1\int_\Omega |\nabla\dagger^n(\mathrm{y},\eta,\mathfrak{s})|^2\,\mathrm{d}\mathrm{y}\,\mathrm{d}\eta + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta}\int_0^1\int_\Omega |\nabla\dagger^n(\mathrm{y},\eta,\mathfrak{s})|^2\,\mathrm{d}\mathrm{y}\,\mathrm{d}\eta = 0. \tag{59}$$

Therefore, we obtain

$$\frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\int_0^1\int_\Omega |\nabla \dot{\uparrow}^n(\mathrm{y},\eta,\mathfrak{s})|^2\,\mathrm{d}\mathrm{y}\,\mathrm{d}\eta+\frac{1}{2}\int_\Omega\Big(|\nabla \dot{\uparrow}^n(\mathrm{y},1,\mathfrak{s})|^2-|\nabla \dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\Big)\,\mathrm{d}\mathrm{y}=0. \tag{60}$$

Integrating (60) with respect to $\mathfrak{s}$, we have

$$\begin{aligned}\frac{\tau}{2}\int_0^1\int_\Omega |\nabla \dot{\uparrow}^n(\mathrm{y},\eta,\mathfrak{s})|^2\,\mathrm{d}\mathrm{y}\,\mathrm{d}\eta+\frac{1}{2}\int_0^{\mathfrak{s}}\int_\Omega |\nabla \dot{\uparrow}^n(\mathrm{y},1,\varrho)|^2\,\mathrm{d}\mathrm{y}\mathrm{d}\varrho&\\ =\frac{\tau}{2}\int_0^1\int_\Omega |\nabla \dot{\uparrow}^n(\mathrm{y},\eta,0)|^2\,\mathrm{d}\mathrm{y}\,\mathrm{d}\eta&\\ +\frac{1}{2}\int_0^{\mathfrak{s}}\|\nabla \dot{\varkappa}^n(\varrho)\|_2^2\,\mathrm{d}\varrho.&\end{aligned} \tag{61}$$

Summing (57) and (61), we obtain

$$\begin{aligned}&\frac{1}{2}\int_\Omega|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\,\mathrm{d}\mathrm{y}+\frac{1}{2}\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^2+\left(\mu_1-\frac{1}{2}\right)\int_0^{\mathfrak{s}}\|\nabla\dot{\varkappa}^n(\varrho)\|_2^2\,\mathrm{d}\varrho\\
&\quad+\frac{1}{2}\|\nabla\dot{\varkappa}^n(\mathfrak{s})\|_2^2+g(0)\int_0^{\mathfrak{s}}\|\Delta\dot{\varkappa}^n(\varrho)\|_2^2\,\mathrm{d}\varrho\\
&\quad+\frac{\tau}{2}\int_0^1\int_\Omega|\nabla\dot{\uparrow}^n(\mathrm{y},\eta,\mathfrak{s})|^2\,\mathrm{d}\mathrm{y}\,\mathrm{d}\eta+\frac{1}{2}\int_0^{\mathfrak{s}}\int_\Omega|\nabla\dot{\uparrow}^n(\mathrm{y},1,\varrho)|^2\,\mathrm{d}\mathrm{y}\,\mathrm{d}\varrho\\
&=-\frac{l}{2}\int_\Omega|\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^{l-2}\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{d}\mathrm{y}\\
&\quad+\frac{1}{2}\int_\Omega|\varkappa_{1n}(\mathrm{y})|^l|\dot{\varkappa}^n(\mathrm{y},0)|^2\,\mathrm{d}\mathrm{y}+\frac{1}{2}\|\Delta\varkappa_{1n}\|_2^2\\
&\quad+\frac{1}{2}\|\nabla\dot{\varkappa}^n(0)\|_2^2+\frac{\tau}{2}\int_0^1\int_\Omega|\nabla\dot{\uparrow}^n(\mathrm{y},\eta,0)|^2\,\mathrm{d}\mathrm{y}\,\mathrm{d}\eta\\
&\quad-(p-2)\int_\Omega|\nabla\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-4}(\nabla\varkappa^n(\mathrm{y},\mathfrak{s})\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s}))(\nabla\varkappa^n(\mathrm{y},\mathfrak{s})\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s}))\,\mathrm{d}\mathrm{y}\\
&\quad-\int_0^{\mathfrak{s}}\int_\Omega|\nabla\varkappa^n(\mathrm{y},\varrho|^{p-2}\nabla\dot{\varkappa}^n(\mathrm{y},\varrho)\nabla\dot{\varkappa}^n(\mathrm{y},\varrho)\,\mathrm{d}\mathrm{y}\,\mathrm{d}\varrho\\
&\quad+g(0)\Big\{\int_\Omega\Delta\varkappa^n(\mathrm{y},\mathfrak{s})\Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{d}\mathrm{y}\\
&\quad-\int_\Omega\Delta\varkappa_{0n}\Delta\varkappa_{1n}\,\mathrm{d}\mathrm{y}\Big\}+\int_0^{\mathfrak{s}}\int_\Omega g'(\mathfrak{s}-\zeta)\Delta\varkappa^n(\zeta)\Delta\dot{\varkappa}^n(\mathfrak{s})\,\mathrm{d}\mathrm{y}\,\mathrm{d}\zeta\\
&\quad-g'(0)\int_0^{\mathfrak{s}}\int_\Omega\Delta\varkappa^n(\mathrm{y},\varrho)\Delta\dot{\varkappa}^n(\mathrm{y},\varrho)\,\mathrm{d}\mathrm{y}\,\mathrm{d}\varrho\\
&\quad-\int_0^{\mathfrak{s}}\int_0^{\varrho}\int_\Omega g''(\varrho-\zeta)\Delta\varkappa^n(\zeta)\Delta\dot{\varkappa}^n(\varrho)\,\mathrm{d}\mathrm{y}\,\mathrm{d}\zeta\,\mathrm{d}\varrho\\
&\quad-\mu_2\int_0^{\mathfrak{s}}\int_\Omega\nabla\dot{\uparrow}^n(\mathrm{y},1,\varrho)\nabla\dot{\varkappa}^n(\mathrm{y},\varrho)\,\mathrm{d}\mathrm{y}\,\mathrm{d}\varrho\\
&\quad+(p-1)\int_0^{\mathfrak{s}}\int_\Omega\dot{\varkappa}(\mathrm{y},\varrho)|\varkappa^n(\mathrm{y},\varrho)|^{p-2}\ln|\varkappa^n(\mathrm{y},\varrho)|\dot{\varkappa}^n(\mathrm{y},\varrho)\mathrm{d}\mathrm{y}\,\mathrm{d}\varrho\\
&\quad+\int_0^{\mathfrak{s}}\int_\Omega\dot{\varkappa}(\mathrm{y},\varrho)|\varkappa^n(\mathrm{y},\varrho)|^{p-2}\dot{\varkappa}^n(\mathrm{y},\varrho)\mathrm{d}\mathrm{y}\,\mathrm{d}\varrho.
\end{aligned} \tag{62}$$

Now, we estimate each term on the right-hand side of (62). Taking into account the Cauchy–Schwarz and Hölder inequalities, we have for any $\sigma > 0$

$$\begin{aligned}
&\int_\Omega |\nabla \varkappa^n(\mathrm{y},\mathfrak{s})|^{p-4}(\nabla \varkappa^n(\mathrm{y},\mathfrak{s})\nabla \dot{\varkappa}^n(\mathrm{y},\mathfrak{s}))(\nabla \varkappa^n(\mathrm{y},\mathfrak{s})\nabla \ddot{\varkappa}^n(\mathrm{y},\mathfrak{s}))\,\mathrm{dy}\\
&\qquad\le \int_\Omega |\nabla \varkappa^n(\mathrm{y},\varrho)|^{p-2}\nabla \dot{\varkappa}^n(\mathrm{y},\varrho)\nabla \ddot{\varkappa}^n(\mathrm{y},\varrho)\,\mathrm{dy}\\
&\qquad\le \|\nabla \varkappa^n(\mathfrak{s})\|_{2(p-1)}^{p-2}\|\nabla \dot{\varkappa}^n(\mathfrak{s})\|_{2(p-1)}\|\nabla \ddot{\varkappa}^n(\mathfrak{s})\|_2\\
&\qquad\le c_2^{p-2}\|\Delta \varkappa^n(\mathfrak{s})\|_2^{p-2}\left\{\frac{c_2^2\kappa_1^2}{4\sigma}+\sigma\|\nabla \ddot{\varkappa}^n(\mathfrak{s})\|_2^2\right\}\\
&\qquad\le c_2^{p-2}\kappa_0^{p-2}\left\{\frac{c_2^2\kappa_1^2}{4\sigma}+\sigma\|\nabla \ddot{\varkappa}^n(\mathfrak{s})\|_2^2\right\},
\end{aligned}\tag{63}$$

$$\begin{aligned}
\left|\int_\Omega \Delta \varkappa^n(\mathrm{y},\mathfrak{s})\Delta \dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\right| &\le \frac{1}{4\sigma}\|\Delta \varkappa^n(\mathfrak{s})\|_2^2+\sigma\|\Delta \dot{\varkappa}^n(\mathfrak{s})\|_2^2\\
&\le \frac{\kappa_0^2}{4\sigma}+\sigma\|\Delta \dot{\varkappa}^n(\mathfrak{s})\|_2^2,
\end{aligned}\tag{64}$$

and

$$\left|\int_\Omega \Delta \varkappa_{0n}\Delta \varkappa_{1n}\,\mathrm{dy}\right| \le \frac{1}{2}\|\Delta \varkappa_{0n}\|_2^2+\frac{1}{2}\|\Delta \varkappa_{1n}\|_2^2,\tag{65}$$

Moreover,

$$\begin{aligned}
&\left|\int_0^{\mathfrak{s}}\int_\Omega g'(\mathfrak{s}-\zeta)\Delta \varkappa^n(\zeta)\Delta \dot{\varkappa}^n(\mathfrak{s})\,\mathrm{dy}\,\mathrm{d}\zeta\right|\\
&\quad\le \sigma\int_\Omega |\Delta \dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\,\mathrm{dy}+\frac{1}{4\sigma}\int_\Omega\left(\int_0^{\mathfrak{s}} g'(\mathfrak{s}-\zeta)\Delta \varkappa^n(\zeta)\,\mathrm{d}\zeta\right)^2\mathrm{dy}\\
&\quad\le \sigma\int_\Omega |\Delta \dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\,\mathrm{dy}\\
&\qquad+\frac{1}{4\sigma}\int_\Omega\left(\int_0^{\mathfrak{s}} g'(\mathfrak{s}-\zeta)\Big(|\Delta \varkappa^n(\zeta)-\Delta \varkappa^n(\mathfrak{s})|+|\Delta \varkappa^n(\mathfrak{s})|\Big)\,\mathrm{d}\zeta\right)^2\mathrm{dy}.
\end{aligned}$$

Using Hypothesis (H1), we obtain

$$\begin{aligned}
&\int_\Omega\left(\int_0^{\mathfrak{s}} g'(\mathfrak{s}-\zeta)(|\Delta \varkappa^n(\zeta)-\Delta \varkappa^n(\mathfrak{s})|+|\Delta \varkappa^n(\mathfrak{s})|)\,\mathrm{d}\zeta\right)^2\mathrm{dy}\\
&\quad\le \int_\Omega\left(\int_0^{\mathfrak{s}} g'(\mathfrak{s}-\zeta)(|\Delta \varkappa^n(\zeta)-\Delta \varkappa^n(\mathfrak{s})|)\,\mathrm{d}\zeta\right)^2\mathrm{dy}\\
&\qquad+\int_\Omega\left(\int_0^{\mathfrak{s}} g'(\mathfrak{s}-\zeta)|\Delta \varkappa^n(\mathfrak{s})|\,\mathrm{d}\zeta\right)^2\mathrm{dy}\\
&\qquad+2\int_\Omega\left(\int_0^{\mathfrak{s}} g'(\mathfrak{s}-\zeta)\big(|\Delta \varkappa^n(\zeta)-\Delta \varkappa^n(\mathfrak{s})|\big)\,\mathrm{d}\zeta\right)\\
&\qquad\times\int_\Omega\left(\int_0^{\mathfrak{s}} g'(\mathfrak{s}-\zeta)|\Delta \varkappa^n(\mathfrak{s})|\,\mathrm{d}\zeta\right)\mathrm{dy}\\
&\quad\le \left(1+\frac{1}{\sigma}\right)\int_\Omega\left(\int_0^{\mathfrak{s}} g'(\mathfrak{s}-\zeta)\Big(|\Delta \varkappa^n(\zeta)-\Delta \varkappa^n(\mathfrak{s})|\Big)\,\mathrm{d}\zeta\right)^2\mathrm{d}\zeta+(1+\sigma)\\
&\qquad\times\int_\Omega\left(\int_0^{\mathfrak{s}} g'(\mathfrak{s}-\zeta)|\Delta \varkappa^n(\mathfrak{s})|\,\mathrm{d}\zeta\right)^2\mathrm{dy}\\
&\quad\le \left(1+\frac{1}{\sigma}\right)\int_\Omega\left(\int_0^{\mathfrak{s}} g'(\zeta)\,\mathrm{d}\zeta\right)\int_0^{\mathfrak{s}} g'(\mathfrak{s}-\zeta)|\Delta \varkappa^n(\zeta)-\Delta \varkappa^n(\mathfrak{s})|^2\,\mathrm{d}\zeta\,\mathrm{dy}
\end{aligned}$$

$$
\begin{aligned}
&+(1+\sigma)\int_{\Omega}\left(\int_0^{\mathfrak{s}} g'(\zeta)\,\mathrm{d}\zeta\right)^2|\Delta\varkappa^n(\mathfrak{s})|^2\,\mathrm{d}\zeta\,\mathrm{d}y\\
&\leq g(0)\left(1+\frac{1}{\sigma}\right)(g'\circ\Delta\varkappa^n)(\mathfrak{s})+g^2(0)(1+\sigma)\|\Delta\varkappa^n(\mathfrak{s})\|_2^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left|\int_0^{\mathfrak{s}}\int_{\Omega} g'(\mathfrak{s}-\zeta)\Delta\varkappa^n(\zeta)\Delta\dot{\varkappa}^n(\mathfrak{s})\,\mathrm{d}y\,\mathrm{d}\zeta\right|\\
&\qquad\leq\sigma\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^2+g^2(0)\frac{1+\sigma}{4\sigma}\|\Delta\varkappa^n(\mathfrak{s})\|_2^2\\
&\qquad\leq\sigma\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^2+g^2(0)\frac{1+\sigma}{4\sigma}\kappa_0^2.
\end{aligned}
\tag{66}
$$

By (H2) and using the similar arguments as the estimate (66), we obtain

$$
\begin{aligned}
&\left|\int_0^{\mathfrak{s}}\int_0^{\varrho}\int_{\Omega} g''(\varrho-\zeta)\Delta\varkappa^n(\zeta)\Delta\dot{\varkappa}^n(\varrho)\,\mathrm{d}y\,\mathrm{d}\zeta\,\mathrm{d}\varrho\right|\\
&\qquad\leq\xi_2\left|\int_0^{\mathfrak{s}}\int_0^{\eta}\int_{\Omega} g(\varrho-\zeta)\Delta\varkappa^n(\zeta)\Delta\dot{\varkappa}^n(\varrho)\,\mathrm{d}y\,\mathrm{d}\zeta\,\mathrm{d}\varrho\right|\\
&\qquad\leq\xi_2\left\{\sigma\int_0^{\mathfrak{s}}\|\Delta\dot{\varkappa}^n(\varrho)\|_2^2\,\mathrm{d}\varrho+(1-\vartheta)\left(1+\frac{1}{\sigma}\right)\int_0^{\mathfrak{s}}(g\circ\Delta\varkappa^n)(\varrho)\,\mathrm{d}\varrho\right.\\
&\qquad\quad\left.+(1+\sigma)(1-\vartheta)^2\int_0^{\mathfrak{s}}\|\Delta\varkappa^n(\varrho)\|_2^2\,\mathrm{d}\varrho\right\}\\
&\qquad\leq\xi_2\left\{\sigma\int_0^{\mathfrak{s}}\|\Delta\dot{\varkappa}^n(\varrho)\|_2^2\,\mathrm{d}\varrho\right.\\
&\qquad\quad\left.+(1-\vartheta)\left(1+\frac{1}{\sigma}\right)\int_0^{\mathfrak{s}}(g\circ\Delta\varkappa^n)(\varrho)\,\mathrm{d}\varrho+\kappa_0^2(1+\sigma)(1-\vartheta)^2\delta\right\},
\end{aligned}
\tag{67}
$$

On one hand, using hypothesis (H2) with (44), we obtain

$$
\xi_3\int_0^{\mathfrak{s}}(g\circ\Delta\varkappa^n)(\varrho)\,\mathrm{d}\varrho\leq-\int_0^{\mathfrak{s}}(g'\circ\Delta\varkappa^n)(\varrho)\,\mathrm{d}\varrho\leq 2\mathring{d}.
\tag{68}
$$

Substituting (68) in (67), we conclude

$$
\begin{aligned}
&\left|\int_0^{\mathfrak{s}}\int_0^{\varrho}\int_{\Omega} g''(\varrho-\zeta)\Delta\varkappa^n(\zeta)\Delta\dot{\varkappa}^n(\varrho)\,\mathrm{d}y\,\mathrm{d}\zeta\,\mathrm{d}\varrho\right|\\
&\qquad\leq\sigma\xi_2\int_0^{\mathfrak{s}}\|\Delta\dot{\varkappa}^n(\varrho)\|_2^2\,\mathrm{d}\varrho\\
&\qquad\quad+2\frac{\xi_2}{\xi_3}(1-\vartheta)\left(1+\frac{1}{\sigma}\right)\mathring{d}+\xi_2\kappa_0^2(1+\sigma)(1-\vartheta)^2\delta.
\end{aligned}
\tag{69}
$$

Choosing $\beta=1$ in (35) and after using generalised Young's inequality with exponents $1/2p$, $p-1/2p$ and $1/2$, we will establish the following estimate for

$$
\int_0^{\mathfrak{s}}\int_{\Omega}\dot{\varkappa}^n(y,\varrho)|\varkappa^n(y,\varrho)|^{p-2}\ln|\varkappa^n(y,\varrho)|\ddot{\varkappa}^n(y,\varrho)\,\mathrm{d}y\,\mathrm{d}\varrho,
$$

indeed,

$$\begin{aligned}
&\int_0^{\mathfrak{s}}\int_\Omega \dot{\varkappa}^n(\mathrm{y},\varrho)|\varkappa^n(\mathrm{y},\varrho)|^{p-2}\ln|\varkappa^n(\mathrm{y},\varrho)|\ddot{\varkappa}^n(\mathrm{y},\varrho)\,\mathrm{dy}\,\mathrm{d}\varrho\\
&\quad\le \frac{1}{e(p-2)}\int_0^{\mathfrak{s}}\int_{\Omega_1}|\dot{\varkappa}^n(\mathrm{y},\varrho)||\ddot{\varkappa}^n(\mathrm{y},\varrho)|\,\mathrm{dy}\,\mathrm{d}\varrho\\
&\qquad+\int_0^{\mathfrak{s}}\int_{\Omega_2}|\dot{\varkappa}^n(\mathrm{y},\varrho)||\varkappa^n(\mathrm{y},\varrho)|^{p-1}|\ddot{\varkappa}^n(\mathrm{y},\varrho)|\,\mathrm{dy}\,\mathrm{d}\varrho\\
&\quad\le \frac{1}{4e(p-2)\sigma}\int_0^{\mathfrak{s}}\int_{\Omega_1}|\dot{\varkappa}^n(\mathrm{y},\varrho)|^2\,\mathrm{dy}\,\mathrm{d}\varrho\\
&\qquad+\frac{\sigma}{e(p-2)}\int_0^{\mathfrak{s}}\int_{\Omega_1}|\ddot{\varkappa}^n(\mathrm{y},\varrho)|^2\,\mathrm{dy}\,\mathrm{d}\varrho\\
&\qquad+\frac{1}{2}\int_0^{\mathfrak{s}}\int_{\Omega_2}|\ddot{\varkappa}^n(\mathrm{y},\varrho)|^2\,\mathrm{dy}\,\mathrm{d}\varrho+\frac{1}{2p}\int_0^{\mathfrak{s}}\int_{\Omega_2}|\dot{\varkappa}^n(\mathrm{y},\varrho)|^{2p}\,\mathrm{dy}\,\mathrm{d}\varrho\\
&\qquad+\frac{p-1}{2p}\int_0^{\mathfrak{s}}\int_{\Omega_2}|\varkappa^n(\mathrm{y},\varrho)|^{2p}\,\mathrm{dy}\,\mathrm{d}\varrho\\
&\quad\le c_*^2\frac{\delta}{4e(p-2)\sigma}\kappa_1^2+c_*^2\left(\frac{1}{2}+\frac{\sigma}{e(p-2)}\right)\int_0^{\mathfrak{s}}\|\nabla\ddot{\varkappa}^n(\varrho)\|_2^2\,\mathrm{d}\varrho\\
&\qquad+c_*^{2p}\frac{\delta}{2p}\kappa_1^{2p}+c_1^{2p}\frac{p-1}{2p}\kappa_0^{2p}.
\end{aligned}\tag{70}$$

On one hand, as estimate (63), we obtain

$$\begin{aligned}
&\int_0^{\mathfrak{s}}\int_\Omega \dot{\varkappa}(\mathrm{y},\varrho)|\varkappa^n(\mathrm{y},\varrho)|^{p-2}\ddot{\varkappa}^n(\mathrm{y},\varrho)\mathrm{dy}\,\mathrm{d}\varrho\\
&\qquad\le c_1^{p-2}\kappa_0^{p-2}\left\{\sigma c_1^2\int_0^{\mathfrak{s}}\|\Delta\dot{\varkappa}^n(\varrho)\|_2^2\,\mathrm{d}\varrho+\frac{1}{4\sigma}c_*^2\int_0^{\mathfrak{s}}\|\nabla\ddot{\varkappa}^n(\varrho)\|_2^2\,\mathrm{d}\varrho\right\},
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\int_0^{\mathfrak{s}}\int_\Omega \nabla\dagger^n(\mathrm{y},1,\varrho)\nabla\ddot{\varkappa}^n(\mathrm{y},\varrho)\,\mathrm{dy}\,\mathrm{d}\varrho\\
&\qquad\le\sigma\int_0^{\mathfrak{s}}\int_\Omega|\nabla\dagger^n(\mathrm{y},1,\varrho)|^2\,\mathrm{dy}\,\mathrm{d}\varrho+\frac{1}{4\sigma}\int_0^{\mathfrak{s}}\|\nabla\ddot{\varkappa}^n(\varrho)\|_2^2\,\mathrm{d}\varrho.
\end{aligned}\tag{71}$$

Using Hölder's inequality with exponents $1/2$, $1/l+1$ and $l-1/2(l+1)$, we obtain

$$\begin{aligned}
&\int_0^{\mathfrak{s}}\int_\Omega|\ddot{\varkappa}^n(\mathrm{y},\varrho)|^2\dot{\varkappa}^n(\mathrm{y},\varrho)|\dot{\varkappa}^n(\mathrm{y},\varrho)|^{l-2}\ddot{\varkappa}^n(\mathrm{y},\varrho)\,\mathrm{dy}\,\mathrm{d}\varrho\\
&\quad\le\int_0^{\mathfrak{s}}\left(\int_\Omega|\ddot{\varkappa}^n(\mathrm{y},\varrho)|^{2(l+1)}\,\mathrm{dy}\right)^{\frac{1}{l+1}}\left(\int_\Omega|\dot{\varkappa}^n(\mathrm{y},\varrho)|^{2(l+1)}\,\mathrm{dy}\right)^{\frac{l-1}{2(l+1)}}\\
&\qquad\times\left(\int_\Omega|\ddot{\varkappa}^n(\mathrm{y},\varrho)|^2\,\mathrm{dy}\right)^{\frac{1}{2}}\mathrm{d}\varrho\\
&\quad\le c_*^{l+1}\kappa_1^{l-1}\int_0^{\mathfrak{s}}\|\nabla\ddot{\varkappa}^n(\varrho)\|_2^3\,\mathrm{d}\varrho.
\end{aligned}$$

Testing (27) by $\alpha_{in}''(\mathfrak{s})$ and choosing $\mathfrak{s} = 0$, we obtain

$$\begin{aligned}\int_\Omega |\nabla \ddot{\varkappa}^n(\mathrm{y},0)|^2 = &\int_\Omega \varkappa_{0n}(\mathrm{y})|\varkappa_{0n}(\mathrm{y})|^{p-2}\ln|\varkappa_{0n}(\mathrm{y})|\ddot{\varkappa}^n(\mathrm{y},0)\,\mathrm{dy}\\ &-\int_\Omega |\varkappa_{1n}(\mathrm{y})|^l|\ddot{\varkappa}^n(\mathrm{y},0)|^2\,\mathrm{dy}\\ &-\int_\Omega |\nabla\varkappa_{0n}(\mathrm{y})|^{p-2}\nabla\varkappa_{0n}(\mathrm{y})\nabla\ddot{\varkappa}^n(\mathrm{y},0)\,\mathrm{dy}\\ &-\int_\Omega \Delta^2\varkappa_{0n}(\mathrm{y})\ddot{\varkappa}^n(\mathrm{y},0)\,\mathrm{dy}\\ &+\int_\Omega\int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\Delta^2\varkappa_{0n}(\zeta)\ddot{\varkappa}^n(\mathrm{y},0)\,\mathrm{dy}\\ &-\mu_1\int_\Omega \nabla\varkappa_{1n}(\mathrm{y})\nabla\ddot{\varkappa}^n(\mathrm{y},0)\,\mathrm{dy}-\mu_2\int_\Omega \nabla\top_{0n}\nabla\ddot{\varkappa}^n(\mathrm{y},0)\,\mathrm{dy}.\end{aligned} \tag{72}$$

Taking (29) and (30) into account with a large n equality (72) yields

$$\|\nabla\ddot{\varkappa}^n(\mathrm{y},0)\|_2^2 \le c_3. \tag{73}$$

We insert (63)–(65), (66), (69), (70), (71) and (73) in (62) to obtain

$$\begin{aligned}\frac{1}{2}\quad &\int_\Omega |\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l|\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s}|^2\,\mathrm{dy}\\ &+\left(\frac{1}{2}-g(0)\sigma-\sigma\right)\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^2+\left(\frac{1}{2}-c_2^{p-2}\kappa_0^{p-2}\sigma\right)\|\nabla\ddot{\varkappa}^n(\mathfrak{s})\|_2^2\\ &+\left(\mu_1-c_*^2(p-1)\left(\frac{1}{2}+\frac{\sigma}{e(p-2)}\right)-\frac{\mu_2}{4\sigma}-\frac{c_1^{p-2}\kappa_0^{p-2}c_*^2}{4\sigma}\right)\\ &\times\int_0^{\mathfrak{s}}\|\nabla\ddot{\varkappa}^n(\varrho)\|_2^2\,\mathrm{d}\varrho+\frac{\tau}{2}\int_0^1\int_\Omega|\nabla\dot{\top}^n(\mathrm{y},\eta,\mathfrak{s})|^2\,\mathrm{dy}\,\mathrm{d}\eta\\ &+\left(g(0)-\frac{g'(0)}{4\sigma}-\sigma\xi_2-c_1^{p-2}\kappa_0^{p-2}c_1^2\sigma\right)\int_0^{\mathfrak{s}}\|\Delta\dot{\varkappa}^n(\varrho)\|_2^2\,\mathrm{d}\varrho\\ &+\left(\frac{1}{2}-\mu_2\sigma\right)\int_0^{\mathfrak{s}}\int_\Omega|\nabla\dot{\top}^n(\mathrm{y},1,\varrho)|^2\,\mathrm{dy}\,\mathrm{d}\varrho\\ &\le A(\delta)+\frac{l}{2}c_*^{l+1}\kappa_1^{l-1}\int_0^{\mathfrak{s}}\|\nabla\ddot{\varkappa}^n(\varrho)\|_2^3\,\mathrm{d}\varrho.\end{aligned} \tag{74}$$

Now, we choose σ small enough so that

$$\frac{1}{2}-g(0)\sigma-\sigma>0,\quad g(0)-\frac{g'(0)}{4\sigma}-\sigma\xi_2-c_1^{p-2}\kappa_0^{p-2}c_1^2\sigma>0,\quad \frac{1}{2}-\mu_2\sigma>0. \tag{75}$$

After that, according to (10), we may choose μ_1 large enough such that

$$\mu_1-c_*^2(p-1)\left(\frac{1}{2}+\frac{\sigma}{e(p-2)}\right)-\frac{\mu_2}{4\sigma}-\frac{c_1^{p-2}\kappa_0^{p-2}c_*^2}{4\sigma}>0. \tag{76}$$

Putting $z(\mathfrak{s}) = \|\ddot{\varkappa}^n(\varrho)\|_2^2$, Inequality (74) becomes

$$z(\mathfrak{s}) \le C_1 + C_2\int_0^{\mathfrak{s}} z^{\frac{3}{2}}(\varrho)d\varrho. \tag{77}$$

Differentiating both sides of the equation $w(\mathfrak{s}) = C_1 + C_2\int_0^{\mathfrak{s}} w^{\frac{3}{2}}(\varrho)d\varrho$, we obtain

$$w'(\mathfrak{s}) = C_2 w^{\frac{3}{2}}(\mathfrak{s}),\qquad w(0) = C_1. \tag{78}$$

A simple integration of (78) gives

$$w(\mathfrak{s}) = \left[C_1^{\frac{-1}{2}} - \frac{1}{2}C_2\mathfrak{s}\right]^{-2}. \tag{79}$$

So, by a standard comparison theorem, we obtain $z(\mathfrak{s}) \leq w(\mathfrak{s})$. Although $w(\mathfrak{s})$ blows up in finite time, there exists a time $0 < \delta < \delta_n$ such that $z(\mathfrak{s}) \leq w(\mathfrak{s}) \leq \kappa + 1$, where $\mathfrak{s} \in [0, \delta]$ and $\kappa > 1$ is independent of n. Therefore, (74) and conditions (75) and (76) ensure that

$$\{\dot{\varkappa}^n\}_{n=1}^{\infty} \text{ is bounded in } L^{\infty}(0, T; H^2(\Omega) \cap H_0^1(\Omega)), \tag{80}$$

$$\{\ddot{\varkappa}^n\}_{n=1}^{\infty} \text{ is bounded in } L^{\infty}(0, \delta; H_0^1(\Omega)), \tag{81}$$

$$\{\dot{\top}^n\}_{n=1}^{\infty} \text{ is bounded in } L^{\infty}(0, \delta; H_0^1(\Omega \times [0,1])). \tag{82}$$

Applying Dunford–Pettis' Theorem, we observe from (81) and (82) that there exist subsequences $(\dot{\varkappa}^n)$ and $(\ddot{\varkappa}^n)$ such that

$$\dot{\varkappa}^n \rightharpoonup \dot{\varkappa} \text{ weakly star in } L^{\infty}(0, T; H^2(\Omega) \cap H_0^1(\Omega)), \tag{83}$$

$$\ddot{\varkappa}^n \rightharpoonup \ddot{\varkappa} \text{ weakly star in } L^{\infty}(0, \delta; H_0^1(\Omega)), \tag{84}$$

$$\dot{\top}^n \rightharpoonup \dot{\top} \text{ weakly star in } L^{\infty}(0, \delta; H_0^1(\Omega \times [0,1])). \tag{85}$$

Using (83) and (84) and [20], we deduce that $\dot{\varkappa}^n \to \dot{\varkappa}$ in $C([0, \delta], L^2(\Omega))$, for all $\mathrm{y} \in \bar{\Omega}$, and $\dot{\varkappa}^n \to \dot{\varkappa}$ a.e $\mathrm{y} \in \Omega$, $\forall \mathfrak{s} \geq 0$. Replacing $w_i(\mathrm{y})$ in (27) with $-\Delta w_i(\mathrm{y})$, multiplying by $\alpha_{in}'(\mathfrak{s})$ and summing over i from 1 to n, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\left\{\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2 + \|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^2\right\} \\
&\qquad - \int_\Omega |\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l \ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy} \\
&\qquad - \int_\Omega |\nabla\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\nabla\varkappa^n(\mathrm{y},\mathfrak{s})\nabla\Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy} \\
&\qquad - \int_\Omega \int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\nabla\Delta\varkappa^n(\zeta)\nabla\Delta\dot{\varkappa}^n(\mathfrak{s})\,\mathrm{d}\zeta\,\mathrm{dy} \\
&\qquad + \mu_1\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^2 + \mu_2\int_\Omega \Delta\top^n(\mathrm{y},1,\mathfrak{s})\Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy} \\
&= -\int_\Omega \varkappa^n(\mathrm{y},\mathfrak{s})|\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\ln|\varkappa^n(\mathrm{y},\mathfrak{s})|\Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}.
\end{aligned} \tag{86}$$

Integrating by parts and noting that $\dot{\varkappa}^n = 0$ on $\partial\Omega$, we have

$$\begin{aligned}
&-\int_\Omega |\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l \ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy} \\
&\qquad = l\int_\Omega |\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^{l-1}|\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy} \\
&\qquad\quad + \int_\Omega |\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l\nabla\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy},
\end{aligned} \tag{87}$$

and

$$\begin{aligned}
&l\int_\Omega \ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^{l-1}|\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\,\mathrm{dy} \\
&\qquad = \frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\left\{\int_\Omega |\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l|\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\,\mathrm{dy}\right\} \\
&\qquad\quad - 2\int_\Omega |\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l|\nabla\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}.
\end{aligned} \tag{88}$$

Inserting (88) in (87), we deduce

$$\begin{aligned}
&-\int_\Omega |\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l \dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy} \\
&\qquad = \frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\left\{\int_\Omega |\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l |\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\,\mathrm{dy}\right\} \\
&\qquad\quad -\int_\Omega |\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l \nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}.
\end{aligned} \tag{89}$$

On the other hand,

$$\begin{aligned}
&-\int_\Omega\int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\nabla\Delta\varkappa^n(\zeta)\nabla\Delta\dot{\varkappa}^n(\mathfrak{s})\,\mathrm{d}\zeta\,\mathrm{dy} \\
&\qquad = \frac{1}{2}g(\mathfrak{s})\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2 - \frac{1}{2}(g'\circ\nabla\Delta\varkappa^n)(\mathfrak{s}) \\
&\qquad\quad + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\left\{(g\circ\nabla\Delta\varkappa^n)(\mathfrak{s}) - \left(\int_0^{\mathfrak{s}} g(\zeta)\,\mathrm{d}\zeta\right)\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2\right\}.
\end{aligned} \tag{90}$$

Replacing $\phi_i(\mathrm{y},\eta)$ with $\Delta^2\phi_i(\mathrm{y},\eta)$, multiplying by $\beta_{in}(\mathfrak{s})$ and summing over i from 1 to n, we obtain

$$\frac{\tau}{2}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\int_\Omega |\Delta\top^n(\mathrm{y},\eta,\mathfrak{s})|^2\,\mathrm{dy}\,\mathrm{d}\eta + \frac{1}{2}\int_0^1 \frac{d}{\mathrm{d}\eta}\int_\Omega |\Delta\top^n(\mathrm{y},\eta,\mathfrak{s})|^2\,\mathrm{dy}\,\mathrm{d}\eta = 0. \tag{91}$$

Therefore, we obtain

$$\frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\int_0^1\int_\Omega |\Delta\top^n(\mathrm{y},\eta,\mathfrak{s})|^2\,\mathrm{dy}\,\mathrm{d}\eta + \frac{1}{2}\int_\Omega |\Delta\top^n(\mathrm{y},1,\mathfrak{s})|^2 - |\Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\,\mathrm{dy} = 0. \tag{92}$$

Inserting (89) and (90) in (86) and after summing the result with (92), we deduce

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\left\{\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2 + \|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^2 + 2\int_\Omega |\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l |\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\,\mathrm{dy}\right. \\
&\qquad \left. + (g\circ\nabla\Delta\varkappa^n)(\mathfrak{s}) - \left(\int_0^{\mathfrak{s}} g(\zeta)\,\mathrm{d}\zeta\right)\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2\right\} \\
&\qquad + \frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\int_0^1\int_\Omega |\Delta\top^n(\mathfrak{s},\eta,\mathfrak{s})|^2\,\mathrm{dy}\,\mathrm{d}\eta + \mu_1\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^2 \\
&\qquad + \frac{1}{2}g(\mathfrak{s})\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2 + \frac{1}{2}\int_\Omega |\Delta\top^n(\mathrm{y},1,\mathfrak{s})|^2\,\mathrm{dy} - \frac{1}{2}(g'\circ\nabla\Delta\varkappa^n)(\mathfrak{s}) \\
&= \frac{1}{2}\|\Delta\dot{\varkappa}(\mathfrak{s})\|_2^2 - \int_\Omega \varkappa^n(\mathrm{y},\mathfrak{s})|\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\ln|\varkappa^n(\mathrm{y},\mathfrak{s})|\Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy} \\
&\qquad - \int_\Omega |\nabla\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\nabla\varkappa^n(\mathrm{y},\mathfrak{s})\nabla\Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy} \\
&\qquad + \int_\Omega |\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l \nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy} \\
&\qquad - \mu_2\int_\Omega \Delta\top^n(\mathrm{y},1,\mathfrak{s})\Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}.
\end{aligned} \tag{93}$$

On the other hand, it is convenient to observe that

$$\begin{aligned}
&\int_\Omega |\nabla\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\nabla\varkappa^n(\mathrm{y},\mathfrak{s})\nabla\Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy} \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\left\{\int_\Omega |\nabla\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\nabla\varkappa^n(\mathrm{y},\mathfrak{s})\nabla\Delta\varkappa^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\right\} \\
&\qquad - (p-2)\int_\Omega |\nabla\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-4}(\nabla\varkappa^n(\mathrm{y},\mathfrak{s})\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s}))(\nabla\varkappa^n(\mathrm{y},\mathfrak{s})\nabla\Delta\varkappa^n(\mathrm{y},\mathfrak{s})))\,\mathrm{dy}
\end{aligned}$$

$$+\int_\Omega |\nabla \varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\nabla \dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\nabla\Delta\varkappa^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}.$$

Therefore, (93) is rewritten as

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\Big\{\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2+\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^2+2\int_\Omega|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l|\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\,\mathrm{dy}\\
&\quad+(g\circ\nabla\Delta\varkappa^n)(\mathfrak{s})-\Big(\int_0^{\mathfrak{s}}g(\zeta)\,\mathrm{d}\zeta\Big)\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2\Big\}\\
&\quad+\frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\int_0^1\int_\Omega|\Delta\top^n(\mathrm{y},\eta,\mathfrak{s})|^2\,\mathrm{dy}\,\mathrm{d}\eta\\
&\quad+\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\Big\{\int_\Omega|\nabla\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\nabla\varkappa^n(\mathrm{y},\mathfrak{s})\nabla\Delta\varkappa^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\Big\}\\
&\quad-\frac{1}{2}(g'\circ\nabla\Delta\varkappa^n)(\mathfrak{s})+\frac{1}{2}g(\mathfrak{s})\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2\\
&\quad+\mu_1\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^2+\frac{1}{2}\int_\Omega|\Delta\top^n(\mathrm{y},1,\mathfrak{s})|^2\,\mathrm{dy}\\
&=\frac{1}{2}\|\Delta\dot{\varkappa}(\mathfrak{s})\|_2^2-\int_\Omega\varkappa^n(\mathrm{y},\mathfrak{s})|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^{p-2}\ln|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|\Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\\
&\quad-\int_\Omega|\nabla\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\nabla\dot{\varkappa}^n(\mathrm{y},t\mathfrak{s})\nabla\Delta\varkappa^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\\
&\quad+(p-2)\int_\Omega|\nabla\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-4}(\nabla\varkappa^n(\mathrm{y},\mathfrak{s})\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s}))(\nabla\varkappa^n(\mathrm{y},\mathfrak{s})\nabla\Delta\varkappa^n(\mathrm{y},\mathfrak{s}))\,\mathrm{dy}\\
&\quad+\int_\Omega|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l\nabla\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}-\mu_2\int_\Omega\Delta\top^n(\mathrm{y},1,\mathfrak{s})\Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}.
\end{aligned}\tag{94}$$

Applying Young's inequality and using (80) and (81), we obtain

$$\begin{aligned}
&\Big|\int_\Omega|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l\nabla\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\Big|\\
&\qquad\le\frac{l}{2(l+1)}\int_\Omega|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^{2(l+1)}\,\mathrm{dy}\\
&\qquad\quad+\frac{1}{2}\int_\Omega|\nabla\ddot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\,\mathrm{dy}+\frac{1}{2(l+1)}\int_\Omega|\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^{2(l+1)}\,\mathrm{dy}\\
&\qquad\le\frac{l}{2(l+1)}c_1^{2(l+1)}\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^{2(l+1)}+\frac{1}{2}\|\nabla\ddot{\varkappa}^n(\mathfrak{s})\|_2^2\\
&\qquad\quad+\frac{c_2^{2(l+1)}}{2(l+1)}\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^{2(l+1)}\\
&\qquad\le\frac{\kappa_2^2}{2(l+1)}(lc_1^{2(l+1)}+c_2^{2(l+1)})+\frac{\kappa_3^2}{2},
\end{aligned}\tag{95}$$

and from (45), we obtain

$$\begin{aligned}
&\left|\int_\Omega |\nabla \varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\nabla \dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\nabla\Delta\varkappa^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\right| \\
&\qquad\le \frac{p-2}{2(p-1)}\int_\Omega |\nabla\varkappa^n(\mathrm{y},\mathfrak{s})|^{2(p-1)}\,\mathrm{dy} \\
&\qquad\quad+\frac{1}{2(p-1)}\int_\Omega |\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^{2(p-1)}\,\mathrm{dy}+\frac{1}{2}\int_\Omega |\nabla\Delta\varkappa^n(\mathrm{y},\mathfrak{s})|^2\,\mathrm{dy} \\
&\qquad\le \frac{p-2}{2(p-1)}c_2^{2(p-1)}\|\Delta\varkappa^n(\mathfrak{s})\|_2^{2(p-1)} \\
&\qquad\quad+\frac{c_2^{2(p-1)}}{2(p-1)}\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^{2(p-1)}+\frac{1}{2}\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2 \\
&\qquad\le \frac{p-2}{2(p-1)}c_2^{2(p-1)}\kappa_0^{2(p-1)}+\frac{c_2^{2(p-1)}}{2(p-1)}\kappa_2^{2(p-1)}+\frac{1}{2}\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2.
\end{aligned} \tag{96}$$

Moreover, we obtain

$$\begin{aligned}
&\left|\int_\Omega \varkappa^n(\mathrm{y},\mathfrak{s})|\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\ln|\varkappa^n(\mathrm{y},\mathfrak{s})|\Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\right| \\
&\qquad\le \int_\Omega |\varkappa^n(\mathrm{y},\mathfrak{s},\mathfrak{s})|^p\Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s},\mathfrak{s})\,\mathrm{dy},\mathfrak{s}+\frac{1}{e(p-1)}\int_\Omega \Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy} \\
&\qquad\le \|\varkappa^n(\mathfrak{s})\|_{2p}^p\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2+\frac{\sqrt{|\Omega|}}{e(p-1)}\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2 \\
&\qquad\le c_1^p\|\Delta\varkappa^n(\mathfrak{s})\|_2^p\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2+\frac{\sqrt{|\Omega|}}{e(p-1)}\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2 \\
&\qquad\le \kappa_2\left(c_1^p\kappa_0^p+\frac{\sqrt{|\Omega|}}{e(p-1)}\right),
\end{aligned} \tag{97}$$

and using (45) and (80), we obtain

$$\begin{aligned}
&\left|\int_\Omega |\nabla\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-4}\nabla\varkappa^n((\mathrm{y},\mathfrak{s})\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s}))(\nabla\varkappa^n(\mathrm{y},\mathfrak{s})\nabla\Delta\varkappa^n(\mathrm{y},\mathfrak{s}))\,\mathrm{dy}\right| \\
&\qquad\le \|\nabla\varkappa^n(\mathfrak{s})\|_{2(p-1)}^{p-2}.\|\nabla\dot{\varkappa}^n(\mathfrak{s})\|_{2(p-1)}\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2 \\
&\qquad\le c_2^{p-2}\|\Delta\varkappa^n(\mathfrak{s})\|_2^{p-2}\left(\sigma\|\nabla\dot{\varkappa}^n(\mathfrak{s})\|_{2(p-1)}^2+\frac{1}{4\sigma}\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2\right) \\
&\qquad\le c_2^{p-2}\kappa_0^{p-2}\left(c_2^2\sigma\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^2+\frac{1}{4\sigma}\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2\right) \\
&\qquad\le c_2^{p-2}\kappa_0^{p-2}\left(c_2^2\kappa_2^2\sigma+\frac{1}{4\sigma}\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2\right),
\end{aligned} \tag{98}$$

and

$$\begin{aligned}
\left|\int_\Omega \Delta\top^n(\mathrm{y},1,\mathfrak{s})\Delta\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\right| &\le \frac{1}{4\sigma}\int_\Omega |\Delta\top^n(\mathrm{y},1,\mathfrak{s})|^2\,\mathrm{dy}+\sigma\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^2 \\
&\le \frac{1}{4\sigma}\int_\Omega |\Delta\top^n(\mathrm{y},1,\mathfrak{s})|^2\,\mathrm{dy}+\sigma\kappa_2^2.
\end{aligned} \tag{99}$$

Substituting (95)–(99) in (94), we conclude

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\left\{\left(1-\int_0^{\mathfrak{s}} g(\zeta)\,\mathrm{d}\zeta\right)\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2+\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^2\right.\\
&\qquad\left.+2\int_\Omega|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l|\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\,\mathrm{dy}+(g\circ\nabla\Delta\varkappa^n)(\mathfrak{s})\right\}\\
&\qquad+\frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\int_0^1\int_\Omega|\Delta\top^n(\mathrm{y},\eta,\mathfrak{s})|^2\,\mathrm{dy}\,\mathrm{d}\eta\\
&\qquad+\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\left\{\int_\Omega|\nabla\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\nabla\varkappa^n(\mathrm{y},\mathfrak{s})\nabla\Delta\varkappa^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\right\}\\
&\qquad-\frac{1}{2}(g'\circ\nabla\Delta\varkappa^n)(\mathfrak{s})+\frac{1}{2}g(\mathfrak{s})\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2\\
&\qquad+\mu_1\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^2+\frac{1}{2}\int_\Omega|\Delta\top^n(\mathrm{y},1,\mathfrak{s})|^2\,\mathrm{dy}\\
&\quad\le A_2+\left(\frac{1}{2}+\frac{1}{4\sigma}\right)\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2+\frac{\mu_2}{4\sigma}\int_\Omega|\Delta\top^n(\mathrm{y},1,\mathfrak{s})|^2\,\mathrm{dy},
\end{aligned}\tag{100}$$

for some constant $A_2>0$. On the other hand, we observe that

$$\begin{aligned}
&-\int_\Omega|\nabla\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\nabla\varkappa^n(\mathrm{y},\mathfrak{s})\nabla\Delta\varkappa^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\\
&\qquad\le\|\nabla\varkappa^n(\mathfrak{s})\|_{2(p-1)}^{p-1}\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2\\
&\qquad\le\frac{c_1^{2(p-1)}}{2}\|\Delta\varkappa^n(\mathfrak{s})\|_2^{2(p-1)}+\frac{1}{2}\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2\\
&\qquad\le\frac{c_1^{2(p-1)}\kappa_0^{2(p-1)}}{2}+\frac{1}{2}\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2.
\end{aligned}$$

Therefore,

$$c_1^{2(p-1)}\kappa_0^{2(p-1)}+\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2+\int_\Omega|\nabla\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\nabla\varkappa^n(\mathrm{y},\mathfrak{s})\nabla\Delta\varkappa^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\ge 0.\tag{101}$$

Recalling (H1), using (100) and (101), we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\left\{\left(1-\int_0^{\mathfrak{s}} g(\zeta)\,\mathrm{d}\zeta\right)\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2+\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^2\right.\\
&\qquad\left.+2\int_\Omega|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l|\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\,\mathrm{dy}+(g\circ\nabla\Delta\varkappa^n)(\mathfrak{s})\right\}\\
&\qquad+\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\left\{\int_\Omega|\nabla\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\nabla\varkappa^n(\mathrm{y},\mathfrak{s})\nabla\Delta\varkappa^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\right\}\\
&\qquad+\frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\int_0^1\int_\Omega|\Delta\top^n(\mathrm{y},\eta,\mathfrak{s})|^2\,\mathrm{dy}\,\mathrm{d}\eta\\
&\qquad+\frac{\tilde{\xi}_3}{2}(g\circ\nabla\Delta\varkappa^n)(\mathfrak{s})+\mu_1\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^2\\
&\qquad+\left(\frac{1}{2}-\mu_2\frac{c_*^2}{4\sigma}\right)\int_\Omega|\Delta\top^n(\mathrm{y},1,\mathfrak{s})|^2\,\mathrm{dy}\\
&\quad\le A+\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2+\int_\Omega|\nabla\varkappa^n(\mathrm{y},\mathfrak{s})|^{p-2}\nabla\varkappa^n(\mathrm{y},\mathfrak{s})\nabla\Delta\varkappa^n(\mathrm{y},\mathfrak{s})\,\mathrm{dy}.
\end{aligned}\tag{102}$$

Choosing σ large enough such that $\frac{1}{2}-\mu_2\frac{c_*^2}{4\sigma}>0$, and after applying Gronwall's inequality, we obtain

$$\|\nabla\Delta\varkappa^n(\mathfrak{s})\|_2^2+\|\Delta\dot{\varkappa}^n(\mathfrak{s})\|_2^2+\int_\Omega|\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^l|\nabla\dot{\varkappa}^n(\mathrm{y},\mathfrak{s})|^2\,\mathrm{dy}$$

$$+ (g \circ \nabla \Delta \varkappa^n)(\mathfrak{s}) + \int_0^{\mathfrak{s}} \|\nabla \ddot{\varkappa}(\varrho)\|_2^2 \, \mathrm{d}\varrho \leq A(\delta).$$

At this step, similar as in [22], we have $\psi = |\nabla \varkappa|^{p-2} \nabla \varkappa$. Taking $\theta(\mathfrak{s}) \in \mathcal{D}(0, \delta)$ a nonnegative function, where $\mathcal{D}(0, \delta)$ is the space of C^∞-function with compact support in $(0, \delta)$, and multiplying (27) by $\theta(\mathfrak{s})$. Integrating the obtained result over $(0, \delta)$, we conclude

$$\begin{aligned}
&-\frac{1}{l+1} \int_0^\delta \int_\Omega |\dot{\varkappa}^n(\mathrm{y}, \mathfrak{s})|^l \dot{\varkappa}^n(\mathrm{y}, \mathfrak{s}) w_i(\mathrm{y}) \theta(\mathfrak{s}) \, \mathrm{dy} \, \mathrm{d}\mathfrak{s} \\
&\quad + \int_0^\delta \int_\Omega |\nabla \varkappa^n(\mathrm{y}, \mathfrak{s})|^{p-2} \nabla \varkappa^n(\mathrm{y}, \mathfrak{s}) \nabla w_i(\mathrm{y}) \theta(\mathfrak{s}) \, \mathrm{dy} \, \mathrm{d}\mathfrak{s} \\
&\quad + \int_0^\delta \int_\Omega \Delta \left(\varkappa^n(\mathrm{y}, \mathfrak{s}) - \int_0^{\mathfrak{s}} g(\mathfrak{s} - \zeta) \varkappa^n(\zeta) \, \mathrm{d}\zeta \right) \Delta w_i(\mathrm{y}) \theta(\mathfrak{s}) \, \mathrm{dy} \, \mathrm{d}\mathfrak{s} \\
&\quad + \mu_1 \int_0^\delta \int_\Omega \nabla \dot{\varkappa}^n(\mathrm{y}, \mathfrak{s}) \nabla w_i(\mathrm{y}) \theta(\mathfrak{s}) \, \mathrm{dy} \, \mathrm{d}\mathfrak{s} \\
&\quad + \mu_2 \int_0^\delta \int_\Omega \nabla \top^n(\mathrm{y}, 1, \mathfrak{s}) \nabla w_i(\mathrm{y}) \theta(\mathfrak{s}) \, \mathrm{dy} \, \mathrm{d}\mathfrak{s} \\
&\quad + \int_0^\delta \int_\Omega \nabla \ddot{\varkappa}^n(\mathrm{y}, \mathfrak{s}) \nabla w_i(\mathrm{y}) \theta(\mathfrak{s}) \, \mathrm{dy} \, \mathrm{d}\mathfrak{s} \\
&= \int_0^\delta \int_\Omega \varkappa^n(\mathrm{y}, \mathfrak{s}) |\varkappa^n(\mathrm{y}, \mathfrak{s})|^{p-2} \ln |\varkappa^n(\mathrm{y}, \mathfrak{s})| w_i(\mathrm{y}) \theta(\mathfrak{s}) \, \mathrm{dy} \, \mathrm{d}\mathfrak{s}.
\end{aligned} \tag{103}$$

Multiplying (28) by $\theta(\mathfrak{s})$ and integrating the result over $(0, \delta)$, we obtain

$$\begin{aligned}
&\tau \int_0^\delta \int_0^1 \int_\Omega \dot{\top}^n(\mathrm{y}, \eta, \mathfrak{s}) \phi_i(\mathrm{y}, \eta) \theta(\mathfrak{s}) \, \mathrm{dy} \, \mathrm{d}\eta \, \mathrm{d}\mathfrak{s} \\
&\qquad + \int_0^\delta \int_0^1 \int_\Omega \top_\eta(\mathrm{y}, \eta, \mathfrak{s}) \phi_i(\mathrm{y}, \eta) \theta(\mathfrak{s}) \, \mathrm{dy} \, \mathrm{d}\eta \, \mathrm{d}\mathfrak{s} = 0.
\end{aligned} \tag{104}$$

Recalling convergences (48)–(51), (53) and (83), we can pass to the limit as n goes to ∞ in (103) and (104) to obtain

$$\begin{aligned}
&-\frac{1}{l+1} \int_0^\delta \int_\Omega |\dot{\varkappa}(\mathrm{y}, \mathfrak{s})|^l \dot{\varkappa}(\mathrm{y}, \mathfrak{s}) w(\mathrm{y}) \theta(\mathfrak{s}) \, \mathrm{dy} \, \mathrm{d}\mathfrak{s} \\
&\quad + \int_0^\delta \int_\Omega |\nabla \varkappa(\mathrm{y}, \mathfrak{s})|^{p-2} \nabla \varkappa(\mathrm{y}, \mathfrak{s}) \nabla w(\mathrm{y}) \theta(\mathfrak{s}) \, \mathrm{dy} \, \mathrm{d}\mathfrak{s} \\
&\quad + \int_0^\delta \int_\Omega \Delta \left(\varkappa(\mathrm{y}, \mathfrak{s}) - \int_0^{\mathfrak{s}} g(\mathfrak{s} - \zeta) \varkappa(\zeta) \, \mathrm{d}\zeta \right) \Delta w(\mathrm{y}) \theta(\mathfrak{s}) \, \mathrm{dy} \, \mathrm{d}\mathfrak{s} \\
&\quad + \mu_1 \int_0^\delta \int_\Omega \nabla \dot{\varkappa}(\mathrm{y}, \mathfrak{s}) \nabla w(\mathrm{y}) \theta(\mathfrak{s}) \, \mathrm{dy} \, \mathrm{d}\mathfrak{s} \\
&\quad + \mu_2 \int_0^\delta \int_\Omega \nabla \top(\mathrm{y}, 1, \mathfrak{s}) \nabla w(\mathrm{y}) \theta(\mathfrak{s}) \, \mathrm{dy} \, \mathrm{d}\mathfrak{s} \\
&\quad + \int_0^\delta \int_\Omega \nabla \ddot{\varkappa}(\mathrm{y}, \mathfrak{s}) \nabla w(\mathrm{y}) \theta(\mathfrak{s}) \, \mathrm{dy} \, \mathrm{d}\mathfrak{s} \\
&= \int_0^\delta \int_\Omega \varkappa(\mathrm{y}, \mathfrak{s}) |\varkappa(\mathrm{y}, \mathfrak{s})|^{p-2} \ln |\varkappa(\mathrm{y}, \mathfrak{s})| w(\mathrm{y}) \theta(\mathfrak{s}) \, \mathrm{dy} \, \mathrm{d}\mathfrak{s},
\end{aligned}$$

and

$$\begin{aligned}
&\tau \int_0^\delta \int_0^1 \int_\Omega \dot{\top}(\mathrm{y}, \eta, \mathfrak{s}) \phi(\mathrm{y}, \eta) \theta(\mathfrak{s}) \, \mathrm{dy} \, \mathrm{d}\eta \, \mathrm{d}\mathfrak{s} \\
&\qquad + \int_0^\delta \int_0^1 \int_\Omega \top_\eta(\mathrm{y}, \eta, \mathfrak{s}) \phi_i(\mathrm{y}, \eta) \theta(\mathfrak{s}) \, \mathrm{dy} \, \mathrm{d}\eta \, \mathrm{d}\mathfrak{s} = 0.
\end{aligned}$$

Then, the local existence of weak solutions is established. □

3.2. General Decay of Global Solution

We will derive the decay rate of the energy functional for Cauchy problem (9) by Nakao's method, as in [23].

Theorem 2. *Assume that the assumptions* (H1)–(H3) *hold. Let* $\varkappa_0 \in W$ *be given. The energy associated to Problem* (9) *satisfies for a positive constant* $\sigma > 0$,

$$E(\mathfrak{s}) \leq \begin{cases} E(0)e^{-[\mathfrak{s}-1]^+ \ln\left(\frac{\sigma}{\sigma-1}\right)}, & l = 2, \\ \left(E(0)^{-\frac{l}{2}} + \sigma^{-1}\frac{l}{2}[\mathfrak{s}-1]^+\right)^{\frac{-2}{l}}, & l > 2, \end{cases} \tag{105}$$

Proof. By integrating (15) over $[\mathfrak{s}, \mathfrak{s}+1]$, we obtain

$$\begin{aligned} B^2(\mathfrak{s}) := &-\frac{1}{2}\int_{\mathfrak{s}}^{\mathfrak{s}+1}(g'\circ\Delta u)(\mathfrak{s})\,\mathrm{d}\mathfrak{s} + \frac{1}{2}\int_{\mathfrak{s}}^{\mathfrak{s}+1} g(\mathfrak{s})\|\Delta\varkappa(\mathfrak{s})\|_2^2\,\mathrm{d}\mathfrak{s} \\ &+ \left(\mu_1 - \frac{\xi}{\tau} - \frac{|\mu_2|}{2}\right)\int_{\mathfrak{s}}^{\mathfrak{s}+1}\|\nabla\dot{\varkappa}(\mathfrak{s})\|_2^2\,\mathrm{d}\mathfrak{s} \\ &+ \left(\frac{\xi}{\tau} - \frac{|\mu_2|}{2}\right)\int_{\mathfrak{s}}^{\mathfrak{s}+1}\|\nabla\top(\mathrm{y},1,\mathfrak{s})\|_2^2\,\mathrm{d}\mathfrak{s}, \end{aligned} \tag{106}$$

where $B^2(\mathfrak{s}) = E(\mathfrak{s}) - E(\mathfrak{s}+1)$. By (106), we observe that

$$\begin{aligned} \int_{\mathfrak{s}}^{\mathfrak{s}+1}\int_\Omega |\dot{\varkappa}(\mathrm{y},\mathfrak{s})|^{l+2}\,\mathrm{dy}\,\mathrm{d}\mathfrak{s} &+ \left(\mu_1 - \frac{\xi}{\tau} - \frac{|\mu_2|}{2}\right)\int_{\mathfrak{s}}^{\mathfrak{s}+1}\|\nabla\dot{\varkappa}(\mathfrak{s})\|_2^2\,\mathrm{d}\mathfrak{s} \\ &+ \left(\frac{\xi}{\tau} - \frac{|\mu_2|}{2}\right)\int_{\mathfrak{s}}^{\mathfrak{s}+1}\|\nabla\top(\mathrm{y},1,\mathfrak{s})\|_2^2\,\mathrm{d}\mathfrak{s} \\ &\leq c_*^{l+2}\int_{\mathfrak{s}}^{\mathfrak{s}+1}\|\nabla\dot{\varkappa}(\mathfrak{s})\|_2^2\,\mathrm{d}\mathfrak{s} \\ &\quad + \left(\mu_1 - \frac{\xi}{\tau} - \frac{|\mu_2|}{2}\right)\int_{\mathfrak{s}}^{\mathfrak{s}+1}\|\nabla\dot{\varkappa}(\mathfrak{s})\|_2^2\,\mathrm{d}\mathfrak{s} \\ &\quad + \left(\frac{\xi}{\tau} - \frac{|\mu_2|}{2}\right)\int_{\mathfrak{s}}^{\mathfrak{s}+1}\|\nabla\top(\mathrm{y},1,\mathfrak{s})\|_2^2\,\mathrm{d}\mathfrak{s} \\ &\leq \sigma(\Omega)B^2(\mathfrak{s}). \end{aligned}$$

By mean value theorem, there exists $\mathfrak{s}_1 \in \left[\mathfrak{s}, \mathfrak{s}+\frac{1}{4}\right]$ and $\mathfrak{s}_2 \in \left[\mathfrak{s}+\frac{3}{4}, \mathfrak{s}+1\right]$ such that

$$\|\dot{\varkappa}(\mathfrak{s}_i)\|_{l+2}^{l+2} + \|\nabla\dot{\varkappa}(\mathfrak{s}_i)\|_2^2 + \|\nabla\top(\mathrm{y},1,\mathfrak{s}_i)\|_2^2 \leq 4\sigma(\Omega)B^2(\mathfrak{s}), \qquad i = 1,2. \tag{107}$$

Multiplying the first equation of (9) by $\varkappa(\mathfrak{s})$ and integrating by parts the result over $\Omega \times [\mathfrak{s}_1, \mathfrak{s}_2]$, we obtain

$$\begin{aligned}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_\Omega |\dot{\varkappa}(\mathrm{y},\mathfrak{s})|^l \ddot{\varkappa}(\mathrm{y},\mathfrak{s})\varkappa(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\,\mathrm{d}\mathfrak{s} &+ \int_{\mathfrak{s}_1}^{\mathfrak{s}_2} \|\nabla\varkappa(\mathfrak{s})\|_p^p\,\mathrm{d}\mathfrak{s}\\ &+ \int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_\Omega \Delta^2\left(\varkappa(\mathrm{y},\mathfrak{s}) - \int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\varkappa(\zeta)\,\mathrm{d}\zeta\right)\varkappa(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\,\mathrm{d}\mathfrak{s}\\ &- \mu_1\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_\Omega \Delta\dot{\varkappa}(\mathrm{y},\mathfrak{s})\varkappa(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\,\mathrm{d}\mathfrak{s}\\ &- \mu_2\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_\Omega \Delta\top(\mathrm{y},1,\mathfrak{s})\varkappa(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\,\mathrm{d}\mathfrak{s}\\ &- \int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_\Omega \Delta\ddot{\varkappa}(\mathrm{y},\mathfrak{s})\varkappa(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\,\mathrm{d}\mathfrak{s}\\ &= \int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_\Omega |\varkappa(\mathrm{y},\mathfrak{s})|^p \ln|\varkappa(\mathrm{y},\mathfrak{s})|\,\mathrm{dy}\,\mathrm{d}\mathfrak{s}.\end{aligned} \tag{108}$$

By integrating by parts and using Cauchy–Schwarz inequality, we deduce

$$\begin{aligned}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_\Omega |\dot{\varkappa}(\mathrm{y},\mathfrak{s})|^l \ddot{\varkappa}(\mathrm{y},\mathfrak{s})\varkappa(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\,\mathrm{d}\mathfrak{s} &\le \frac{1}{1+l}\sum_{i=1}^{2}\|\varkappa(\mathfrak{s}_i)\|_{2+l}\|\dot{\varkappa}(\mathfrak{s}_i)\|_{2+l}^{1+l}\\ &\quad + \frac{1}{1+l}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\dot{\varkappa}(\mathfrak{s})\|_{l+2}^{l+2}\,\mathrm{d}\mathfrak{s},\end{aligned}$$

$$\begin{aligned}-\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_\Omega \Delta\ddot{\varkappa}(\mathrm{y},\mathfrak{s})\varkappa(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\,\mathrm{d}\mathfrak{s} &\le \sum_{i=1}^{2}\|\nabla\dot{\varkappa}(\mathfrak{s}_i)\|_2\|\nabla\varkappa(\mathfrak{s}_i)\|_2 + \int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\nabla\dot{\varkappa}(\mathfrak{s})\|_2^2\,\mathrm{d}\mathfrak{s},\\ &\quad - \int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_\Omega \Delta\dot{\varkappa}(\mathrm{y},\mathfrak{s})\varkappa(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\,\mathrm{d}\mathfrak{s}\\ &\le \int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\nabla\dot{\varkappa}(\mathfrak{s})\|_2\|\nabla u(\mathfrak{s})\|_2\end{aligned}$$

$$-\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_\Omega \Delta\top(\zeta,1,\mathfrak{s})\varkappa(\mathrm{y},\mathfrak{s})\,\mathrm{dy}\,\mathrm{d}\mathfrak{s} \le \int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\nabla\top(\mathrm{y},1,\mathfrak{s})\|_2\|\nabla\varkappa(\mathfrak{s})\|_2\,\mathrm{d}\mathfrak{s}. \tag{109}$$

We notice that

$$\begin{aligned}-\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_\Omega \Delta^2\varkappa(\mathrm{y},\mathfrak{s})\int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\varkappa(\mathrm{y},\zeta)\,\mathrm{dy}\,\mathrm{d}\zeta\,\mathrm{d}\mathfrak{s}&\\ = \int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_\Omega \Delta\varkappa(\mathrm{y},\mathfrak{s})\int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)(\Delta\varkappa(\mathrm{y},\mathfrak{s}) - \Delta\varkappa(\mathrm{y},\zeta))\,\mathrm{dy}\,\mathrm{d}\zeta\,\mathrm{d}\mathfrak{s}&\\ - \int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\Delta\varkappa(\mathfrak{s})\|_2^2\int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\,\mathrm{d}\zeta\,\mathrm{d}\mathfrak{s},&\end{aligned} \tag{110}$$

and

$$\begin{aligned}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_\Omega \Delta\varkappa(\mathrm{y},\mathfrak{s})\int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)(\Delta\varkappa(\mathrm{y},\mathfrak{s}) - \Delta\varkappa(\mathrm{y},\zeta))\,\mathrm{dy}\,\mathrm{d}\zeta\,\mathrm{d}\mathfrak{s}&\\ = -\frac{1}{2}\left\{\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\left(\|\Delta\varkappa(\mathfrak{s})\|_2^2 + \|\Delta\varkappa(\zeta)\|_2^2\right)\mathrm{d}\zeta\,\mathrm{d}\mathfrak{s}\right.&\\ \left. - \int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\left(\|\Delta\varkappa(\mathfrak{s}) - \Delta\varkappa(\zeta)\|_2^2\right)\mathrm{d}\zeta\,\mathrm{d}\mathfrak{s}\right\}&\\ + \int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\Delta\varkappa(\mathfrak{s})\|_2^2\int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\,\mathrm{d}\zeta\,\mathrm{d}\mathfrak{s}.&\end{aligned} \tag{111}$$

Combining (110) and (111), we obtain

$$\begin{aligned}
&-\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_\Omega \Delta^2\varkappa(\mathrm{y},\mathfrak{s})\int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\varkappa(\mathrm{y},\zeta)\,\mathrm{dy}\,\mathrm{d}\zeta\,\mathrm{d}\mathfrak{s}\\
&\quad=-\frac{1}{2}\Bigg\{\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\Big(\|\Delta\varkappa(\mathfrak{s})\|_2^2+\|\Delta\varkappa(\zeta)\|_2^2\Big)\,\mathrm{d}\zeta\,\mathrm{d}\mathfrak{s}\\
&\qquad-\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\Big(\|\Delta\varkappa(\mathfrak{s})-\Delta\varkappa(\zeta)\|_2^2\Big)\,\mathrm{d}\zeta\,\mathrm{d}\mathfrak{s}\Bigg\}\\
&\quad=-\frac{1}{2}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\Delta\varkappa(\mathfrak{s})\|_2^2\int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\,\mathrm{d}\zeta\,\mathrm{d}\mathfrak{s}\\
&\qquad-\frac{1}{2}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\|\Delta\varkappa(\zeta)\|_2^2\,\mathrm{d}\zeta\,\mathrm{d}\mathfrak{s}+\frac{1}{2}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}(g\circ\Delta\varkappa)(\mathfrak{s})\,\mathrm{d}\mathfrak{s}.
\end{aligned}\tag{112}$$

Substituting (109) and (112) in (108), adding and subtracting the term

$$\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\Big(1-\int_0^{\mathfrak{s}} g(\zeta)\,\mathrm{d}\zeta\Big)\|\Delta\varkappa(\mathfrak{s})\|_2^2\,\mathrm{d}\mathfrak{s},$$

we conclude

$$\begin{aligned}
\int_{\mathfrak{s}_1}^{\mathfrak{s}_2} I(\mathfrak{s})\,\mathrm{d}\mathfrak{s}&=\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\Big(1-\int_0^{\mathfrak{s}} g(\zeta)\,\mathrm{d}\zeta\Big)\|\Delta\varkappa(\mathfrak{s})\|_2^2\,\mathrm{d}\mathfrak{s}\\
&\quad+\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\nabla\varkappa(\mathfrak{s})\|_p^p\,\mathrm{d}\mathfrak{s}-\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_\Omega|\varkappa(\mathrm{y},\mathfrak{s})|^p\ln|\varkappa(\mathrm{y},\mathfrak{s})|\,\mathrm{dy}\,\mathrm{d}\mathfrak{s}\\
&\le\frac{1}{1+l}\sum_{i=1}^{2}\|\varkappa(\mathfrak{s}_i)\|_{2+l}\|\dot\varkappa(\mathfrak{s}_i)\|_{2+l}^{1+l}+\frac{1}{1+l}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\dot\varkappa(\mathfrak{s})\|_{l+2}^{l+2}\,\mathrm{d}\mathfrak{s}\\
&\quad+\sum_{i=1}^{2}\|\nabla\dot\varkappa(\mathfrak{s}_i)\|_2\|\nabla\varkappa(\mathfrak{s}_i)\|_2+\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\nabla\dot\varkappa(\mathfrak{s})\|_2^2\,\mathrm{d}\mathfrak{s}\\
&\quad+\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\nabla\dot\varkappa(\mathfrak{s})\|_2\|\nabla\varkappa(\mathfrak{s})\|_2\,\mathrm{d}\mathfrak{s}\\
&\quad+\int_{\varkappa_1}^{\varkappa_2}\|\nabla\top(\mathrm{y},1,\mathfrak{s})\|_2\|\nabla\varkappa(\mathfrak{s})\|_2\,\mathrm{d}\mathfrak{s}\\
&\quad+\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\Big(1-\int_0^{\mathfrak{s}} g(\zeta)\,\mathrm{d}\zeta\Big)\|\Delta\varkappa(\mathfrak{s})\|_2^2\,\mathrm{d}\mathfrak{s}\\
&\quad+\frac{1}{2}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\Delta\varkappa(\mathfrak{s})\|_2^2\int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\,\mathrm{d}\zeta\,\mathrm{d}\mathfrak{s}\\
&\quad+\frac{1}{2}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_0^{\mathfrak{s}} g(\mathfrak{s}-\tau)\|\Delta\varkappa(\tau)\|_2^2\,\mathrm{d}\tau\,\mathrm{d}\eta+\frac{1}{2}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}(g\circ\Delta\varkappa)(\mathfrak{s})\,\mathrm{d}\mathfrak{s}.
\end{aligned}\tag{113}$$

Moreover, by (21), we have

$$\begin{aligned}
E(\mathfrak{s})&=\frac{1}{l+2}\|\dot\varkappa(\mathfrak{s})\|_{l+2}^{l+2}+\frac{1}{2}\|\nabla\dot\varkappa(\mathfrak{s})\|_2^2+\frac{1}{2}(g\circ\Delta\varkappa)(\mathfrak{s})\\
&\quad+\xi\int_\Omega\int_0^1|\nabla\top(\mathrm{y},\eta,\mathfrak{s})|^2\,\mathrm{dy}\,\mathrm{d}\zeta\\
&\quad+\Big(\frac{1}{2}-\frac{1}{p}\Big)\Big(1-\int_0^{\mathfrak{s}} g(\zeta)\,\mathrm{d}\zeta\Big)\|\Delta\varkappa(\mathfrak{s})\|_2^2+\frac{1}{p^2}\|\varkappa\|_p^p+\frac{1}{p}I(\varkappa).
\end{aligned}\tag{114}$$

Now, we will estimate the right-hand side of (113). Using (107), (114) and Lemma 2, we obtain

$$\|\dot\varkappa(\mathfrak{s}_i)\|_{l+2}^{l+1}=\|\dot\varkappa(\mathfrak{s}_i)\|_{l+2}^{l-1}\|\dot\varkappa(\mathfrak{s}_i)\|_{l+2}^2\le c_*^{l-1}(l+2)^{\frac{2}{l+2}}E^{\frac{2}{l+2}}(0)\|\nabla\dot\varkappa(\mathfrak{s}_i)\|_2^{l-1},$$

and

$$\|\varkappa(\mathfrak{s}_i)\|_{l+2} \leq c_1\|\Delta\varkappa(\mathfrak{s}_i)\|_2 \leq c_1\sqrt{\frac{2p}{\vartheta(p-2)}}\sup_{\mathfrak{s}_1\leq\eta\leq\mathfrak{s}_2} E^{\frac{1}{2}}(\mathfrak{s}).$$

Therefore,

$$\begin{aligned}\|\varkappa(\mathfrak{s}_i)\|_{l+2}\|\dot{\varkappa}(\mathfrak{s}_i)\|_{l+2}^{l+1} &\leq 4^{\frac{l-1}{2}}c_1\sqrt{\frac{2p}{\vartheta(p-2)}}c_*^{l-1}\sigma^{\frac{l-1}{2}}(\Omega)\\ &\quad\times((l+2)E(0))^{\frac{2}{l+2}}B^{l-1}(\mathfrak{s})\sup_{\mathfrak{s}_1\leq\eta\leq\mathfrak{s}_2}E^{\frac{1}{2}}(\mathfrak{s}),\end{aligned} \tag{115}$$

and

$$\|\nabla\dot{\varkappa}(\mathfrak{s}_i)\|_2\|\nabla\varkappa(\mathfrak{s}_i)\|_2 \leq 4c_2\sqrt{\frac{\sigma(\Omega)p}{\vartheta(p-2)}}B(\mathfrak{s})\sup_{\mathfrak{s}_1\leq\eta\leq\mathfrak{s}_2}E^{\frac{1}{2}}(\mathfrak{s}). \tag{116}$$

Moreover, we have

$$\begin{aligned}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\nabla\dot{\varkappa}(\mathfrak{s})\|_2^2\,\mathrm{d}\mathfrak{s} &\leq \frac{\sigma(\Omega)}{\left(\mu_1-\frac{\xi}{\tau}-\frac{|\mu_2|}{2}\right)}B^2(\mathfrak{s})\\ \int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\dot{\varkappa}(\mathfrak{s})\|_{l+2}^{l+2}\,\mathrm{d}\eta &\leq \sigma(\Omega)B^2(\mathfrak{s}).\end{aligned} \tag{117}$$

By using Hölder's inequality, we observe that

$$\begin{aligned}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\nabla\dot{\varkappa}(\mathfrak{s})\|_2\|\nabla\varkappa(\mathfrak{s})\|_2\,\mathrm{d}\mathfrak{s} &\leq \left(\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\nabla\dot{\varkappa}(\mathfrak{s})\|_2^2\,\mathrm{d}\mathfrak{s}\right)^{\frac{1}{2}}\left(\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\nabla\varkappa(\mathfrak{s})\|_2^2\,\mathrm{d}\mathfrak{s}\right)^{\frac{1}{2}}\\ &\leq c_2\sqrt{2\frac{p\sigma(\Omega)}{\vartheta\left(\mu_1-\frac{\xi}{\tau}-\frac{|\mu_2|}{2}\right)(p-2)}}B(\mathfrak{s})\sup_{\mathfrak{s}_1\leq\eta\leq\mathfrak{s}_2}E^{\frac{1}{2}}(\mathfrak{s}).\end{aligned} \tag{118}$$

Similarly, we obtain

$$\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\nabla\top(\mathrm{y},1,\mathfrak{s})\|_2\|\nabla\varkappa(\mathfrak{s})\|_2\,\mathrm{d}\mathfrak{s} \leq c_2\sqrt{2\frac{p\sigma(\Omega)}{\vartheta\left(\frac{\xi}{\tau}-\frac{|\mu_2|}{2}\right)(p-2)}}B(\mathfrak{s})\sup_{\mathfrak{s}_1\leq\eta\leq\mathfrak{s}_2}E^{\frac{1}{2}}(\mathfrak{s}). \tag{119}$$

Applying Young's inequality for convolution $\|\phi*\psi\|_q \leq \|\phi\|_r\|\psi\|_s$,

$$\frac{1}{q}=\frac{1}{r}+\frac{1}{s}-1, \qquad 1\leq q,r,s\leq\infty,$$

and recalling (19), (23) and (114), we obtain

$$\begin{aligned}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_0^{\mathfrak{s}}g(\mathfrak{s}-\zeta)\|\Delta\varkappa(\zeta)\|_2^2\,\mathrm{d}\zeta\,\mathrm{d}\mathfrak{s} &\leq \int_{\mathfrak{s}_1}^{\mathfrak{s}_2}g(\mathfrak{s})\,\mathrm{d}\mathfrak{s}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\Delta\varkappa(\mathfrak{s})\|_2^2\,\mathrm{d}\mathfrak{s}\\ &\leq \frac{1-\vartheta}{\vartheta}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\left(I(\mathfrak{s})+\int_\Omega|\varkappa(\mathrm{y},\mathfrak{s})|^p\ln|\varkappa(\mathrm{y},\mathfrak{s})|\,\mathrm{d}\mathrm{y}\right)\mathrm{d}\mathfrak{s}\\ &\leq \frac{1-\vartheta}{\vartheta}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\left(I(\mathfrak{s})+c_1^{p+\beta}\|\Delta\varkappa(\mathfrak{s})\|_2^{p+\beta}\right)\mathrm{d}\mathfrak{s}\\ &\leq \frac{1-\vartheta}{\vartheta}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}I(\mathfrak{s})\,\mathrm{d}\mathfrak{s}+c_1^{p+\beta}\frac{1-\vartheta}{\vartheta}\left(\frac{2p}{\vartheta(p-2)}\right)^{\frac{p+\beta}{2}}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}E^{\frac{p+\beta}{2}}(\mathfrak{s})\,\mathrm{d}\mathfrak{s}.\end{aligned}$$

Taking $\beta = 2$ and noting that $\mathfrak{s}_2 - \mathfrak{s}_1 \leq 1$, we have

$$\begin{aligned}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_0^{\mathfrak{s}} & g(\mathfrak{s}-\zeta)\|\Delta\varkappa(\zeta)\|_2^2\,\mathrm{d}\zeta\,\mathrm{d}\mathfrak{s}\\ &\leq \frac{1-\vartheta}{\vartheta}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2} I(\mathfrak{s})\,\mathrm{d}\mathfrak{s} + c_1^{p+2}\frac{1-\vartheta}{\vartheta}\left(\frac{2p}{\vartheta(p-2)}\right)^{\frac{p+2}{2}} E^{\frac{p}{2}}(0)\sup_{\mathfrak{s}_1\leq\eta\leq\mathfrak{s}_2} E(\mathfrak{s}).\end{aligned} \tag{120}$$

On the other hand,

$$\begin{aligned}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\Delta\varkappa(\mathfrak{s})\|_2^2\int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\,\mathrm{d}\zeta\,\mathrm{d}\mathfrak{s} &\leq \int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\frac{2p}{\vartheta(p-2)}E(\mathfrak{s})\int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\,\mathrm{d}\zeta\,\mathrm{d}\mathfrak{s}\\ &\leq \frac{2p}{\vartheta(p-2)}(1-\vartheta)\sup_{\mathfrak{s}_1\leq\eta\leq\mathfrak{s}_2} E(\mathfrak{s}).\end{aligned} \tag{121}$$

Exploiting (120), (121) and the fact $(x-y)^2 \leq 2(x^2+y^2)$ for any $x, y \in \mathbb{R}$, we arrive at

$$\begin{aligned}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}(g\circ\Delta\varkappa)(\mathfrak{s})\,\mathrm{d}\mathfrak{s} &= \int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\|\Delta\varkappa(\zeta)-\Delta\varkappa(\eta)\|_2^2\,\mathrm{d}\zeta\,\mathrm{d}\mathfrak{s}\\ &\leq 2\Bigg\{\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\|\Delta\varkappa(\zeta)\|_2^2\,\mathrm{d}\zeta\,\mathrm{d}\mathfrak{s}\\ &\quad+\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\Delta\varkappa(\mathfrak{s})\|_2^2\int_0^{\mathfrak{s}} g(\mathfrak{s}-\zeta)\,\mathrm{d}\zeta\,\mathrm{d}\mathfrak{s}\Bigg\}\\ &\leq 2\frac{1-\vartheta}{\vartheta}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2} I(\mathfrak{s})\,\mathrm{d}\mathfrak{s} + \frac{2(1-\vartheta)}{\vartheta}\Bigg\{c_1^{p+2}\left(\frac{2p}{\vartheta(p-2)}\right)^{\frac{p+2}{2}} E^{\frac{p}{2}}(0)\\ &\quad+\frac{2p}{p-2}\Bigg\}\sup_{\mathfrak{s}_1\leq\eta\leq\mathfrak{s}_2} E(\mathfrak{s}).\end{aligned} \tag{122}$$

Replacing (115)–(119), (120)–(122) in (113), we can infer

$$\begin{aligned}\frac{2\vartheta-1}{\vartheta}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2} I(\mathfrak{s})\,\mathrm{d}\mathfrak{s} \leq \sigma_0\Big\{&\Big(B^{l-1}(\mathfrak{s})+B(\mathfrak{s})\Big)\sup_{\mathfrak{s}_1\leq\eta\leq\mathfrak{s}_2} E^{\frac{1}{2}}(\mathfrak{s})\\ &+B^2(\mathfrak{s})+\sup_{\mathfrak{s}_1\leq\eta\leq\mathfrak{s}_2} E(\mathfrak{s})\Big\}.\end{aligned} \tag{123}$$

On the other hand, by integrating (114) over $(\mathfrak{s}_1, \mathfrak{s}_2)$ and using (123), we obtain

$$\begin{aligned}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2} E(\mathfrak{s})\,\mathrm{d}\mathfrak{s} &= \frac{1}{l+2}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\dot{\varkappa}(\mathfrak{s})\|_{l+2}^{l+2}\,\mathrm{d}\mathfrak{s}+\frac{1}{2}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\nabla\dot{\varkappa}(\mathfrak{s})\|_2^2\,\mathrm{d}\mathfrak{s}\\ &\quad+\frac{1}{2}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}(g\circ\Delta\varkappa)(\mathfrak{s})\,\mathrm{d}\mathfrak{s}+\xi\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\int_\Omega\int_0^1|\nabla\top(\mathrm{y},\eta,\mathfrak{s})|^2\,\mathrm{d}\eta\,\mathrm{d}\mathrm{y}\,\mathrm{d}\mathfrak{s}\\ &\quad+\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\left(1-\int_0^{\mathfrak{s}} g(\zeta)\,\mathrm{d}\zeta\right)\|\Delta\varkappa(\mathfrak{s})\|_2^2\,\mathrm{d}\mathfrak{s}\\ &\quad+\frac{1}{p^2}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\|\varkappa(\mathfrak{s})\|_p^p\,\mathrm{d}\mathfrak{s}+\frac{1}{p}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2} I(\mathfrak{s})\,\mathrm{d}\mathfrak{s}\\ &\leq \sigma_0'\Big\{\Big(B^{l-1}(\mathfrak{s})+B(\mathfrak{s})\Big)\sup_{\mathfrak{s}\leq\eta\leq\mathfrak{s}+1} E^{\frac{1}{2}}(\mathfrak{s})+B^2(\mathfrak{s})+\sup_{\mathfrak{s}\leq\eta\leq\mathfrak{s}+1} E(\mathfrak{s})\Big\}.\end{aligned} \tag{124}$$

Because $\mathfrak{s}_2 - \mathfrak{s}_1 \geq 1/2$, we conclude by applying the mean theorem that

$$E(\mathfrak{s}) \leq 2(\mathfrak{s}_2-\mathfrak{s}_1)E(\mathfrak{s}_2) \leq 2\int_{\mathfrak{s}_1}^{\mathfrak{s}_2} E(\mathfrak{s})d\mathfrak{s}. \tag{125}$$

Morever, by integrating (16) over $(\mathfrak{s}, \mathfrak{s}_2)$, recalling (106) and using (125), we have

$$\begin{aligned} E(\mathfrak{s}) &\leq E(\mathfrak{s}_2) - \frac{1}{2}\int_{\mathfrak{s}}^{\mathfrak{s}_2} (g' \circ \Delta\varkappa)(\mathfrak{s})\, \mathrm{d}\mathfrak{s} + \frac{1}{2}\int_{\mathfrak{s}}^{\mathfrak{s}_2} g(\mathfrak{s}) \|\nabla\varkappa(\mathfrak{s})\|_2^2\, \mathrm{d}\mathfrak{s} \\ &\quad + k_0 \int_{\mathfrak{s}}^{\mathfrak{s}_2} \|\nabla\dot{\varkappa}(\mathfrak{s})\|_2^2\, \mathrm{d}\mathfrak{s} + k_1 \int_{\mathfrak{s}}^{\mathfrak{s}_2} \|\nabla\top(\mathrm{y}, 1, \mathfrak{s})\|_2^2\, \mathrm{d}\mathfrak{s} \\ &\leq 2\int_{\mathfrak{s}_1}^{\mathfrak{s}_2} E(\mathfrak{s})\, \mathrm{d}\mathfrak{s} + B^2(\mathfrak{s}). \end{aligned} \tag{126}$$

Combining (124) with (126) and using the nonincreasing property of $E(\mathfrak{s})$, we obtain

$$E(\mathfrak{s}) \leq \sigma_1' \Big\{ \Big(B^{l-1}(\mathfrak{s}) + B(\mathfrak{s}) \Big) E^{\frac{1}{2}}(\mathfrak{s}) + B^2(\mathfrak{s}) + E(\mathfrak{s}) \Big\}.$$

After utilizing Young's inequality, we have

$$E(\mathfrak{s}) \leq \sigma \Big\{ B^{2(l-1)}(\mathfrak{s}) + B^2(\mathfrak{s}) \Big\}. \tag{127}$$

By taking $l = 2$, we have $E(\mathfrak{s}) \leq \sigma B^2(\mathfrak{s}) = \sigma(E(\mathfrak{s}) - E(\mathfrak{s}+1))$. Utilising Lemma 5, we conclude that

$$E(\mathfrak{s}) \leq E(0) e^{-[\mathfrak{s}-1]^+ \ln\left(\frac{\sigma}{\sigma-1}\right)}.$$

By taking $l > 2$, using (127) and because $B^2(\mathfrak{s}) \leq E(\mathfrak{s}) \leq E(0)$, we have

$$\begin{aligned} E^{\frac{l+2}{2}}(\mathfrak{s}) &\leq \sigma' B^{l+2}(\mathfrak{s}) \Big\{ B^{(l-2)(l+2)}(\mathfrak{s}) + 1 \Big\} \\ &\leq \sigma' E^{\frac{l}{2}}(0) B^2(\mathfrak{s}) \Big\{ E^{\frac{(l-2)(l+2)}{2}}(0) + 1 \Big\} \\ &\leq \sigma(E(\mathfrak{s}) - E(\mathfrak{s}+1)). \end{aligned}$$

Utilizing Lemma 5 with $\alpha = \dfrac{l}{2}$, we conclude that

$$E(\mathfrak{s}) \leq \left(E(0)^{-\frac{l}{2}} + \sigma^{-1} \frac{l}{2} [\mathfrak{s}-1]^+ \right)^{\frac{-2}{l}}.$$

Therefore, we completed the proof of Theorem (105). □

4. An Example and Numerical Study

In this section, we present an illustrative example which guarantees our main results. Furthermore, we provide nice algorithms which help us to calculate all the numerical results.

Example 1. *Consider the initial-boundary value problem of nonlinear wave equation with strong damping and logarithmic source terms.*

$$\begin{cases} |\dot{\varkappa}|^l \ddot{\varkappa} - \Delta_p \varkappa(\mathrm{y}, \mathfrak{s}) + \Delta^2 \left(\varkappa(\mathfrak{s}) - \displaystyle\int_0^{\mathfrak{s}} \frac{\varkappa(\zeta)}{\exp(\mathfrak{s}-\zeta)+2}\, \mathrm{d}\zeta \right) & \\ \qquad -\mu_1 \Delta\dot{\varkappa} - \mu_2 \Delta\dot{\varkappa}(\mathrm{y}, \mathfrak{s}-\zeta) - \Delta\ddot{\varkappa} & \\ \qquad = \varkappa(\mathrm{y}, \mathfrak{s}) |\varkappa(\mathrm{y}, \mathfrak{s})|^{p-2} \ln|\varkappa(\mathrm{y}, \mathfrak{s})|, & \mathrm{y} \in \Omega,\ \mathfrak{s} > 0, \\ \varkappa = \Delta\varkappa = 0, & \mathrm{y} \in \partial\Omega,\ \mathfrak{s} > 0, \\ \varkappa(\mathrm{y}, 0) = 0, \quad \dot{\varkappa}\big|_{(\mathrm{y},0)} = \sqrt[3]{\mathrm{y}^2}, & \mathrm{y} \in \Omega, \\ \dot{\varkappa}\big|_{(\mathrm{y},\mathfrak{s}-\zeta)} = \frac{\mathfrak{s}-\zeta+2}{\mathrm{y}+5}, & \mathrm{y} \in \Omega,\ 0 < \mathfrak{s} < \zeta, \end{cases} \tag{128}$$

for $\mathfrak{s} > 0$, $p = 2$ *and* $l = 0.4$. *Clearly,* $g(\mathfrak{s}) = \frac{1}{\exp(\mathfrak{s})+2}$, $\varkappa_0(\mathrm{y}) = 0$, $\varkappa_1(\mathrm{y}) = \sqrt[3]{\mathrm{y}^2}$, $h_0(\mathrm{y}, \mathfrak{s}) = \frac{\mathfrak{s}+2}{\mathrm{y}+5}$ *and*

$$\vartheta = 1 - \int_0^{\infty} g(\zeta)\, \mathrm{d}\zeta \simeq 0.45069 > 0.$$

Consider positive constants $\xi_1 = 0.2, \xi_2 = 1.9, \xi_3 = 0.25$*; then, we have*

$$-\xi_1 \leq g'(\mathfrak{s}) \leq -\xi_3 g(\mathfrak{s}), \qquad 0 \leq g''(\mathfrak{s}) \leq \xi_2 g(\mathfrak{s}),$$

whenever $\mathfrak{s} \geq 0$. *Table 1 shows the numerical results. Figures 1–5 show graphical representation of the variables. Therefore, assumptions* (H1)–(H3) *hold and so the Cauchy problem* (128) *possesses a unique global weak solution* $\varkappa \in L^{\infty}([0, \infty), H^3(\Omega) \cap H_0^1(\Omega))$. *See Figure 6a,b.*

Table 1. Numerical values of $1 - \int_0^{\mathfrak{s}} g(\zeta)\, \mathrm{d}\zeta, g'(\mathfrak{s}), -\xi_3 g(\mathfrak{s}), g''(\mathfrak{s}), \xi_2 g(\mathfrak{s})$ in Example 1 for $\mathfrak{s} > 0$.

$\mathfrak{s}$	$1 - \int_0^{\mathfrak{s}} g(\zeta)\, \mathrm{d}\zeta$	$g'(\mathfrak{s})$	$-\xi_3 g(\mathfrak{s})$	$g''(\mathfrak{s})$	$\xi_2 g(\mathfrak{s})$
0.00	0.633333	1.000000	−0.111111	−0.037037	−0.083333
1.00	0.402689	0.726416	−0.122103	0.018588	−0.052985
2.00	0.202363	0.570466	−0.083820	0.048110	−0.026627
3.00	0.086029	0.498155	−0.041178	0.033720	−0.011320
4.00	0.033570	0.468682	−0.017044	0.015840	−0.004417
5.00	0.012632	0.457387	−0.006560	0.006386	−0.001662
6.00	0.004686	0.453166	−0.002454	0.002430	−0.000617
7.00	0.001729	0.451605	−0.000909	0.000905	−0.000228
8.00	0.000637	0.451029	−0.000335	0.000335	−0.000084
9.00	0.000234	0.450817	−0.000123	0.000123	−0.000031
10.00	0.000086	0.450739	−0.000045	0.000045	−0.000011
11.00	0.000032	0.450711	−0.000017	0.000017	−0.000004
12.00	0.000012	0.450700	−0.000006	0.000006	−0.000002
13.00	0.000004	0.450696	−0.000002	0.000002	−0.000001
14.00	0.000002	0.450695	−0.000001	0.000001	0.000000
15.00	0.000001	0.450694	0.000000	0.000000	0.000000
16.00	0.000000	0.450694	0.000000	0.000000	0.000000
17.00	0.000000	0.450694	0.000000	0.000000	0.000000
18.00	0.000000	0.450694	0.000000	0.000000	0.000000
19.00	0.000000	0.450694	0.000000	0.000000	0.000000
20.00	0.000000	0.450694	0.000000	0.000000	0.000000

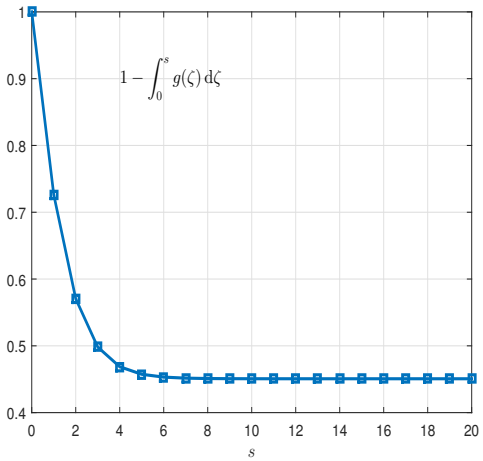


Figure 1. Two-dimensional graph of $1 - \int_0^{\mathfrak{s}} g(\zeta)\, \mathrm{d}\zeta$ for $\mathfrak{s} > 0$ in Example 1.

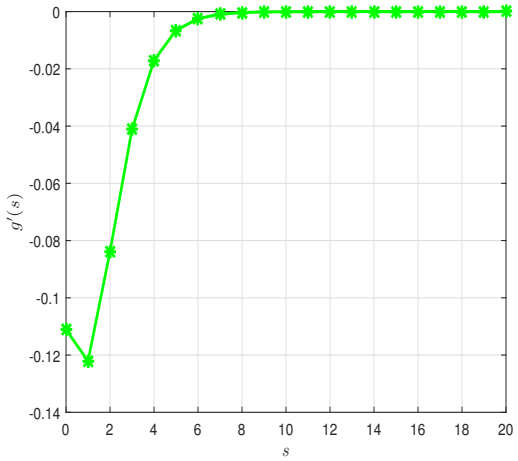


Figure 2. Two-dimensional graph of $g'(\mathfrak{s})$ for $\mathfrak{s} > 0$ in Example 1.

In addition, by using Equation (11)*, the energy functional associated to Problem* (128) *is obtained as*

$$\begin{aligned} E(\mathfrak{s}) = {} & \frac{1}{l+2}\|\dot{\varkappa}(\mathfrak{s})\|_{l+2}^{l+2} + \|\nabla\dot{\varkappa}(\mathfrak{s})\|_2^2 + \frac{1}{2}\left(1 - \int_0^{\mathfrak{s}} g(\zeta)\,\mathrm{d}\zeta\right)\|\Delta\varkappa(\mathfrak{s})\|_2^2 \\ & + \frac{1}{p}\|\nabla\varkappa(\mathfrak{s})\|_p^p + \frac{1}{2}(g\circ\Delta\varkappa)(\mathfrak{s}) \\ & - \frac{1}{p}\int_\Omega |\varkappa(\zeta,\mathfrak{s})|^p \ln|\varkappa(\zeta,\mathfrak{s})|\,\mathrm{d}\zeta + \frac{1}{p^2}\|\varkappa(\mathfrak{s})\|_p^p \\ & + \xi\int_\Omega\int_0^1 |\nabla\top(\zeta,\eta,\mathfrak{s})|^2\,\mathrm{d}\eta\mathrm{d}\zeta, \end{aligned}$$

where $(g\circ\phi)(\mathfrak{s}) = \int_0^{\mathfrak{s}} \frac{1}{\exp(\mathfrak{s}-\zeta)+2}\|\phi(\mathfrak{s}) - \phi(\zeta)\|_2^2\,\mathrm{d}\zeta$.

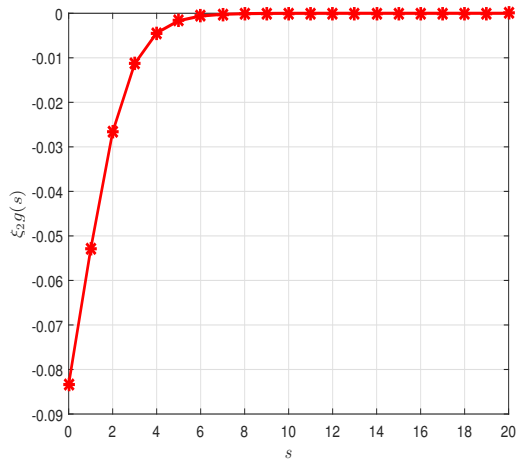


Figure 3. Two-dimensional graph of $-\xi_3 g(\mathfrak{s})$ for $\mathfrak{s} > 0$ in Example 1.

In addition, the solution satisfies the energy inequality

$$\begin{aligned} & E(\mathfrak{s}) - \frac{1}{2}\int_0^{\mathfrak{s}} (g'\circ\Delta\varkappa)(\zeta)\,\mathrm{d}\zeta + \frac{1}{2}\int_0^{\mathfrak{s}} g(\zeta)\|\Delta\varkappa(\zeta)\|_2^2\,\mathrm{d}\zeta \\ & \quad + \left(\mu_1 - \frac{\xi}{\tau} - \frac{|\mu_2|}{2}\right)\int_0^{\mathfrak{s}} \|\nabla\dot{\varkappa}(\zeta)\|_2^2\,\mathrm{d}\zeta \\ & \quad + \left(\frac{\xi}{\tau} - \frac{|\mu_2|}{2}\right)\int_0^{\mathfrak{s}} \|\nabla\top(\mathrm{y},1,\zeta)\|_2^2\,\mathrm{d}\zeta \le E(0). \end{aligned}$$

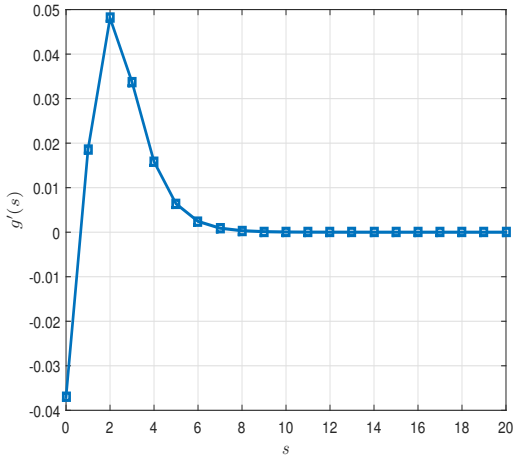


Figure 4. Two-dimensional graph of $g''(\mathfrak{s})$ for $\mathfrak{s} > 0$ in Example 1.

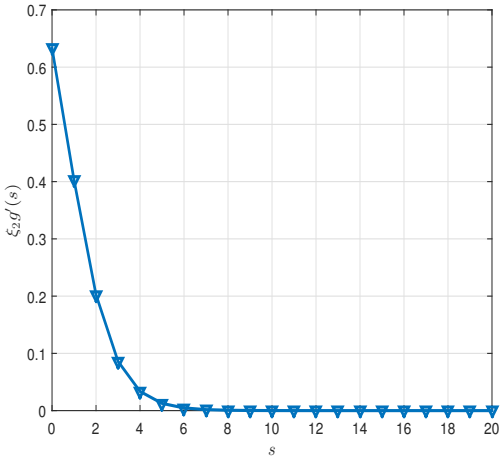
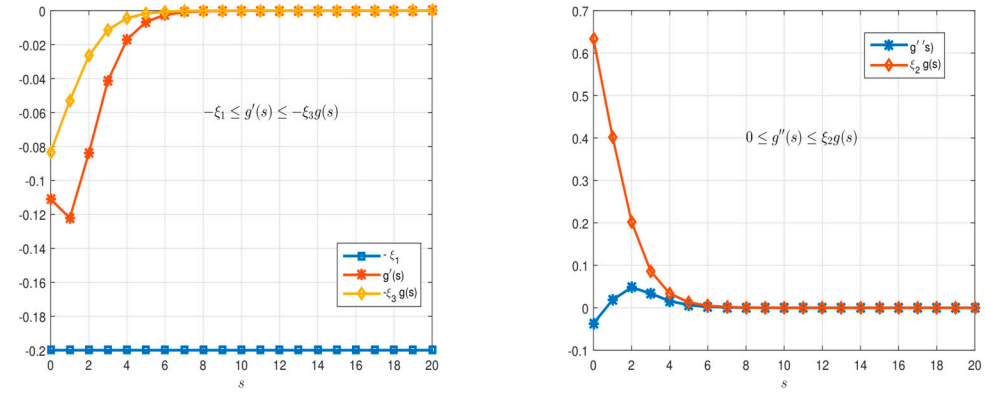


Figure 5. Two-dimensional graph of $\xi_2 g(\mathfrak{s})$ for $\mathfrak{s} > 0$ in Example 1.



(a) $-\xi_1 \le g'(\mathfrak{s}) \le -\xi_3 g(\mathfrak{s})$ **(b)** $0 \le g''(\mathfrak{s}) \le \xi_2 g(\mathfrak{s})$

Figure 6. Graphical representation of (H1) and (H2) for $\mathfrak{s} \ge 0$ in Example 1.

5. Conclusions

Evolution problems with an interior logarithmic source have a wide range of applications in physics, such as nuclear physics, optics and geophysics. In recent years, there have been many works concerning the global existence and stabilisation of the wave equation, including a constant delay or time-varying delay. However, to the best of our knowledge, there is no decay result for the nonlinear p-Laplacian viscoelastic Petrovesky equation with a delay term and logarithmic nonlinearity. In this paper, we study the global existence of solutions for a delayed viscoelastic-type Petrovesky wave equation with a p-Laplacian operator and logarithmic source under some appropriate conditions. By making some essential assumptions on the memory kernel function and exponents p, l, we proved that the rate of decay of the total energy is exponential or polynomial, depending on the exponent l. The illustrative examples are designed to validate the theoretical findings. The equation under the equation studies and evolution problems with an interior logarithmic source answer the question as to what will happen if one replaces the power source term by other source terms, for example, the logarithmic source term. This paper provides an affirmative reply to such a situation.

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References

1. Barrow, J.D.; Parsons, P. Inflationary models with logarithmic potentials. *Phys. Rev. D* **1995**, *52*, 5576. [CrossRef] [PubMed]
2. Górka, P. Logarithmic Klein–Gordon equation. *Acta Phys. Pol.* **2009**, *40*, 3477–3482.
3. Zhao, K. Global stability of a novel nonlinear diffusion online game addiction model with unsustainable control. *AIMS Math.* **2022**, *7*, 20752–20766. [CrossRef]
4. Sattinger, D.H. On global solution of nonlinear hyperbolic equations. *Arch. Ration. Mech. Anal.* **1968**, *30*, 148–172. [CrossRef]
5. Messaoudi, S.A. Blow up and global existence in a nonlinear viscoelastic wave equation. *Math. Nachrichten* **2003**, *260*, 58–66. [CrossRef]
6. Gazzola, F.; Squassina, M. Global solutions and finite time blow up for damped semilinear wave equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **2006**, *23*, 185–207. [CrossRef]
7. Wang, Y.J. A suffiecient condition for finite time blow up of the nonlinear Klein-Gordon equations with arbitrarily positive initial energy. *Proc. Am. Math. Soc.* **2008**, *136*, 3477–3482. [CrossRef]
8. Wu, Y.; Xue, X. Uniform decay rate estimates for a class of quasilinear hyperbolic equations with nonlinear damping and source terms. *Appl. Anal.* **2013**, *92*, 1169–1178. [CrossRef]
9. Pişkin, E.; Boulaaras, S.; Irkil, N. Qualitative analysis of solutions for the p-Laplacian hyperbolic equation with logarithmic nonlinearity. *Math. Methods Appl. Sci.* **2021**, *44*, 4654–4672. [CrossRef]
10. Ma, L.; Fang, Z.B. Eenrgy decay estimates and infinite blow-up phenomena for a strongly damped semilinear wave equation with logarithmic nonlinear source. *Math. Methods Appl. Sci.* **2018**, *41*, 2639–2653. [CrossRef]
11. Di, H.; Shang, Y.; Song, Z. Initial boundary value problems for a class of strongly damped semilinear wave equations with logarithmic nonlinearity. *Nonlinear Anal. Real World Appl.* **2020**, *51*, 102968. [CrossRef]
12. Ha, T.G.; Park, S.H. Blow-up phenomena for a viscoelastic wave equation with strong damping and logarithmic nonlinearity. *Adv. Differ. Equ.* **2020**, *2020*, 1–17. [CrossRef]
13. Menglan, L. *The Lifespan of Solutions for a Viscoelastic Wave Equation with a Strong Damping and Logarithmic Nonlinearity*; Evolution Equations & Control Theory: Paris, France, 2021.
14. Zang, A.; Fu, Y. Interpolation inequalities for derivatives in variable exponent Lebesgue–Sobolev spaces. *Nonlinear Anal.* **2008**, *269*, 3629–3636. [CrossRef]

15. Adams, R.A. Sobolev Spaces. In *Pure and Applied Mathematics*; Academic Press: Cambridge, MA, USA, 1975; p. 65.
16. Boulaaras, S. Existence of positive solutions for a new class of Kirchhoff parabolic systems. *Rocky Mt. J. Math.* **2020**, *50*, 445–454. [CrossRef]
17. Nakao, M. Asymptotic stability of the bounded or almost periodic solution of the wave equation with nonlinear dissipation term. *J. Math. Anal. Appl.* **1977**, *58*, 336–343. [CrossRef]
18. Datko, R.; Lagnese, J.; Polis, M.P. An example on the effect of time delays in boundary feedback stabilization of wave equations. *SIAM J. Control Optim.* **1986**, *24*, 152–156. [CrossRef]
19. Kafini, M.; Messaoudi, S. Local existence and blow up of solutions to a logarithmic non linear wave equation with delay. *Appl. Anal.* **2019**, *99*, 530–547.
20. Simon, J. Compact sets in the space $L^p(O,T,B)$. *Ann. Mat. Pura Ed. Appl.* **1986**, *146*, 65–96. [CrossRef]
21. Lions, J.L. *Quelques Méthodes de Résolution des Problémes aux Limites non Linéaires*; Dunod: Paris, France, 1969.
22. Cao, Y.; Liu, C. Initial boundary value problem for a mixed pseudo-parabolic p-Laplacian type equation with logarithmic nonlinearity. *Electron. J. Differ. Equ.* **2018**, *2018*, 1–19.
23. Nakao, M. A difference inequality and its application to nonlinear evolution equations. *J. Math. Soc. Jpn.* **1978**, *30*, 747–762. [CrossRef]