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*Article*

# Exponential Stability of Impulsive Neutral Stochastic Functional Differential Equations

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**Abstract:** This paper focuses on the problem of the *p*th moment and almost sure exponential stability of impulsive neutral stochastic functional differential equations (INSFDEs). Based on the Lyapunov function and average dwell time (ADT), two sufficient criteria for the exponential stability of INSFDEs are derived, which manifest that the result obtained in this paper is more convenient to be used than those Razumikhin conditions in former literature. Finally, two numerical examples and simulations are given to verify the validity of our result.

**Keywords:** impulse; exponential stability; neutral stochastic functional differential equations; average dwell time

**MSC:** 60H10; 34F05; 34K45

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## 1. Introduction

As is known to all, neutral stochastic functional differential equations already have attracted much consideration in engineering and science since they can be applied in many fields such as medical, economic, physics and so on [1–7]. The stability analysis is one of the most important topics in NSFDEs and has already been studied by some classical methods: for example, the Lyapunov–Krasovskii [8], the Lyapunov functional [9,10] and the Razumikhin-type theorem [11,12]. Furthermore, many NSFDEs are often influenced by some impulsive phenomenon which can stabilize or disturb the original system. In fact, when the continuous part of the system is unstable, the impulse can play a positive effect on stabilizing the unstable system [13]; on the contrary, when the continuous part of the system is stable, the impulse can also induce the instability to disturb the original system [14]. Thus, it is extremely necessary and meaningful to investigate the effect of impulse on stability.

The theory of INSFDEs has attracted many authors' interest over the past few years, and a lot of literature has been published. For instance, without thinking the stochastic item, several criteria of the uniformly asymptotically stability have been studied in [15] by using the classical Razumikhin method. By making the use of contraction, mapping principle and generalized Gronwall–Bellmain inequality, the global exponential stability of a special delayed impulsive neutral differential equation has been studied in [16]. Under the condition of the Euler–Maruyama method, the framework of exponential stability of a special delayed impulsive neutral system has been studied in [17]. While in [18], a new sufficient criterion for the *p*th moment exponential stability has been investigated under the ADT condition. In addition, readers can refer [19–30] to learn more about the investigation of the stability analysis for stochastic functional differential equations.

However, as far as we know, only a few papers reported on the general *p*th moment and almost sure exponential stability for INSFDEs, for which sporadic results are expected. For example, in [18], the authors investigate the *p*th moment exponential stability of INSFDEs by using the ADT condition, but the paper only considers the systems with stable continuous stochastic dynamic and unstable discrete dynamics; the other situation, i.e., the system with an unstable continuous part of the stochastic dynamics and stable discrete

dynamics, is not considered. In [19], the paper is concerned with the exponential stability of INSFDEs by using the classical Razumikhin technique, but sometimes, it is not easy for us to verify the Razumikhin conditions. The previous literature makes the theoretical results more conservative, which motivates us to close such a gap in this paper.

According to the above discussion, the further investigation about the stability analysis of INSFDEs is absolutely necessary. In this paper, by using Lyapunov function and ADT condition, a Lyapunov theorem is established to handle the $p$th moment and almost sure exponential of INSFDEs. The results proposed in this paper are suitable for stabilizing impulses or destabilizing impulses; on the other hand, under the ADT condition, the time sequence with eventually uniformly bounded impulse frequency, that is to say, neither $t_{k+1} - t_k > l_1$ nor $t_{k+1} - t_k < l_2$ are needed ($l_1, l_2$ are positive constants), which is very different from the former literature. Moreover, a useful corollary is obtained by using impulsive condition and some inequality techniques. The remainder of the paper is organized as follows. Some definitions and assumptions are going to be firstly introduced in Section 2. The new conditions are proposed to guarantee the $p$th moment and almost sure exponential stability of INSFDEs in Section 3. In Section 4, we shall also provide two examples and simulations to verify the effectiveness of our result. Finally, we will give a conclusion to enhance the continuity of this paper in the last section.

## 2. Preliminaries

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an $m$-dimensional Brownian motion defined on the probability space. Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{R}_+$ denote the set of positive real numbers, $\mathbb{R}^n$ the $n$-dimensions real Euclidean space and $\mathbb{R}^{n\times m}$ the space of $n \times m$ real matrices. For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm, and $\mathbb{E}$ represents the mathematical expectation operator. If $A$ is a vector or matrix, its transpose is denoted by $A^T$.

Let $\tau > 0$, $C([-\tau, 0]; \mathbb{R}^n)$ denote the family of all continuous $\mathbb{R}^n$-valued functions $\phi$ on $[-\tau, 0]$. $PC([-\tau, 0]; \mathbb{R}^n) = \{\varphi : [-\tau, 0] \to \mathbb{R}^n | \varphi(t^+), \varphi(t^-)$ exists and $\varphi(t^+) = \varphi(t)\}$, where $\varphi(t^+)$ and $\varphi(t^-)$ denote the right-hand and left-hand limits of function $\varphi(t)$ at $t$. $PC^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n)$ denotes the family of all $\mathcal{F}_t$-measurable $PC([-\tau, 0]; \mathbb{R}^n)$-valued random processes $\varphi = \{\varphi(\theta) : -\tau \leq \theta \leq 0\}$ such that $\|\varphi\|^p = \sup_{-\tau\leq\theta\leq 0}\mathbb{E}|\varphi(\theta)|^p < \infty$. $PC^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ denotes the family of all $\mathcal{F}_{t_0}$ measurable bounded $PC([-\tau, 0]; \mathbb{R}^n)$-valued functions.

The INSFDE we considered in the paper is as follows

$$\begin{cases} d[x(t) - D(x_t)] = f(t, x(t), x_t)dt + g(t, x(t), x_t)dB(t), t \neq t_k; \\ \Delta x(t_k) = I_k(t_k, x(t_k^-)), k \in \mathbb{N}; \\ x(t) = \xi, t \in [t_0 - \tau, t_0]. \end{cases} \tag{1}$$

Here, the initial value $\xi \in PC^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$, $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ is the system state, $x_t = x(t + \theta)$. Moreover, $D : PC^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n, f : [t_0, \infty) \times \mathbb{R}^n \times PC^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n, g : [t_0, \infty) \times \mathbb{R}^n \times PC^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^{n\times n}$ and $I_k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. $\Delta x(t_k) = x(t_k) - x(t_k^-)$. The fixed moments impulse times $t_k, k \in \mathbb{N}$ satisfy $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < \cdots, t_k \to \infty$, as $k \to \infty$.

**Assumption 1.** *Assume that there is a constant $\kappa \in (0, 1)$ such that for any $\phi \in PC^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n)$*

$$\mathbb{E}|D(\phi)|^p \leq \kappa^p \sup_{-\tau\leq\theta\leq 0} \mathbb{E}|\phi(\theta)|^p. \tag{2}$$

**Remark 1.** *Assumption 1 mainly gives a limitation to neutral term. In fact, the neutral term can be regarded as a perturbation for the stochastic functional differential equations (SFDEs), which*

*can disturb the original system. So, the coefficient* $\kappa$ *is chosen from* $(0,1)$ *in order to reduce the disturbance of neutral term.*

Throughout the paper, we always assume that $f, g, I_k$ satisfied the needful condition for the global existence and uniqueness of solution for all $t \geq t_0$, denoted by $x(t) = x(t, t_0, \xi)$, which is continuous on the right-hand side and limited on the left-hand side. In addition, we assume that $D(0) \equiv 0, f(t,0,0) \equiv 0, g(t,0,0) \equiv 0$ and $I_k(t,0) \equiv 0, t \in [t_0, \infty), k \in \mathbb{N}$, which implies that $x(t) = 0$ is a trivial solution. For convenience, we denote $\tilde{x}(t) = x(t) - D(x_t)$ for $t \geq t_0$ and $\tilde{x}(\phi) = \phi(0) - D(\phi)$ for $\phi \in PC^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n)$. Let $C^{1,2}([t_{k-1}, t_k) \times \mathbb{R}^n; \mathbb{R}_+), k \in \mathbb{N}$ denotes the non-negative valued and continuous twice differential for $x$ and once for $t$. For $V \in C^{1,2}([t_{k-1}, t_k) \times \mathbb{R}^n; \mathbb{R}_+), k \in \mathbb{N}$, we define the following operator for Equation (1).

$$\begin{aligned} LV(t, \phi(0), \phi) = {} & V_t(t, \tilde{x}(\phi)) + V_{\tilde{x}}(t, \tilde{x}(\phi)) f(t, \phi(0), \phi) \\ & + \frac{1}{2} \text{trace}\Big[ g^T(t, \phi(0), \phi) V_{\tilde{x}\tilde{x}}(t, \tilde{x}(\phi)) g(t, \phi(0), \phi) \Big]. \end{aligned}$$

$$V_t(t, \tilde{x}(\phi)) = \frac{\partial V(t, \tilde{x}(\phi))}{\partial t}, V_{\tilde{x}}(t, \tilde{x}(\phi)) = \left( \frac{\partial V(t, \tilde{x}(\phi))}{\partial \tilde{x}_1}, \cdots, \frac{\partial V(t, \tilde{x}(\phi))}{\partial \tilde{x}_n} \right),$$
$$V_{\tilde{x}\tilde{x}}(t, \tilde{x}(\phi)) = \left( \frac{\partial^2 V(t, \tilde{x}(\phi))}{\partial \tilde{x}_k \partial \tilde{x}_l} \right)_{n \times n}.$$

**Definition 1.** *The trivial solution of (1) is said to be pth moment exponentially stable if there exist two positive constants* $\lambda, M$ *such that*

$$\mathbb{E}|x(t, t_0, \xi)|^p \leq M \|\xi\|^p e^{-\lambda(t - t_0)}, t \geq t_0$$

*for any initial value* $\xi \in PC^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$.

**Remark 2** ([1])**.** *When* $p = 2$*, it is often said to be exponentially stable in mean square.*

**Definition 2.** *The trivial solution of (1) is said to be almost surely exponentially stable if there exist* $\lambda > 0$ *such that*

$$\limsup_{t \to \infty} \frac{1}{t} \ln|x(t, t_0, \xi)| \leq -\lambda, t \geq t_0 \qquad \text{a.s.} \tag{3}$$

*for any initial value* $\xi \in PC^b_{\mathcal{F}_0}([t_0 - \tau, t_0]; \mathbb{R}^n)$.

**Definition 3** ([31–33])**.** *Let* $\{t_k, k \in \mathbb{N}\}$ *denote the impulsive sequence and* $N(t, s)$ *represent the number of instant* $t_k$ *on the semiopen interval* $(s, t]$*, if*

$$\frac{t - s}{T_a} - N_0 \leq N(t, s) \leq \frac{t - s}{T_a} + N_0.$$

*For* $T_a > 0, N_0 > 0$*, then* $T_a$ *and* $N_0$ *are named ADT and elasticity number, respectively.*

**Lemma 1** ([5])**.** *Suppose Assumption 1 holds. Then, for any* $p \geq 1$ *and* $\phi \in PC^p_{\mathcal{F}_t}([-\tau, 0], \mathbb{R}^n)$*, we have*

$$\mathbb{E}|\phi(0) - D(\phi)|^p \leq (1 + \kappa)^p \sup_{-\tau \leq \theta \leq 0} \mathbb{E}|\phi(\theta)|^p,$$

*and*

$$\mathbb{E}|\phi(0)|^p \leq \kappa \sup_{-\tau\leq\theta\leq 0}\mathbb{E}|\phi(\theta)|^p + \frac{1}{(1-\kappa)^{p-1}}\mathbb{E}|\phi(0)-D(\phi)|^p.$$

## 3. Main Results

In this part, by Lyapunov function and ADT condition, we will establish a framework to check both the $p$th moment and almost sure exponential stability. The criteria proposed in this section are more general than the previous result in [16].

### *3.1. pth Moment Exponential Stability*

**Theorem 1.** *Let $p \geq 1$, $\alpha_1 \in \mathbb{R}$, $\mu_2 \geq 0$, $c_1$, $c_2$, $\alpha_2$, $\mu_1$, $h$ be positive numbers. Suppose that Assumption 1 holds and there exists a function $V \in C^{1,2}([t_{k-1}, t_k) \times \mathbb{R}^n; \mathbb{R}_+), k \in \mathbb{N}$ such that*

(H1) $c_1|x|^p \leq V(t,x) \leq c_2|x|^p$, *for any* $(t,x) \in [t_0,\infty) \times \mathbb{R}^n$;

(H2) *For any* $t \geq t_0, t \neq t_k, k \in \mathbb{N}, \phi \in PC^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^n), \mathbb{E}LV(t,\phi(0),\phi) \leq \alpha_1\mathbb{E}V(t,\tilde{x}(\phi)) + \alpha_2 \sup_{-\tau\leq\theta\leq 0}\mathbb{E}V(t+\theta,\phi(\theta))$;

(H3) *For* $k \in \mathbb{N}, \phi \in PC^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^n), \mathbb{E}V(t_k,\tilde{x}(\phi)) \leq \mu_1\mathbb{E}V(t_k^-,\tilde{x}(\phi)) + \mu_2 \sup_{-\tau\leq\theta\leq 0}\mathbb{E}V(t_k^- + \theta,\phi(\theta))$;

(H4) $[c_1(1-\kappa)^p - c_2\mu_2\beta h]\left(\alpha_1 + \frac{\ln\mu_1}{T_a}\right) + c_2\alpha_2\beta < 0, h = \lim_{t\to\infty}\sum_{\iota=1}^{N(t,t_0)} e^{(\alpha_1+\frac{\ln\mu_1}{T_a})(t-t_\iota)}, \alpha_1 + \frac{\ln\mu_1}{T_a} < 0$ *and* $\beta = \max\left\{e^{-N_0\ln\mu_1}, e^{N_0\ln\mu_1}\right\}$.

*Then, the trivial solution of (1) is pth moment exponentially stable, where $T_a$ and $N_0$ are positive constants defined in Definition 3.*

**Proof.** For $t \in [t_{k-1}, t_k), k \in \mathbb{N}$, by applying the Dynkin's [34] formula and condition (H2) we have

$$\begin{aligned}
&e^{-\alpha_1(t-t_{k-1})}\mathbb{E}V(t,\tilde{x}(t))\\
&= \mathbb{E}V(t_{k-1},\tilde{x}(t_{k-1})) + \int_{t_{k-1}}^{t} -\alpha_1 e^{-\alpha_1(s-t_{k-1})}\mathbb{E}V(s,\tilde{x}(s)) + e^{-\alpha_1(s-t_{k-1})}\mathbb{E}LV(s,\tilde{x}(s))ds\\
&\leq \mathbb{E}V(t_{k-1},\tilde{x}(t_{k-1})) + \alpha_2\int_{t_{k-1}}^{t} e^{-\alpha_1(s-t_{k-1})}\sup_{-\tau\leq\theta\leq 0}\mathbb{E}V(s+\theta,x(s+\theta))ds,
\end{aligned}$$

which implies

$$\begin{aligned}
\mathbb{E}V(t,\tilde{x}(t)) \leq\ & e^{\alpha_1(t-t_{k-1})}\mathbb{E}V(t_{k-1},\tilde{x}(t_{k-1}))\\
&+\alpha_2\int_{t_{k-1}}^{t} e^{\alpha_1(t-s)}\sup_{-\tau\leq\theta\leq 0}\mathbb{E}V(s+\theta,x(s+\theta))ds. \qquad (4)
\end{aligned}$$

For $t = t_k, k \in \mathbb{N}$, from condition (H3) and (4), we deduce that

$$\begin{aligned}
\mathbb{E}V(t_k,\tilde{x}(t_k)) &\le \mu_1\mathbb{E}V(t_k^-,\tilde{x}(t_k^-)) + \mu_2 \sup_{-\tau\le\theta\le 0}\mathbb{E}V(t_k^-+\theta,x(t_k^-+\theta))\\
&\le \mu_1\Big[e^{\alpha_1(t_k-t_{k-1})}\mathbb{E}V(t_{k-1},\tilde{x}(t_{k-1}))\\
&\quad +\alpha_2\int_{t_{k-1}}^{t_k} e^{\alpha_1(t_k-s)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(s+\theta,x(s+\theta))ds\Big]\\
&\quad +\mu_2\sup_{-\tau\le\theta\le 0}\mathbb{E}V(t_k^-+\theta,x(t_k^-+\theta))\\
&= \mu_1\mathbb{E}V(t_{k-1},\tilde{x}(t_{k-1}))e^{\alpha_1(t_k-t_{k-1})}\\
&\quad +\mu_1\alpha_2\int_{t_{k-1}}^{t_k} e^{\alpha_1(t_k-s)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(s+\theta,x(s+\theta))ds\\
&\quad +\mu_2\sup_{-\tau\le\theta\le 0}\mathbb{E}V(t_k^-+\theta,x(t_k^-+\theta)). \qquad (5)
\end{aligned}$$

Thus, for $t\in[t_0,t_1)$, by inequality (4), we obtain

$$\mathbb{E}V(t,\tilde{x}(t)) \le e^{\alpha_1(t-t_0)}\mathbb{E}V(t_0,\tilde{x}(t_0)) + \alpha_2\int_{t_0}^{t} e^{\alpha_1(t-s)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(s+\theta,x(s+\theta))ds.$$

Then, by inequality (5), we have

$$\begin{aligned}
\mathbb{E}V(t_1,\tilde{x}(t_1)) &\le \mu_1\mathbb{E}V(t_0,\tilde{x}(t_0))e^{\alpha_1(t_1-t_0)} + \mu_1\alpha_2\int_{t_0}^{t_1} e^{\alpha_1(t_1-s)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(s+\theta,x(s+\theta))ds\\
&\quad +\mu_2\sup_{-\tau\le\theta\le 0}\mathbb{E}V(t_1^-+\theta,x(t_1^-+\theta)). \qquad (6)
\end{aligned}$$

For $t\in[t_1,t_2)$, according to the inequality (4) and (6), we obtain

$$\begin{aligned}
\mathbb{E}V(t,\tilde{x}(t)) &\le e^{\alpha_1(t-t_1)}\mathbb{E}V(t_1,\tilde{x}(t_1)) + \alpha_2\int_{t_1}^{t} e^{\alpha_1(t-s)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(s+\theta,x(s+\theta))ds\\
&\le e^{\alpha_1(t-t_1)}\Big[\mu_1\mathbb{E}V(t_0,\tilde{x}(t_0))e^{\alpha_1(t_1-t_0)}\\
&\quad +\mu_1\alpha_2\int_{t_0}^{t_1} e^{\alpha_1(t_1-s)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(s+\theta,x(s+\theta))\\
&\quad +\mu_2\sup_{-\tau\le\theta\le 0}\mathbb{E}V(t_1^-+\theta,x(t_1^-+\theta))\Big]\\
&\quad +\alpha_2\int_{t_1}^{t} e^{\alpha_1(t-s)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(s+\theta,x(s+\theta))ds\\
&= \mu_1 e^{\alpha_1(t-t_0)}\mathbb{E}V(t_0,\tilde{x}(t_0)) + \mu_1\alpha_2\int_{t_0}^{t_1} e^{\alpha_1(t-s)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(s+\theta,x(s+\theta))ds\\
&\quad +\alpha_2\int_{t_1}^{t} e^{\alpha_1(t-s)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(s+\theta,x(s+\theta))ds\\
&\quad +\mu_2 e^{\alpha_1(t-t_1)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(t_1^-+\theta,x(t_1^-+\theta)).
\end{aligned}$$

Similiar as the deduction of inequality (5), we obtain

$$\begin{aligned}\mathbb{E}V(t_2,\tilde{x}(t_2)) &\le \mu_1\mathbb{E}V(t_1,\tilde{x}(t_1))e^{\alpha_1(t_2-t_1)}+\mu_1\alpha_2\int_{t_1}^{t_2}e^{\alpha_1(t_2-s)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(s+\theta,x(s+\theta))ds\\ &\quad+\mu_2\sup_{-\tau\le\theta\le 0}\mathbb{E}V(t_2^-+\theta,x(t_2^-+\theta))\\ &\le \mu_1^2e^{\alpha_1(t_2-t_0)}\mathbb{E}V(t_0,\tilde{x}(t_0))+\mu_1^2\alpha_2\int_{t_0}^{t_1}e^{\alpha_1(t_2-s)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(s+\theta,x(s+\theta))ds\\ &\quad+\mu_1\mu_2e^{\alpha_1(t_2-t_1)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(t_1^-+\theta,x(t_1^-+\theta))\\ &\quad+\mu_1\alpha_2\int_{t_1}^{t_2}e^{\alpha_1(t_2-s)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(s+\theta,x(s+\theta))ds\\ &\quad+\mu_2\sup_{-\tau\le\theta\le 0}\mathbb{E}V(t_2^-+\theta,x(t_2^-+\theta)).\end{aligned}\tag{7}$$

For $t\in[t_2,t_3)$, by the same method, together with (4) and (7), we obtain

$$\begin{aligned}\mathbb{E}V(t,\tilde{x}(t)) &\le \mu_1^2e^{\alpha_1(t-t_0)}\mathbb{E}V(t_0,\tilde{x}(t_0))+\mu_1^2\alpha_2\int_{t_0}^{t_1}e^{\alpha_1(t-s)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(s+\theta,x(s+\theta))ds\\ &\quad+\mu_1\alpha_2\int_{t_1}^{t_2}e^{\alpha_1(t-s)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(s+\theta,x(s+\theta))ds\\ &\quad+\alpha_2\int_{t_2}^{t}e^{\alpha_1(t-s)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(s+\theta,x(s+\theta))ds\\ &\quad+\mu_1\mu_2e^{\alpha_1(t-t_1)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(t_1^-+\theta,x(t_1^-+\theta))\\ &\quad+\mu_2e^{\alpha_1(t-t_2)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(t_2^-+\theta,x(t_2^-+\theta)).\end{aligned}\tag{8}$$

By induction, we obtain from $t\in[t_{k-1},t_k),k\in\mathbb{N}$

$$\begin{aligned}\mathbb{E}V(t,\tilde{x}(t)) &\le \mu_1^{N(t,t_0)}e^{\alpha_1(t-t_0)}\mathbb{E}V(t_0,\tilde{x}(t_0))\\ &\quad+\alpha_2\int_{t_0}^{t}\mu_1^{N(t,s)}e^{\alpha_1(t-s)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(s+\theta,x(s+\theta))ds\\ &\quad+\sum_{\iota=1}^{N(t,t_0)}\mu_1^{N(t,t_\iota)}\mu_2e^{\alpha_1(t-t_\iota)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(t_\iota^-+\theta,x(t_\iota^-+\theta)).\end{aligned}\tag{9}$$

With the help of Lemma 1, it yields from (9) that

$$\begin{aligned}\mathbb{E}|x(t)|^p &\le \kappa\sup_{-\tau\le\theta\le 0}\mathbb{E}|x(t+\theta)|^p+\frac{1}{c_1(1-\kappa)^{p-1}}\mathbb{E}V(t,\tilde{x}(t))\\ &\le \kappa\sup_{-\tau\le\theta\le 0}\mathbb{E}|x(t+\theta)|^p+\frac{\beta\mathbb{E}V(t_0,\tilde{x}(t_0))}{c_1(1-\kappa)^{p-1}}e^{\left(\alpha_1+\frac{\ln\mu_1}{T_a}\right)(t-t_0)}\\ &\quad+\frac{\alpha_2\beta}{c_1(1-\kappa)^{p-1}}\int_{t_0}^{t}e^{\left(\alpha_1+\frac{\ln\mu_1}{T_a}\right)(t-s)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(s+\theta,x(s+\theta))ds\\ &\quad+\frac{\mu_2\beta}{c_1(1-\kappa)^{p-1}}\sum_{\iota=1}^{N(t,t_0)}e^{\left(\alpha_1+\frac{\ln\mu_1}{T_a}\right)(t-t_\iota)}\sup_{-\tau\le\theta\le 0}\mathbb{E}V(t_\iota^-+\theta,x(t_\iota^-+\theta)).\end{aligned}\tag{10}$$

Based on the condition (H4), inequality $[c_1(1-\kappa)^p - c_2\mu_2\beta h]\left(\alpha_1 + \frac{\ln\mu_1}{T_a}\right) + c_2\alpha_2\beta < 0$ and continuity, one can choose $M > 0$ and a small enough constant $\lambda > 0$ such that

$$\begin{aligned} &\kappa e^{\lambda\tau} - \frac{c_2\alpha_2\beta e^{\lambda\tau}}{c_1(1-\kappa)^{p-1}\left(\alpha_1 + \frac{\ln\mu_1}{T_a} + \lambda\right)} + \frac{c_2\mu_2\beta e^{\lambda\tau}h}{c_1(1-\kappa)^{p-1}} \le 1, \\ &M > -\frac{(1+\kappa)^p\left(\alpha_1 + \frac{\ln\mu_1}{T_a} + \lambda\right)}{\alpha_2 e^{\lambda\tau}}. \end{aligned} \tag{11}$$

By the definition of $M$, we have

$$\mathbb{E}|x(t)|^p \le \|\xi\|^p \le M\|\xi\|^p e^{-\lambda(t-t_0)}, t \in [t_0-\tau, t_0] \tag{12}$$

Next, we shall claim that

$$\mathbb{E}|x(t)|^p \le M\|\xi\|^p e^{-\lambda(t-t_0)}, \qquad t \in [t_0, \infty). \tag{13}$$

Suppose the above inequality (13) is not true; thus, there must exists $t^* \in [t_0, \infty)$ such that

$$\mathbb{E}|x(t)|^p \le M\|\xi\|^p e^{-\lambda(t-t_0)}, t \in [t_0, t^*], \quad \mathbb{E}|x(t^*)|^p \ge M\|\xi\|^p e^{-\lambda(t^*-t_0)}. \tag{14}$$

By inequality (10) and (14), we deduce that

$$\begin{aligned} \mathbb{E}|x(t^*)|^p \le{}& \kappa \sup_{-\tau\le\theta\le 0}\mathbb{E}|x(t^*+\theta)|^p + \frac{\beta c_2(1+\kappa)^p\|\xi\|^p}{c_1(1-\kappa)^{p-1}}e^{\left(\alpha_1+\frac{\ln\mu_1}{T_a}\right)(t^*-t_0)} \\ &+ \frac{c_2\alpha_2\beta}{c_1(1-\kappa)^{p-1}}\int_{t_0}^{t^*} e^{\left(\alpha_1+\frac{\ln\mu_1}{T_a}\right)(t^*-s)}\sup_{-\tau\le\theta\le 0}\mathbb{E}|x(s+\theta)|^p ds \\ &+ \frac{c_2\mu_2\beta}{c_1(1-\kappa)^{p-1}}\sum_{\iota=1}^{N(t^*,t_0)} e^{\left(\alpha_1+\frac{\ln\mu_1}{T_a}\right)(t^*-t_\iota)}\sup_{-\tau\le\theta\le 0}\mathbb{E}\left|x(t_\iota^- +\theta)\right|^p. \end{aligned} \tag{15}$$

There are two cases that need to be considered.

**Case 1.** $\sup_{-\tau\le\theta\le 0}\mathbb{E}|x(t^*+\theta)|^p = \sup_{-\tau\le\theta<0}\mathbb{E}|x(t^*+\theta)|$. In this case, we obtain

$$\begin{aligned} \mathbb{E}|x(t^*)|^p \le{}& \kappa \sup_{-\tau\le\theta<0}\mathbb{E}|x(t^*+\theta)|^p + \frac{\beta c_2(1+\kappa)^p\|\xi\|^p}{c_1(1-\kappa)^{p-1}}e^{\left(\alpha_1+\frac{\ln\mu_1}{T_a}\right)(t^*-t_0)} \\ &+ \frac{c_2\alpha_2\beta}{c_1(1-\kappa)^{p-1}}\int_{t_0}^{t^*} e^{\left(\alpha_1+\frac{\ln\mu_1}{T_a}\right)(t^*-s)}\sup_{-\tau\le\theta\le 0}\mathbb{E}|x(s+\theta)|^p ds \\ &+ \frac{c_2\mu_2\beta}{c_1(1-\kappa)^{p-1}}\sum_{\iota=1}^{N(t^*,t_0)} e^{\left(\alpha_1+\frac{\ln\mu_1}{T_a}\right)(t^*-t_\iota)}\sup_{-\tau\le\theta\le 0}\mathbb{E}\left|x(t_\iota^- +\theta)\right|^p \\ \le{}& \kappa e^{\lambda\tau}M\|\xi\|^p e^{-\lambda(t^*-t_0)} + \frac{\beta c_2(1+\kappa)^p\|\xi\|^p}{c_1(1-\kappa)^{p-1}}e^{\left(\alpha_1+\frac{\ln\mu_1}{T_a}\right)(t^*-t_0)} \\ &+ \frac{c_2\alpha_2\beta e^{\lambda\tau}M\|\xi\|^p}{c_1(1-\kappa)^{p-1}}\int_{t_0}^{t^*} e^{\left(\alpha_1+\frac{\ln\mu_1}{T_a}\right)(t^*-s)}e^{-\lambda(s-t_0)}ds \\ &+ \frac{c_2\mu_2\beta e^{\lambda\tau}M\|\xi\|^p}{c_1(1-\kappa)^{p-1}}\sum_{\iota=1}^{N(t^*,t_0)} e^{\left(\alpha_1+\frac{\ln\mu_1}{T_a}\right)(t^*-t_\iota)-\lambda(t_\iota-t_0)} \\ \le{}& \left[A - \frac{C}{\alpha_1+\frac{\ln\mu_1}{T_a}+\lambda} + Dh\right]M\|\xi\|^p e^{-\lambda(t^*-t_0)} \\ &+ \left[B + \frac{CM}{\alpha_1+\frac{\ln\mu_1}{T_a}+\lambda}\right]\|\xi\|^p e^{(\alpha_1+\frac{\ln\mu_1}{T_a})(t^*-t_0)}, \end{aligned} \tag{16}$$

where

$$A = \kappa e^{\lambda\tau}, B = \frac{\beta c_2 (1+\kappa)^p}{c_1 (1-\kappa)^{p-1}}, C = \frac{c_2 \alpha_2 \beta e^{\lambda\tau}}{c_1 (1-\kappa)^{p-1}}, D = \frac{c_2 \mu_2 \beta e^{\lambda\tau}}{c_1 (1-\kappa)^{p-1}}.$$

**Case 2.** $\sup_{-\tau\le\theta\le 0} \mathbb{E}|x(t^*+\theta)|^p = \mathbb{E}|x(t^*)|$. In this case, we have

$$\begin{aligned}
\mathbb{E}|x(t^*)|^p \le{}& \frac{\beta c_2 (1+\kappa)^p \|\xi\|^p}{c_1 (1-\kappa)^p} e^{(\alpha_1 + \frac{\ln\mu_1}{T_a})(t^*-t_0)} + \frac{c_2 \alpha_2 \beta e^{\lambda\tau} M \|\xi\|^p}{c_1 (1-\kappa)^p} \int_{t_0}^{t^*} e^{(\alpha_1 + \frac{\ln\mu_1}{T_a})(t^*-s) - \lambda(s-t_0)} ds \\
&+ \frac{c_2 \mu_2 \beta M \|\xi\|^p e^{\lambda\tau}}{c_1 (1-\kappa)^p} \sum_{\iota=1}^{N(t^*,t_0)} e^{(\alpha_1 + \frac{\ln\mu_1}{T_a})(t^*-t_\iota) - \lambda(t_\iota - t_0)} \\
={}& \left[ C'h - \frac{B'}{\alpha_1 + \frac{\ln\mu_1}{T_a} + \lambda} \right] M \|\xi\|^p e^{-\lambda(t^*-t_0)} \\
&+ \left[ A' + \frac{B'M}{\alpha_1 + \frac{\ln\mu_1}{T_a} + \lambda} \right] \|\xi\|^p e^{(\alpha_1 + \frac{\ln\mu_1}{T_a})(t^*-t_0)},
\end{aligned} \tag{17}$$

where

$$A' = \frac{\beta c_2 (1+\kappa)^p}{c_1 (1-\kappa)^p}, B' = \frac{c_2 \alpha_2 \beta e^{\lambda\tau}}{c_1 (1-\kappa)^p}, C' = \frac{c_2 \mu_2 \beta e^{\lambda\tau}}{c_1 (1-\kappa)^p}.$$

Therefore, based on the analysis of inequality (16) and (17) and the technique proposed in [21], we know that inequality (11) implies that $\mathbb{E}|x(t^*)|^p < M\|\xi\|^p e^{-\lambda(t^*-t_0)}$, which is a contradiction. So (13) is proven, and the trivial solution of Equation (1) is $p$th moment exponentially stable. □

**Remark 3.** *Theorem 1 can be regarded as a complement of literature* [2] *in impulsive cases. In fact, this theorem tells us that the impulse can potentially destroy the stability of INSFDEs when* $\mu_1 > 1$ *and can also be beneficial to the stability when* $0 < \mu_1 < 1$.

**Remark 4.** *The condition* $\alpha_1 + \frac{\ln\mu_1}{T_a} < 0$ *guarantees that the series* $\lim_{t\to\infty} \sum_{\iota=1}^{N(t,t_0)} e^{(\alpha_1 + \frac{\ln\mu_1}{T_a})(t-t_\iota)}$ *is convergent, if* $\varepsilon > 0$, *set* $\varepsilon = \inf_{k\in\mathbb{N}}\{t_k - t_{k-1}\}$, *then* $h = \lim_{t\to\infty} \sum_{\iota=0}^{N(t,t_0)} e^{(\alpha_1 + \frac{\ln\mu_1}{T_a})(t-t_\iota)}$ *can be redefined as* $h = \sum_{k=0}^{\infty} e^{(\alpha_1 + \frac{\ln\mu_1}{T_a})k\varepsilon}$.

**Remark 5.** *In Theorem 1, we define a Lyapunov function* $V(t, x(t) - D(x_t))$, *and we should point out that the Lyapunov function can not only be affected by the impulse condition defined in Equation (1), but also be influenced by a neutral item when* $t = t_k$, *which is the crucial different from the traditional impulsive systems.*

### *3.2. Almost Sure Exponential Stability*

**Lemma 2** ([1])**.** *Suppose that there is a positive constant* $\kappa \in (0,1)$ *such that*

$$|D(\phi) - D(\varphi)| \le \kappa \sup_{-\tau\le\theta\le 0} |\phi(\theta) - \varphi(\theta)|, \quad \phi, \varphi \in C([-\tau, 0]; \mathbb{R}^n). \tag{18}$$

*Let* $z : [t_0 - \tau, \infty) \to \mathbb{R}^n$ *be a continuous function and define* $z_t = \{z(t+\theta) : -\tau \le \theta \le 0\}$ *for all* $t \ge t_0$. *Let* $0 < \lambda < \left(0, \min\left\{ \frac{1}{\tau}\ln\frac{1}{\kappa}, \frac{\alpha_2}{1-\mu} \right\}\right)$ *and* $H > 0$. *Then*

$$|z(t) - D(z_t)|^p \le He^{-\lambda(t-t_0)}, \ t \ge t_0 \tag{19}$$

*implies that*

$$\limsup_{t\to\infty}\frac{1}{t}\ln|z(t)| \le -\frac{\lambda}{p}. \tag{20}$$

**Remark 6.** *Lemma 2 tells us that if $|z(t) - D(z_t)|$ is exponentially stable; then, we can deduce that $z(t)$ is also exponentially stable.*

**Assumption 2.** *Suppose the impulsive instants $t_k$ satisfy*

$$\delta = \inf_{1\le k<\infty}\{t_k - t_{k-1}\} > 0 \tag{21}$$

**Theorem 2.** *Let $p \ge 2$, suppose that Assumption 2 holds and all the conditions of Theorem 1 are satisfied. If there exists positive constants $L,\ N,\ L_1$ such that for all $\phi \in PC^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^n)$,*

$$\mathbb{E}(|f(t,\phi(0),\phi)|^p + |g(t,\phi(0),\phi)|^p) \le L\mathbb{E}|\phi(0)|^p + N\sup_{-\tau\le\theta\le 0}\mathbb{E}|\phi(\theta)|^p, \tag{22}$$

*and*

$$|I_k(t,\phi(0))| \le L_1|\phi(0)|, \tag{23}$$

*then for any $\xi \in PC^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$*

$$\limsup_{t\to\infty}\frac{1}{t}\ln|x(t,t_0,\xi)| \le -\frac{\lambda}{p}. \tag{24}$$

**Proof.** For $t \ge t_0 + \tau$, we deduce that

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le h\le\tau}|x(t+h) - D(x_{t+h})|^p\right) \le{}& 5^{p-1}\mathbb{E}|x(t) - D(x_t)|^p + 5^{p-1}E\left[\int_t^{t+\tau}|f(s,x(s),x_s)|ds\right]^p\\ &+5^{p-1}\mathbb{E}\left[\sup_{0\le h\le\tau}\left|\int_t^{t+h}g(s,x(s),x_s)dB(s)\right|^p\right]\\ &+5^{p-1}\mathbb{E}\left[\sum_{t\le t_k\le t+h}\left|D(x_{t_k^-}) - D(x_{t_k})\right|\right]^p\\ &+5^{p-1}\mathbb{E}\left[\sum_{t\le t_k\le t+h}\left|I_k(t_k,x(t_k^-))\right|\right]^p.\end{aligned} \tag{25}$$

By Assumption 1, we obtain

$$\begin{aligned}\mathbb{E}|x(t) - D(x_t)|^p &\le 2^{p-1}\mathbb{E}\left[|x(t)|^p + \kappa^p\|x(t+\theta)\|^p\right]\\ &\le 2^{p-1}M\|\xi\|^p\left[1 + \kappa^p e^{\lambda\tau}\right]e^{-\lambda(t-t_0)}.\end{aligned} \tag{26}$$

By Hölder inequality, combining (22), we obtain

$$\begin{aligned}\mathbb{E}\left[\int_t^{t+\tau}|f(s,x(s),x_s)|ds\right]^p &\le \tau^{p-1}\int_t^{t+\tau}\mathbb{E}|f(s,x(s),x_s)|^p ds\\ &\le L\tau^{p-1}\int_t^{t+\tau}\mathbb{E}|x(s)|^p ds + N\tau^{p-1}\int_t^{t+\tau}\sup_{-\tau\le\theta\le 0}\mathbb{E}|x(s+\theta)|^p ds\\ &\le L\tau^{p-1}M\|\xi\|^p\int_t^{t+\tau}e^{-\lambda(s-t_0)}ds\\ &\quad +N\tau^{p-1}M\|\xi\|^p e^{\lambda\tau}\int_t^{t+\tau}e^{-\lambda(s-t_0)}ds\\ &\le \left(LM\|\xi\|^p\tau^p + NM\|\xi\|^p\tau^p e^{\lambda\tau}\right)e^{-\lambda(t-t_0)}. \end{aligned} \tag{27}$$

According to the Burkholder–Davis–Gundy inequality, we have

$$\mathbb{E}\left[\sup_{0\le h\le\tau}\left|\int_t^{t+h}g(s,x(s),x_s)dB(s)\right|^p\right] \le C_q\mathbb{E}\left[\int_t^{t+\tau}|g(s,x(s),x_s)ds|^2\right]^{\frac{p}{2}}, \tag{28}$$

where $C_q$ is a positive constant. By (22) and Hölder inequality, we obtain

$$\mathbb{E}\left[\sup_{0\le h\le\tau}\left|\int_t^{t+h}g(s,x(s),x_s)dB(s)\right|^p\right] \le C_q\tau^{\frac{p}{2}}\left(LM\|\xi\|^p + NM\|\xi\|^p e^{\lambda\tau}\right)e^{-\lambda(t-t_0)}. \tag{29}$$

By inequality (23), we have

$$\begin{aligned}\mathbb{E}\left[\sum_{t\le t_k\le t+h}\left|D(x_{t_k}) - D(x_{t_k^-})\right|\right]^p &\le \mathbb{E}\left[\sum_{t-\tau\le t_k\le t+\tau}\kappa\left|x(t_k)-x(t_k^-)\right|\right]^p\\ &\le \left[\frac{2\tau\kappa}{\delta}\right]^p\sup_{t-\tau\le t_k\le t+\tau}\left(L_1^p\mathbb{E}|x(t_k)|^p\right)\\ &\le \left[\frac{2\tau\kappa}{\delta}\right]^p Me^{\lambda\tau}\|\xi\|^p L_1^p e^{-\lambda(t-t_0)}\end{aligned} \tag{30}$$

and

$$\begin{aligned}\mathbb{E}\left[\sum_{t\le t_k\le t+h}\left|I_k(t_k,x(t_k^-))\right|\right]^p &\le \left[\frac{\tau}{\delta}\right]^p\sup_{t\le t_k\le t+\tau}(L_1|x(t_k)|)^p\\ &\le \left[\frac{\tau}{\delta}\right]^p M\|\xi\|^p L_1^p e^{-\lambda(t-t_0)},\end{aligned} \tag{31}$$

where $\left[\frac{\tau}{\delta}\right], \frac{\tau}{\delta}\in\mathbb{R}$ is the maximum integer not more than $\frac{\tau}{\delta}$. Substituting (26), (27), (29), (30), and (31) into (25), we have for $t\ge t_0+\tau$,

$$\mathbb{E}\left(\sup_{0\le h\le\tau}|x(t+h)-D(x_{t+h})|^p\right) \le \hat{H}e^{-\lambda(t-t_0)}, \tag{32}$$

where $\hat{H}$ is a positive constant. Let $n\ge 1$ and $\varepsilon\in(0,\lambda)$ be arbitrary. Inequality (32) implies that

$$\mathbb{P}\left\{\omega:\sup_{0\le h\le\tau}|x(t_0+n\tau+h)-D(x_{t_0+n\tau+h})|^p > e^{-(\lambda-\varepsilon)n\tau}\right\} \le \hat{H}e^{-\varepsilon n\tau}. \tag{33}$$

By the Borel–Cantelli lemma, one can choose a positive constant $n_0(\omega)$, then for almost all $\omega \in \Omega, n \geq n_0(\omega)$,

$$\sup_{0\leq h\leq \tau} \left|x(t_0+n\tau+h) - D(x_{t_0+n\tau+h})\right|^p \leq e^{-(\lambda-\varepsilon)n\tau}. \tag{34}$$

Then, for $t \geq t_0 + n\tau, n \geq n_0(\omega)$ and for almost all $\omega \in \Omega$

$$|x(t) - D(x_t)|^p \leq e^{-(\lambda-\varepsilon)(t-\tau-t_0)}. \tag{35}$$

However, $|x(t) - D(x_t)|^p$ is finite on $[t_0, t_0 + n\tau]$. Thus, we conclude that for almost all $\omega \in \Omega$, there exist a finite positive number $H$ such that for any $t \geq t_0$,

$$|x(t) - D(x_t)|^p \leq He^{-(\lambda-\varepsilon)(t-t_0)}. \tag{36}$$

Therefore, by Lemma 2

$$\limsup_{t\to\infty} \frac{1}{t}\ln|x(t)| \leq -\frac{\lambda-\varepsilon}{p}, \quad \text{a.s.} \tag{37}$$

Then, let $\varepsilon \to 0$. the proof is completed. □

In the next part, we will extend the results we obtained to the impulsive neutral stochastic differential delay equations (INSDDEs).

Consider the following INSDDE

$$\begin{cases} d[x(t) - \bar{D}(x(t) - u(t))] = \bar{f}(t, x(t), x(t - u(t)))dt \\ \qquad\qquad + \bar{g}(t, x(t), x(t - u(t)))dB(t), t \neq t_k, \\ \Delta x(t_k) = I_k(t_k, x(t_k^-)), k \in \mathbb{N}, \\ x(t) = \xi(t), t \in [t_0 - \tau, t_0], \end{cases} \tag{38}$$

where $u : [t_0, \infty) \to [0, \tau]$, $\bar{D} : \mathbb{R}^n \to \mathbb{R}^n$, $\bar{f} : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \bar{g} : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ are Borel measurable functions with $\bar{D}(0) = 0, \bar{f}(t, 0, 0) = 0$ and $\bar{g}(t, 0, 0) = 0$ for all $t \geq t_0$. This is a special case of Equation (1) with $D(\phi) = \bar{D}(\phi(-u(t)))$, $f(t, \phi(0), \phi(\theta)) = \bar{f}(t, \phi(0), \phi(-u(t)))$ and $g(t, \phi(0), \phi(\theta)) = \bar{g}(t, \phi(0), \phi(-u(t)))$ for all $(t, \phi) \in [t_0, \infty) \times PC^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n)$. If $V \in C^{1,2}([t_{k-1}, t_k) \times \mathbb{R}^n; \mathbb{R}_+), k \in \mathbb{N}$; for the special case, the operator $L$ defined in (1) becomes

$$LV(t, X, Z) = V_t(t, Y) + V_Y(t, Y)\hat{f}(t, X, Z) + \frac{1}{2}\text{trace}\left[\hat{g}^T(t, X, Z)V_{YY}(t, Y)\hat{g}(t, X, Z)\right]. \tag{39}$$

where $X = \phi(0), Z = \phi(-h(t))$ and $Y = X - \hat{D}(Z)$. The above result is applied to (38) and the following useful corollary is obtained.

**Corollary 1.** *Let $p \geq 1$, $\eta_1 \in \mathbb{R}$,$c_1$, $c_2$, $\eta_2$, $\gamma_1$, $\gamma_2$ be positive numbers. Assume that there exists a constant $\kappa \in (0, 1)$ such that*

$$\left|\hat{D}(y)\right|^p \leq \kappa|y|^p, y \in \mathbb{R}^n \tag{40}$$

*and there exists a function $V \in C^{1,2}([t_{k-1}, t_k) \times \mathbb{R}^n; \mathbb{R}_+), k \in \mathbb{N}$ such that*

(H1) $c_1|x|^p \leq V(t, x) \leq c_2|x|^p$, *for all* $(t, x) \in [t_0, \infty) \times \mathbb{R}^n$;

(H2) *for all* $t \geq t_0, t \neq t_k, k \in \mathbb{N}$ *and* $X, Z \in \mathbb{R}^n, LV(t, X, Z) \leq \eta_1 V(t, X) + \eta_2 V(t - h(t), Z)$;

(H3) *for all* $k \in \mathbb{R}^n, X, Z \in \mathbb{R}^n, V(t_k, Y) \leq \gamma_1 V(t_k^-, X) + \gamma_2 V(t_k^- - h(t), Z)$;

(H4) $[c_1(1-\kappa)^p - c_2\bar{\mu}_2\bar{\beta}\bar{h}]\left(\bar{\alpha}_1 + \frac{\ln\bar{\mu}_1}{T_a}\right) + c_2\bar{\alpha}_2\bar{\beta} < 0, \bar{h} = \lim_{t\to\infty}\sum_{\iota=1}^{N(t,t_0)} e^{(\bar{\alpha}_1 + \frac{\ln\bar{\mu}_1}{T_a})(t-t_\iota)}, \bar{\alpha}_1 + \frac{\ln\bar{\mu}_1}{T_a} < 0$ *and* $\bar{\beta} = \max\left\{e^{-N_0\ln\bar{\mu}_1}, e^{N_0\ln\bar{\mu}_1}\right\}$, *where $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\mu}_1, \bar{\mu}_2$ will be defined in the*

*following process; then, the trivial solution of Equation (38) is pth moment exponentially stable.*

**Proof.** For any $p \geq 1$, by inequality (40) and Lemma 1, we can obtain

$$V(t,X) \geq \frac{c_1 V(t,Y)}{c_2(1+\kappa)^{p-1}} - \kappa V(t-h(t),Z)$$

and

$$V(t,X) \leq \kappa \frac{c_2}{c_1} V(t-h(t),Z) + \frac{c_2}{c_1(1-\kappa)^{p-1}} V(t,Y).$$

If $\eta_1 > 0$, in this case, for any $t \in [t_{k-1}, t_k), k \in \mathbb{N}$, we have

$$\begin{aligned} LV(t,X,Z) &\leq \eta_1 V(t,X) + \eta_2 V(t-h(t),Z) \\ &\leq \eta_1 \left[ \kappa \frac{c_2}{c_1} V(t-h(t),Z) + \frac{c_2}{c_1(1-\kappa)^{p-1}} V(t,Y) \right] + \eta_2 V(t-h(t),Z) \\ &= \frac{c_2 \eta_1}{c_1(1-\kappa)^{p-1}} V(t,Y) + \left[ \eta_2 + \kappa \eta_1 \frac{c_2}{c_1} \right] V(t-h(t),Z). \end{aligned} \tag{41}$$

If $\eta_1 < 0$, in this case, for any $t \in [t_{k-1}, t_k), k \in \mathbb{N}$, we obtain

$$\begin{aligned} LV(t,X,Z) &\leq \eta_1 V(t,X) + \eta_2 V(t-h(t),Z) \\ &\leq \eta_1 \left[ \frac{c_1 V(t,Y)}{c_2(1+\kappa)^{p-1}} - \kappa V(t-h(t),Z) \right] + \eta_2 V(t-h(t),Z) \\ &\leq \frac{c_1 \eta_1}{c_2(1+\kappa)^{p-1}} V(t,Y) + (\eta_2 - \kappa \eta_1) V(t-h(t),Z). \end{aligned} \tag{42}$$

For $t = t_k$, we obtain

$$\begin{aligned} V(t_k,Y) &\leq \gamma_1 V(t_k^-,X) + \gamma_2 V(t_k^- - h(t_k),Z) \\ &\leq \gamma_1 \left[ \kappa \frac{c_2}{c_1} V(t_k - h(t_k),Z) + \frac{c_2}{c_1(1-\kappa)^{p-1}} V(t_k,Y) \right] + \gamma_2 V(t_k^- - h(t_k),Z) \\ &\leq \frac{c_2 \gamma_1}{c_1(1-\kappa)^{p-1}} V(t_k^-,Y) + \left( \gamma_1 \kappa \frac{c_2}{c_1} + \gamma_2 \right) V(t_k^- - h(t_k),Z). \end{aligned} \tag{43}$$

Let $\bar{\mu}_1 = \frac{c_2 \gamma_1}{c_1(1-\kappa)^{p-1}}$ and $\bar{\mu}_2 = \gamma_1 \kappa \frac{c_2}{c_1} + \gamma_2$. If $\eta_1 > 0$, choose $\bar{\alpha}_1 = \frac{c_2 \eta_1}{c_1(1-\kappa)^{p-1}}$ and $\bar{\alpha}_2 = \eta_2 + \kappa \eta_1 \frac{c_2}{c_1}$; If $\eta_1 < 0$, choose $\bar{\alpha}_1 = \frac{c_1 \eta_1}{c_2(1+\kappa)^{p-1}}$ and $\bar{\alpha}_2 = \eta_2 - \kappa \eta_1$. Then, by condition (H4) and Theorem 1, we obtain that the trivial solution of Equation (38) is $p$th moment exponentially stable. Of course, if the equation satisfies the inequality (21)–(23), the trivial solution is also almost surely exponentially stable. □

## 4. Examples

In this section, we shall give two examples and simulations to show the effectiveness of the results obtained in this paper.

**Example 1.** *Consider the one-dimensional INSDDE*

$$\begin{cases} d[x(t) - 0.1x(t-1)] = [1.2x(t) + 0.2x(t-1)]dt \\ \qquad\qquad\qquad + [0.4x(t) + 0.3x(t-1)]dB(t), t \neq k, k \in \mathbb{N} \\ x(k) = 0.2x(k^-). \end{cases} \tag{44}$$

By applying Corollary 1, we shall show that the zero solution of Equation (44) is exponentially stable in mean square.

One can choose a simple Lyapunov function

$$V(t, x(t)) = |x(t)|^2,$$

then (H1) is satisfied with $p = 2$ and $c_1 = c_2 = 1$.

By calculating the operator $LV$, utilizing Itô's formula and Corollary 1, we have

$$\begin{aligned} LV &= 2[x(t) - 0.1x(t-1)][1.2x(t) + 0.2x(t-1)] + [0.4x(t) + 0.3x(t-1)]^2 \\ &\leq 2.76x^2(t) + 0.25x^2(t-1) \\ &\leq 3.07|x(t) - 0.1x(t-1)|^2 + 0.526x^2(t-1). \end{aligned}$$

Thus, one can verify that (H2) is satisfied by choosing $\bar{\alpha}_1 = 3.07$ and $\bar{\alpha}_2 = 0.526$.

For $t = k$, we compute that

$$|x(k) - 0.1x(k-1)|^2 = 0.04|x(k^-) - 0.1x(k^- - 1)|^2, \tag{45}$$

which implies that (H3) holds by letting $\bar{\mu}_1 = 0.04$ and $\bar{\mu}_2 = 0$.

The impulsive time sequence is given by $\zeta = \{t_k | t_k = t_{k-1} + 1, t_0 = 0, k \in \mathbb{N}\}$ with $N_0 = 1, T_a = 1, \bar{\beta} = 0.04$ and $\bar{h} = 0$, which illustrates that (H4) is satisfied.

According to Corollary 1, the trivial solution of (44) is exponentially stable in mean square. Set $\xi = 0.6$; the simulation result is presented in Figures 1 and 2. For linear INSDDE, the conditions of Theorem 2 are obviously satisfied, so the trivial is also almost surely exponentially stable.

**Remark 7.** *In Example 1, the impulses play a positive effect on stabilizing the unstable dynamical system. Thus, we need to guarantee that the impulses must be frequent enough to ensure their amplitude is suitable for the growth rate of LV.*

**Remark 8.** *It should be pointed out that $\alpha_1 = 3.07 > 0$ and $\alpha_2 = 0.526 > 0$ imply the operator LV cannot be controlled by nonpositive constants. In such a situation, those results in* [16] *cannot be applied because they need to keep the operator LV to be negative.*

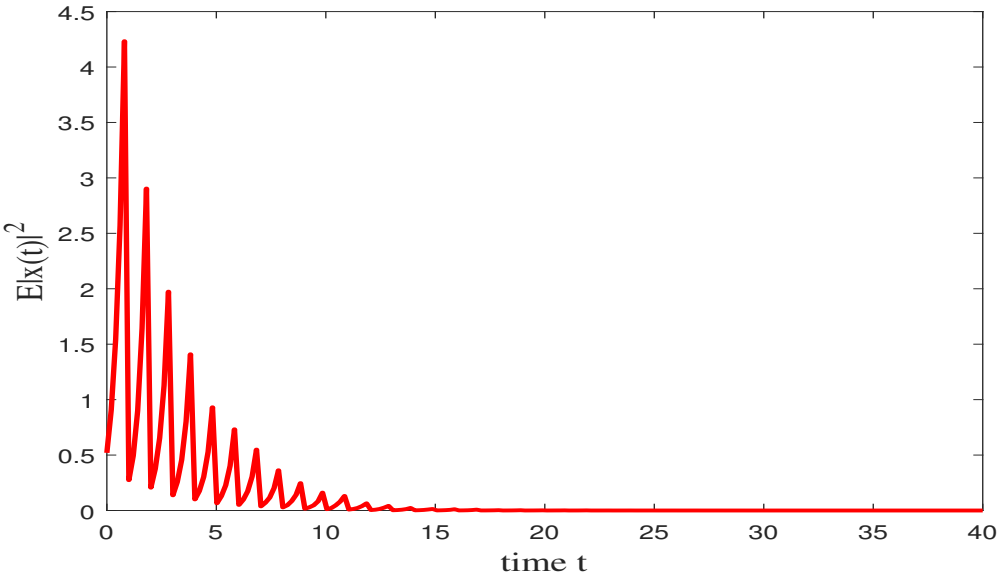



**Figure 1.** State trajectories of (44) with impulsive control.

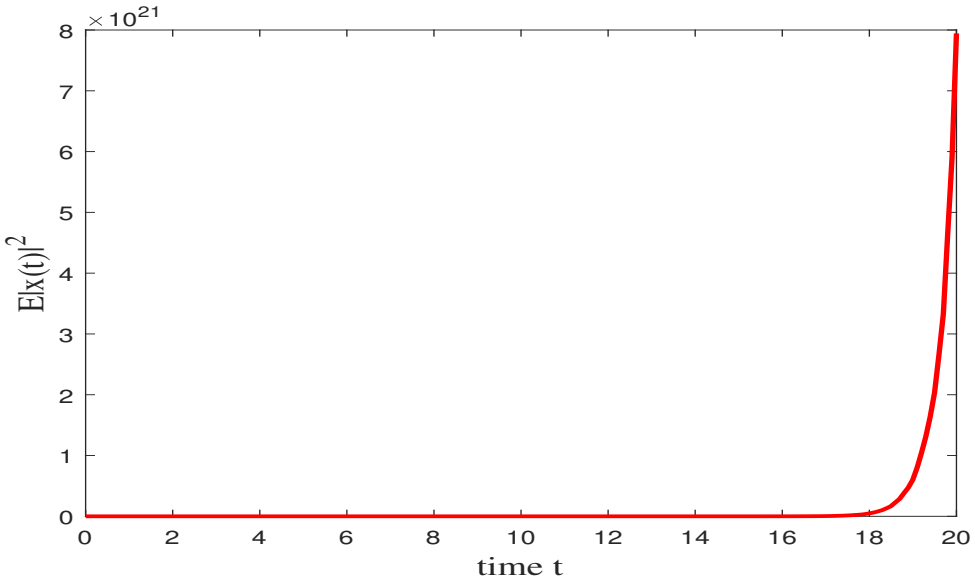



**Figure 2.** State trajectories of (44) without impulsive control.

**Example 2.** *Consider the following INSDDE*

$$\begin{cases} d[x(t) - Hx(t-\tau)] = Ax(t)dt + Bx(t-\tau)dB(t), \\ x(t_k) = Cx(t_k^-), \end{cases} \tag{46}$$

*where*

$$H = \begin{pmatrix} 0.05 & 0 \\ 0 & 0.05 \end{pmatrix}, A\begin{pmatrix} -0.9 & 0.3 \\ 0.3 & -0.9 \end{pmatrix}, B = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, C = \begin{pmatrix} 1.2 & 0 \\ 0 & 1.2 \end{pmatrix}, x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

*and* $\tau = 0.06$.

We conclude from Corollary 1 that the zero solution of Equation (46) is exponentially stable in mean square. Just as the former example, one can choose the following Lyapunov function

$$V(t, x(t)) = |x(t)|^2,$$

where condition (H2) implies $p = 2$ and $c_1 = c_2 = 1$.

By calculating the operator $LV$, utilizing Itô formula and Corollary 1, we have

$$\begin{aligned} LV &\leq 2[x_1(t) - 0.05x_1(t-\tau), x_2(t) - 0.05x_2(t-\tau)]\begin{pmatrix} -0.9 & 0.3 \\ 0.3 & -0.9 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \\ &\quad + \frac{1}{2}\left[\left(x_1^T(t-\tau), x_2^T(t-\tau)\right)\begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}\begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}\begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix}\right] \\ &\leq -1.2|x(t) - 0.05x(t-\tau)|^2 + 0.13|x(t-\tau)|^2. \end{aligned}$$

Thus, one can verify that (H2) is satisfied by choosing $\bar{\alpha}_1 = -1.2$ and $\bar{\alpha}_2 = 0.13$.

For $t = t_k$, we compute that

$$\begin{aligned} |x(t_k) - 0.05x(t_k - \tau)|^2 &\leq 1.5\left|x(t_k^-)\right|^2 + 0.0625|x(t_k - \tau)|^2 \\ &\leq 1.58|x(t_k) - 0.05x(t_k - \tau)|^2 + 0.137|x(t_k - \tau)|^2, \end{aligned}$$

then (H3) holds by letting $\bar{\mu}_1 = 1.58$ and $\bar{\mu}_2 = 0.137$.

The impulsive time sequence is given by $\{\varepsilon, 2\varepsilon, \cdots, N_0\varepsilon, N_0T_a + N_0\varepsilon, \cdots\}$ [35] with $\varepsilon = 0.5, N_0 = 1, T_a = 5, \bar{\beta} = 1.58$ and $\bar{h} = 1.7$, which illustrates that (H4) is satisfied.

According to Corollary 1, the trivial solution of (46) is exponentially stable in mean square. Of course, the trivial solution of (46) is also almost surely exponentially stable, the simulation is presented in Figure 3 in which the initial value is $[0.6, -0.4]$.

**Remark 9.** *We find that the results in* [14,16] *are invalid for this example because the impulse condition proposed in* [14,16] *is too restrictive to be applied. Meanwhile, the impulse condition mentioned in this paper can be satisfied easily.*

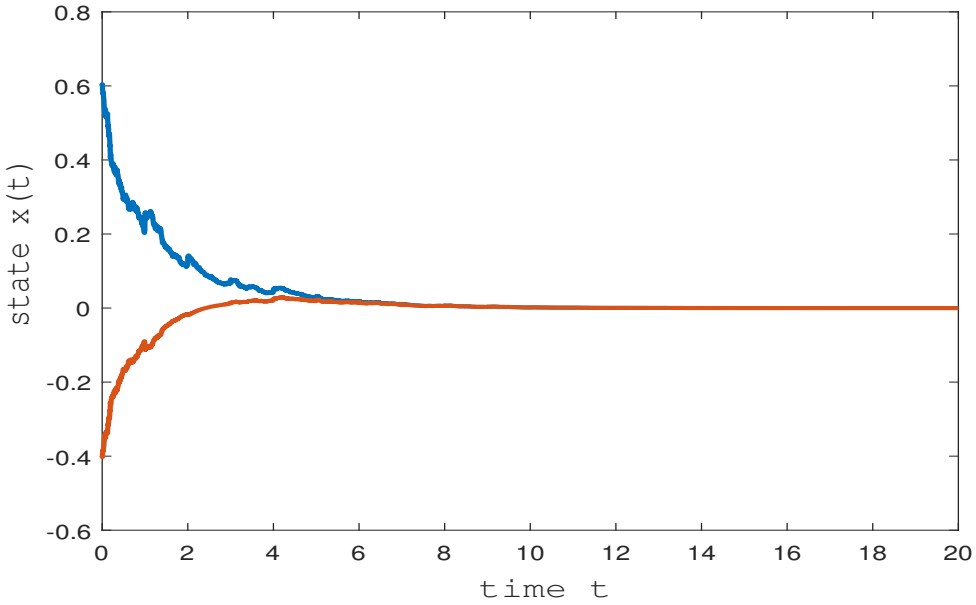



**Figure 3.** State trajectories of (46) under destabilizing impulsive perturbation.

## 5. Conclusions

With the help of the ADT condition, less constructive conditions for the $p$th moment and almost sure exponential stability have been derived for INSFDEs. The method proposed in this paper can be further used to investigate the time-varying systems, which will be studied in the near future.

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