mathematics

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Solving System of Mixed Variational Inclusions Involving Generalized Cayley Operator and Generalized Yosida Approximation Operator with Error Terms in q-Uniformly Smooth Space

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Abstract: In this paper, we solve a system of mixed variational inclusions involving a generalized Cayley operator and the generalized Yosida approximation operator. An iterative algorithm is suggested to discuss the convergence analysis. We have shown that our system admits a unique solution by using the properties of q-uniformly smooth Banach space, and we discuss the convergence criteria for sequences generated by iterative algorithm. Two examples are constructed, and an application is provided.

Keywords: system; inclusion; operator; solution; convergence

MSC: 40H09; 47J40

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Citation: Ahmad, R.; Ishtyak, M.; Rajpoot, A.K.; Wang, Y. Solving System of Mixed Variational Inclusions Involving Generalized Cayley Operator and Generalized Yosida Approximation Operator with Error Terms in q-Uniformly Smooth Space. *Mathematics* **2022**, *10*, 4131. https://doi.org/10.3390/math10214131

Academic Editor: Enrique Alfonso Sánchez Pérez

Received: 29 September 2022
Accepted: 2 November 2022
Published: 5 November 2022

1. Introduction

By using the clever transformation, Baiocchi [1] investigated the fact that the free boundary value problem associated with seepage through an earth dam is equivalent to a class of variational inequality. Variational inequalities have a tremendous impact in this field, and consequently variational inequalities are applied rather than other methods.

The study of variational inequality theory is twofold. On the one hand, it reveals the basic facts regarding the qualitative behaviour of solutions related to many nonlinear boundary value problems. On the other hand, it produces effective numerical methods to solve free and moving boundary value problems (see for more details [2–5]).

It is well known that projection methods are not applicable to solve variational inclusion problems and such an issue was solved by Hassouni and Moudafi [6] by using the resolvent operator technique. Generalized resolvent operators were introduced by several authors by using accretive operators, H-accretive operators, m-accretive operators, etc. (see [7–9]). The system of variational inclusions can be regarded as a natural extension of the system of variational inequalities. Many problems related to mathematical and convex analysis, biological sciences, image recovery processes, biomedical sciences, elasticity, data compression, mechanics, computer programming, and mathematical physics, etc., can be worked out by using the framework of system of variational inclusions. Due to their applications, system of variational inclusions (inequalities) were considered and studied by many authors, that is, by Pang [10], Cohen and Chaplais [11], Bianchi [12], Ansari and Yao [13], Yan et al. [14], etc. For more details on variational inclusions and their systems, we refer to [15–27] and references therein.

It is well-known that monotone operators can be structured into single-valued Lipschitzian monotone operators through a process known as the Yosida approximation process. The applications of the Yosida approximation operator can be found while dealing with wave equations, heat equations, linearized equations of coupled sound, and heat flow, etc.

(see [28–30]). The Cayley transform is a mapping between skew symmetric matrices and special orthogonal matrices. In real, complex, and quaternionic analysis, many applications of the Cayley transform can be found (see [31–34]).

Compared with other normed spaces, Banach spaces have the advantage that it is easy to obtain the convergence of a sequence of vectors. That is why we choose a q-uniformly smooth Banach space in order to achieve better results.

Conjoining the above facts, in this paper, we study a system of mixed variational inclusions which involve a generalized Cayley operator and a generalized Yosida approximation operator in a q-uniformly smooth Banach space. To obtain convergence result, we define an algorithm with error terms to take into account inexact computation. We prove that our system admits a unique solution, and we discuss convergence criteria for sequences generated by algorithm.

In support of our system, we provide an example. Moreover, another example is constructed to show that the generalized Cayley operator is Lipschitz-continuous and the generalized Yosida approximation operator is Lipschitz-continuous, as well as strongly accretive. Lipschitz continuity of both the operators is shown in Figures in Section 5, respectively. An application is also given.

2. Fundamental Concepts

Let $\tilde{E}$ be real Banach space with its topological dual $\tilde{E}^*$. The norm on $\tilde{E}$ is denoted by $\|\cdot\|$, duality pairing between $\tilde{E}$ and $\tilde{E}^*$ by $\langle\cdot,\cdot\rangle$ and the class of subsets of $\tilde{E}$ by $2^{\tilde{E}}$.

It is well known that generalized duality mapping $J_q : \tilde{E} \to 2^{\tilde{E}^*}$, is defined by

$$J_q(x) = \left\{ f \in \tilde{E} : \langle x, f\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1} \right\}, \text{ for all } x \in \tilde{E}.$$

For $q = 2$, generalized duality mapping reduces to normalized duality mapping. Note that if $\tilde{E}$ is uniformly smooth then J_q is single-valued.

A Banach space $\tilde{E}$ is called q-uniformly smooth if

$$\rho_{\tilde{E}}(t) \le kt^q,$$

where $q > 1$, $k > 0$ are constants and $\rho_{\tilde{E}}(t)$ is the modulus of smoothness.

The following result of Xu [35] is important to prove the main result.

Lemma 1. *Let $\tilde{E}$ be a real, uniformly smooth Banach space. Then $\tilde{E}$ is q-uniformly smooth if and only if there exists a constant C_q such that for all $x, y \in \tilde{E}$,*

$$\|x + y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + C_q\|y\|^q.$$

Definition 1 ([36])**.** *Let $P : \tilde{E} \to \tilde{E}$ be a mapping. Then, P is called*

(i) *accretive, if*

$$\langle P(x) - P(y), J_q(x - y)\rangle \ge 0, \text{ for all } x, y \in \tilde{E},$$

(ii) *strictly accretive, if*

$$\langle P(x) - P(y), J_q(x - y)\rangle > 0, \text{ for all } x, y \in \tilde{E},$$

and the equality holds if and only if $x = y$,

(iii) *strongly accretive, if there exists a constant $r > 0$ such that*

$$\langle P(x) - P(y), J_q(x - y)\rangle \ge r\|x - y\|^q, \text{ for all } x, y \in \tilde{E},$$

(iv) *Lipschitz-continuous, if there exists a constant $\lambda_P > 0$, such that*

$$\|P(x) - P(y)\| \le \lambda_P\|x - y\|, \text{ for all } x, y \in \tilde{E},$$

(*v*) *ξ_P-expansive, if there exists a constant $\xi_P > 0$ such that*

$$\|P(x) - P(y)\| \geq \xi_P \|x - y\|, \text{ for all } x, y \in \tilde{E}.$$

Definition 2 ([36])**.** *A multi-valued mapping $M : \tilde{E} \to 2^{\tilde{E}}$ is said to be accretive if for all $x, y \in \tilde{E}$,*

$$\langle u - v, J_q(x - y) \rangle \geq 0, \text{ for all } u \in M(x),\ v \in M(y).$$

Definition 3 ([7])**.** *Let $P : \tilde{E} \to \tilde{E}$ be a mapping. A multi-valued mapping $M : \tilde{E} \to 2^{\tilde{E}}$ is said to be P-accretive if M is accretive and $[P + \lambda M](\tilde{E}) = \tilde{E}$, for all $\lambda > 0$.*

Definition 4 ([7])**.** *Let $P : \tilde{E} \to \tilde{E}$ be single-valued mapping and $M : \tilde{E} \to 2^{\tilde{E}}$ be P-accretive multi-valued mapping. The generalized resolvent operator $R^M_{P,\lambda} : \tilde{E} \to \tilde{E}$ is defined as*

$$R^M_{P,\lambda}(x) = [P + \lambda M]^{-1}(x), \textit{ for all } x \in \tilde{E}.$$

Theorem 1 ([36])**.** *Let $P : \tilde{E} \to \tilde{E}$ be strongly accretive operator with constant r and $M : \tilde{E} \to 2^{\tilde{E}}$ be P-accretive multi-valued mapping. Then,*

$$\left\| R^M_{P,\lambda}(x) - R^M_{P,\lambda}(y) \right\| \leq \frac{1}{r} \|x - y\|, \textit{ for all } x, y \in \tilde{E}.$$

That is, the generalized resolvent operator $R^M_{P,\lambda}$ is Lipschitz-continuous.

Definition 5 ([36])**.** *The generalized Cayley operator $C^M_{P,\lambda} : \tilde{E} \to \tilde{E}$ is defined as*

$$C^M_{P,\lambda} = \left[2R^M_{P,\lambda} - P \right](x), \textit{ for all } x \in \tilde{E} \text{ and } \lambda > 0.$$

Definition 6 ([36])**.** *The generalized Yosida approximation operator $Y^M_{P,\lambda} : \tilde{E} \to \tilde{E}$ is defined as*

$$Y^M_{P,\lambda} = \frac{1}{\lambda} \left[P - R^M_{P,\lambda} \right](x), \textit{ for all } x \in \tilde{E} \text{ and } \lambda > 0.$$

Now, we prove that the generalized Cayley operator is Lipschitz-continuous and the generalized Yosida approximation operator is strongly accretive as well as Lipschitz-continuous.

Proposition 1. *The generalized Cayley operator is λ_C-Lipschitz-continuous, where $\lambda_C = \left(\frac{2 + \lambda_P r}{r} \right), \lambda_P, r > 0$ are constants and $P : \tilde{E} \to \tilde{E}$ is λ_P-Lipschitz-continuous mapping.*

Proof. For any $x, y \in \tilde{E}$ and using the Lipschitz continuity of $R^M_{P,\lambda}$ and P, we evaluate

$$\begin{aligned}
\left\| C^M_{P,\lambda}(x) - C^M_{P,\lambda}(y) \right\| &= \left\| \left[2R^M_{P,\lambda}(x) - P(x) \right] - \left[2R^M_{P,\lambda}(y) - P(y) \right] \right\| \\
&= \left\| 2\left[R^M_{P,\lambda}(x) - R^M_{P,\lambda}(y) \right] - [P(x) - P(y)] \right\| \\
&\leq 2\left\| R^M_{P,\lambda}(x) - R^M_{P,\lambda}(y) \right\| + \|P(x) - P(y)\| \\
&\leq \frac{2}{r}\|x - y\| + \lambda_P \|x - y\| \\
&= \frac{2 + \lambda_P r}{r} \|x - y\|.
\end{aligned}$$

Thus,

$$\left\|C^M_{P,\lambda}(x) - C^M_{P,\lambda}(y)\right\| \leq \lambda_C\|x-y\|.$$

That is, the generalized Cayley operator is λ_C-Lipschitz-continuous. □

Proposition 2. *The generalized Yosida approximation operator $Y^M_{P,\lambda} : \tilde{E} \to \tilde{E}$ is*

(*i*) *λ_Y-Lipschitz-continuous, where $\lambda_Y = \left(\dfrac{\lambda_P r + 1}{\lambda r}\right), \lambda_P, r, \lambda > 0$ are constants and P is λ_P-Lipschitz-continuous,*

(*ii*) *δ_Y-strongly accretive, where $\delta_Y = \left(\dfrac{r^2-1}{\lambda r}\right), r, \lambda > 0$ are constants, $r^2 > 1$ and P is r-strongly accretive.*

Proof. (*i*) For any $x, y \in \tilde{E}, \lambda > 0$ and using the Lipschitz continuity of P and $R^M_{P,\lambda}$, we have

$$\begin{aligned}
\left\|Y^M_{P,\lambda}(x) - Y^M_{P,\lambda}(y)\right\| &= \frac{1}{\lambda}\left\|\left[P(x) - R^M_{P,\lambda}(x)\right] - \left[P(y) - R^M_{P,\lambda}(y)\right]\right\| \\
&= \frac{1}{\lambda}\|P(x) - P(y)\| + \frac{1}{\lambda}\left\|R^M_{P,\lambda}(x) - R^M_{P,\lambda}(y)\right\| \\
&\leq \frac{\lambda_P}{\lambda}\|x-y\| + \frac{1}{\lambda\, r}\|x-y\| \\
&= \left(\frac{\lambda_P r + 1}{\lambda r}\right)\|x-y\|,
\end{aligned}$$

that is,

$$\left\|Y^M_{P,\lambda}(x) - Y^M_{P,\lambda}(y)\right\| \leq \lambda_Y\|x-y\|.$$

Thus, the generalized Yosida approximation operator $Y^M_{P,\lambda}$ is λ_Y-Lipschitz-continuous.

(*ii*) For any $x, y \in \tilde{E}$ and using the Lipschitz continuity of $R^M_{P,\lambda}$, we have

$$\begin{aligned}
\left\langle Y^M_{P,\lambda}(x) - Y^M_{P,\lambda}(y), J_q(x-y)\right\rangle &= \frac{1}{\lambda}\left\langle\left[P(x) - R^M_{P,\lambda}(x)\right] - \left[P(y) - R^M_{P,\lambda}(y)\right], J_q(x-y)\right\rangle \\
&= \frac{1}{\lambda}\Big[\langle P(x) - P(y), J_q(x-y)\rangle \\
&\qquad -\left\langle R^M_{P,\lambda}(x) - R^M_{P,\lambda}(y), J_q(x-y)\right\rangle\Big] \\
&\geq \frac{1}{\lambda}\left[r\|x-y\|^q - \left\|R^M_{P,\lambda}(x) - R^M_{P,\lambda}(y)\right\|^q\right] \\
&\geq \frac{1}{\lambda}\left[r\|x-y\|^q - \frac{1}{r}\|x-y\|^q\right] \\
&= \left(\frac{r^2-1}{\lambda r}\right)\|x-y\|^q,
\end{aligned}$$

that is,

$$\left\langle Y^M_{P,\lambda}(x) - Y^M_{P,\lambda}(y), J_q(x-y)\right\rangle \geq \delta_Y\|x-y\|^q.$$

Thus, the generalized Yosida approximation operator $Y^M_{P,\lambda}$ is δ_Y-strongly accretive. □

Lemma 2 ([37])**.** *Let $\{a_n\}_{n=1}^{\infty}$ be a non-negative real sequence satisfying*

$$a_{n+1} \leq (1-\alpha_n)a_n + \sigma_n,$$

where $\alpha_n \in [0,1], \sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sigma_n = O(\alpha_n)$. *Then* $\lim_{n\to\infty} a_n = 0$.

3. Framework of the Problem and Fixed-Point Formulation

Let $\tilde{E}$ be real Banach space. Let $A, B, g, P_1, P_2 : \tilde{E} \to \tilde{E}$ be single-valued mappings and $M, N : \tilde{E} \to 2^{\tilde{E}}$ be multi-valued mappings. Let for $\lambda, \rho > 0$, $C^M_{P_1,\lambda} : \tilde{E} \to \tilde{E}$ be a generalized Cayley operator associated with the generalized resolvent operator $R^M_{P_1,\lambda} = [P_1 + \lambda M]^{-1}$ and $Y^N_{P_2,\rho} : \tilde{E} \to \tilde{E}$ be a generalized Yosida approximation operator associated with the generalized resolvent operator $R^N_{P_2,\rho} = [P_2 + \rho N]^{-1}$. We consider following system of mixed variational inclusions involving generalized Cayley operator and generalized Yosida approximation operator.

Find $(x, y) \in \tilde{E} \times \tilde{E}$, such that

$$\begin{aligned} 0 &\in C^M_{P_1,\lambda}(A(x)) + M(g(y)), \\ 0 &\in Y^N_{P_2,\rho}(B(y)) + N(g(x)). \end{aligned} \tag{1}$$

For suitable choices of operators involved in system (1), one can obtain many previously studied systems of variational inclusions and variational inequalities.

The following Lemma ensures the equivalence between system of mixed variational inclusions involving generalized Cayley operator and generalized Yosida approximation operator (1) and a system of equations.

Lemma 3. *The system of mixed variational inclusions involving generalized Cayley operator and the generalized Yosida approximation operator* (1) *admits a solution* $(x, y) \in \tilde{E} \times \tilde{E}$ *if and only if the following equations are satisfied:*

$$g(y) = R^M_{P_1,\lambda}\Big[P_1(g(y)) - \lambda C^M_{P_1,\lambda}(A(x))\Big], \tag{2}$$

$$g(x) = R^N_{P_2,\rho}\Big[P_2(g(x)) - \rho Y^N_{P_2,\rho}(B(y))\Big]. \tag{3}$$

Proof. From Equation (2), we have

$$g(y) = R^M_{P_1,\lambda}\Big[P_1(g(y)) - \lambda C^M_{P_1,\lambda}(A(x))\Big].$$

By using the definition of generalized resolvent operator $R^M_{P_1,\lambda}$, we obtain

$$\begin{aligned} g(y) = [P_1 + \lambda M]^{-1}\Big[P_1(g(y)) - \lambda C^M_{P_1,\lambda}(A(x))\Big], \\ P_1(g(y)) + \lambda M(g(y)) = P_1(g(y)) - \lambda C^M_{P_1,\lambda}(A(x)), \\ \lambda M(g(y)) = -\lambda C^M_{P_1,\lambda}(A(x)), \end{aligned}$$

which impies that

$$0 \in C^M_{P_1,\lambda}(A(x)) + M(g(y)).$$

In a similar way, by using Equation (3) and definition of generalized resolvent operator $R^N_{P_2,\rho}$, we obtain

$$0 \in Y^M_{P_2,\rho}(B(y)) + N(g(x)).$$

□

Theorem 2. *Let* $\tilde{E}$ *be a q-uniformly smooth Banach space. Let* $A, B, g, P_1, P_2 : \tilde{E} \to \tilde{E}$ *be single-valued mappings such that* A *is* λ_A*-Lipschitz-continuous,* B *is* λ_B*-Lipschitz-continuous and* ξ_B*-expansive,* g *is* λ_g*-Lipschitz-continuous,* P_1 *is* λ_{P_1}*-Lipschitz-continuous and* r_1*-strongly accretive,* P_2 *is* λ_{P_2}*-Lipschitz-continuous and* r_2*-strongly accretive. Suppose that* $M, N : \tilde{E} \to 2^{\tilde{E}}$

be multi-valued mappings and generalized resolvent operators $R^M_{P_1,\lambda}$ *and* $R^N_{P_2,\rho}$ *are* $\dfrac{1}{r_1}$ *and* $\dfrac{1}{r_2}$*-Lipschitz-continuous, respectively. Suppose that generalized Cayley operator* $C^M_{P_1,\lambda}$ *is* λ_C*-Lipschitz-continuous, generalized Yosida approximation operator* $Y^N_{P_2,\rho}$ *is* λ_Y*-Lipschitz-continuous and* δ_Y*-strongly accretive with respect to B. Suppose that the following conditions are satisfied:*

$$\left.\begin{aligned} &0 < \left(\frac{\lambda_g r_1 + \lambda_{P_1}\lambda_g}{r_1}\right) + \left(\tfrac{1}{r_2}\sqrt[q]{[\lambda_B^q - q\rho\delta_Y\xi_B^q + \rho q C_q \lambda_Y^q \lambda_B^q]}\right) < 1, \\ &0 < \frac{\lambda\lambda_C\lambda_A}{r_1} + \left(\frac{\lambda_g r_2 + \lambda_{P_2}\lambda_g}{r_2}\right) < 1, \end{aligned}\right\}, \tag{4}$$

where $\lambda_C = \frac{2+\lambda_{P_1} r}{r}, \delta_Y = \frac{r^2-1}{\lambda r}$ *and* $\lambda_Y = \frac{\lambda_{P_2} r+1}{\lambda r}$*. Then, the system of mixed variational inclusions involving generalized Cayley operator and the generalized Yosida approximation operator* (1) *admits a unique solution.*

Proof. For each $(x,y) \in \tilde{E} \times \tilde{E}$, we define the mappings

$$\begin{aligned} P_\lambda(x,y) &= g(y) - R^M_{P_1,\lambda}\Big[P_1(g(y)) - \lambda C^M_{P_1,\lambda}(A(x))\Big], \\ \text{and} \quad P_\rho(x,y) &= g(x) - R^N_{P_2,\rho}\Big[P_2(g(x)) - \rho Y^N_{P_2,\rho}(B(y))\Big]. \end{aligned}$$

By using the Lipschitz continuity of $g, R^M_{P_1,\lambda}, P_1, A, C^M_{P_1,\lambda}$ and strong accretiveness of P_1, for $(x_1,y_1),(x_2,y_2) \in \tilde{E} \times \tilde{E}$, we obtain

$$\begin{aligned} \|P_\lambda(x_1,y_1) - P_\lambda(x_2,y_2)\| &= \Big\|\Big(g(y_1) - R^M_{P_1,\lambda}\Big[P_1(g(y_1)) - \lambda C^M_{P_1,\lambda}(A(x_1))\Big]\Big) \\ &\quad -\Big(g(y_2) - R^M_{P_1,\lambda}\Big[P_1(g(y_2)) - \lambda C^M_{P_1,\lambda}(A(x_2))\Big]\Big)\Big\| \\ &= \Big\|g(y_1) - g(y_2) - \Big(R^M_{P_1,\lambda}\Big[P_1(g(y_1)) - \lambda C^M_{P_1,\lambda}(A(x_1))\Big] \\ &\quad -R^M_{P_1,\lambda}\Big[P_1(g(y_2)) - \lambda C^M_{P_1,\lambda}(A(x_2))\Big]\Big)\Big\| \\ &\leq \|g(y_1) - g(y_2)\| + \Big\|R^M_{P_1,\lambda}\Big[P_1(g(y_1)) - \lambda C^M_{P_1,\lambda}(A(x_1))\Big] \\ &\quad -R^M_{P_1,\lambda}\Big[P_1(g(y_2)) - \lambda C^M_{P_1,\lambda}(A(x_2))\Big]\Big\| \\ &\leq \|g(y_1) - g(y_2)\| + \frac{1}{r_1}\Big\|P_1(g(y_1)) - P_1(g(y_2)) \\ &\quad -\lambda\Big[C^M_{P_1,\lambda}(A(x_1)) - C^M_{P_1,\lambda}(A(x_2))\Big]\Big\| \\ &\leq \|g(y_1) - g(y_2)\| + \frac{1}{r_1}\|P_1(g(y_1)) - P_1(g(y_2))\| \\ &\quad +\frac{\lambda}{r_1}\Big\|C^M_{P_1,\lambda}(A(x_1)) - C^M_{P_1,\lambda}(A(x_2))\Big\| \\ &\leq \lambda_g\|y_1 - y_2\| + \frac{\lambda_{P_1}}{r_1}\|g(y_1) - g(y_2)\| \\ &\quad +\frac{\lambda\lambda_C}{r_1}\|A(x_1) - A(x_2)\| \\ &\leq \lambda_g\|y_1 - y_2\| + \frac{\lambda_{P_1}\lambda_g}{r_1}\|y_1 - y_2\| + \frac{\lambda\lambda_C\lambda_A}{r_1}\|x_1 - x_2\| \\ &\leq \left(\frac{\lambda_g r_1 + \lambda_{P_1}\lambda_g}{r_1}\right)\|y_1 - y_2\| + \frac{\lambda\lambda_C\lambda_A}{r_1}\|x_1 - x_2\|. \end{aligned} \tag{5}$$

By using the same arguments as for (5), we obtain

$$\begin{aligned}
\|P_\rho(x_1,y_1)-P_\rho(x_2,y_2)\| &= \Big\|\Big(g(x_1)-R^N_{P_2,\rho}\Big[P_2(g(x_1))-\rho Y^N_{P_2,\rho}(B(y_1))\Big]\Big)\\
&\quad -\Big(g(x_2)-R^N_{P_2,\rho}\Big[P_2(g(x_2))-\rho Y^N_{P_2,\rho}(B(y_2))\Big]\Big)\Big\|\\
&= \Big\|g(x_1)-g(x_2)-\Big(R^N_{P_2,\rho}\Big[P_2(g(x_1))-\rho Y^N_{P_2,\rho}(B(y_1))\Big]\\
&\quad -R^N_{P_2,\rho}\Big[P_2(g(x_2))-\rho Y^N_{P_2,\rho}(B(y_2))\Big]\Big)\Big\|\\
&\le \|g(x_1)-g(x_2)\|+\Big\|R^N_{P_2,\rho}\Big[P_2(g(x_1))-\rho Y^N_{P_2,\rho}(B(y_1))\Big]\\
&\quad -R^N_{P_2,\rho}\Big[P_2(g(x_2))-\rho Y^N_{P_2,\rho}(B(y_2))\Big]\Big\|\\
&\le \|g(x_1)-g(x_2)\|+\frac{1}{r_2}\Big\|P_2(g(x_1))-P_2(g(x_2))\\
&\quad -\rho\Big[Y^N_{P_2,\rho}(B(y_1))-Y^N_{P_2,\rho}(B(y_2))\Big]\Big\|\\
&= \|g(x_1)-g(x_2)\|+\frac{1}{r_2}\Big\|P_2(g(x_1))-P_2(g(x_2))\\
&\quad +[B(y_1)-B(y_2)]-[B(y_1)-B(y_2)]\\
&\quad -\rho\Big[Y^N_{P_2,\rho}(B(y_1))-Y^N_{P_2,\rho}(B(y_2))\Big]\Big\|\\
&\le \|g(x_1)-g(x_2)\|+\frac{1}{r_2}\|P_2(g(x_1))-P_2(g(x_2))\|\\
&\quad +\frac{1}{r_2}\|B(y_1)-B(y_2)\|\\
&\quad +\frac{1}{r_2}\Big\|[B(y_1)-B(y_2)]-\rho\Big[Y^N_{P_2,\rho}(B(y_1))-Y^N_{P_2,\rho}(B(y_2))\Big]\Big\|\\
&\le \lambda_g\|x_1-x_2\|+\frac{\lambda_{P_2}}{r_2}\|g(x_1)-g(x_2)\|+\frac{\lambda_B}{r_2}\|y_1-y_2\|\\
&\quad +\frac{1}{r_2}\Big\|[B(y_1)-B(y_2)]-\rho\Big[Y^N_{P_2,\rho}(B(y_1))-Y^N_{P_2,\rho}(B(y_2))\Big]\Big\|\\
&\le \lambda_g\|x_1-x_2\|+\frac{\lambda_{P_2}\lambda_g}{r_2}\|x_1-x_2\|+\frac{\lambda_B}{r_2}\|y_1-y_2\|\\
&\quad +\frac{1}{r_2}\Big\|[B(y_1)-B(y_2)]-\rho\Big[Y^N_{P_2,\rho}(B(y_1))-Y^N_{P_2,\rho}(B(y_2))\Big]\Big\|.
\end{aligned} \tag{6}$$

Applying the Lipschitz continuity and expansiveness of B, Lipschitz continuity and strongly accretiveness of $Y^N_{P_2,\rho}$ with respect to B, we evaluate

$$\begin{aligned}
&\Big\|B(y_1)-B(y_2)-\rho\Big[Y^N_{P_2,\rho}(B(y_1))-Y^N_{P_2,\rho}(B(y_2))\Big]\Big\|^q\\
\le\ & \|B(y_1)-B(y_2)\|^q-q\rho\Big\langle Y^N_{P_2,\rho}(B(y_1))-Y^N_{P_2,\rho}(B(y_2)),J_q(B(y_1)-B(y_2))\Big\rangle\\
&+\rho C_q\Big\|Y^N_{P_2,\rho}(B(y_1))-Y^N_{P_2,\rho}(B(y_2))\Big\|^q\\
\le\ & \lambda_B^q\|y_1-y_2\|^q-q\rho\delta_Y\|B(y_1)-B(y_2)\|^q+\rho C_q\lambda_Y^q\|B(y_1)-B(y_2)\|^q\\
\le\ & \lambda_B^q\|y_1-y_2\|^q-q\rho\delta_Y\xi_B^q\|y_1-y_2\|^q+\rho C_q\lambda_Y^q\lambda_B^q\|y_1-y_2\|^q\\
=\ & [\lambda_B^q-q\rho\delta_Y\xi_B^q+\rho C_q\lambda_Y^q\lambda_B^q]\|y_1-y_2\|^q.
\end{aligned}$$

It follows that

$$\Big\|B(y_1)-B(y_2)-\rho\Big[Y^N_{P_2,\rho}(B(y_1))-Y^N_{P_2,\rho}(B(y_2))\Big]\Big\| = \sqrt[q]{\lambda_B^q-q\rho\delta_Y\xi_B^q+\rho C_q\lambda_Y^q\lambda_B^q}\|y_1-y_2\|. \tag{7}$$

Combining (6) and (7), we obtain

$$\begin{aligned}\|P_\rho(x_1,y_1)-P_\rho(x_2,y_2)\| &\leq \lambda_g\|x_1-x_2\|+\frac{\lambda_{P_2}\lambda_g}{r_2}\|x_1-x_2\|+\frac{\lambda_B}{r_2}\|y_1-y_2\| \\ &\quad+\frac{1}{r_2}\left[\sqrt[q]{\lambda_B^q-q\rho\delta_Y\xi_B^q+\rho C_q\lambda_Y^q\lambda_B^q}\|y_1-y_2\|\right] \\ &=\left(\frac{\lambda_g r_2+\lambda_{P_2}\lambda_g}{r_2}\right)\|x_1-x_2\| \\ &\quad+\frac{1}{r_2}\left(\lambda_B+\sqrt[q]{\lambda_B^q-q\rho\delta_Y\xi_B^q+\rho C_q\lambda_Y^q\lambda_B^q}\right)\|y_1-y_2\|.\end{aligned} \tag{8}$$

Adding (5) and (8), we have

$$\begin{aligned} & \|P_\lambda(x_1,y_1)-P_\lambda(x_2,y_2)\|+\|P_\rho(x_1,y_1)-P_\rho(x_2,y_2)\| \\ \leq\ & \left(\frac{\lambda_g r_1+\lambda_{P_1}\lambda_g}{r_1}\right)\|y_1-y_2\|+\frac{\lambda\lambda_C\lambda_A}{r_1}\|x_1-x_2\| \\ & +\left(\frac{\lambda_g r_2+\lambda_{P_2}\lambda_g}{r_2}\right)\|x_1-x_2\| \\ & +\frac{1}{r_2}\left(\lambda_B+\sqrt[q]{\lambda_B^q-q\rho\delta_Y\xi_B^q+\rho C_q\lambda_Y^q\lambda_B^q}\right)\|y_1-y_2\| \\ =\ & \left[\left(\frac{\lambda_g r_1+\lambda_{P_1}\lambda_g}{r_1}\right)+\frac{1}{r_2}\left(\lambda_B+\sqrt[q]{\lambda_B^q-q\rho\delta_Y\xi_B^q+\rho C_q\lambda_Y^q\lambda_B^q}\right)\right]\|y_1-y_2\| \\ & +\left[\frac{\lambda\lambda_C\lambda_A}{r_1}+\left(\frac{\lambda_g r_2+\lambda_{P_2}\lambda_g}{r_2}\right)\right]\|x_1-x_2\|.\end{aligned}$$

It follows that

$$\|P_\lambda(x_1,y_1)-P_\lambda(x_2,y_2)\|+\|P_\rho(x_1,y_1)-P_\rho(x_2,y_2)\|\leq\mathcal{V}(\theta)\left[\|x_1-x_2\|+\|y_1-y_2\|\right],$$

where

$$\begin{aligned}\mathcal{V}(\theta) &= \max\left\{\left[\left(\frac{\lambda_g r_1+\lambda_{P_1}\lambda_g}{r_1}\right)+\frac{1}{r_2}\left(\lambda_B+\sqrt[q]{\lambda_B^q-q\rho\delta_Y\xi_B^q+\rho C_q\lambda_Y^q\lambda_B^q}\right)\right],\right. \\ &\qquad \left.\left[\frac{\lambda\lambda_C\lambda_A}{r_1}+\left(\frac{\lambda_g r_2+\lambda_{P_2}\lambda_g}{r_2}\right)\right]\right\}.\end{aligned}$$

Because $\tilde{E}\times\tilde{E}$ is a Banach space, we define $\tilde{P}_{\lambda,\rho}:\tilde{E}\times\tilde{E}\to\tilde{E}\times\tilde{E}$, such that

$$\tilde{P}_{\lambda,\rho}(x,y)=\big(P_\lambda(x,y),P_\rho(x,y)\big),\ \text{for all}\ \ (x,y)\in\tilde{E}\times\tilde{E}.$$

Condition (4) implies that $0<\mathcal{V}(\theta)<1$. Thus,

$$\|\tilde{P}_{\lambda,\rho}(x_1,y_1)-\tilde{P}_{\lambda,\rho}(x_2,y_2)\|\leq\mathcal{V}(\theta)\|(x_1,y_1)-(x_2,y_2)\|. \tag{9}$$

From (9) it is clear that $\tilde{P}_{\lambda,\rho}$ is a contraction mapping. By using the Banach contraction principle, it follows that there exists a unique $(x,y)\in\tilde{E}\times\tilde{E}$, such that

$$\tilde{P}_{\lambda,\rho}(x,y)=(x,y).$$

That is,

$$\begin{aligned} g(y) &= R^M_{P_1,\lambda}\left[P_1(g(y))-\lambda C^M_{P_1,\lambda}(A(x))\right], \\ g(x) &= R^N_{P_2,\rho}\left[P_2(g(x))-\rho Y^N_{P_2,\rho}(B(y))\right].\end{aligned}$$

By Lemma 3, we conclude that (x, y) is the unique solution of system of mixed variational inclusions involving the generalized Cayley operator and the generalized Yosida approximation operator (1). □

4. Algorithm and Convergence Result

An algorithm is designed to establish convergence result for system of mixed variational inclusions involving the generalized Cayley operator and the generalized Yosida approximation operator (1).

By using Lemma 3, we suggest the following iterative algorithm for solving system (1).

Now, we prove convergence of sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 1.

Algorithm 1 Iterative algorithm for solving system (1).

For initial points $x_0, y_0 \in \tilde{E}$, let

$$g(y_1) = (1-\alpha_n)g(y_0) + \alpha_n R^M_{P_1,\lambda}[P_1(g(y_0)) - \lambda C^M_{P_1,\lambda}(A(x_0))], \tag{10}$$

$$g(x_1) = (1-\alpha_n)g(x_0) + \alpha_n R^N_{P_2,\rho}[P_2(g(x_0)) - \rho Y^N_{P_2,\rho}(B(y_0))]. \tag{11}$$

For next iterative points $x_1, y_1 \in \tilde{E}$, let

$$g(y_2) = (1-\alpha_n)g(y_1) + \alpha_n R^M_{P_1,\lambda}[P_1(g(y_1)) - \lambda C^M_{P_1,\lambda}(A(x_1))], \tag{12}$$

$$g(x_2) = (1-\alpha_n)g(x_1) + \alpha_n R^N_{P_2,\rho}[P_2(g(x_1)) - \rho Y^N_{P_2,\rho}(B(y_1))]. \tag{13}$$

Continuing in the same manner, compute sequences $\{x_n\}$ and $\{y_n\}$ by the scheme:

$$g(y_{n+1}) = (1-\alpha_n)g(y_n) + \alpha_n R^M_{P_1,\lambda}[P_1(g(y_n)) - \lambda C^M_{P_1,\lambda}(A(x_n))] + \alpha_n e_n, \tag{14}$$

$$g(x_{n+1}) = (1-\alpha_n)g(x_n) + \alpha_n R^N_{P_2,\rho}[P_2(g(x_n)) - \rho Y^N_{P_2,\rho}(B(y_n))] + \alpha_n r_n, \tag{15}$$

where $\lambda, \rho > 0$ are constants, $\{e_n\} \in \tilde{E}$ and $\{r_n\} \in \tilde{E}$ are sequences introduced as error terms for inexact computation such that $\lim\limits_{n\to\infty}\|e_n\| = 0 = \lim\limits_{n\to\infty}\|r_n\|$ and $\{\alpha_n\}$ is a sequence such that $0 < \alpha_n \le 1, \forall n \ge 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$.

Theorem 3. *Let all the conditions of Theorem 2 be satisfied and additionally if the following conditions are satisfied:*

(i) $\lim\limits_{n\to\infty}\|e_n\| = 0 = \lim\limits_{n\to\infty}\|r_n\|$,

(ii) $0 < \alpha_n \le 1$ *and* $\sum_{n=0}^{\infty}\alpha_n = +\infty$, *for all* n,

(iii) $\lambda_g \le \delta_g$ *and* $\delta_g \ge 1$,

(iv) g *is* δ_g*-strongly accretive,*

then sequences $\{x_n\}$ and $\{y_n\}$ defined by Algorithm 1 converge strongly to x and y, respectively, where (x, y) is the unique solution of system of mixed variational inclusions involving the generalized Cayley operator and the generalized Yosida approximation operator (1).

Proof. It follows from Theorem 2 that the system of mixed variational inclusions involving the generalized Cayley operator and the generalized Yosida approximation operator (1) has a unique solution $(x, y) \in \tilde{E} \times \tilde{E}$. By Lemma 3, we have

$$g(y^*) = (1-\alpha_n)g(y^*) + \alpha_n R^M_{P_1,\lambda}[P_1(g(y^*)) - \lambda C^M_{P_1,\lambda}(A(x^*))], \tag{16}$$

$$\text{and} \quad g(x^*) = (1-\alpha_n)g(x^*) + \alpha_n R^N_{P_2,\rho}[P_2(g(x^*)) - \rho Y^N_{P_2,\rho}(B(y^*))]. \tag{17}$$

By using (14) of Algorithm 1, (16) and Lipschitz continuity of $g, R^M_{P_1,\lambda}, P_1, A, C^M_{P_1,\lambda}$, we evaluate

$$
\begin{aligned}
\|g(y_{n+1})-g(y^*)\| &= \Big\|(1-\alpha_n)g(y_n)+\alpha_n R^M_{P_1,\lambda}\Big[P_1(g(y_n))-\lambda C^M_{P_1,\lambda}(A(x_n))\Big]+\alpha_n e_n \\
&\quad -\Big[(1-\alpha_n)g(y^*)+\alpha_n R^M_{P_1,\lambda}\Big[P_1(g(y^*))-\lambda C^M_{P_1,\lambda}(A(x^*))\Big]\Big]\Big\| \\
&\leq (1-\alpha_n)\|g(y_n)-g(y^*)\|+\alpha_n\Big\|R^M_{P_1,\lambda}\Big[P_1(g(y_n))-\lambda C^M_{P_1,\lambda}(A(x_n))\Big] \\
&\quad -R^M_{P_1,\lambda}\Big[P_1(g(y^*))-\lambda C^M_{P_1,\lambda}(A(x^*))\Big]\Big\|+\alpha_n\|e_n\| \\
&\leq (1-\alpha_n)\lambda_g\|y_n-y^*\|+\frac{\alpha_n}{r_1}\Big\|[P_1(g(y_n))-P_1(g(y^*))] \\
&\quad -\lambda\Big[C^M_{P_1,\lambda}(A(x_n))-C^M_{P_1,\lambda}(A(x^*))\Big]\Big\|+\alpha_n\|e_n\| \\
&\leq (1-\alpha_n)\lambda_g\|y_n-y^*\|+\frac{\alpha_n}{r_1}\|P_1(g(y_n))-P_1(g(y^*))\| \\
&\quad +\frac{\alpha_n}{r_1}\lambda\Big\|C^M_{P_1,\lambda}(A(x_n))-C^M_{P_1,\lambda}(A(x^*))\Big\|+\alpha_n\|e_n\| \\
&\leq (1-\alpha_n)\lambda_g\|y_n-y^*\|+\frac{\alpha_n}{r_1}\lambda_{P_1}\|g(y_n)-g(y^*)\| \\
&\quad +\frac{\alpha_n}{r_1}\lambda\lambda_C\|A(x_n)-A(x^*)\|+\alpha_n\|e_n\| \\
&\leq (1-\alpha_n)\lambda_g\|y_n-y^*\|+\frac{\alpha_n}{r_1}\lambda_{P_1}\lambda_g\|y_n-y^*\| \\
&\quad +\frac{\alpha_n}{r_1}\lambda\lambda_C\lambda_A\|x_n-x^*\|+\alpha_n\|e_n\| \\
&= \Big[(1-\alpha_n)\lambda_g+\frac{\alpha_n}{r_1}\lambda_{P_1}\lambda_g\Big]\|y_n-y^*\| \\
&\quad +\frac{\alpha_n}{r_1}\lambda\lambda_C\lambda_A\|x_n-x^*\|+\alpha_n\|e_n\|.
\end{aligned} \tag{18}
$$

By using (15) of Algorithm 1, (17) and Lipschitz continuity of g, $R^N_{P_2,\rho}$, P_2, B, we evaluate

$$
\begin{aligned}
\|g(x_{n+1})-g(x^*)\| &= \Big\|(1-\alpha_n)g(x_n)+\alpha_n R^N_{P_2,\rho}\Big[P_2(g(x_n))-\rho Y^N_{P_2,\rho}(B(y_n))\Big]+\alpha_n r_n \\
&\quad -\Big[(1-\alpha_n)g(x^*)+\alpha_n R^N_{P_2,\rho}\Big[P_2(g(x^*))-\rho Y^N_{P_2,\rho}(B(y^*))\Big]\Big]\Big\| \\
&\leq (1-\alpha_n)\|g(x_n)-g(x^*)\|+\alpha_n\Big\|R^N_{P_2,\rho}\Big[P_2(g(x_n))-\rho Y^N_{P_2,\rho}(B(y_n))\Big] \\
&\quad -R^N_{P_2,\rho}\Big[P_2(g(x^*))-\rho Y^N_{P_2,\rho}(B(y^*))\Big]\Big\|+\alpha_n\|r_n\| \\
&\leq (1-\alpha_n)\lambda_g\|x_n-x^*\|+\frac{\alpha_n}{r_2}\Big\|[P_2(g(x_n))-P_2(g(x^*))] \\
&\quad -\rho\Big[Y^N_{P_2,\rho}(B(y_n))-Y^N_{P_2,\rho}(B(y^*))\Big]\Big\|+\alpha_n\|r_n\| \\
&\leq (1-\alpha_n)\lambda_g\|x_n-x^*\|+\frac{\alpha_n}{r_2}\lambda_{P_2}\|g(x_n)-g(x^*)\| \\
&\quad +\frac{\alpha_n}{r_2}\Big\|B(y_n)-B(y^*)-[B(y_n)-B(y^*)] \\
&\quad -\rho\Big[Y^N_{P_2,\rho}(B(y_n))-Y^N_{P_2,\rho}(B(y^*))\Big]\Big\|+\alpha_n\|r_n\| \\
&\leq (1-\alpha_n)\lambda_g\|x_n-x^*\|+\frac{\alpha_n}{r_2}\lambda_{P_2}\|g(x_n)-g(x^*)\| \\
&\quad +\frac{\alpha_n}{r_2}\|B(y_n)-B(y^*)\|+\frac{\alpha_n}{r_2}\Big\|[B(y_n)-B(y^*)] \\
&\quad -\rho\Big[Y^N_{P_2,\rho}(B(y_n))-Y^N_{P_2,\rho}(B(y^*))\Big]\Big\|+\alpha_n\|r_n\| \\
&\leq (1-\alpha_n)\lambda_g\|x_n-x^*\|+\frac{\alpha_n}{r_2}\lambda_{P_2}\lambda_g\|x_n-x^*\|+\frac{\alpha_n}{r_2}\lambda_B\|y_n-y^*\| \\
&\quad +\frac{\alpha_n}{r_2}\Big\|B(y_n)-B(y^*)-\rho\Big[Y^N_{P_2,\rho}(B(y_n))-Y^N_{P_2,\rho}(B(y^*))\Big]\Big\|+\alpha_n\|r_n\|.
\end{aligned} \tag{19}
$$

By using the same arguments as for (7), we have

$$
\Big\|B(y_n)-B(y^*)-\rho\Big[Y^N_{P_2,\rho}(B(y_n))-Y^N_{P_2,\rho}(B(y^*))\Big]\Big\| = \sqrt[q]{\lambda_B^q-q\rho\delta_Y\xi_B^q+\rho C_q\lambda_Y^q\lambda_B^q}\,\|y_n-y^*\|. \tag{20}
$$

By combining (19) and (20), we have

$$\begin{aligned}\|g(x_{n+1})-g(x^*)\| \quad &\leq \quad \left[(1-\alpha_n)\lambda_g+\frac{\alpha_n}{r_2}\lambda_{P_2}\lambda_g\right]\|x_n-x^*\| \\ &+\left[\frac{\alpha_n}{r_2}\lambda_B+\frac{\alpha_n}{r_2}\sqrt[q]{\lambda_B^q-q\rho\delta_Y\xi_B^q+\rho C_q\lambda_Y^q\lambda_B^q}\right]\|y_n-y^*\|+\alpha_n\|r_n\|.\end{aligned} \tag{21}$$

By accretiveness of g with constant δ_g, we have

$$\begin{aligned}\|g(y_{n+1})-g(y^*)\|\|y_{n+1}-y^*\|^{q-1} \quad &= \quad \|g(y_{n+1})-g(y^*)\|\|J_q(y_{n+1}-y^*)\| \\ &\geq \quad \langle g(y_{n+1})-g(y^*),J_q(y_{n+1}-y^*)\rangle \\ &\geq \quad \delta_g\|y_{n+1}-y^*\|^q,\end{aligned}$$

which implies that

$$\|y_{n+1}-y^*\|\leq\frac{1}{\delta_g}\|g(y_{n+1})-g(y^*)\|. \tag{22}$$

Similarly,

$$\|x_{n+1}-x^*\|\leq\frac{1}{\delta_g}\|g(x_{n+1})-g(x^*)\|. \tag{23}$$

From (18) and (22), we have

$$\begin{aligned}\|y_{n+1}-y^*\| \quad &\leq \quad \frac{1}{\delta_g}\|g(y_{n+1})-g(y^*)\|\leq\frac{1}{\delta_g}\left[(1-\alpha_n)\lambda_g+\frac{\alpha_n}{r_1}\lambda_{P_1}\lambda_g\right]\|y_n-y^*\| \\ &+\frac{1}{\delta_g}\frac{\alpha_n}{r_1}\lambda\lambda_C\lambda_A\|x_n-x^*\|+\frac{1}{\delta_g}\alpha_n\|e_n\|.\end{aligned} \tag{24}$$

From (21) and (23), we have

$$\begin{aligned}\|x_{n+1}-x^*\| \quad &\leq \quad \frac{1}{\delta_g}\|g(x_{n+1})-g(x^*)\|\leq\frac{1}{\delta_g}\left[(1-\alpha_n)\lambda_g+\frac{\alpha_n}{r_2}\lambda_{P_2}\lambda_g\right]\|x_n-x^*\| \\ &+\frac{1}{\delta_g}\left[\frac{\alpha_n}{r_2}\lambda_B+\frac{\alpha_n}{r_2}\sqrt[q]{\lambda_B^q-q\rho\delta_Y\xi_B^q+\rho C_q\lambda_Y^q\lambda_B^q}\right]\|y_n-y^*\| \\ &+\frac{1}{\delta_g}\alpha_n\|r_n\|.\end{aligned} \tag{25}$$

Adding (24) and (25), we have

$$
\begin{aligned}
\|y_{n+1}-y^*\|+\|x_{n+1}-x^*\| &\le \left[\frac{1}{\delta_g}\left[(1-\alpha_n)\lambda_g+\frac{\alpha_n}{r_1}\lambda_{P_1}\lambda_g\right]\|y_n-y^*\|\right.\\
&\quad\left.+\frac{1}{\delta_g}\frac{\alpha_n}{r_1}\lambda\lambda_C\lambda_A\|x_n-x^*\|+\frac{1}{\delta_g}\alpha_n\|e_n\|\right]\\
&\quad+\left[\frac{1}{\delta_g}\left[(1-\alpha_n)\lambda_g+\frac{\alpha_n}{r_2}\lambda_{P_2}\lambda_g\right]\|x_n-x^*\|\right.\\
&\quad+\frac{1}{\delta_g}\left[\frac{\alpha_n}{r_2}\lambda_B+\frac{\alpha_n}{r_2}\sqrt[q]{\lambda_B^q-q\rho\delta_Y\xi_B^q+\rho C_q\lambda_Y^q\lambda_B^q}\right]\|y_n-y^*\|\\
&\quad\left.+\frac{1}{\delta_g}\alpha_n\|r_n\|\right]\\
&= \left[\frac{1}{\delta_g}\left[(1-\alpha_n)\lambda_g+\frac{\alpha_n}{r_1}\lambda_{P_1}\lambda_g\right]\right.\\
&\quad\left.+\frac{1}{\delta_g}\left[\frac{\alpha_n}{r_2}\lambda_B+\frac{\alpha_n}{r_2}\sqrt[q]{\lambda_B^q-q\rho\delta_Y\xi_B^q+\rho C_q\lambda_Y^q\lambda_B^q}\right]\right]\|y_n-y^*\|\\
&\quad+\left[\frac{1}{\delta_g}\frac{\alpha_n}{r_1}\lambda\lambda_C\lambda_A+\frac{1}{\delta_g}\left[(1-\alpha_n)\lambda_g+\frac{\alpha_n}{r_2}\lambda_{P_2}\lambda_g\right]\right]\|x_n-x^*\|\\
&\quad+\frac{\alpha_n}{\delta_g}\left[\|e_n\|+\|r_n\|\right]\\
&= \frac{\lambda_g}{\delta_g}\left[(1-\alpha_n(1-\theta))\right]\|y_n-y^*\|\\
&\quad+\frac{\lambda_g}{\delta_g}\left[(1-\alpha_n(1-\theta'))\right]\|x_n-x^*\|\\
&\quad+\frac{\alpha_n}{\delta_g}\left[\|e_n\|+\|r_n\|\right],
\end{aligned}
\tag{26}
$$

where $\theta=\frac{\lambda_{P_1}}{r_1}+\frac{1}{\lambda_g r_2}\left(\lambda_B+\sqrt[q]{\lambda_B^q-q\rho\delta_Y\xi_B^q+\rho C_q\lambda_Y^q\lambda_B^q}\right)$ and $\theta'=\frac{\lambda_{P_2}}{r_2}+\frac{\lambda\lambda_C\lambda_A}{\lambda_g r_1}$.

Because $\lambda_g\le\delta_g$ and $\delta_g\ge 1$, (26) becomes

$$
\begin{aligned}
\|y_{n+1}-y^*\|+\|x_{n+1}-x^*\| &\le [(1-\alpha_n(1-\theta))]\|y_n-y^*\|+[(1-\alpha_n(1-\theta'))]\|x_n-x^*\|\\
&\quad+\alpha_n(\|e_n\|+\|r_n\|)\\
&\le \xi(\theta)\left[\|y_n-y^*\|+\|x_n-x^*\|\right]+\alpha_n(\|e_n\|+\|r_n\|)\\
&= \xi(\theta)\left[\|y_n-y^*\|+\|x_n-x^*\|\right]+(1-\theta)\alpha_n\frac{(\|e_n\|+\|r_n\|)}{(1-\theta)}\\
&= \xi(\theta)\left[\|y_n-y^*\|+\|x_n-x^*\|\right]+\sigma_n,
\end{aligned}
\tag{27}
$$

where $\xi(\theta)=\max\{(1-\alpha_n(1-\theta)),(1-\alpha_n(1-\theta'))\}$ and $\sigma_n=(1-\theta)\alpha_n\dfrac{(\|e_n\|+\|r_n\|)}{(1-\theta)}$.

Applying condition (i), we have $\dfrac{\|e_n\|+\|r_n\|}{1-\theta}\to 0$, as $n\to\infty$. Hence $\sigma_n=O[(1-\theta)\alpha_n]$. Thus, all the conditions of Lemma 2 are satisfied.

We conclude that $(x_n,y_n)\to(x,y)\in\tilde{E}\times\tilde{E}$, as $n\to\infty$. This completes the proof. □

5. Numerical Example

We construct following example in support of system (1). It is shown that $(0,0)$ is the unique solution of system (1).

Example 1. *Let $\tilde{E}=\mathbb{R}$, $A,B,P_1,P_2,g:\mathbb{R}\to\mathbb{R}$ be single-valued mappings and $M,N:\mathbb{R}\to 2^{\mathbb{R}}$ be multi-valued mappings such that $A(x)=13x$, $B(y)=7y$, $P_1(x)=\frac{6}{5}x$, $P_2(x)=\frac{9}{4}x$, $M(z)=\left\{\frac{1}{10}z\right\}$, $N(z)=\left\{\frac{2}{3}z\right\}$, $g(x)=2x$ and $g(y)=2y$, for all $x,y,z\in\mathbb{R}$.*

For $\lambda = 1$ and $\rho = 1$, we evaluate the generalized resolvent operators

$$R^M_{P_1,\lambda}(x) = [P_1 + \lambda M]^{-1}(x) = \frac{10}{13}x, \; R^N_{P_2,\rho}(x) = [P_2 + \rho N]^{-1}(x) = \frac{12}{35}x, \text{ for all } x \in \mathbb{R}.$$

Consequently, the generalized Cayley operators and the generalized Yosida approximation operators are calculated.

$$\begin{aligned} C^M_{P_1,\lambda}(x) &= \left[2R^M_{P_1,\lambda} - P_1\right](x) = \frac{22}{65}x, \\ \text{and} \quad Y^N_{P_2,\rho}(x) &= \frac{1}{\rho}\left[P_2 - R^N_{P_2,\rho}\right](x) = \frac{267}{140}x, \text{ for all } x \in \mathbb{R}. \end{aligned}$$

Further, we calculate

$$\begin{aligned} C^M_{P_1,\lambda}\Big(A(x) + M\big(g(y)\big)\Big) &= C^M_{P_1,\lambda}(13x) + M(2y) \\ &= \tfrac{22}{65}(13x) + \tfrac{1}{10}(2y) \\ &= \tfrac{22}{5}x + \tfrac{1}{5}y \\ &= \tfrac{1}{5}(22x + y), \end{aligned} \tag{28}$$

$$\begin{aligned} \text{and} \quad Y^N_{P_2,\rho}\Big(B(y) + N\big(g(x)\big)\Big) &= Y^N_{P_2,\rho}(7y) + N(2x) \\ &= \tfrac{267}{140}(7y) + \tfrac{2}{3}(2x) \\ &= \tfrac{267}{20}y + \tfrac{4}{3}x. \end{aligned} \tag{29}$$

For (28) and (29), we have the following matrix representation.

$$\begin{bmatrix} \frac{22}{5} & \frac{1}{5} \\ \frac{267}{20} & \frac{4}{3} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}, \text{ for all } x, y \in \mathbb{R}.$$

It is clear from above matrix representation that $(0,0)$ is the solution of system of mixed variational inclusion involving the generalized Cayley operator and the generalized Yosida approximation operator (1).

In continuation of Example 1, we construct another example showing that the generalized Cayley operator is Lipschitz continuous and generalized Yosida approximation operator is Lipschitz-continuous as well as strongly accretive. Lipschitz continuity for both operators is also shown by graphs.

Example 2. *Let $\widetilde{E} = \mathbb{R}$ and all the mappings remain same as in Example 1. That is,*

$$\begin{aligned} R^M_{P_1,\lambda}(x) &= [P_1 + \lambda M]^{-1}(x) = \frac{10}{13}x, \\ R^N_{P_2,\rho}(x) &= [P_2 + \rho N]^{-1}(x) = \frac{12}{35}x, \\ C^M_{P_1,\lambda}(x) &= \left[2R^M_{P_1,\lambda} - P_1\right](x) = \frac{22}{65}x, \\ Y^N_{P_2,\rho}(x) &= \frac{1}{\rho}\left[P_2 - R^N_{P_2,\rho}\right](x) = \frac{267}{140}x. \end{aligned}$$

Because $P_1(x) = \frac{6}{5}x$, we have

$$\begin{aligned} \|P_1(x) - P_1(y)\| &= \left\|\frac{6}{5}x - \frac{6}{5}y\right\| \\ &= \frac{6}{5}\|x - y\| \\ &\leq \frac{7}{5}\|x - y\|, \end{aligned}$$

that is, P_1 is $\lambda_{P_1} = \frac{7}{5}$-Lipschitz continuous. Moreover,

$$\begin{aligned}\langle P_1(x) - P_1(y), x - y\rangle &= \langle \frac{6}{5}x - \frac{6}{5}y, x - y\rangle \\ &= \frac{6}{5}\|x - y\|^2 \\ &\geq \|x - y\|^2,\end{aligned}$$

that is, P_1 is $r_1 = 1$-strongly accretive.

Similarly, for $P_2(x) = \frac{9}{4}x$, one can easily prove that P_2 is $\lambda_{P_2} = \frac{9}{4}$-Lipschitz continuous and $r_2 = 2$-strongly accretive. Furthermore,

$$\begin{aligned}\|C^M_{P_1,\lambda}(x) - C^M_{P_1,\lambda}(y)\| &= \left\|\frac{22}{65}x - \frac{22}{65}y\right\| \\ &= \frac{22}{65}\|x - y\| \\ &\leq \frac{17}{7}\|x - y\|,\end{aligned}$$

that is, generalized Cayley operator is $\lambda_C = \frac{17}{7}$-Lipschitz-continuous, where $\lambda_C = \frac{2+\lambda_{P_1}r_1}{r_1} = \frac{2+1.\frac{7}{5}}{\frac{7}{5}} = \frac{17}{7}$. Furthermore,

$$\begin{aligned}\|Y^N_{P_2,\rho}(x) - Y^N_{P_2,\rho}(y)\| &= \left\|\frac{267}{140}x - \frac{267}{140}y\right\| \\ &= \frac{267}{140}\|x - y\| \\ &\leq \frac{22}{9}\|x - y\|,\end{aligned}$$

that is, the generalized Yosida approximation operator is $\lambda_Y = \frac{22}{9}$-Lipschitz-continuous, where $\lambda_Y = \frac{\lambda_{P_2}r_2+1}{\lambda r_2} = \frac{2\cdot\frac{9}{4}+1}{1\cdot\frac{9}{4}} = \frac{22}{9}$. Moreover,

$$\begin{aligned}\langle Y^M_{P_2,\rho}(x) - Y^M_{P_2,\rho}(y), x - y\rangle &= \langle \frac{267}{140}x - \frac{267}{140}y, x - y\rangle \\ &= \frac{267}{140}\langle x - y, x - y\rangle \\ &= \frac{267}{140}\|x - y\|^2 \\ &\geq \frac{65}{36}\|x - y\|^2,\end{aligned}$$

that is, generalized Yosida approximation operator is $\delta_Y = \frac{65}{36}$-strongly accretive, where $\delta_Y = \frac{r_2^2-1}{\lambda r_2} = \frac{\left(\frac{9}{4}\right)^2-1}{1\cdot\frac{9}{4}} = \frac{65}{36}$.

It is a well-known fact that for a Lipschitz-continuous function, there exists a double cone whose origin can be moved along the graph so that the whole graph always stays outside the double cone. The following figures (Figures 1 and 2) demonstrate the Lipschitz continuity of generalized Cayley operator and generalized Yosida approximation operator calculated above, respectively.

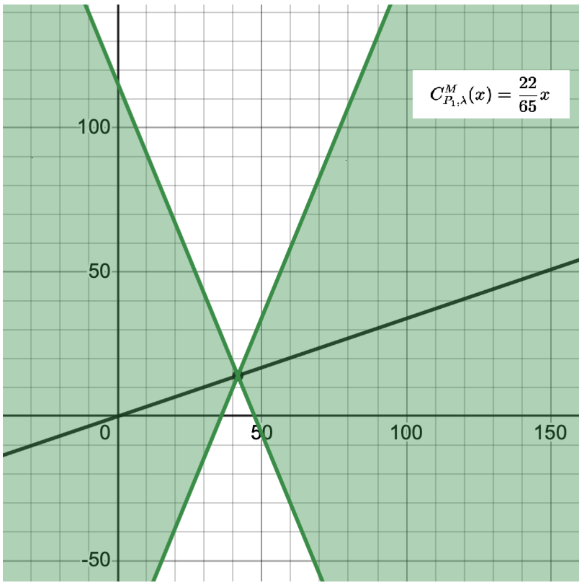

Figure 1. Graph of Lipschitz-continuous of generalized Cayley operator.

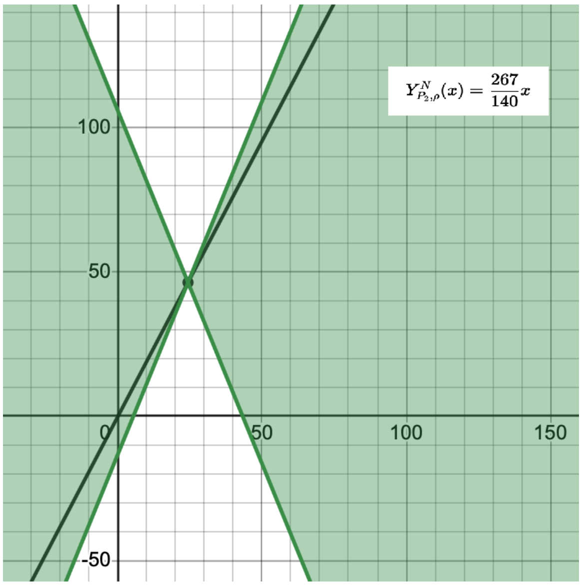

Figure 2. Graph of Lipschitz-continuous of generalized Yosida approximation operator.

6. Application

A dynamical system is a system that changes over time according to a set of fixed rules and determine how one state of the system moves to another state. On the other hand, a dynamical system describes the disequilibrium adjustment processes which may produce important transient phenomenon prior to the achievement of steady state. Dynamical system is a generalization of classical mechanics where the equation of motion postulated directly and are not constrained to be Euler–Lagrange equations of a least action principle.

Dynamical system theory has been applied in the field of neuroscience, cognitive development, equation of motion, electronic circuits, chaotic system (double pendulum), etc.

As an application of system of mixed variational inclusions involving the generalized Cayley operator and the generalized Yosida approximation operator (1), we mention a system of resolvent dynamical systems.

By using Lemma 3, we suggest the following system of resolvent dynamical systems:

$$\begin{aligned} \frac{\mathrm{d}(A(x))}{\mathrm{d}t} &= \xi_1\Big[R^M_{P_1,\lambda}\Big\{g(y) - \lambda C^M_{P_1,\lambda}(A(x))\Big\} - g(y)\Big], x(t_0) = t_0 \in \tilde{E} \text{ over } [t_0, \infty), \\ \frac{\mathrm{d}(B(y))}{\mathrm{d}t} &= \xi_2\Big[R^N_{P_2,\rho}\Big\{g(x) - \rho Y^N_{P_2,\rho}(B(y))\Big\} - g(x)\Big], y(t_0) = t_0 \in \tilde{E} \text{ over } [t_0, \infty), \end{aligned} \quad (30)$$

where ξ_1 and ξ_2 are parameters.

It can be shown easily that by using the techniques of Noor [38], the Gronwall lemma and Lyapunov function, which are the trajectory of the solution of the system of resolvent dynamical systems (30), converge globally exponentially to the unique solution of system of mixed variational inclusions involving the generalized Cayley operator and the generalized Yosida approximation operator (1).

7. Conclusions

It is well known that the Cayley operator, the Yosida approximation operator, and a system of variational inclusions are application oriented. This paper is focused on the study of a system of mixed variational inclusions involving the generalized Cayley operator and the generalized Yosida approximation operator in q-uniformly smooth Banach space. We obtain the unique solution of our system, and we discuss the convergence criteria by suggesting an iterative algorithm. Two examples are provided with an application.

The novelty of work lies in the fact that our results are refinement of previously known results (see for example [8,9,13,18,26,28,33,36]).

Our results can be extended further and may be useful for other scientists.

Author Contributions: Conceptualization, R.A. and Y.W.; methodology, R.A. and M.I.; software, A.K.R.; validation, R.A. and M.I.; formal analysis, R.A.; investigation, Y.W.; resources, R.A.; data curation, Y.W.; writing—original draft preparation, A.K.R.; writing—review and editing, R.A. and M.I.; visualization, M.I.; supervision, R.A.; project administration, Y.W.; funding acquisition, Y.W. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Science Foundation of China (Grant no. 12171435).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: All authors are thankful to all referees for their valuable suggestions which improve this paper a lot.

Conflicts of Interest: The authors declare no conflict of interest.

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