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Generalized Spacelike Normal Curves in Minkowski Three-Space

Yusra Tashkandy [1], Walid Emam [1], Clemente Cesarano [2,*], M. M. Abd El-Raouf [3] and Ayman Elsharkawy [4,*]

[1] Department of Statistics and Operations Research, Faculty of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia;
[2] Section of Mathematics, International Telematic University Uninettuno, CorsoVittorio Emanuele II, 39, 00186 Roma, Italy
[3] Basic and Applied Science Institute, Arab Academy for Science, Technology and Maritime Transport (AASTMT), Montaza 2, Alexandria 21532, Egypt
[4] Department of Mathematics, Faculty of Science, Tanta University, Tanta 31511, Egypt
* Correspondence: c.cesarano@uninettunouniversity.net (C.C.); ayman_ramadan@science.tanta.edu.eg (A.E.)

Abstract: Equiform geometry is considered an extension of other geometries. Furthermore, an equiform frame is a generalization of the Frenet frame. In this study, we begin by defining the term "equiform parameter (EQP)", "equiform frame", and "equiform formulas (EQF)" in regard to the Minkowski three-space. Second, we define spacelike normal curves (SPN) in Minkowski three-space and present a variety of descriptions of these curves with equiform spacelike (EQS) or equiform timelike (EQN) principal normals in Minkowski three-space. Third, we discuss the implications of these findings. Finally, an example is given to illustrate our theoretical results.

Keywords: Minkowski three-space; equiform frame; equiform equations; equiform curvatures; equiform normal curves

MSC: 53A35; 53C50

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1. Introduction

Numerous mathematicians, including [1–4], have designed and analyzed curves in Minkowski space during the past 20 years. It is well known that an arc-length-parameterized differentiable curve in Minkowski three-space E_1^3 has orthonormal Frenet frames which are the tangent ($\mathbb{T}$), principal normal ($\mathbb{N}$), and binormal vectors ($\mathbb{B}$), respectively. Moreover, the Minkowski three-space has three distinct planes that are the rectifying plane ($\mathbb{TB}$), the osculating plane ($\mathbb{TN}$), and the normal plane ($\mathbb{NB}$), respectively, [5]. The term "normal curve" refers to a unit-speed curve in E_1^3 space, whose position vector always resides in its normal plane, in other words, if $\varrho(s)$ can be written as $\varrho(s) = f(s)\mathbb{N}(s) + g(s)\mathbb{B}(s)$ for some differentiable functions f and g of s in $I \subset R$. The definition of a normal curve suffices [6,7] for a few descriptions of timelike, spacelike, and null normal curves that occupy the Minkowski 3-space.

The equiform frame is more useful and general than the Frenet frame. The equiform geometry of space curves and surfaces has been the subject of a significant amount of attention and investigation from a large number of mathematicians. This is due to the fact that it is thought that equiform geometry is an extension of other geometries. This is because of the significant amount of attention that it has garnered in the area of mathematics. As a consequence of this, a significant number of studies have been carried out on the equiform geometry in many spaces [8–10]. The authors in [11] defined the equiform differential geometry of curves in G4-space.

This paper aims to study the normal curves with regard to the equiform frame and describe the EQS normal curves with principal normals that are also EQS or EQT. The article is structured as follows: In Section 2, we provide context for the topic by introducing the Frenet frame and Frenet equations along a spacelike unit-speed curve in Minkowski three-space. In Section 3, for spacelike curves with either a spacelike or a timelike principal normal vector in Minkowski space E_1^3, we provide the EQP as well as the EQF for EQS curves in Minkowski space E_1^3. In Section 4, we describe some EQS normal curves with principal normals that are also EQS or EQT.

2. Preliminaries

The Lorentz–Minkowski space E_1^3 is the Euclidean three-space E^3 with the metric π determined by

$$\pi = -dx_1^2 + dx_2^2 + dx_3^2, \tag{1}$$

such that the coordinate system for E_1^3 is (x_1, x_2, x_3). It is said that a vector v in E_1^3 is spacelike if $\pi(v,v) > 0$ or $v = 0$, timelike if $\pi(v,v) < 0$, and null (lightlike) if $\pi(v,v) = 0$ and $v \neq 0$. For instance, a nonlightlike vector v's norm may be calculated using the formula $\| v \| = \sqrt{|\pi(v,v)|}$. The vector v is referred to as a unit vector if $\| v \| = 1$. If $\pi(u,v) = 0$, two vectors u, v are said to be orthogonal. Any $u, v \in E_1^3$ has the Lorentzian vector product of u and v described in [2,6,7,12] by

$$u \wedge v = (u_3 v_2 - u_2 v_3,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1). \tag{2}$$

If all of the velocity vectors on an arbitrary curve ϱ in E_1^3 are timelike, spacelike, or null (lightlike), the curve is said to be timelike, spacelike, or null (lightlike). The non-null curve ϱ is said to be unit speed, if $\pi(\acute{\varrho}, \acute{\varrho}) = \pm 1$. The tangent, principal normal, and binormal vectors are labeled as $\mathbb{T}$, $\mathbb{N}$, and $\mathbb{B}$, respectively. The moving Frenet frame along the $\varrho(s)$ curve is denoted by $\mathbb{T}, \mathbb{N}$, and $\mathbb{B}$. Frenet's formulas depend on the curve's causal character. The following Frenet formulas are provided in [2,3,6] for an arbitrary curve $\varrho(s)$ in the space E_1^3.

The Frenet equations in the case where ϱ is a spacelike curve with a spacelike or a timelike principal normal are defined by

$$\begin{bmatrix} \dot{\mathbb{T}} \\ \dot{\mathbb{N}} \\ \dot{\mathbb{B}} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 \\ -\epsilon\kappa_1 & 0 & \kappa_2 \\ 0 & \kappa_2 & 0 \end{bmatrix} \begin{bmatrix} \mathbb{T} \\ \mathbb{N} \\ \mathbb{B} \end{bmatrix}, \tag{3}$$

such that $\pi(\mathbb{N},\mathbb{N}) = \epsilon = \pm 1, \pi(\mathbb{T},\mathbb{T}) = 1,\ \pi(\mathbb{B},\mathbb{B}) = -\epsilon$ and $\pi(\mathbb{T},\mathbb{N}) = \pi(\mathbb{N},\mathbb{B}) = \pi(\mathbb{T},\mathbb{B}) = 0$.

The Frenet equations in the case where ϱ is a spacelike curve with a null (lightlike) principal normal are given by

$$\begin{bmatrix} \dot{\mathbb{T}} \\ \dot{\mathbb{N}} \\ \dot{\mathbb{B}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \kappa_2 & 0 \\ -1 & 0 & \kappa_2 \end{bmatrix} \begin{bmatrix} \mathbb{T} \\ \mathbb{N} \\ \mathbb{B} \end{bmatrix}, \tag{4}$$

such that $\pi(\mathbb{T},\mathbb{T}) = 1,\ \pi(\mathbb{N},\mathbb{N}) = 0,\ \pi(\mathbb{B},\mathbb{B}) = 0,\ \pi(\mathbb{T},\mathbb{N}) = \pi(\mathbb{T},\mathbb{B}) = 0$ and $\pi(\mathbb{N},\mathbb{B}) = 1$.

The only two potential values for κ_1 in this situation are $\kappa_1 = 0$. In all other cases and whenever ϱ is a straight line, $\kappa_1 = 1$. If $\varrho(s)$ is parameterized by s, then

$$\kappa_1 = \|\varrho^{\cdot\cdot}(s)\| \quad , \quad \kappa_2 = \|B^{\cdot}(s)\|.$$

The pseudo-Riemannian cone, pseudo-Riemannian lightlike sphere, and pseudo-Riemannian hyperbolic space are each provided, respectively, as

$$S_1^2(m,r) = \{u \in E_1^3 : \pi(u-m, u-m) = r^2\},$$

$$H_0^2(m,r) = \{u \in E_1^3 : \pi(u-m, u-m) = -r^2\},$$
$$C(m) = \{u \in E_1^3 : \pi(u-m, u-m) = 0\},$$

such that $r > 0$ is a constant and m is a fixed point in E_1^3 [6,7].

3. EQF for EQS Curves in E_1^3

In this section, we first introduce the equiform geometry in Minkowski space E_1^3. Next, we present the EQF in the case of EQS curves.

Definition 1 ([8–10])**.** *Let $\varrho : I \to E_1^3$ be a unit-speed curve in Minkowski space. The EQP of $\varrho(s)$ is defined by*

$$\sigma = \int \frac{ds}{\rho} = \int \kappa_1 ds,$$

such that the radius of the curvature of the curve $\rho = \frac{1}{\kappa_1}$.

Then, we obtain

$$\frac{ds}{d\sigma} = \rho. \tag{5}$$

Let us write the equiform frame for the curve ϱ in terms of the equiform invariant parameter σ in the E_1^3 space. The vector

$$\mathfrak{T} = \frac{d\varrho}{d\sigma},$$

is known as the curve's equiform tangent vector. From Equation (3), we get

$$\mathfrak{T} = \frac{d\varrho}{d\sigma} = \frac{d\varrho}{ds}\frac{ds}{d\sigma} = \frac{d\varrho}{ds}\rho = \rho\mathbb{T}. \tag{6}$$

The equiform principal normal vector and the equiform binormal vector are defined by

$$\mathfrak{N} = \rho\mathbb{N} \ , \qquad \mathfrak{B} = \rho\mathbb{B}. \tag{7}$$

Then, we quickly demonstrate that each of $\mathfrak{T}, \mathfrak{N}$, and $\mathfrak{B}$ is not an orthonormal frame of the curve ϱ but rather an equiform invariant orthogonal frame.

(I) Using Equations (1), (3), and (4), if ϱ has a principal normal $\mathfrak{N}$ that is either EQS or EQT, then

$$\begin{aligned} \mathfrak{T}' = \frac{d\mathfrak{T}}{d\sigma} &= \frac{d(\rho\mathbb{T})}{d\sigma} \\ &= \frac{d\rho}{ds}\frac{ds}{d\sigma}\mathbb{T} + \rho\frac{d\mathbb{T}}{ds}\frac{ds}{d\sigma} \\ &= \dot{\rho}\mathfrak{T} + \rho^2\kappa_1\mathbb{N} \\ &= \dot{\rho}\mathfrak{T} + \mathfrak{N}. \end{aligned}$$

Similarly, $\mathfrak{N}' = -\epsilon\mathfrak{T} + \dot{\rho}\mathfrak{N} + \frac{\kappa_2}{\kappa_1}\mathfrak{B}$

$\mathfrak{B}' = \frac{\kappa_2}{\kappa_1}\mathfrak{N} + \dot{\rho}\mathfrak{B}.$

Definition 2 ([8–10])**.** *The curve's first equiform curvature of ϱ is described by $K_1 : I \to R$, which is defined by*

$$K_1 = \dot{\rho}.$$

Definition 3 ([8–10])**.** *The curve's second equiform curvature of ϱ is described by the function $K_2 : I \to R$, which is defined by*

$$K_2 = \frac{\kappa_2}{\kappa_1}.$$

The total curvature, the EQP σ for closed curves, is crucial to the global differential geometry of Euclidean space. Additionally, K_2 is a canonical curvature that has a fascinating geometric meaning. By definitions (1) and (2), the EQF in case (I) become

$$\begin{bmatrix} \mathfrak{T}' \\ \mathfrak{N}' \\ \mathfrak{B}' \end{bmatrix} = \begin{bmatrix} K_1 & 1 & 0 \\ -\epsilon & K_1 & K_2 \\ 0 & K_2 & K_1 \end{bmatrix} \begin{bmatrix} \mathfrak{T} \\ \mathfrak{N} \\ \mathfrak{B} \end{bmatrix}, \tag{8}$$

such that $\pi(\mathfrak{T}, \mathfrak{T}) = \rho^2$, $\pi(\mathfrak{N}, \mathfrak{N}) = \epsilon\rho^2$, $\pi(\mathfrak{B}, \mathfrak{B}) = -\epsilon\rho^2$, and $\pi(\mathfrak{T}, \mathfrak{N}) = \pi(\mathfrak{T}, \mathfrak{B}) = \pi(\mathfrak{N}, \mathfrak{B}) = 0$.

Corollary 1. *If $\varrho(\sigma)$ is an EQS curve and $\mathfrak{N}$ is either an EQS or EQT principal normal, then the equiform curvatures are provided, respectively, by*

$$K_1 = \frac{\pi(\mathfrak{T}, \mathfrak{T}')}{\rho^2} = \frac{\epsilon\pi(\mathfrak{N}, \mathfrak{N}')}{\rho^2} = \frac{-\epsilon\pi(\mathfrak{B}, \mathfrak{B}')}{\rho^2},$$

$$K_2 = \frac{-\epsilon\pi(\mathfrak{N}', \mathfrak{B})}{\rho^2} = \frac{\epsilon\pi(\mathfrak{B}', \mathfrak{N})}{\rho^2}.$$

(II) If $\mathfrak{N}$ is an equiform null principal normal and ϱ is an EQS, there are only two possible values for κ_1 in this situation: $\kappa_1 = 0$ when ϱ is a straight line and $\kappa_1 = 1$ in all other circumstances. The uniform parameter σ is not specified when ϱ is a straight line. Consequently, if ϱ is not a straight line, $s = \sigma$, and $K_1 = 0$ will apply. Hence, the EQF for nonstraight line curves are identical to Equation (2). Accordingly, the research results in [6] hold when the space is uniform, and the principal normal curve is uniformly null.

4. Main Results

The concept of an equiform normal curve (EQN) in Minkowski space E_1^3 is presented in this section, along with certain characterization theorems for EQS normal curves in E_1^3. Along this section, we assume that $\varrho(\sigma)$ is an EQS curve and $\mathfrak{N}$ is either an EQS or EQT principal normal in E_1^3.

Definition 4. *If the position vector $\varrho(\sigma)$ always resides on the equiform normal plane spanned by $\{\mathfrak{N}, \mathfrak{B}\}$, then we say that the curve is an equiform normal curve (EQN) in Minkowski three-space.*

Theorem 1. *If $K_1(\sigma) > 0$ and $K_2(\sigma) \neq 0$, $\forall\, \sigma \in I \subset R$, and $\varrho = \varrho(\sigma)$ is an EQN, the following outcomes are therefore satisfied:*
(i) The equiform curvatures $K_1(\sigma)$ and $K_2(\sigma)$ have the following relation:

$$\frac{K_1}{K_2} = \frac{1}{K_1}[K_2 - (\frac{K_1}{K_2})'].$$

(ii) The equiform binormal with the position vector of the curve and the component of the equiform principal normal are each provided by

$$\pi(\varrho, \mathfrak{N}) = -\rho^2,$$

$$\pi(\varrho, \mathfrak{B}) = \frac{-\rho^2}{K_1}[K_2 - (\frac{K_1}{K_2})'].$$

(iii) If the position vector of $\varrho(\sigma)$ is a lightlike vector, then ϱ is located on $C(m)$, and $K_1(\sigma)$ and $K_2(\sigma)$ satisfy

$$K_1(\sigma) = \pm K_2(\sigma).$$

On the other hand, if $\varrho(\sigma)$ has equiform curvatures $K_1(\sigma) > 0$, $K_2(\sigma) \neq 0$, $\forall\, \sigma \in I \subset R$ and one of the requirements (i), (ii), or (iii) is satisfied, then ϱ is either an EQN or congruent to one.

Proof. Assume $\varrho(\sigma)$ is an EQN in E_1^3. Then,

$$\varrho(\sigma) = \lambda(\sigma)\mathfrak{N}(\sigma) + \mu(\sigma)\mathfrak{B}(\sigma).$$

With the use of (6) and by differentiation in the light of σ, we get

$$-\lambda\epsilon = 1, \quad \lambda' + \lambda K_1 + \mu K_2 = 0, \quad \lambda K_2 + \mu' + \mu K_1 = 0. \tag{9}$$

We get the following from the first and second relation of (7):

$$\lambda = -\epsilon, \quad \mu = \epsilon\frac{K_1}{K_2}. \tag{10}$$

Thus,

$$\varrho(\sigma) = -\epsilon\mathfrak{N} + \epsilon\frac{K_1}{K_2}\mathfrak{B}. \tag{11}$$

Furthermore, we get the following equation from the third relation of (9) and using (8):

$$\left(\frac{K_1}{K_2}\right)' + \frac{K_1^2}{K_2} - K_2 = 0. \tag{12}$$

The equality is thus satisfied by the equiform curvatures.

$$\frac{K_1}{K_2} = \frac{1}{K_1}[K_2 - (\frac{K_1}{K_2})']. \tag{13}$$

Thus, we have established (i). Next, by substituting (13) into (11), we get

$$\varrho(\sigma) = -\epsilon\mathfrak{N} + \frac{\epsilon}{K_1}[K_2 - (\frac{K_1}{K_2})']\mathfrak{B}. \tag{14}$$

Therefore, we can simply see that from (14).

$$\pi(\varrho, \varrho) = \epsilon\rho^2 - \frac{\epsilon\rho^2}{K_1^2}[K_2 - (\frac{K_1}{K_2})']^2, \tag{15}$$

$$\pi(\varrho, \mathfrak{N}) = -\rho^2, \tag{16}$$

$$\pi(\varrho, \mathfrak{B}) = \frac{-\rho^2}{K_1}[K_2 - (\frac{K_1}{K_2})']. \tag{17}$$

As a result, we have established (ii).

Now, consider that ϱ has an equiform null (lightlike) position vector and is a spacelike normal curve. Then, $\pi(\varrho, \varrho) = 0$ is obtained. When we substitute (15), we get

$$\frac{1}{K_1^2}[K_2 - (\frac{K_1}{K_2})']^2 = 1,$$

$$\frac{1}{K_1}[K_2 - (\frac{K_1}{K_2})'] = \pm 1.$$

When we substitute into (13), we get

$$\frac{K_1}{K_2} = \pm 1.$$

Thus,

$$K_1 = \pm K_2. \tag{18}$$

Contrarily, let us take a look at the vector

$$m = \varrho(\sigma) + \epsilon \mathfrak{N} - \epsilon \frac{K_1}{K_2} \mathfrak{B}.$$

By differentiating this with respect to σ and using the related EQF (8) with Equation (12), then it follows: $m' = 0$ and hence $m =$ constant. Therefore,

$$\pi(\varrho - m, \varrho - m) = \epsilon \rho^2 - \epsilon \rho^2 (\frac{K_1}{K_2})^2 = \epsilon \rho^2 (1 - (\frac{K_1}{K_2})^2) = 0.$$

This indicates that ϱ is located on $C(m)$. As a result, we have established (iii).

Conversely, suppose that statement (i) is satisfied. Therefore, the equiform curvatures satisfy the equality

$$\frac{K_1}{K_2} = \frac{1}{K_1} [K_2 - (\frac{K_1}{K_2})'].$$

Next, we have

$$\left(\frac{K_1}{K_2}\right)' + \frac{K_1^2}{K_2} - K_2 = 0.$$

By using EQF (8), we obtain

$$\frac{d}{d\sigma} \left[\varrho(\sigma) + \epsilon \mathfrak{N} - \epsilon \frac{K_1}{K_2} \mathfrak{B} \right] = 0.$$

As a result, ϱ is congruent with an EQN.

Let us assume (ii) is satisfied. After that,

$$\pi(\varrho, \mathfrak{N}) = -\rho^2, \qquad \pi(\varrho, \mathfrak{B}) = \frac{-\rho^2}{K_1} [K_2 - (\frac{K_1}{K_2})'].$$

By differentiating the equation $\pi(\varrho, \mathfrak{N}) = -\rho^2$ in light of σ, we have

$$\pi(\mathfrak{T}, \mathfrak{N}) + \pi(\varrho, \mathfrak{N}') = -2\rho\rho'.$$

Therefore,

$$-\epsilon \pi(\varrho, \mathfrak{T}) - K_1 \rho^2 + K_2 (\frac{-\rho^2}{K_1} [K_2 - (\frac{K_1}{K_2})']) = -2\rho\rho',$$

which indicates that $\pi(\varrho, \mathfrak{T}) = 0$ and that ϱ is an EQN.

Let us suppose that statement (iii) is satisfied. Consequently, ϱ is located on the lightlike cone $C(m)$, whose vertex is at m, m is constant, and whose equiform curvatures $K_1(\sigma)$ and $K_2(\sigma)$ satisfy $K_1 = \pm K_2$. This implies that

$$\pi(\varrho - m, \varrho - m) = 0.$$

Using Equation (8) and a three-time differentiation of the above equation in light of σ, we have

$$\pi(\varrho - m,\ \mathfrak{T}) = 0,\ \ \pi(\varrho - m,\ \mathfrak{N}) = -\rho^2,\ \ \pi(\varrho - m,\ \mathfrak{B}) = -\rho^2 \frac{K_1}{K_2}.$$

Consequently,

$$\varrho - m = -\epsilon\mathfrak{N} + \epsilon\frac{K_1}{K_2}\mathfrak{B}.$$

This indicates that curve ϱ is congruent to an EQN up to a translation for vector m. Set $m = 0$, the proof is therefore complete since $\pi(\varrho, \varrho) = 0$ was quickly discovered using $K_1 = \pm K_2$. □

Theorem 2. *Assume that $\varrho = \varrho(\sigma)$ with equiform curvatures $K_1(\sigma) > 0$, $K_2(\sigma) \neq 0$, $\forall$ $\sigma \in I \subset R$, with a non-null equiform principal normal $\mathfrak{N}$ and with a non-null position vector. Then, (i) The position vector ϱ must be an EQS vector in order for the curve ϱ to be on $S^2_1(m, r)$ and satisfy*

$$\frac{K_1}{K_2} = \pm\sqrt{1 - \frac{r^2}{\epsilon\rho^2}}\ ,\quad \frac{r^2}{\epsilon\rho^2} < 1.$$

(ii) The position vector ϱ must be an EQT vector in order for the curve ϱ to be on $H^2_0(m, r)$ and satisfy

$$\frac{K_1}{K_2} = \pm\sqrt{1 + \frac{r^2}{\epsilon\rho^2}}.$$

Proof. First, let us assume that $r \in R^+$, $\pi(\varrho, \varrho) = r^2$. When we substitute Equation (13), we get

$$\epsilon\rho^2 - \frac{\epsilon\rho^2}{K_1^2}[K_2 - (\frac{K_1}{K_2})']^2 = r^2.$$

Thus

$$\frac{1}{K_1}[K_2 - (\frac{K_1}{K_2})'] = \pm\sqrt{1 - \frac{r^2}{\epsilon\rho^2}}. \tag{19}$$

When we change (13) for (19), we get

$$\frac{K_1}{K_2} = \pm\sqrt{1 - \frac{r^2}{\epsilon\rho^2}}. \tag{20}$$

Take a look at the vector m,

$$m = \varrho(\sigma) + \epsilon\mathfrak{N} - \epsilon\frac{K_1}{K_2}\mathfrak{B}.$$

Using the related (8), we differentiate this in light of σ and $m' = 0$ and therefore m is constant. Then,

$$\begin{aligned}\pi(\varrho - m, \varrho - m) &= \epsilon\rho^2 - \epsilon\rho^2(\frac{K_1}{K_2})^2 \\ &= \epsilon\rho^2 - \epsilon\rho^2(1 - \frac{r^2}{\epsilon\rho^2}) \\ &= r^2.\end{aligned}$$

Then, ϱ is on $S^2_1(m, r)$, which has m as its center and $r \in R^+$ as its radius.

On the other hand, suppose (18) is true and that ϱ lies on $S^2_1(m, r)$. Consequently, $\pi(\varrho - m, \varrho - m) = r^2$, $r \in R^+$. Using the EQF and differentiating this in light of σ three times, we have

$$\varrho - m = -\epsilon\mathfrak{N} + \epsilon\frac{K_1}{K_2}\mathfrak{B}.$$

Consequently, up to a vector m translation, a normal curve with an equal slope, ϱ, is congruent. Specifically, let us set $m = 0$. Therefore, (18) yields $\pi(\varrho, \varrho) = r^2$. Statement (i) is proved.

(ii) is similar to (i). $\square$

5. An Example

Let $\varrho(s)$ be a unit-speed spacelike curve with a spacelike normal vector in E_1^3 as in Figure 1.

$$\varrho(s) = (s,\ s\sin(\ln s),\ s\cos(\ln s)),$$

The Frenet frame of the curve $\varrho(s)$ is given by:

$$\mathbb{T}(s) = (1,\ \sin(\ln s) + \cos(\ln s),\ \cos(\ln s) - \sin(\ln s)),$$

$$\mathbb{N}(s) = \frac{1}{\sqrt{2}}\,(0,\ \cos(\ln s) - \sin(\ln s),\ -\sin(\ln s) - \cos(\ln s)),$$

$$\mathbb{B} = \frac{1}{\sqrt{2}}(2,\ \sin(\ln s) + \cos(\ln s),\ \cos(\ln s) - \sin(\ln s)).$$

Thus, the first and the second curvatures are, respectively, given by

$$\kappa_1 = \frac{\sqrt{2}}{s}, \quad \kappa_2 = \frac{1}{s}.$$

Therefore, the equiform parameter σ is given by

$$\sigma = \int \frac{ds}{\rho} = \sqrt{2}\ln s,$$

Furthermore, the first and the second equiform curvatures are, respectively, given by:

$$K_1 = \frac{1}{\sqrt{2}}, \quad K_2 = \frac{1}{\sqrt{2}}.$$

Further,

$$\varrho(\sigma) = (e^{\frac{\sigma}{\sqrt{2}}},\ e^{\frac{\sigma}{\sqrt{2}}}\sin(\frac{\sigma}{\sqrt{2}}),\ e^{\frac{\sigma}{\sqrt{2}}}\cos(\frac{\sigma}{\sqrt{2}})).$$

Here, the equiform frame of $\varrho(\sigma)$ is given by:

$$\mathfrak{T}(\sigma) = \frac{e^{\frac{\sigma}{\sqrt{2}}}}{\sqrt{2}}(1,\ \sin(\frac{\sigma}{\sqrt{2}}) + \cos(\frac{\sigma}{\sqrt{2}}),\ \cos(\frac{\sigma}{\sqrt{2}}) - \sin(\frac{\sigma}{\sqrt{2}})),$$

$$\mathfrak{N}(\sigma) = \frac{e^{\frac{\sigma}{\sqrt{2}}}}{2}(0,\ \cos(\frac{\sigma}{\sqrt{2}}) - \sin(\frac{\sigma}{\sqrt{2}}),\ -\sin(\frac{\sigma}{\sqrt{2}}) - \cos(\frac{\sigma}{\sqrt{2}})),$$

$$\mathfrak{B}(\sigma) = \frac{e^{\frac{\sigma}{\sqrt{2}}}}{2}(0,\ \cos(\frac{\sigma}{\sqrt{2}}) + \sin(\frac{\sigma}{\sqrt{2}}),\ -\sin(\frac{\sigma}{\sqrt{2}}) + \cos(\frac{\sigma}{\sqrt{2}})).$$

Since $\pi(\mathfrak{T}, \mathfrak{T}) > 0$ and $\pi(\mathfrak{N}, \mathfrak{N}) > 0$, $\varrho(\sigma)$ is an EQS with an EQS principal normal $\mathfrak{N}$. Furthermore, $\pi(\varrho, \mathfrak{T}) = 0$. Therefore, $\varrho(\sigma)$ is an EQS normal curve with an EQS normal vector $\mathfrak{N}$.

By finding $\frac{K_1}{K_2}$, we obtain:

$$\frac{K_1}{K_2} = 1,$$

$$\frac{1}{K_1}[K_2 - (\frac{K_1}{K_2})'] = 1.$$

Hence, Equation (13) is satisfied.
Moreover,

$$\pi(\varrho,\ \mathfrak{N}) = -\frac{e^{\frac{2\sigma}{\sqrt{2}}}}{2} = -\rho^2,$$

$$\pi(\varrho,\ \mathfrak{B}) = -\frac{e^{\frac{2\sigma}{\sqrt{2}}}}{2} = \frac{-\rho^2}{K_1}[K_2 - (\frac{K_1}{K_2})'].$$

Thus, Equations (16) and (17) are satisfied.

$$K_1 = K_2, \quad \pi(\varrho, \varrho) = 0.$$

Thus, Equation (18) is satisfied and therefore, all results in Theorem 1 are satisfied.

```
In[8]:= ParametricPlot3D[{s, s Sin[Log[s]], s Cos[ Log[s]]}, {s, 1, 100}]
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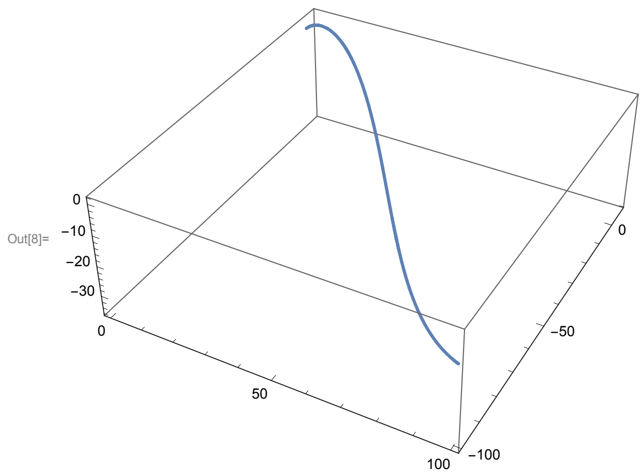

Figure 1. An EQS curve with an EQS normal satisfying the results.

6. Conclusions

In this paper, we defined the "equiform parameter (EQP)", "equiform frame" and "equiform formulas (EQF)" in Minkowski three-space. Further, we defined spacelike normal curves (SPN) in Minkowski three-space and presented a variety of descriptions of these curves with equiform spacelike (EQS) or equiform timelike (EQN) principal normals in Minkowski three-space. Furthermore, we gave some characterizations of these curves in Minkowski three-space.

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