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A Novel Formulation of the Fractional Derivative with the Order $\alpha \geq 0$ and without the Singular Kernel

Hassan Kamil Jassim [1,*] and Mohammed A. Hussein [2,3]

[1] Department of Mathematics, University of Thi-Qar, Nasiriyah 64001, Iraq
[2] Scientific Research Center, Thi Qar University, Thi-Qar 64001, Iraq
[3] Scientific Research Center, Al-Ayen University, Thi-Qar 64001, Iraq
* Correspondence: hassankamil@utq.edu.iq

Abstract: A new definition of fractional derivative (NFD) with order $\alpha \geq 0$, is developed in this paper. The new derivative has a smooth kernel that takes on two different representations for the temporal and spatial variables. The advantage of the proposed approach over traditional local theories and fractional models with a singular kernel lies in the possibility that there is a class of problems capable of describing scale-dependent fluctuations and material heterogeneities. Moreover, it has been shown that the NFD converges to the classical derivative faster than some other fractional derivatives.

Keywords: new fractional derivative; new fractional integral; integral transforms; existence and uniqueness

MSC: 26A33; 34A12; 34A34; 35R11

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1. Introduction

In a letter dated 30th September 1695, L'Hopital wrote to Leibniz asking him particular notation he has used in his publication for the nth derivative of a function $\frac{D^n f(x)}{Dx^n}$, i.e., what would the result be if $n = 1/2$. Leibniz's response: "an apparent paradox from which, one day, useful consequences will be drawn. Since 1695, and after L'Hopital's question regarding the order, Leibniz was the first to begin in this direction. L. Euler (1730) suggested using a relationship for negative or non-integer (rational) values. Riemann (1847) utilized a generalization of a Taylor series for obtaining a formula for fractional order. Riemann introduced an arbitrary "complementary" function because he did not fix the lower bound of integration. This disadvantage he could not solve. From here, the initialized fractional calculus emerged in the latter half of the twentieth century. M. Caputo (1967) introduces a definition of the fractional derivative with a singular kernel. Subsequently, M. Caputo and M. Fabrizio (2015) presented a definition of fractional derivative without a singular kernel [1–3].

Fractional calculus (FC) is an incipient tool in the field of mathematics with strong execution in the diverse areas of science and engineering. FC is defined as the generalization of classical calculus where we study the integral and differential operators of fractional order and even can be lengthened to a complex set. In the past few decades, many mathematical minds have strengthened this concept and designed various fractional differential and integral operators [4,5].

Fractional calculus has progressed dramatically in recent decades, as seen by the numerous mathematical volumes devoted to the subject (e.g., Caputo [6], Katugampola et al. [7], Yang et al. [8], and Atangana et al. [9]) and by the notable diffusion as shown by the many conferences dedicated to it and the plethora of articles and non-mathematical journals (e.g., H. K. Jassim et al. [10], Losada et al. [11], Peter et al. [12]).

Apart from mathematics and physics, the usage of fractional-order derivatives has expanded to many other sectors of study (e.g., Laskin [13], Baleanu et al. [14,15]) such as biology (e.g., Cesarone et al. [16], Caputo and Cametti [17]), economics (e.g., Caputo [18]),

demographics (e.g., Jumarie [19]), geophysics (e.g., Iaffaldano [20]), medicine (e.g., El Sahed [21]) and bioengineering (e.g., Magin [22]). However, some have expressed dissatisfaction with the rather convoluted mathematical statement of its concept, as well as the resulting complexities in the solution of FPDES.

The novel definition of the fractional derivative is based on two alternative representations of the temporal and spatial variables. With certain simplifications in the equations and calculations, the first representation works on time variables, where the real powers occurring in the solutions of the typical fractional derivative are converted to integer powers. The Laplace transform is appropriate in this context. Because the second representation is connected to the spatial variables, it is more straightforward to deal with the Fourier transform for this non-local fractional derivative. This paper includes various examples and simulations of how these new derivatives behave when applied to classical functions.

2. New Definition of Fractional Derivative

Different definitions of fractional derivatives have been proposed by Riemann, Liouville, Riesz, Grünwald, Letnikov, and Marchaud during the history of fractional calculus. Here, we mention some of them:

Definition 1. ([6,7]). Let $f \in H^1(a,b)$, $a < b$, $\alpha \in (0,1]$, then the definition of Caputo–Fabrizio in the Caputo sense is:

$$ {}^{\mathcal{CFC}}_{a}\mathcal{D}^{\alpha}_{t} f(t) = \frac{\beta(\alpha)}{1-\alpha} \int_a^t f'(\mathcal{T}) \exp\left(-\alpha \frac{t-\mathcal{T}}{1-\alpha}\right) d\mathcal{T}, $$

where $\beta(\alpha)$ is a normalization function, such that $\beta(0) = \beta(1) = 1$.

Definition 2. ([23]). Let $f \in H^1(a,b)$, $a < b$, $\alpha \in (0,1]$, then the definition of Atangana–Baleanu in the Caputo sense is:

$$ {}^{\mathcal{ABC}}_{a}\mathcal{D}^{\alpha}_{t} f(t) = \frac{\beta(\alpha)}{1-\alpha} \int_a^t f'(\mathcal{T}) \mathrm{E}_{\alpha}\left(-\alpha \frac{(t-\mathcal{T})^{\alpha}}{1-\alpha}\right) d\mathcal{T}, $$

where $\mathrm{E}_{\alpha}\left(-\alpha \frac{(t-\mathcal{T})^{\alpha}}{1-\alpha}\right) = \sum_{j=0}^{\infty} \frac{-\alpha \frac{(t-\mathcal{T})^{j\alpha}}{1-\alpha}}{\Gamma(j\alpha+1)}$ is the Mittag–Leffler function.

Definition 3. ([6,23]). The Laplace transform of a function $v(t), t > 0$ is defined as:

$$ \mathcal{L}[v(t)] = \mathcal{F}(s) = \int_0^{\infty} v(t) e^{-st} dt \,, $$

where s can be either real or complex.

Definition 4. ([7,24]). The Sumudu transform is defined over the set of functions:

$$ \mathcal{A} = \left\{ v(t) \middle| \exists \mathcal{M}, \tau_1 \,, \tau_2 > 0, |v(t)| < \mathcal{M} e^{\frac{|t|}{\tau_j}} \,, \; if \; t \in (-1)^j \times [0,\infty) \right\}, $$

by the following formula:

$$ \mathcal{S}[v(t)] = \mathcal{G}(s) = \int_0^{\infty} v(st) e^{-t} dt \,, \;\; s \in (\tau_1 \,, \tau_2). $$

Now, we show the Laplace and Sumudu transform of fractional derivatives [6,7,23]:

$$\mathcal{L}\left[{}^{\mathcal{CFC}}_{a}\mathcal{D}^{\alpha}_{t}f(t)\right] = \frac{\beta(\alpha)}{(1-\alpha)s+\alpha}(s\mathcal{F}(s)-f(0)),$$

$$\mathcal{L}\left[{}^{\mathcal{ABC}}_{a}\mathcal{D}^{\alpha}_{t}f(t)\right] = \frac{\beta(\alpha)}{(1-\alpha)s^{\alpha}+\alpha}\left(s^{\alpha}\mathcal{F}(s)-s^{\alpha-1}f(0)\right),$$

$$\mathcal{S}\left[{}^{\mathcal{CFC}}_{a}\mathcal{D}^{\alpha}_{t}f(t)\right] = \frac{\beta(\alpha)}{1-\alpha+\alpha s}(\mathcal{G}(s)-f(0)),$$

$$\mathcal{S}\left[{}^{\mathcal{ABC}}_{a}\mathcal{D}^{\alpha}_{t}f(t)\right] = \frac{\beta(\alpha)}{1-\alpha+\alpha s^{\alpha}}(\mathcal{G}(s)-f(0)).$$

In 1967, Caputo [25] modified the Riemann–Liouville fractional derivative to present a new fractional derivative:

$${}^{\mathcal{C}}_{a}\mathcal{D}^{\alpha}_{t}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t} f^{(n)}(\mathcal{T})(t-\mathcal{T})^{n-\alpha-1}d\mathcal{T}, \quad n-1<\alpha\leq n, \tag{1}$$

where $n=\alpha$ is integer number, $\alpha\geq 0$ is real number, $a\in(-\infty,t)$, $t>\mathcal{T}$, $f\in H^{n}(a,b)$ and $a>b$. The Laplace and Sumudu transforms of the Caputo derivative are [1,23,24]:

$$\mathcal{L}\left[{}^{\mathcal{C}}_{a}\mathcal{D}^{\alpha}_{t}f(t)\right] = s^{\alpha}\mathcal{F}(s)-\sum_{j=0}^{n-1}s^{\alpha-j-1}\mathcal{D}^{j}_{t}f(0),$$

$$\mathcal{S}\left[{}^{\mathcal{C}}_{a}\mathcal{D}^{\alpha}_{t}f(t)\right] = s^{-\alpha}\mathcal{G}(s)-\sum_{j=0}^{n-1}s^{j-\alpha}\mathcal{D}^{j}_{t}f(0).$$

Now, by replacing the function $(t-\mathcal{T})^{n-\alpha-1}$ with the function $\mathrm{e}^{-\mathcal{M}_{\alpha}(t-\mathcal{T})}$ and $\frac{1}{\Gamma(n-\alpha)}$ with $\mathcal{M}_{\alpha}$, the following new definition of fractional time derivative can be obtained:

$${}^{\mathcal{M}}_{a}\mathcal{D}^{\alpha}_{t}f(t) = \mathcal{M}_{\alpha}\int_{a}^{t} f^{(n)}(\mathcal{T})\mathrm{e}^{-\mathcal{M}_{\alpha}(t-\mathcal{T})}\,d\mathcal{T}, \; n-1<\alpha\leq n, \tag{2}$$

where, $\mathcal{M}_{\alpha}$ is a function of α, such that $\lim_{\alpha\to n}\mathcal{M}_{\alpha}=\infty$.

To prove that the new derivative is a generalization of the differential operator, it must be shown that the new derivative is the differential operator for all $\alpha\to n$.

Suppose that $\mathcal{K}_{\alpha}=\mathcal{M}_{\alpha}\mathrm{e}^{-\mathcal{M}_{\alpha}(t-\mathcal{T})}$ is the kernel of the new derivative, if $t=\mathcal{T}$ and using the definition of $\mathcal{M}_{\alpha}$, we obtain:

$$\lim_{\alpha\to n}\mathcal{K}_{\alpha} = \lim_{\alpha\to n}\mathcal{M}_{\alpha} = \infty, \tag{3}$$

but, when $t>\mathcal{T}$ and using L'Hospital's rule, we obtain:

$$\lim_{\alpha\to n}\mathcal{K}_{\alpha} = \lim_{\alpha\to n}\mathcal{M}_{\alpha}\,\mathrm{e}^{-\mathcal{M}_{\alpha}(t-\mathcal{T})} = 0, \tag{4}$$

from the foregoing, it seems clear that the kernel of the new derivative represents the Dirac delta function, in other words:

$$\lim_{\alpha\to n}\mathcal{K}_{\alpha} = \delta(t-\mathcal{T}), \tag{5}$$

using Equation (5), and the properties of the Dirac delta function, we obtain:

$$\lim_{\alpha \to n} {}^{\mathcal{M}}_{a}\mathcal{D}^{\alpha}_{t} f(t) = \int_{a}^{t} f^{(n)}(\mathcal{T})\delta(t-\mathcal{T})\, d\mathcal{T} = \mathcal{D}^{n}_{t} f(t). \tag{6}$$

Theorem 1. *Let us* ${}^{\mathcal{M}}_{a}\mathcal{D}^{\alpha}_{t}$ *is the new derivative of order* $\alpha \geq 0$, $a > a$, $\lambda \in \mathbb{R}$, $f, g \in H^{n}(a, b)$ *and* $a > b$ *then the following relationships are correct:*

$${}^{\mathcal{M}}_{a}\mathcal{D}^{\alpha}_{t}\,(f(t) \pm g(t)) = {}^{\mathcal{M}}_{a}\mathcal{D}^{\alpha}_{t}\, f(\ a) \pm {}^{\mathcal{M}}_{a}\mathcal{D}^{\alpha}_{t} g(\ a) \tag{7}$$

$${}^{\mathcal{M}}_{a}\mathcal{D}^{\alpha}_{t}\,(\lambda f(t)) = \lambda\, {}^{\mathcal{M}}_{a}\mathcal{D}^{\alpha}_{t}\, f(t) \tag{8}$$

Proof. Using the properties of integration, this theorem can easily be proven. □

Lemma 1. Suppose that ${}^{\mathcal{M}}_{a}\mathcal{D}^{\alpha}_{t}$ is the new derivative of order $\alpha \geq 0$, $t > a$ and $n,\ m$ are integer numbers, then the following relationships are correct:

$${}^{\mathcal{M}}_{a}\mathcal{D}^{\alpha}_{t}\, c = \left[1 - \mathrm{e}^{-\mathcal{M}_{\alpha}(t-a)}\right] D^{n}_{\mathcal{T}}(c) \text{ c is constant} \tag{9}$$

$${}^{\mathcal{M}}_{a}\mathcal{D}^{\alpha}_{t}\, t^{m} = \frac{m!}{(m-n)!}\left[\sum_{k=0}^{m-n}(-1)^{k}\frac{t^{m-n-k}}{\mathcal{M}^{k}_{\alpha}} - \sum_{k=0}^{m-n}(-1)^{k}\frac{a^{m-n-k}}{\mathcal{M}^{k}_{\alpha}} e^{-\mathcal{M}_{\alpha}(t-a)}\right],\ m \geq n \tag{10}$$

$${}^{\mathcal{M}}_{a}\mathcal{D}^{\alpha}_{t}\, t^{m} = 0,\ \ m < n \tag{11}$$

Proof.

1. Using the definition of the new fractional derivative and the properties of the integral, this relationship can be easily proven.
2. Using Equation (2) and since $m \geq n$,the following relationship can be obtained:

$${}^{\mathcal{M}}_{a}\mathcal{D}^{\alpha}_{t}\, t^{m} = \mathcal{M}_{\alpha}\int_{t}^{a} \frac{m!}{(m-n)!}\mathcal{T}^{m-n}\mathrm{e}^{-\mathcal{M}_{\alpha}(t-\mathcal{T})}\, d\mathcal{T}$$

using part integral, the above relationship can be written as follows:

$$\begin{aligned} {}^{\mathcal{M}}_{a}\mathcal{D}^{\alpha}_{t}\, t^{m} &= \tfrac{m!}{(m-n)!}\left[\textstyle\sum_{k=0}^{m-n}(-1)^{k}\frac{\mathcal{T}^{m-n-k}}{\mathcal{M}^{k}_{\alpha}}\mathrm{e}^{-\mathcal{M}_{\alpha}(t-\mathcal{T})}\right]_{a}^{t} \\ &= \tfrac{m!}{(m-n)!}\left[\sum_{k=0}^{m-n}(-1)^{k}\tfrac{t^{m-n-k}}{\mathcal{M}^{k}(\alpha)} - \sum_{k=0}^{m-n}(-1)^{k}\tfrac{a^{m-n-k}}{\mathcal{M}^{k}_{\alpha}} e^{-\mathcal{M}_{\alpha}(t-a)}\right] \end{aligned}$$

3. Using the definition of the new derivative and since $m < n$, then $D^{n}_{\mathcal{T}}\mathcal{T}^{m} = 0$; thus, Equation (11) is true. □

Lemma 2. Assume that ${}^{\mathcal{M}}_{a}\mathcal{D}^{\alpha}_{t}$ is the new derivative of order $\alpha \geq 0$, $t > a$ and λ is the real number then, the following relationships are correct:

$${}^{\mathcal{M}}_{a}\mathcal{D}^{\alpha}_{t}\, e^{\lambda t} = \frac{\lambda^{n}\,\mathcal{M}_{\alpha}}{\lambda + \mathcal{M}_{\alpha}}\left[e^{\lambda t} - e^{\lambda t + \mathcal{M}_{\alpha}(a-t)}\right], \tag{12}$$

$$ {}^{\mathcal{M}}_{a}\mathcal{D}^{\alpha}_{t} \sin(\lambda t) = \frac{\lambda^n}{\mathcal{M}_\alpha^2 + \lambda^2} \begin{pmatrix} \mathcal{M}_\alpha^2 \sin\left(\lambda t + \frac{n\pi}{2}\right) - \lambda \mathcal{M}_\alpha \cos\left(\lambda t + \frac{n\pi}{2}\right) \\ -\mathcal{M}_\alpha^2 \sin\left(\frac{n\pi}{2}\right) e^{-\mathcal{M}_\alpha t} + \lambda \mathcal{M}_\alpha \cos\left(\frac{n\pi}{2}\right) e^{-\mathcal{M}_\alpha t} \end{pmatrix}, \tag{13} $$

$$ {}^{\mathcal{M}}_{a}\mathcal{D}^{\alpha}_{t} \cos(\lambda t) = \frac{\lambda^n}{\mathcal{M}_\alpha^2 + \lambda^2} \begin{pmatrix} \mathcal{M}_\alpha^2 \cos\left(\lambda t + \frac{n\pi}{2}\right) + \lambda \mathcal{M}_\alpha \sin\left(\lambda t + \frac{n\pi}{2}\right) \\ -\mathcal{M}_\alpha^2 \sin\left(\frac{n\pi}{2}\right) e^{-\mathcal{M}_\alpha t} - \lambda \mathcal{M}_\alpha \sin\left(\frac{n\pi}{2}\right) e^{-\mathcal{M}_\alpha t} \end{pmatrix}, \tag{14} $$

Proof.

1. By definition of new fractional derivative and integral of the exponential function, it is possible to prove the following relationship:

$$ \begin{aligned} {}^{\mathcal{M}}_{a}\mathcal{D}^{\alpha}_{t}\, e^{\lambda t} &= \mathcal{M}_\alpha \int_a^t \lambda^n e^{\lambda \mathcal{T}} \mathrm{e}^{-\mathcal{M}_\alpha (t-\mathcal{T})}\, d\mathcal{T}, \\ &= \tfrac{\lambda^n \mathcal{M}_\alpha}{\lambda + \mathcal{M}_\alpha} \left[\mathrm{e}^{\lambda \mathcal{T} - \mathcal{M}_\alpha (t-\mathcal{T})} \right]_a^t, \\ &= \tfrac{\lambda^n\, \mathcal{M}_\alpha}{\lambda + \mathcal{M}_\alpha} \left[e^{\lambda t} - e^{\lambda a + \mathcal{M}_\alpha (a-t)} \right], \end{aligned} $$

2. Relationship No. 2 can easily be proven using the new derivative definition and the integral of the sine function:

$$ \begin{aligned} {}^{\mathcal{M}}_{a}\mathcal{D}^{\alpha}_{t}\, \sin(\lambda t) &= \mathcal{M}_\alpha \int_a^t \lambda^n \sin\left(\lambda \mathcal{T} + \tfrac{n\pi}{2}\right) \mathrm{e}^{-\mathcal{M}_\alpha (t-\mathcal{T})}\, d\mathcal{T}, \\ = \tfrac{\lambda^n \mathcal{M}_\alpha}{\lambda^2 + \mathcal{M}_\alpha^2} &\left[\mathrm{e}^{-\mathcal{M}_\alpha (t-\mathcal{T})} \left(\mathcal{M}_\alpha \sin\left(\lambda \mathcal{T} + \tfrac{n\pi}{2}\right) + \lambda \cos\left(\lambda \mathcal{T} + \tfrac{n\pi}{2}\right) \right) \right]_a^t, \\ &= \tfrac{\lambda^n}{\mathcal{M}_\alpha^2 + \lambda^2} \begin{pmatrix} \mathcal{M}_\alpha^2 \sin\left(\lambda t + \frac{n\pi}{2}\right) - \lambda \mathcal{M}_\alpha \cos\left(\lambda t + \frac{n\pi}{2}\right) \\ -\mathcal{M}_\alpha^2 \sin\left(\frac{n\pi}{2}\right) e^{-\mathcal{M}_\alpha t} + \lambda \mathcal{M}_\alpha \cos\left(\frac{n\pi}{2}\right) e^{-\mathcal{M}_\alpha t} \end{pmatrix} \end{aligned} $$

3. Similarly, relationship No. 3 can be proved. □

Now, we define the corresponding fractional integral to the new fractional derivative and discuss some of its properties.

Consider the next fractional differential equation, $\alpha \geq 0$:

$$ {}^{\mathcal{M}}_{a}\mathcal{D}^{\alpha}_{t}\, f(t) = g(t), \qquad t > a, \tag{15} $$

by definition of the new derivative, Equation (15) becomes:

$$ \mathcal{M}_\alpha \int_a^t f^{(n)}(\mathcal{T}) \mathrm{e}^{-\mathcal{M}_\alpha (t-\mathcal{T})}\, d\mathcal{T} = g(t), \qquad t > a, \tag{16} $$

or equivalently:

$$ \int_a^t f^{(n)}(\mathcal{T}) \mathrm{e}^{\mathcal{M}_\alpha \mathcal{T}}\, d\mathcal{T} = \frac{1}{\mathcal{M}_\alpha} g(t) \mathrm{e}^{\mathcal{M}_\alpha t}, \qquad t > a, \tag{17} $$

By differentiating both sides of Equation (17) with respect to t, the following relationship can be obtained:

$$ f^{(n)}(t) \mathrm{e}^{\mathcal{M}_\alpha t} = \frac{1}{\mathcal{M}_\alpha} g'(t) \mathrm{e}^{\mathcal{M}_\alpha t} + g(t) \mathrm{e}^{\mathcal{M}_\alpha t}, \qquad t > a, \tag{18} $$

By integrating both sides of Equation (18) from 0 to t, we obtain:

$$f(t) = \frac{1}{\mathcal{M}_\alpha} I_t^{n-1} g(t) + I_t^n g(t) + \sum_{k=0}^{n-1} \frac{(t-a)^k}{k!} f^{(k)}(t), \qquad t > a, \tag{19}$$

Thus, as a result, the fractional integral corresponding to the new fractional derivative of the order $\alpha \geq 0$ is expected to be defined as follows:

$${}^{\mathcal{M}}_{a} I_t^{\alpha} g(t) = \frac{1}{\mathcal{M}_\alpha} I_t^{n-1} g(t) + I_t^n g(t), \qquad n-1 < \alpha \leq n, \tag{20}$$

Theorem 2. *Let us* ${}^{\mathcal{M}}_{a}\mathcal{D}_t^{\alpha}$ *is the new derivative,* ${}^{\mathcal{M}}_{a} I_t^{\alpha}$ *is the new integral,* $f(t) \in H^n(a,b)$, $a > b$, $\alpha \geq 0$, *and* $L\{.\}$ *is the Laplace transform, then the following relationships are correct:*

$${}^{\mathcal{M}}_{a}\mathcal{D}_t^{\alpha}\, {}^{\mathcal{M}}_{a} I_t^{\alpha}\, f(t) = f(t) - f(a) e^{-\mathcal{M}_\alpha (t-a)} \tag{21}$$

$${}^{\mathcal{M}}_{a} I_t^{\alpha}\, {}^{\mathcal{M}}_{a}\mathcal{D}_t^{\alpha}\, f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^k}{(k)!} f^{(k)}(a). \tag{22}$$

Proof.

1. From Equations (2) and (20), we obtain:

$$\begin{aligned} {}^{\mathcal{M}}_{a}\mathcal{D}_t^{\alpha}\, {}^{\mathcal{M}}_{a} I_t^{\alpha}\, f(t) &= \mathcal{M}_\alpha \int_a^t D_{\mathcal{S}}^n \left({}^{\mathcal{M}}_{a} I_t^{\alpha} f(\mathcal{S})\right) e^{-\mathcal{M}_\alpha (t-\mathcal{S})} d\mathcal{S} \\ &= \int_a^t D_{\mathcal{S}}^n \left(I_{\mathcal{S}}^{n-1} f(\mathcal{S}) + \mathcal{M}_\alpha I_{\mathcal{S}}^n f(\mathcal{S}) \right) e^{-\mathcal{M}_\alpha (t-\mathcal{S})} d\mathcal{S}, \\ &= \int_a^t f'(\mathcal{S}) e^{-\mathcal{M}_\alpha (t-\mathcal{S})} d\mathcal{S} + \mathcal{M}_\alpha \int_a^t f(\mathcal{S}) e^{-\mathcal{M}_\alpha (t-\mathcal{S})} d\mathcal{S}, \end{aligned}$$

by using the integral by part of the last relationship becomes as follows:

$$\begin{aligned} {}^{\mathcal{M}}_{a}\mathcal{D}_t^{\alpha}\, {}^{\mathcal{M}}_{a} I_t^{\alpha}\, f(t) &= f(\mathcal{S}) e^{-\mathcal{M}_\alpha (t-\mathcal{S})} |_a^t \ldots - \mathcal{M}_\alpha \int_a^t f(\mathcal{S}) e^{-\mathcal{M}_\alpha (t-\mathcal{S})} d\mathcal{S} \\ &\quad + \mathcal{M}_\alpha \int_a^t f(\mathcal{S}) e^{-\mathcal{M}_\alpha (t-\mathcal{S})} d\mathcal{S}, \\ &= f(t) - f(a) e^{-\mathcal{M}_\alpha (t-a)}. \end{aligned}$$

2. Using Equations (2) and (20), we obtain:

$$\begin{aligned} {}^{\mathcal{M}}_{a} I_t^{\alpha} {}^{\mathcal{M}}_{a}\mathcal{D}_t^{\alpha}\, f(t) &= \tfrac{1}{\mathcal{M}(\alpha)} I_{\mathcal{S}}^{n-1} {}^{\mathcal{M}}_{a}\mathcal{D}_t^{\alpha} f(\mathcal{S}) + I_{\mathcal{S}}^n {}^{\mathcal{M}}_{a}\mathcal{D}_t^{\alpha} f(\mathcal{S}), \\ = I_t^n D_t \left(\int_a^t D_{\mathcal{S}}^n (f(\mathcal{S})) e^{-\mathcal{M}_\alpha (t-\mathcal{S})} d\mathcal{S} \right) &+ \mathcal{M}_\alpha I_t^n \left(\int_a^t D_{\mathcal{S}}^n (f(\mathcal{S})) e^{-\mathcal{M}_\alpha (t-\mathcal{S})} d\mathcal{S} \right) \end{aligned}$$

by the Leibniz integral rule and properties of differential operator the above relationship can be written as follows:

$$\begin{aligned} {}^{\mathcal{M}}_{a} I_t^{\alpha} {}^{\mathcal{M}}_{a}\mathcal{D}_t^{\alpha}\, f(t) &= I_t^n D_t^n\, f(t) - \mathcal{M}_\alpha I_t^n \left(\int_t^n D_t^n (f(\mathcal{S})) e^{-\mathcal{M}_\alpha (t-\mathcal{S})} d\mathcal{S} \right. \\ &\quad + \mathcal{M}_\alpha I_t^n \left(\int_a^t D_{\mathcal{S}}^n (f(\mathcal{S})) e^{-\mathcal{M}_\alpha (t-\mathcal{S})} d\mathcal{S} \right. \\ &= f(t) - \sum_{k=0}^{n-1} \tfrac{(t-a)^k}{(k)!} f^{(k)}(a) \end{aligned}$$

□

Lemma 3. Let ${}^{\mathcal{M}}_{a} I_t^{\alpha}$ is the new integral of order, $E_{\cdot,\cdot}(.)$ is the Mitteg–Leffler function, $\alpha \geq 0$, $t > a$, $\beta > -1$, $\lambda > 0$, and k is constant, then the following relationships are correct:

$$
\begin{aligned}
{}^{\mathcal{M}}_{a}I^{\alpha}_{t}\, k &= \frac{k}{\mathcal{M}_{\alpha}\Gamma(n)}t^{n-1} + \frac{k}{\Gamma(n+1)}t^{n},\\
{}^{\mathcal{M}}_{a}I^{\alpha}_{t}\, t^{\beta} &= \frac{\Gamma(\beta+1)}{\mathcal{M}_{\alpha}\Gamma(n+\beta)}t^{n+\beta-1} + \frac{\Gamma(\beta+1)}{\Gamma(n+\beta+1)}t^{n+\beta},\\
{}^{\mathcal{M}}_{a}I^{\alpha}_{t}\, e^{\lambda t} &= \frac{e^{\lambda t}}{\mathcal{M}_{\alpha}\lambda^{\mathrm{n}-1}} + \frac{e^{\lambda t}}{\lambda^{\mathrm{n}}},\\
{}^{\mathcal{M}}_{a}I^{\alpha}_{t}\sin(\lambda t) &= \frac{1}{\mathcal{M}_{\alpha}}\lambda t^{n}E_{2,n+1}\left(-(\lambda t)^{2}\right) + \lambda t^{n+1}E_{2,n+2}\left(-(\lambda t)^{2}\right),\\
{}^{\mathcal{M}}_{a}I^{\alpha}_{t}\cos(\lambda t) &= \frac{1}{\mathcal{M}_{\alpha}}t^{n-1}E_{2,n}\left(-(\lambda t)^{2}\right) + t^{n}E_{2,n+1}\left(-(\lambda t)^{2}\right).
\end{aligned}
$$

Proof. From [1] and using the new integral:

$$
{}^{\mathcal{M}}_{a}I^{\alpha}_{t}k = \frac{1}{\mathcal{M}_{\alpha}}I^{n-1}_{t}k + I^{n}_{t}k,
$$

from nth integral of constant:

$$
{}^{\mathcal{M}}_{a}I^{\alpha}_{t}k = \frac{k}{\mathcal{M}_{\alpha}\Gamma(n)}t^{n-1} + \frac{k}{\Gamma(n+1)}t^{n}.
$$

$$
{}^{\mathcal{M}}_{a}I^{\alpha}_{t}t^{\beta} = \frac{1}{\mathcal{M}_{\alpha}}I^{n-1}_{t}t^{\beta} + I^{n}_{t}t^{\beta}
$$

by using nth integral of t^{β},

$$
{}^{\mathcal{M}}_{a}I^{\alpha}_{t}\, t^{\beta} = \frac{\Gamma(\beta+1)}{\mathcal{M}_{\alpha}\Gamma(n+\beta)}t^{n+\beta-1} + \frac{\Gamma(\beta+1)}{\Gamma(n+\beta+1)}t^{n+\beta}.
$$

$$
{}^{\mathcal{M}}_{a}I^{\alpha}_{t}e^{\lambda t} = \frac{1}{\mathcal{M}_{\alpha}}I^{n-1}_{t}e^{\lambda t} + I^{n}_{t}e^{\lambda t},
$$

by using nth integral of exponential function:

$$
{}^{\mathcal{M}}_{a}I^{\alpha}_{t}\, e^{\lambda t} = \frac{e^{\lambda t}}{\mathcal{M}_{\alpha}\lambda^{\mathrm{n}-1}} + \frac{e^{\lambda t}}{\lambda^{\mathrm{n}}}.
$$

$$
{}^{\mathcal{M}}_{a}I^{\alpha}_{t}\sin(\lambda t) = \frac{1}{\mathcal{M}_{\alpha}}I^{n-1}_{t}\sin(\lambda t) + I^{n}_{t}\sin(\lambda t)
$$

by using nth integral of sine function:

$$
{}^{\mathcal{M}}_{a}I^{\alpha}_{t}\sin(\lambda t) = \frac{1}{\mathcal{M}_{\alpha}}\lambda t^{n}E_{2,n+1}\left(-(\lambda t)^{2}\right) + \lambda t^{n+1}E_{2,n+2}\left(-(\lambda t)^{2}\right).
$$

$$
{}^{\mathcal{M}}_{a}I^{\alpha}_{t}\cos(\lambda t) = \frac{1}{\mathcal{M}_{\alpha}}I^{n-1}_{t}\cos(\lambda t) + I^{n}_{t}\cos(\lambda t)
$$

by using nth integral of cosine function:

$$
{}^{\mathcal{M}}_{a}I^{\alpha}_{t}\cos(\lambda t) = \frac{1}{\mathcal{M}_{\alpha}}t^{n-1}E_{2,n}\left(-(\lambda t)^{2}\right) + t^{n}E_{2,n+1}\left(-(\lambda t)^{2}\right).
$$

□

3. Integral Transforms of the New Derivative

In 2021, Hossein Jafari introduced a new integral transform [26]: let $f(t)$ be an integrable function at $t \geq 0$, $\mathcal{P}(s) \neq 0$ and $q(s)$ are positive real functions, then the new integral transform of $f(t)$ is given by:

$$\mathcal{J}\{f(t)\}(s) = \mathcal{F}(s) = \mathcal{P}(s) \int_0^{\infty} f(t) e^{-q(s)t} dt. \tag{23}$$

Theorem 3. *Let us ${}^{\mathcal{M}}_{0}\mathcal{D}^{\alpha}_{t} f(t)$ is the new derivative of $f(t)$ of order $\alpha \geq 0$, $t > 0$ and $\mathcal{J}\{f(t)\}$ is the Jafari transform of $f(t)$, then the Jafari transform of the new derivative of $f(t)$ is given by:*

$$\mathcal{J}\left\{{}^{\mathcal{M}}_{0}\mathcal{D}^{\alpha}_{t} f(t)\right\}(s) = \left[\frac{\mathcal{M}_{\alpha}}{q(s) + \mathcal{M}_{\alpha}}\right]\left[q^{n}(s)\mathcal{J}\{f(t)\} - \mathcal{P}(s)\sum_{k=0}^{n-1} q^{k}(s) f^{(n-k-1)}(0)\right]. \tag{24}$$

Proof. Using the new definition of the fractional derivative and the Jafri transform, the following can be obtained:

$$\mathcal{J}\left\{{}^{\mathcal{M}}_{0}\mathcal{D}^{\alpha}_{t} f(t)\right\}(s) = \mathcal{M}_{\alpha}\mathcal{J}\left\{\int_0^{t} f^{(n)}(\mathcal{T})\mathrm{e}^{-\mathcal{M}_{\alpha}(t-\mathcal{T})}\, d\mathcal{T}\right\}, \tag{25}$$

by the convolution theorem of the Jafari transform, and the Jafari transform of nth derivative and exponential function [26], Equation (25) can be written as follows:

$$\begin{aligned}
&\mathcal{J}\{{}^{\mathcal{M}}_{0}\mathcal{D}^{\alpha}_{t} f(t)\}(s) = \tfrac{\mathcal{M}_{\alpha}}{\mathcal{P}(s)}\,\mathcal{J}\left\{f^{(n)}(\mathcal{T})\right\}\mathcal{J}\{\mathrm{e}^{-\mathcal{M}_{\alpha}t}\} \\
&= \tfrac{\mathcal{M}_{\alpha}}{\mathcal{P}(s)}\left[q^{n}(s)\mathcal{J}\{f(t)\} - \mathcal{P}(s)\sum_{k=0}^{n-1} q^{k}(s) f^{(n-k-1)}(0)\right]\left[\tfrac{\mathcal{P}(s)}{q(s)+\mathcal{M}_{\alpha}}\right] \\
&= \left[\tfrac{\mathcal{M}_{\alpha}}{q(s)+\mathcal{M}_{\alpha}}\right]\left[q^{n}(s)\mathcal{J}\{f(t)\} - \mathcal{P}(s)\sum_{k=0}^{n-1} q^{k}(s) f^{(n-k-1)}(0)\right]
\end{aligned}$$

□

Theorem 4. *Suppose that $t > 0$ and $\mathcal{J}\{f(t)\}$ is the Jafari transform of $f(t)$ and ${}^{\mathcal{M}}_{0}\mathcal{D}^{\alpha}_{t} f(t)$ is the new derivative of $f(t)$ of order $\alpha \geq 0$, then the integral transforms of $f(t)$ is given by:*

$$L\left\{{}^{\mathcal{M}}_{0}\mathcal{D}^{\alpha}_{t} f(t)\right\}(s) = \left[\frac{\mathcal{M}_{\alpha}}{s + \mathcal{M}_{\alpha}}\right]\left[s^{n}L\{f(t)\} - \sum_{k=0}^{n-1} s^{k} f^{(n-k-1)}(0)\right] \tag{26}$$

$L\{.\}$ is the Laplace transform:

$$L_{\alpha}\left\{{}^{\mathcal{M}}_{0}\mathcal{D}^{\alpha}_{t} f(t)\right\}(s) = \left[\frac{\mathcal{M}_{\alpha}}{s^{\frac{1}{\alpha}} + \mathcal{M}_{\alpha}}\right]\left[s^{\frac{n}{\alpha}}L_{\alpha}\{f(t)\} - \sum_{k=0}^{n-1} s^{\frac{k}{\alpha}} f^{(n-k-1)}(0)\right] \tag{27}$$

$L_{\alpha}\{.\}$ is the α—Laplace transform:

$$S\left\{{}^{\mathcal{M}}_{0}\mathcal{D}^{\alpha}_{t} f(t)\right\}(s) = \left[\frac{\mathcal{M}_{\alpha}}{1 + s\mathcal{M}_{\alpha}}\right]\left[s^{1-n}S\{f(t)\} - \sum_{k=0}^{n-1} s^{-k} f^{(n-k-1)}(0)\right] \tag{28}$$

$S\{.\}$ is the Sumudu transform:

$$A\left\{{}_0^{\mathcal{M}}\mathcal{D}_t^{\alpha} f(t)\right\}(s) = \left[\frac{\mathcal{M}_\alpha}{1+\mathcal{M}_\alpha}\right]\left[A\{f(t)\} - \sum_{k=0}^{n-1} s^{-1} f^{(n-k-1)}(0)\right] \tag{29}$$

$A\{.\}$ is the Aboodh transform:

$$HJ\left\{{}_0^{\mathcal{M}}\mathcal{D}_t^{\alpha} f(t)\right\}(s) = \left[\frac{\mathcal{M}_\alpha}{s^2+\mathcal{M}_\alpha}\right]\left[s^{2n} HJ\{f(t)\} - \sum_{k=0}^{n-1} s^{2k+1} f^{(n-k-1)}(0)\right] \tag{30}$$

$HJ\{.\}$ is the Pourreza transform:

$$E\left\{{}_0^{\mathcal{M}}\mathcal{D}_t^{\alpha} f(t)\right\}(s) = \left[\frac{\mathcal{M}_\alpha}{s^{-1}+\mathcal{M}_\alpha}\right]\left[s^{-n} E\{f(t)\} - \sum_{k=0}^{n-1} s^{1-k} f^{(n-k-1)}(0)\right] \tag{31}$$

$E\{.\}$ is the Elzaki transform:

$$M\left\{{}_0^{\mathcal{M}}\mathcal{D}_t^{\alpha} f(t)\right\}(s) = \left[\frac{\mathcal{M}_\alpha}{s+\mathcal{M}_\alpha}\right]\left[s^{n} M\{f(t)\} - \sum_{k=0}^{n-1} s^{k+2} f^{(n-k-1)}(0)\right] \tag{32}$$

$M\{.\}$ is the Mohand transform:

$$Sa\left\{{}_0^{\mathcal{M}}\mathcal{D}_t^{\alpha} f(t)\right\}(s) = \left[\frac{\mathcal{M}_\alpha}{s^{-1}+\mathcal{M}_\alpha}\right]\left[s^{-n} Sa\{f(t)\} - \sum_{k=0}^{n-1} s^{-k-2} f^{(n-k-1)}(0)\right] \tag{33}$$

$Sa\{.\}$ is the Sawi transform:

$$K\left\{{}_0^{\mathcal{M}}\mathcal{D}_t^{\alpha} f(t)\right\}(s) = \left[\frac{\mathcal{M}_\alpha}{s^{-1}+\mathcal{M}_\alpha}\right]\left[s^{-n} K\{f(t)\} - \sum_{k=0}^{n-1} s^{-k} f^{(n-k-1)}(0)\right] \tag{34}$$

$K\{.\}$ is the Kamal transform:

$$G\left\{{}_0^{\mathcal{M}}\mathcal{D}_t^{\alpha} f(t)\right\}(s) = \left[\frac{\mathcal{M}_\alpha}{s^{-1}+\mathcal{M}_\alpha}\right]\left[s^{-n} G\{f(t)\} - \sum_{k=0}^{n-1} s^{-k+\alpha} f^{(n-k-1)}(0)\right] \tag{35}$$

$G\{.\}$ is the G-transform:

$$N\left\{{}_0^{\mathcal{M}}\mathcal{D}_t^{\alpha} f(t)\right\}(s) = \left[\frac{\mathcal{M}_\alpha}{s+u\mathcal{M}_\alpha}\right]\left[\frac{s^n}{u^{n-1}} N\{f(t)\} - \sum_{k=0}^{n-1} \frac{s^k}{u^k} f^{(n-k-1)}(0)\right] \tag{36}$$

$N\{.\}$ is the Natural transform:

$$H\left\{{}_0^{\mathcal{M}}\mathcal{D}_t^{\alpha} f(t)\right\}(s) = \left[\frac{\mathcal{M}_\alpha}{su^{-1}+\mathcal{M}_\alpha}\right]\left[\frac{s^n}{u^n} H\{f(t)\} - \sum_{k=0}^{n-1} \frac{s^k}{u^k} f^{(n-k-1)}(0)\right] \tag{37}$$

$H\{.\}$ is the Shehu transform:

Proof. By theorem 2, and the relationships between the Jafari transform and integral transforms [26], this theorem can be easily proved. □

4. Initial Value Problems with New Derivative

In this section, the existence and uniqueness of the initial value problems solution with the new fractional derivative are presented and discussed.

Lemma 4. In the initial value problem below, let f be the solution and $\alpha \geq 0$:

$$ {}^{\mathcal{M}}_{0}\mathcal{D}^{\alpha}_{t} f(t) = 0, \qquad t \geq 0. \tag{38}$$

Then,

$$ f(t) = \sum_{k=0}^{n-1} \frac{t^k}{(k)!} f^{(k)}(0). \tag{39}$$

Proof. From Equation (19), it can be noticed that the solution of Equation (38) must satisfy Equation (39) for all $t \geq 0$. □

Proposition 1. Suppose that $\alpha \geq 0$, $n-1 < \alpha \leq n$ and $t \geq 0$. Then, the following initial value problem's unique solution:

$$ {}^{\mathcal{M}}_{0}\mathcal{D}^{\alpha}_{t} f(t) = \xi(t), \qquad t \geq 0, \tag{40}$$

$$ f^{(k)}(0) = f_0^{(k)} \in \mathbb{R}, \tag{41}$$

is given by:

$$ f(t) = \sum_{k=0}^{n-1} \frac{t^k}{(k)!} f^{(k)}(0) + \frac{1}{\mathcal{M}_\alpha}\, I_t^{n-1}\xi(t) + I_t^n \xi(t), \qquad t \geq 0, \tag{42}$$

Proof. Assume there are two solutions to the initial value problem Equations (40) and (41), f_1 *and* f_2. As a result of this assumption:

$$ {}^{\mathcal{M}}_{0}\mathcal{D}^{\alpha}_{t} f_1\,(t) - {}^{\mathcal{M}}_{0}\mathcal{D}^{\alpha}_{t} f_2(t) = {}^{\mathcal{M}}_{0}\mathcal{D}^{\alpha}_{t}(f_1 - f_2)(t) = 0, \text{ and } (f_1 - f_2)(0) = 0$$

□

Thus, $f_1(t) - f_2(t) = 0$ can be easily obtained using Lemma 4. That is $f_1(t) = f_2(t)$ for all $t \geq 0$.

From Equation (19), it is easy to conclude that Equation (42) is a solution to the initial value problem in Equations (40) and (41).

In addition, when t is substituted by 0 in Equation (42), it can be written as $f(t) = \sum_{k=0}^{n-1} \frac{t^k}{(k)!} f_0^{(k)}$.

Proposition 2. Let $\alpha \geq 0$, $n-1 < \alpha \leq n$, $T \geq 0$. Then, the following nonlinear fractional differential equation:

$$ {}^{\mathcal{M}}_{0}\mathcal{D}^{\alpha}_{t} f(t) = \xi(t, f(t)), \qquad t \in [0, T], \tag{43}$$

$$ f^k(0) = f_0^k, \qquad k = 0, 1, 2, \ldots \tag{44}$$

has a unique solution, if $\xi(t, f(t))$ satisfies the Lipschitz condition with respect to f with constant $\omega > 0$ such that $\omega\left(\frac{1}{\mathcal{M}_\alpha} + \frac{T}{n}\right)\frac{T^{n-1}}{(n-1)!} < 1$.

Proof. Assume that $\vartheta[0,\ T]$ is the space of all continuous functions defined on the interval $[0, T]$ and has the standard supremum norm:

$$ \|f\| = \sup_{t \in [0,T]} |f(t)|, \qquad for\ all \quad f \in \vartheta[0,\ T]$$

□

Consider the operator $\mathcal{O} : \vartheta[0,\ T] \to \vartheta[0,\ T]$ defined by:

$$\mathcal{O}f(t) = \sum_{k=0}^{n-1} \frac{t^k}{(k)!} f_0^k + \frac{1}{\mathcal{M}_\alpha}\ I_t^{n-1}\xi(t, f(t)) + I_t^n \xi(t, f(t)), \qquad for\ all\ \ f \in \vartheta[0,\ T],$$

From Equation (19), Obtaining a fixed point of the operator $\mathcal{O}$ is analogous to coming up with a solution of Equations (43) and (44) in $\vartheta[0,\ T]$.

Now, it must be proven that $\mathcal{O}$ is a contractive mapping, so take $f_1, f_2 \in\ \vartheta[0, T]$ and $t \in [0, T]$:

$$\begin{aligned}\|\mathcal{O}f_1(t) - \mathcal{O}f_2(t)\| &= \left|\tfrac{1}{\mathcal{M}_\alpha}\ I_t^{n-1}\xi(t, f_1(t)) + I_t^n\xi(t, f_1(t)) - \tfrac{1}{\mathcal{M}_\alpha}\ I_t^{n-1}\xi(t, f_2(t)) - I_t^n\xi(t, f_2(t))\right|,\\ &\leq \tfrac{1}{\mathcal{M}_\alpha}\ I_t^{n-1}(|\xi(t, f_1(t)) - \xi(t, f_2(t))|) + I_t^n(|\xi(t, f_1(t)) - \xi(t, f_2(t))|),\end{aligned}$$

since $\xi(t, f(t))$ satisfies Lipschitz condition with respect to f with constant $\omega > 0$,

$$\begin{aligned}&\leq \tfrac{1}{\mathcal{M}_\alpha}\ I_t^{n-1}(|\omega(f_1(t) - f_2(t))|) + I_t^n(|\omega(f_1(t) - f_2(t))|),\\ &\leq \omega\left(\tfrac{1}{\mathcal{M}_\alpha} + \tfrac{T}{n}\right)\tfrac{T^{n-1}}{(n-1)!} f_1 - f_2\end{aligned}$$

Hence, it produces that $\mathcal{O}$ is a contractive mapping. As a result, the solution to the initial value problem in Equations (43) and (44) is unique.

5. Illustrative Examples

In this section, we solve fractional differential equations with the new fractional operator by taking $\mathcal{M}_\alpha = \Gamma^2(n - \alpha)$.

Example 1. Suppose the fractional differential equation with the new fractional operator, $0 < \alpha \leq 1$, Figure 1, Table 1:

$$^{\mathcal{M}}_{0}\mathcal{D}_t^\alpha \mathcal{Y}(t) + \mathcal{Y}(t) = 0, \qquad \mathcal{Y}(0) = 1, \qquad t \geq 0, \tag{45}$$

By taking LT to both sides of Equation (45), produces the following:

$$\mathrm{L}\{\mathcal{Y}(t)\} = \frac{\Gamma^2(1-\alpha)}{\mathcal{s} + \mathcal{s}\Gamma^2(1-\alpha) + \Gamma^2(1-\alpha)}, t, \mathcal{s} \geq 0, \tag{46}$$

By using the inverse LT to both sides of Equation (46):

$$\mathcal{Y}_{\mathcal{M}}(t) = \frac{\Gamma^2(1-\alpha)}{1 + \Gamma^2(1-\alpha)}\ e^{-\frac{\Gamma^2(1-\alpha)}{1+\Gamma^2(1-\alpha)}t}, \qquad t \geq 0, \tag{47}$$

Equation (45) with the Caputo–Fabrizio–Caputo operator is formulated as follows:

$$^{CFC}_{0}\mathcal{D}_t^\alpha \mathcal{Y}(t) + \mathcal{Y}(t) = 0, \qquad \mathcal{Y}(0) = 1, \qquad t \geq 0, \tag{48}$$

Similarly, the solution to Equation (48) is given by:

$$\mathcal{Y}_{CFC}(t) = \frac{1}{1-\alpha}\ e^{-\frac{\alpha}{1-\alpha}t}, \qquad t \geq 0, \tag{49}$$

Equation (45) with the Atangana–Baleanu–Caputo operator formulated as follows,

$$^{ABC}_{0}\mathcal{D}_t^\alpha \mathcal{Y}(t) + \mathcal{Y}(t) = 0, \qquad \mathcal{Y}(0) = 1, \qquad t \geq 0, \tag{50}$$

Similarly, the solution to Equation (48) is given by:

$$\mathcal{Y}_{ABC}(t) = \frac{1}{1-\alpha} E_{\alpha}\left(-\frac{\alpha}{1-\alpha} t^{\alpha}\right), \qquad t \geq 0, \tag{51}$$

Equation (45) with the Caputo operator formulated as follows:

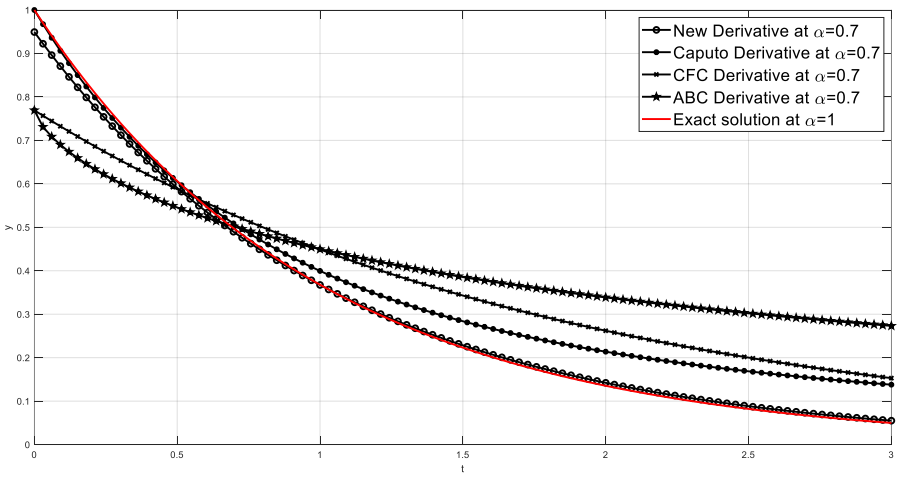
$${}_{0}^{C}\mathcal{D}_{t}^{\alpha}\mathcal{Y}(t) + \mathcal{Y}(t) = 0, \qquad \mathcal{Y}(0) = 1, \qquad t \geq 0, \tag{52}$$

Similarly, the solution to Equation (48) is given by:

$$\mathcal{Y}_{C}(t) = E_{\alpha}(-t), \qquad t \geq 0, \tag{53}$$

when choosing $\alpha = 1$, the exact solution to Equations (45), (48), (50) and (52) will be:

$$\mathcal{Y}(t) = e^{-t}, \qquad t \geq 0. \tag{54}$$



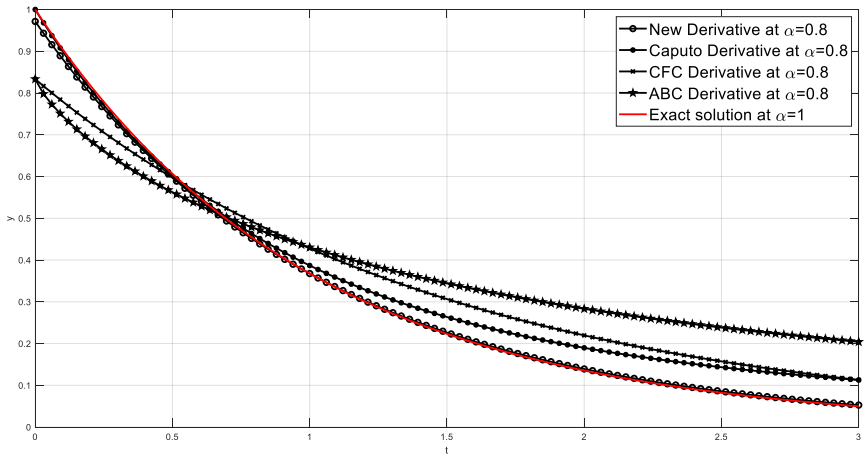


Figure 1. *Cont.*

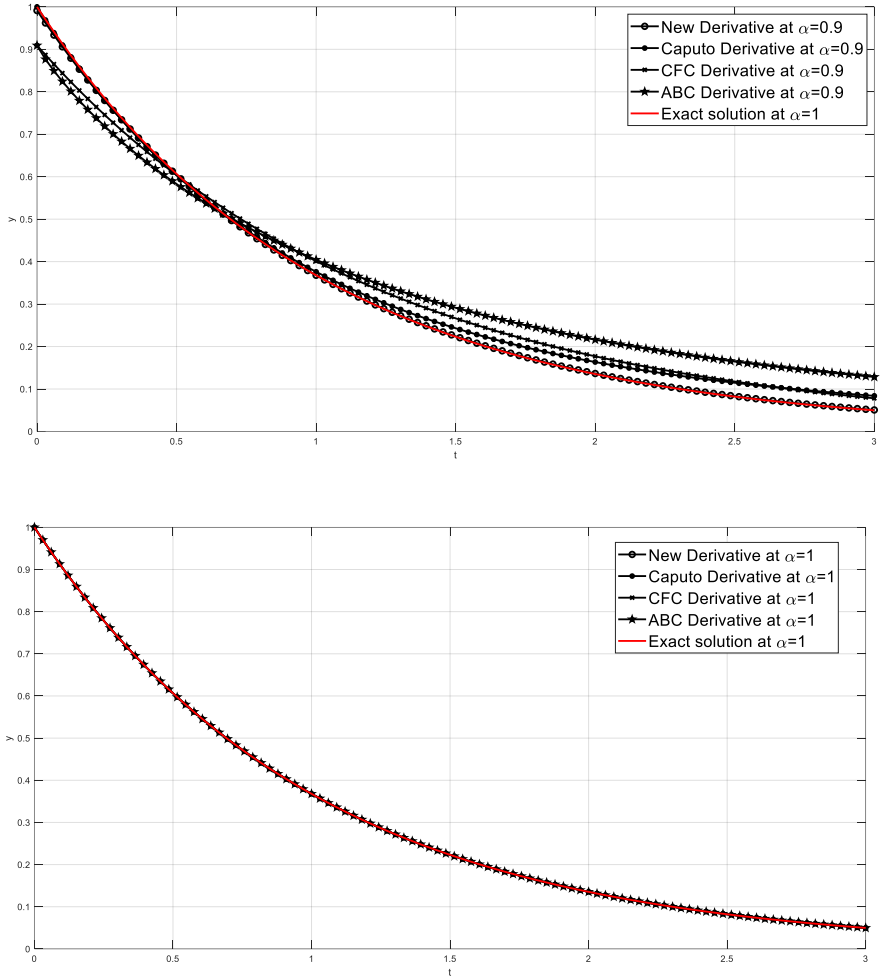


Figure 1. The graphs of the exact solutions among different values of t,α for the exact solution of Equation (45), exact solution of Equation (48), exact solution of Equation (50), exact solution of Equation (52) and the exact solution of Equations (45), (48), (50) and (52) at $\alpha = 1$.

Table 1. The values of the exact solutions at $\alpha = 0.9$ and absolute error of Equations (45), (48), (50), (52) and (54) at different values of t.

t	$y_{\mathcal{M}}$	y_{CFC}	y_C	y_C	y	$\lvert y - y_{\mathcal{M}}\rvert$	$\lvert y - y_{CFC}\rvert$	$\lvert y - y_{ABC}\rvert$	$\lvert y - y_C\rvert$
0.2143	0.8016	0.8024	0.7629	0.7367	0.8071	0.0055	0.0047	0.0442	0.0704
0.4286	0.6482	0.6473	0.6402	0.6165	0.6514	0.0032	0.0041	0.0112	0.0349
0.6429	0.5241	0.5253	0.5373	0.5226	0.5258	0.0016	0.0005	0.0115	0.0032
0.8571	0.4238	0.4289	0.4509	0.4469	0.4244	0.0005	0.0046	0.0265	0.0226
1.0714	0.3427	0.3525	0.3784	0.3851	0.3425	0.0002	0.0100	0.0358	0.0426
1.2857	0.2771	0.2917	0.3175	0.3339	0.2765	0.0007	0.0152	0.0411	0.0575
1.5000	0.2241	0.2431	0.2664	0.2913	0.2231	0.0010	0.0200	0.0433	0.0681
1.7143	0.1812	0.2041	0.2236	0.2554	0.1801	0.0011	0.0240	0.0435	0.0754
1.9286	0.1465	0.1726	0.1876	0.2252	0.1454	0.0012	0.0273	0.0423	0.0798
2.1429	0.1185	0.1471	0.1575	0.1994	0.1173	0.0012	0.0298	0.0401	0.0821
2.3571	0.0958	0.1264	0.1321	0.1775	0.0947	0.0011	0.0317	0.0375	0.0828
2.5714	0.0775	0.1094	0.1109	0.1586	0.0764	0.0010	0.0330	0.0345	0.0822
2.7857	0.0626	0.0954	0.0931	0.1424	0.0617	0.0010	0.0338	0.0314	0.0808
3.0000	0.0507	0.0839	0.0781	0.1284	0.0498	0.0009	0.0341	0.0283	0.0786

Example 2. Suppose Burger's equation with the new fractional operator and $0 < \alpha \leq 1$, Figure 2, Table 2:

$$ {}^{\mathcal{M}}_{0}\mathcal{D}^{\alpha}_{t}u(x,t) + uu_x = u_{xx}, \qquad u(x,0) = x, \tag{55} $$

By using Sumudu transform decomposition method:

$$ \frac{\Gamma^2(1-\alpha)}{1+s\Gamma^2(1-\alpha)}(\mathcal{S}(u) - x) = u_{xx} - uu_x, \qquad ,t,s \geq 0, \tag{56} $$

By taking the inverse Sumudu transform to both sides of Equation (49):

$$ u_{n+1} = x + \mathcal{S}^{-1}\left[\frac{1+t\Gamma^2(1-\alpha)}{\Gamma^2(1-\alpha)}\mathcal{S}\left(\sum_{n=0}^{\infty} u_{nxx} - \sum_{n=0}^{\infty} A_n\right)\right], \tag{57} $$

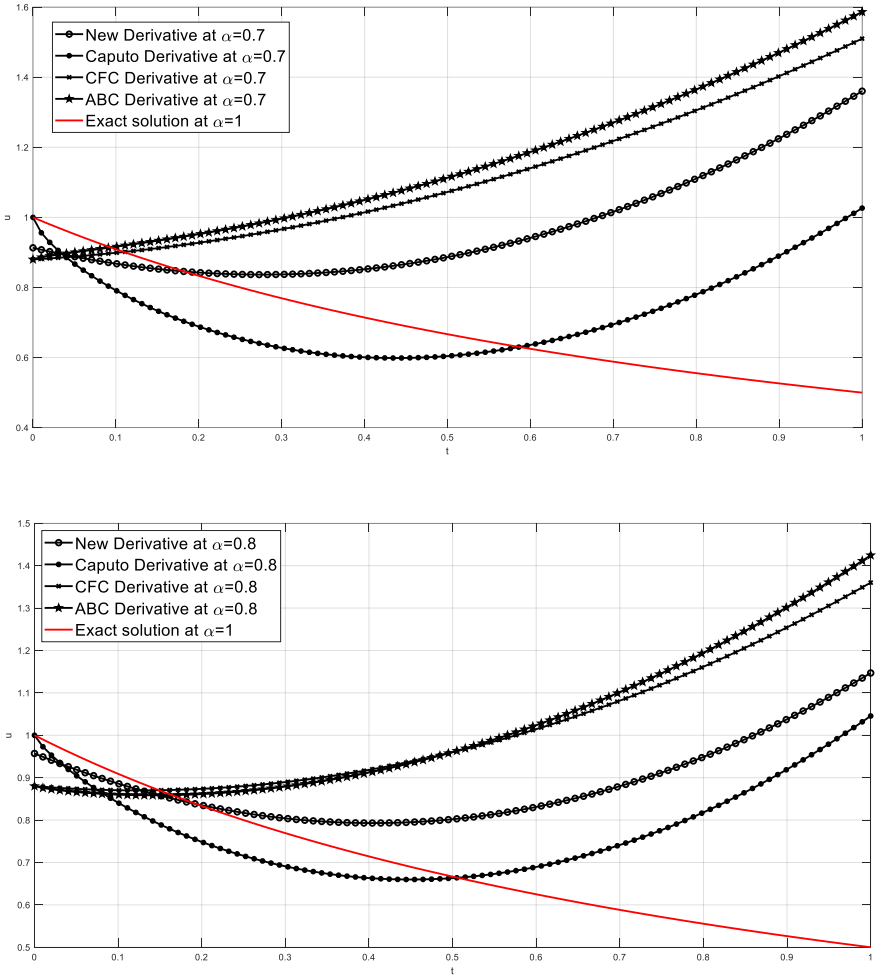


Figure 2. *Cont.*

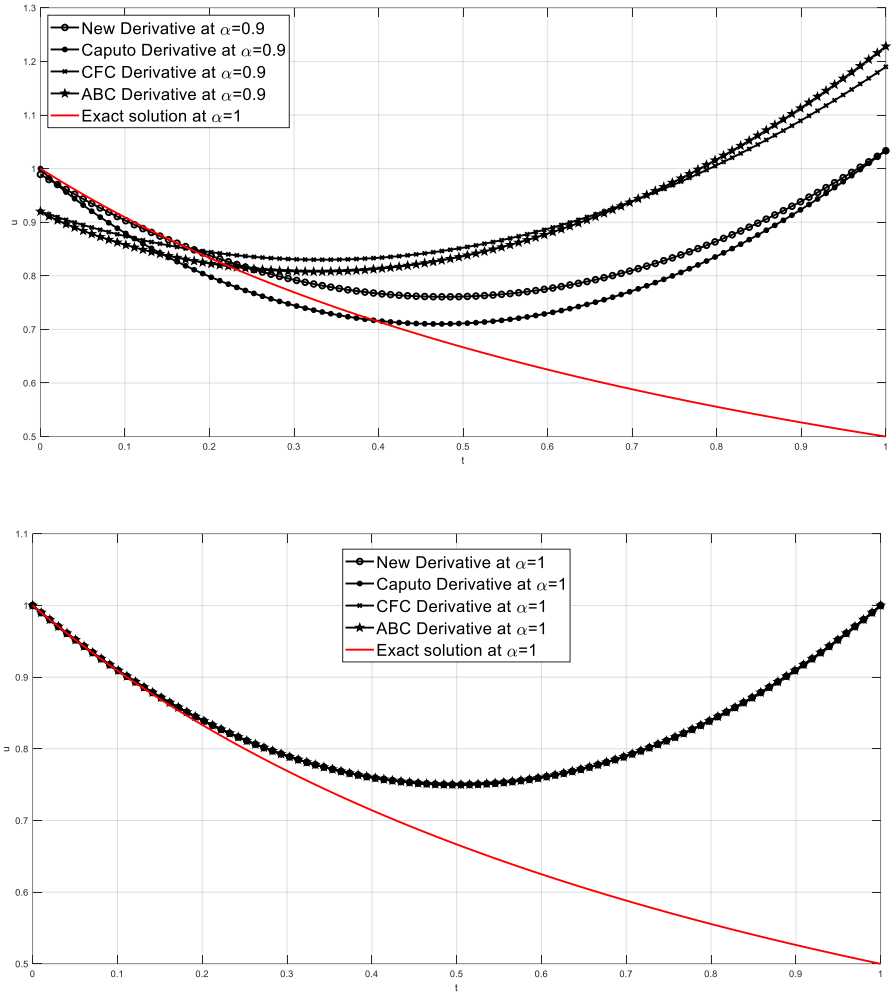


Figure 2. The graphs of the approximate and exact solutions choosing different values of t,α and $\alpha = 1$, for approximate solutions of Equations (55), (59), (62) and (64) and their exact solutions at $\alpha = 1$.

Now, with the algorithm of the Sumudu transform decomposition method (STDM):

$$u_0 = x,$$

$$u_1 = -x\left(\frac{1}{\Gamma^2(1-\alpha)} + t\right),$$

$$u_2 = x\left(\frac{2}{\Gamma^4(1-\alpha)} + \frac{4}{\Gamma^2(1-\alpha)}t + t^2\right),$$

$$u_3 = -x\left(\frac{5}{\Gamma^6(1-\alpha)} + \frac{15}{\Gamma^4(1-\alpha)}t + \frac{8}{\Gamma(1-\alpha)}t^2 + t^3\right),$$

thus, the approximate solution of Equation (53) is given by:

$$u_{\mathcal{M}}(x,t) = x\left(\begin{array}{c}\left[1 - \frac{1}{\Gamma^2(1-\alpha)} + \frac{2}{\Gamma^4(1-\alpha)} - \frac{5}{\Gamma^6(1-\alpha)}\right] + \left[-1 + \frac{4}{\Gamma^2(1-\alpha)} - \frac{15}{\Gamma^4(1-\alpha)}\right]t \\ + \left[1 - \frac{8}{\Gamma^2(1-\alpha)}\right]t^2 + t^3 + \ldots\end{array}\right), \quad (58)$$

Equation (53) with the Caputo–Fabrizio–Caputo operator is formulated as follows, where $0 < \alpha \leq 1$:

$$^{CFC}_{0}\mathcal{D}^{\alpha}_{t}u(x,t) + uu_x = u_{xx}, \qquad u(x,0) = x, \tag{59}$$

In a similar way, the approximate solution to Equation (58) is given by:

$$u_{CFC}(x,t) = x\left[\left(2 - 3\alpha + 2\alpha^2\right) + \left(3\alpha - 4\alpha^2\right)t + \alpha^2 t^2 + \ldots\right] \tag{60}$$

Equation (53) with the Atangana–Baleanu–Caputo operator is formulated as follows, where $0 < \alpha \leq 1$:

$$^{ABC}_{0}\mathcal{D}^{\alpha}_{t}u(x,t) + uu_x = u_{xx}, \qquad u(x,0) = x, \tag{61}$$

Similarly, the approximate solution to Equation (58) is given by:

$$u_{ABC}(x,t) = x\left[\left(2 - 3\alpha + 2\alpha^2\right) + \left(3\alpha - 4\alpha^2\right)\frac{x^{\alpha}}{\Gamma(\alpha+1)} + 2\alpha^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots\right] \tag{62}$$

Equation (53) with the Caputo operator is formulated as follows, where $0 < \alpha \leq 1$,

$$^{C}_{0}\mathcal{D}^{\alpha}_{t}u(x,t) + uu_x = u_{xx}, \qquad u(x,0) = x, \tag{63}$$

Similarly, the approximate solution to Equation (58) is given by:

$$u_C(x,t) = x\left[1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + 2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots\right], \tag{64}$$

when choosing $\alpha = 1$, the approximate solution to Equations (55), (59), (61) and (63) will be:

$$u(x,t) = x\left(1 - t + t^2 - \ldots\right), \tag{65}$$

Therefore, the exact solution of Equations (55), (59), (61) and (63) is given by:

$$u(x,t) = \frac{x}{1+t}. \tag{66}$$

Table 2. The values of the approximate and exact solutions at $\alpha = 1$ and absolute error of Equations (55), (59), (62), (64) and (66) at different values of t, x.

x	t	$u_{\mathcal{M}}$	u_{CFC}	u_{ABC}	u_C	u	$\lvert u - u_{\mathcal{M}}\rvert$	$\lvert u - u_{\mathcal{M}}\rvert$	$\lvert u - u_{\mathcal{M}}\rvert$	$\lvert u - u_{\mathcal{M}}\rvert$
0.0714	0.0714	0.0667	0.0667	0.0667	0.0667	0.0667	0.0000	0.0000	0.0000	0.0000
0.1429	0.1429	0.1254	0.1254	0.1254	0.1254	0.1250	0.0004	0.0004	0.0004	0.0004
0.2143	0.2143	0.1782	0.1782	0.1782	0.1782	0.1765	0.0017	0.0017	0.0017	0.0017
0.2857	0.2857	0.2274	0.2274	0.2274	0.2274	0.2222	0.0052	0.0052	0.0052	0.0052
0.3571	0.3571	0.2751	0.2751	0.2751	0.2751	0.2632	0.0120	0.0120	0.0120	0.0120
0.4286	0.4286	0.3236	0.3236	0.3236	0.3236	0.3000	0.0236	0.0236	0.0236	0.0236
0.5000	0.5000	0.3750	0.3750	0.3750	0.3750	0.3333	0.0417	0.0417	0.0417	0.0417
0.5714	0.5714	0.4315	0.4315	0.4315	0.4315	0.3636	0.0679	0.0679	0.0679	0.0679
0.6429	0.6429	0.4953	0.4953	0.4953	0.4953	0.3913	0.1040	0.1040	0.1040	0.1040
0.7143	0.7143	0.5685	0.5685	0.5685	0.5685	0.4167	0.1518	0.1518	0.1518	0.1518
0.7857	0.7857	0.6534	0.6534	0.6534	0.6534	0.4400	0.2134	0.2134	0.2134	0.2134
0.8571	0.8571	0.7522	0.7522	0.7522	0.7522	0.4615	0.2906	0.2906	0.2906	0.2906
0.9286	0.9286	0.8670	0.8670	0.8670	0.8670	0.4815	0.3855	0.3855	0.3855	0.3855
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.5000	0.5000	0.5000	0.5000	0.5000

Remark. It is clear from this study that the new derivative has fractional order $\alpha \geq 0$, which is not similar to the Caputo–Fabrizio derivative and Atangana–Baleanu derivative, where

$0 < \alpha \leq 1$. In addition, it can be noticed from the study that the solutions of differential equations with a new derivative are better than that obtained with the Caputo–Fabrizio derivative and Atangana–Baleanu derivative.

6. Conclusions

In this paper, a novel new definition of fractional derivative for order $\alpha \geq 0$, is introduced and discussed. The results of this study showed that the solution of differential equations with the new fractional derivative exists and is unique. The new fractional derivative converges to the classical derivative faster than the other fractional derivatives such as the Caputo–Fabrizio, Atangana–Baleanu, and Caputo fractional derivatives, which is proven by the analysis presented in this paper.

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