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Article

The Structure of Local Rings with Singleton Basis and Their Enumeration

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Abstract: A local ring is an associative ring with unique maximal ideal. We associate with each Artinian local ring with singleton basis four invariants (positive integers) p, n, s, t. The purpose of this article is to describe the structure of such rings and classify them (up to isomorphism) with the same invariants. Every local ring with singleton basis can be constructed over its coefficient subring by a certain polynomial called the associated polynomial. These polynomials play significant role in the enumeration.

Keywords: local rings; chain rings; isomorphism classes; Galois rings

MSC: 16L30; 13B05; 12E20; 16P20

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1. Introduction

We consider only associative Artinian ring with identity. A local ring is a ring whose zero divisors form an ideal. It turned out that a ring is a local ring if and only if it has a unique maximal ideal. For the application of Artinian local rings in different areas, see [1–3].

Suppose R is a finite local ring, then (cf. [4]) $|R| = p^{ms}$, $|J(R)| = p^{(m-1)s}$, $R/J(R) \cong GF(p^s)$ and the characteristic of R is p^n, where $J(R)$ is the maximal (Jacobson) ideal, p is the characteristic of residue field and m, n, s are positive integers with $1 \leq n \leq m$. The positive integers p, n, s, m are called invariants of R. An interesting case is when $m = n$,

$$R \cong \mathbb{Z}_{p^n}[x]/(g(x)),$$

where $g(x)$ is a monic polynomial of degree s over $\mathbb{Z}_{p^n}$ and irreducible modulo p. In such case, R is a commutative chain ring and $R = \mathbb{Z}_{p^n}[a]$, where a is an element of R of multiplicative order $p^s - 1$, $J(R) = pR$ and $Aut(R)$ is a cyclic group of order s, see [5–7] for more details on chain rings. These rings have a lot in common with Galois fields, and thus they are called Galois rings and denoted by $GR(p^n, s)$ (cf. [8]). Moreover, Galois rings are uniquely determined by the invariants p, n, r.

Let R be a local ring, a commutative chain subring S of R is called a coefficient subring if

$$R = S + J(R) \text{ and } R/J(R) \cong S/pS.$$

When R is finite local ring, S is a Galois subring of the form $GR(p^n, s)$ which is maximal Galois subring of R (cf. [9]). While if R is an Artinian local duo ring (two-sided ideal ring) with absolutely algebraic residue field, i.e., algebraic over its prime subfield $R/J(R)$, then [5] R has a coefficient subring S. Actually, in this case, S is a union of ascending chain of Galois subrings of R and it is called a generalized Galois ring. It turned out that S plays an important role in the structure of a local ring.

Assume that R is either a finite local ring or Artinian local duo ring in which its residue field is absolutely algebraic, and suppose also S is its coefficient subring. The following facts are in [5,10]. There exist $\pi_1, \pi_2, \ldots, \pi_k$ in $J(R)$ and $\sigma_1, \sigma_2, \ldots, \sigma_k$ in $Aut(S)$ such that

$$R = S \oplus_{i=1}^{k} S\pi_i \tag{1}$$

as $S-$modules and $\pi_i\, r = r^{\sigma_i}\pi_i$ for each r in S and for all $i = 1, 2, \ldots, k$. Moreover, the automorphisms $\sigma_1, \sigma_2, \ldots, \sigma_k$ are uniquely determined by R and S. Let us call the set $\{\pi_1, \pi_2, \ldots, \pi_k\}$ a distinguished basis of R over S.

In this study, we assume that R is a local ring which is either finite or Artinian duo in which the residue field is absolutely algebraic. We also assume that R has a singleton basis, i.e., $k = 1$ and S is its coefficient subring. From Equation (1),

$$R = S \oplus S\pi \tag{2}$$

In Section 3, we determine fully the structure of R as a quotient of a ring of skew polynomials by a certain ideal. Furthermore, Section 4 gives the classification of finite local rings up to isomorphism in certain cases. Finally, in Section 5, we obtain the number of non-isomorphic classes of Artinian local duo rings where the residue field is absolutely algebraic.

2. Preliminaries

This section introduces several notations and some basic facts. During this article, R is a local ring which is either finite or Artinian duo in which the residue field is absolutely algebraic. Suppose also R has singleton basis $\{\pi\}$ and σ is the associated automorphism with respect to the coefficient subring S. Since π^2 is not a unit in R, then π^2 can be written as:

$$\pi^2 = p^d u\lambda + p^e u_1\lambda_1\pi, \tag{3}$$

where $u, u_1, \lambda, \lambda_1$ are units in S and d, e are positive integers. Assume that t is the least positive integer such that $p^t\pi = 0$. If R is finite, then $|R| = p^{(n+t)s}$ and $S = GR(p^n, s) = \mathbb{Z}_{p^n}[a]$, where a is an element of S with order $p^s - 1$. On the other hand, if R is not finite. Let $S' = GR(p^n, s)$ be the Galois subring of R generated by $\mathbb{Z}_{p^n}$, u, u_1, λ and λ_1. Suppose also

$$R' = S' \oplus \pi S'.$$

Then, R' is a finite local ring with singleton basis. We call R' the associated ring of R. The set

$$\Gamma(s) =< a > \cup\{0\} = \{0, a, a^2, \ldots, a^{p^s-1}\}$$

is called the Teichmuller set of S if R is finite and of S' otherwise. In either case, we call the integers p, n, s, t *invariants* of R. Note that $t = n - 1$, $u, u_1 \in \Gamma(s)$ and λ, λ_1 are in $1 + pS$ or in $1 + pS'$ depending on wither R is finite or not, respectively.

Suppose $S[x,\sigma]$ is the skew polynomials ring over S determined by σ. The elements of $S[x,\sigma]$ are of the form:

$$\sum_i r_i x^i,$$

$r_i \in S$ and $xr = r^\sigma x$ for every r in S. We also denote K the residue field; $K = S/pS$. Let $S^\sigma, (S')^\sigma$ are the fixed Galois subring of S and S' by σ, respectively. Assume that each S^σ and $(S')^\sigma$ has the form $\mathbb{Z}_{p^n}[b] = GR(p^n, s')$, where $b \in< a >$ of order $p^{s'} - 1$ and $s'|s$. Let T_R denote all (S, π) that satisfies the aforementioned conditions.

All notations mentioned above keep their meaning throughout the paper.

Definition 1. *Let R be a commutative local ring and $\overline{R} = R/J(R)$. For any $g(x)$ in $R[x]$, let $\overline{g}(x)$ denote the natural image in $\overline{R}[x]$. The ring R is called a Hensel ring if it has the following property: Given any monic polynomial $g(x)$ in $R[x]$, if $\overline{g}(x) = \lambda(x)\mu(x)$ for some relatively prime polynomials $\lambda(x)$, $\mu(x)$ in $\overline{R}[x]$, then there exist monic polynomials $f(x)$, $h(x)$ in $R[x]$ such that $g(x) = f(x)h(x)$, $\overline{f}(x) = \lambda(x)$ and $\overline{h}(x) = \mu(x)$.*

Remark 1. *By Hensel Lemma [11], any commutative Artinian local ring is a Hensel ring.*

Lemma 1. *Let R be a local ring with singleton basis with invariants p, n, s, t and $n > 1$. Then $(S,\theta) \in T_R$ if and only if*
$$\theta = \omega(p^{n-t}\gamma + \pi),$$
where ω is an element of $< a >$, $\gamma = \sum_{i=0}^{t-1} p^i a_i$ and a_i are elements of $\Gamma(s)$ with $p^{n-t}\gamma$ is zero if R is non-commutative.

Proof. Suppose that $(S,\theta) \in T_R$, then $R = S + S\theta$. Also θ can be written as $\theta = s_1 + s_2\pi$, where s_1 and s_2 are elements of S. Since $p^t\theta = 0$, then $s_1 = p^{n-t}s$ for some $s \in S$. Moreover, $s_2 \notin pS$ or otherwise $\theta \in pR$ which is impossible. By the same reasoning, we have $s_2 = w \in < a >$. Thus,
$$\theta = p^{n-t}s + w\pi.$$
Now if R is non-commutative, we get
$$p^{n-t}as + wa\pi = a\theta = p^{n-t}a^\sigma s + wa^\sigma\pi.$$
As $p^{n-t}as = p^{n-t}a^\sigma s$ and $a \neq a^\sigma$, then $p^{n-t}s = 0$. Therefore,
$$\theta = \begin{cases} w(p^{n-t}\gamma + \pi), & \text{if R is commutative,} \\ w\pi, & \text{otherwise,} \end{cases} \tag{4}$$
where $\gamma = w^{-1}s$. $\square$

Conversely, let θ be as in Equation (4), then one can see that $R = S \oplus S\theta$ and $s^\sigma\theta = \theta s$ which means that (S,θ) is a distinguished basis, and hence (S,θ) in T_R.

3. The Structure of a Local Ring with Singleton Basis

Proposition 1. *Let R be a local ring with singleton basis with invariants p, n, s, t. Then, an associated polynomial of R with respect to S is either x^2 or $x^2 - p^d u\lambda$ or $x^2 - p^e u_1\lambda_1 x - p^d u\lambda$, where u, u_1 are elements of $< a >$, and λ, λ_1 are elements of $1 + pS$, and d, t, e are positive integers such that $d + t \geq n$ and $e < t$.*

Proof. Let (S, π) be an element of T_R. Then, $R = S \oplus S\pi$, $\pi s = s^\sigma\pi$ for each s in S. Assume $\pi^2 = p\zeta + \zeta_1\pi$, where ζ and ζ_1 are elements of S. If ζ_1 is a unit in S, then $\pi = p(\pi - \zeta_1)^{-1}\zeta$ is an element of pR. Now,
$$\begin{aligned} R &= S + S\pi = S + pR = S + p(S + S\pi) = S + pS + pS\pi \\ &= S + p^2R \\ &= S + p^2(S + S\pi) \\ &= S + p^3R = \dots \end{aligned}$$
We continue in this way until we get $R = S + p^nR = S$ which contradicts to the assumption that R has a singleton basis. Hence, ζ_1 is an element of pS and subsequently π^2 is an element of pR. Thus, $\pi^2 = p^d u\lambda + p^e u_1\lambda_1\pi$. Now,
$$\pi^2 a = (p^d u\lambda + p^e u_1\lambda_1\pi)a = p^d u\lambda a + p^e u_1\lambda_1 a^\sigma\pi.$$
But
$$\pi^2 a = a^{\sigma^2}\pi^2 = a^{\sigma^2}(p^d u\lambda + p^e u_1\lambda_1\pi) = p^d u\lambda a^{\sigma^2} + p^e u_1\lambda_1 a^{\sigma^2}\pi.$$
We distinguish the following cases:

(1) $p^e u_1\lambda_1\pi = 0$ and $\pi^2 = p^d u\lambda = 0$, and hence the order of σ is not specified. Thus, in such case x^2 is the associated polynomial of R.

(2) $p^e u_1\lambda_1\pi = 0$ and $\pi^2 = p^d u\lambda \neq 0$. Hence, $\sigma^2 = Id_S$ and $n > 1$. Also in such case $\pi^2 = p^d u\lambda$ is in the center of R, and hence $p^d u\lambda = p^d u^\sigma\lambda^\sigma$. Thus, $x^2 - p^d u\lambda$ is an

associated polynomial of R, where u is an element of $< a^\sigma >$ and $p^d\lambda$ is an element of S^σ.

(3) $p^e u_1\lambda_1\pi \neq 0$ and $p^d u\lambda \neq 0$, then $p^e u_1\lambda_1 a^\sigma\pi = p^e u_1\lambda_1 a^{\sigma^2}\pi$. Which implies that $a^\sigma = a^{\sigma^2}$, and consequently $\sigma = Id_S$. Moreover, R is commutative with $n > 1$, and therefore $x^2 - p^e u_1\lambda_1 x - p^d u\lambda$ is an associated polynomial of R. Note that in Cases (2) and (3) $p^t\pi^2 = p^{t+d}u\lambda$ and $p^t\pi^2 = p^{t+d}u\lambda + p^{t+e}u_1\lambda_1\pi$ implies that $p^{t+d}u = 0$, and hence $t + d \geq n$. Also in Case (3) if $e \geq t$, then $\pi^2 = p^d u\lambda + p^e u_1\lambda_1\pi = p^d u\lambda$. Thus, we can assume in such case that $e < t$.

□

Construction A

Assume the following:

$$T_1 = S[x, \eta]/(x^2, p^t x), \tag{5}$$

$$T_2 = S[x, \sigma]/(x^2 - p^d u\lambda, p^t x), \text{where } u \text{ and } p^d\lambda \text{ are elements of } S^\sigma, \tag{6}$$

$$T_3 = S[x]/(x^2 - p^e u_1\lambda_1\, x - p^d u\lambda, p^t x). \tag{7}$$

Furthermore, suppose that

$$\begin{cases} \theta_1 = x + (x^2, p^t x), \\ \theta_2 = x + (x^2 - p^d u\lambda, p^t x), \\ \theta_3 = x + (x^2 - p^e u_1\lambda_1 x - p^d u\lambda, p^t x). \end{cases}$$

It is easy to see that $T_i = S \oplus S\theta_i$. Since S is a subring of T_i, Char $(T_i) = p^n$. Obviously, $pS \oplus S\theta_i$ is nilpotent and $T_i/(\,pS \oplus S\theta_i) \cong GR(p^n, s)$. Thus, $J(T_i) = pS \oplus S\theta_i$, and subsequently T_i's are local rings with singleton basis with invariants p, n, s, t.

Theorem 1. *The ring R is a local ring with singleton basis with invariants p, n, s, t if and only if it is isomorphic to one of the rings given by Construction A.*

Proof. Let us define the following maps ψ_1, ψ_2 and ψ_3 from R to T_1, T_2 and T_3, respectively (given by Construction A) as follows

$$\psi_i(s_0 + s_1\pi) = \begin{cases} s_0 + s_1\theta_i, & \text{if } \pi^2 = 0, \\ s_0 + s_1\theta_i, & \text{if } p^e u_1\lambda_1\pi = 0 \text{ and } \pi^2 = p^d u\lambda \neq 0, \\ s_0 + s_1\theta_i, & \text{if } \pi^2 = p^e u_1\lambda_1\pi + p^d u\lambda \text{ with } p^d u\lambda \neq 0 \text{ and } p^e u_1\lambda_1\pi \neq 0, \end{cases}$$

where $i = 1, 2, 3$. Then, it is obvious that ψ_1, ψ_2 and ψ_3 are isomorphisms from R to T_1, T_2 and T_3, respectively. □

We call the polynomial $g(x)$, mentioned in Construction A, the associated of R with respect to S and if ρ is an element of $Aut(S)$, then $\rho(g(x))$ is a polynomial over S with ρ affecting only the coefficients of $g(x)$.

Example 1. *Let $S = \mathbb{Z}_{3^4}[a]$ be a Galois ring of the form $GR(3^4, 5)$, where a is an element of S of multiplicative order $3^5 - 1$ and consider the ring $R = S[x, \eta]/(x^2, 3^3 x)$, where η is the automorphisms of S of order 5, i.e., η is the Frobenius map,*

$$\eta(a_0 + a_1 p + a_2 p^2 + a_3 p^3) = a_0^p + a_1^p p + a_2^p p^2 + a_3^p p^3,$$

where $a_0, a_1, a_2, a_3 \in \Gamma(s)$. Then, the unique maximal ideal of R is $(3, x)$; which means R is local. As the order of η is 5, then R is non-commutative. By Theorem 1, R is with singleton basis and has invariants $p = 3$, $n = 4$, $s = 5$, $t = 3$, and associated polynomial x^2.

Example 2. *Suppose that* $S = \mathbb{Z}_{3^3}[a] = GR(3^3, 2)$*, where* a *is an element of* S *of multiplicative order* $3^2 - 1$*. Consider*

$$R = S[x, \sigma]/(x^2 - 3a, 3^2x),$$

where σ *is the automorphisms of* S *of order* 2*, i.e., the Frobenius automorphism. By the same reasoning as in the Example 1,* R *is a non-commutative local ring with singleton basis with invariants* $p = 3$*,* $n = 3$*,* $s = 2$*,* $t = 2$ *with* $d = 1$*. Moreover,* $x^2 - 3a$ *is an associated polynomial of* R *over* S*.*

Example 3. *Let* $S = \mathbb{Z}_{2^3}$ *and consider the ring* $R = \mathbb{Z}_{2^3}[x]/(x^2 - 2u_1x - 2u, 2^2x)$*. Since* $(2, x)$ *is unique maximal ideal of* R*, then* R *is a commutative local ring with singleton basis with invariants* $p = 2$*,* $n = 3$*,* $s = 1$*,* $t = 2$ *with* $d = e = 1$ *and with an associated polnomial* $x^2 - 2u_1x - 2u$ *over* S*. Actually there are two non-isomorphic local rings with singleton basis with invariants* $p = 2$*,* $n = 3$*,* $s = 1$*,* $t = 2$ *and with an associated polnomial* $x^2 - 2u_1x - 2u$*; namely when* $u = u_1 = 1$ *and* $u = -1, u_1 = 1$*.*

4. The Enumeration of Finite Local Rings with Singleton Basis

First, we introduce the following proposition.

Proposition 2. *Let* R*,* T *be local rings with singleton basis with the same invariants* p*,* n*,* s*,* t *and assume that they are constructed over the same coefficient subring* S*. Let* σ*,* τ *be the associated automorphisms of* R*,* T *with respect to* S *respectively and* π *be an element of* R *satisfying an associated polynomial* $g(x)$ *of* R *over* S *with* $(S, \pi) \in T_R$*. Then,* $R \cong T$ *if and only if there exists an automorphism* ρ *of* S *such that* $\rho(g(x))$ *has a root* π' *in* T *with* $(S, \pi') \in T_T$*.*

Proof. Let φ be the isomorphism between R and T. Suppose ρ be the restriction of φ on S. Then, $\pi' = \varphi(\pi)$ is a root of $\rho(g(x))$. Moreover, $(S, \pi') \in T_T$. Conversely, if $\rho(g(x))$ has a root π' in T for some $\rho \in Aut(S)$. Then, the map $\varphi(s_1 + s_2\pi) = \rho(s_1) + \rho(s_2)\pi'$ is an isomorphism from R onto T. □

By using the last proposition and the fact that $Aut(GR(p^n, s))$ is a cyclic group of order s, it is easy to deduce the following result.

Corollary 1. *Finite local rings with singleton basis with the same invariants* p*,* n*,* s*,* t *and with the same associated polynomial* x^2 *are uniquely determined by* p, n, s, t*.*

Remark 2. *Let* R *be a local ring with singleton basis with invariants* p*,* n*,* s*,* t *with* $n > 1$ *and* $\pi^2 = p^d u\lambda$*. Taking into consideration that* $J = pS \oplus S\pi$ *and* $d + t \geq n$*; then we distinguish the following cases:*

(i) *If* π *is element of* $J \setminus J^2$ *and* p *is an element of* $J \setminus J^2$ *. Then,* $\pi^2 = p^2u\lambda$ *and* $t \geq n - 2$*.*
(ii) *If* π *is element of* $J \setminus J^2$ *and* p *is an element of* $J^i \setminus J^{i+1}$ *with* $i \geq 2$*. Then,* $\pi^2 = p^d u\lambda$ *implies that* $p^d \in J^2$*. But* $p \notin J$*. Thus,* $p \in J^2$*, and hence* $\pi^2 = pu\lambda$ *and* $t \geq n - 1$*.*
(iii) *If* π *is element of* $J^i \setminus J^{i+1}$ *with* $i \geq 2$*. Then,* p *is an element of* $J \setminus J^2$*, hence* $\pi^2 = p^{2i}u\lambda$ *and* $t \geq n - 2i$*.*

Proposition 3. *Let* R *be a finite local ring with singleton basis with invariants* p*,* n*,* s*,* t *with* $n > 1$ *and* $x^2 - p^d u\lambda$ *be associated polynomial of* R *with respect to* S*. Then,*

(i) *If* s *is odd, then* R *is commutative.*
(ii) *If* p *is odd, then* $x^2 - p^d u$ *is an associated polynomial of* R *with respect to* S*.*
(iii) *If* p *is even, then* $x^2 - p^d\lambda$ *is an associated polynomial of* R *with respect to* S*.*

Proof.

(i) Let σ be the associated automorphism of R with respect to S, then $\sigma^2 = Id_S$. But s, the order of $Aut(S)$, is odd. Hence $\sigma = Id_S$, and thus R is commutative.

(ii) Since $(2, p) = 1$, let γ be an element of $1 + pS^{\sigma}$ such that $\lambda^{-1} = \gamma^2$. Put $\theta = \gamma\pi$, then $(S, \theta) \in T_R$ and $\theta^2 = (\gamma\pi)^2 = \gamma^2\pi^2 = p^d u$.

(iii) Since $(2, 2^{s'} - 1) = 1$, let δ be an element of $< b >$ such that $u^{-1} = \delta^2$, where $s' = s/k'$ and k' is the order of σ. Put $\theta = \delta\pi$, then $(S, \theta) \in T_R$ and $\theta^2 = (\delta\pi)^2 = \delta^2\pi^2 = p\lambda$.

□

Proposition 4. *Let R, T be finite local rings with singleton basis with the same invariants p, n, s, t with $n > 1$ and assume that they have the same coefficient subring S. Assume σ, τ be the associated automorphisms of R and T respectively, and $x^2 - p^d u\lambda$, $x^2 - p^d v\mu$ be associated polynomials of R and T respectively, where u, v and $p^d\lambda$, $p^d\mu$ are elements of S^{σ}. Then, $R \cong T$ if and only if $x^k - u^{p^i}v^{-1}$ has a root in S^{σ} mod p for some $1 \leqslant i \leqslant s'$ if p is odd, where $k = p^{s'} + 1$, $s' = s/k'$ and k' is the order of σ; while $\lambda^{\rho} = \mu$ in $GR(p^{n-d}, s')$ if p is even, where ρ is the element of $Aut(S^{\sigma})$ determined by $b^{\rho} = b^{p^i}$.*

Proof. Let φ be the isomorphism between R and T. then by the last proposition $\sigma = \tau$, and there exists an automorphism ρ of S^{σ} such that $x^2 - p^d u^{\rho}$ if p is odd ($x^2 - p^d\lambda^{\rho}$ in the case that p is even) has a root π' in T with $(S, \pi') \in T_T$. Let $(S, \theta) \in T_T$ with $\theta^2 = p^d v$ if p is odd (with $\theta^2 = p^d\lambda$ if p is even). Thus, $\pi' = \omega(p^{n-t+l}\zeta + \theta)$, where ζ is an element of $< a > \cup \{0\}$ and ω is an element of $< a >$, with $p^{n-t+l}\zeta$ is zero if R is non-commutative. Since the order of σ is either 1or 2, σ is the only automorphism of S of such order and hence σ is the automorphism of S which sends a either to a or to $a^{p^{s'}}$. Since $u \in < b >$, then $u^{\rho} = u^{p^i}$ for some $1 \leqslant i \leqslant s'$. Let $(S, \pi) \in T_R$. Now in the case that p is odd,

$$\begin{aligned} p^d u^{p^i} &= p^d u^{\rho} = \varphi(\pi^2) = (\varphi(\pi))^2 = (\pi')^2 \\ &= \omega(p^{n-t+l}\zeta + \theta)\omega(p^{n-t+l}\zeta + \theta) \\ &= \omega\theta\omega\theta = (\omega\omega^{\sigma}\theta^2) \\ &= p^d\omega\omega^{\sigma}v = p^d\omega^k v. \end{aligned}$$

Therefore, ω is a root of $x^k - u^{p^i}v^{-1}$ for some $1 \leq i \leqslant s'$. While in the case that p is even,

$$p^d\lambda^{\rho} = p^d\lambda^{\rho} = \varphi(\pi^2) = (\varphi(\pi))^2 = (\pi')^2 = \omega(p^{n-t+l}\zeta + \theta)\omega(p^{n-t+l}\zeta + \theta) = p^d\,\mu.$$

Therefore, $\lambda^{\rho} = \mu$ in $GR(p^{n-d}, s')$, where ρ is the element of $Aut\ S^{\sigma}$ determined by $b^{\rho} = b^{p^i}$. Conversely, in the case that p is odd, assume that $x^k - u^{p^i}v^{-1}$ has a root in S^{σ} mod p for some $1 \leqslant i \leqslant s'$. This root may be lifted to $\alpha\beta$ in S^{σ} (see Theorem 3–12 in [12] because finite commutative local rings are Hensel rings), where α is an element of $< b >$ and β an element of $1 + pS^{\sigma}$. Since $\alpha \in < b >$, $\alpha = \omega^k$, where $\omega \in < a >$. Let $\pi' = \omega\theta$ and θ is a root of $x^2 - p^d v$ with $(S, \theta) \in T_T$. Now,

$$(\pi')^2 = (\omega\theta)^2 = \omega(\omega^{\sigma})\theta^{\,2} = p^d(\omega)^k v = p^d\alpha v = p^d u^{p^i} = p^d u^{\rho},$$

for some automorphism ρ of S. Assume that $(S, \pi) \in T_R$ such that $\pi^2 = p^d u$ and let φ be a mapping from R to T defined by $\varphi(s_0 + s_1\pi) = s_0^{\rho} + s_1^{\rho}\pi'$, where ρ is the automorphism of S determined by $a^{\rho} = a^{p^i}$ for some $1 \leqslant i \leqslant s'$. Now, it is easy to check that $R \cong T$. Conversely, in the case that p is even, assume that $\lambda^{\rho} = \mu$ in $GR(p^{n-d}, s')$ and θ is a root of $x^2 - p^d\mu$ with $(S, \theta) \in T_T$. Then, $\theta^{\,2} = p^d\mu = p^d\lambda^{\rho}$ for the automorphism ρ of S^{σ} determined by $b^{\rho} = b^{p^i}$ for some $1 \leqslant i \leqslant s'$. Assume that $(S, \pi) \in T_R$ such that $\pi^2 = p^d\lambda$ and let φ be a mapping from R to T defined by $\varphi(s_0 + s_1\pi) = s_0^{\rho} + s_1^{\rho}\theta$, where ρ is the automorphism ρ of S determined by $a^{\rho} = a^{p^i}$ for some $1 \leqslant i \leqslant s'$. Now, it is easy to check that $R \cong T$. □

Theorem 2. *Let $N(\ p,\ n,\ s,\ t)$ be the number of mutually non-isomorphic finite local rings with singleton basis with the same invariants $p,\ n,\ s,\ t$ with $n > 1$ and with an associated polynomial of the form $x^2 - p^d u\lambda$. Then,*

$$N(p,n,s,t) = \begin{cases} \sum_{m|z} \frac{\phi(m)}{\tau(m)}, & \text{if } p \neq 2, \\ \frac{1}{s'} \sum_{i=0}^{s'} p^{(i,s')n-d-1}, & \text{if } p = 2, \end{cases}$$

where $z = (p^{s'} - 1, k)$, where ϕ is Euler function and $\tau(m)$ is the order of p in $\mathbb{Z}_m$.

Proof.

(1) Let $p = 2$. For $\lambda, \mu \in 1 + pS^\sigma$, define $\lambda \sim \mu$ if and only if $\lambda^\rho = \mu$, where $S^\sigma = GR(p^{n-d}, s')$. This relation is equivalent to the action of the group $Aut(S^\sigma)$ on the set $\Gamma^*(s')^{n-d-1}$, where $\Gamma^*(s') = \{1, b, b^2, \ldots, b^{p^{s'}-1}\}$. According to Proposition 4, it suffices to compute the equivalence classes. There are $(p^i - 1, p^{s'} - 1) + 1$ elements of $\Gamma^*(r)$ fixed by ρ. Thus, the number of elements fixed by ρ is

$$[(p^i - 1, p^r - 1) + 1]^{n-d-1},$$

but $[(p^i - 1, p^r - 1) + 1] = p^{(i,r)}$, hence,

$$[(p^i - 1, p^r - 1) + 1]^{n-d-1} = p^{(i,s')n-d-1}. \tag{8}$$

Therefore, by Burnside Lemma, the total number of equivalence classes is

$$\frac{1}{s'} \sum_{i=0}^{s'} p^{(i,s')n-d-1}.$$

(2) Let p be odd, $F \cong GF(p^{s'})$ and $K \cong GF(p^s)$. For $u, v \in F$, define $u \sim v$ if and only if $x^k - u^{p^i} v^{-1}$ has a root in K, for some $0 < i \leqslant s'$. This is easily seen to be equivalence relation. In view of the last proposition, the required number of such rings is the same as the number of equivalence classes of this equivalence relation. By a similar argument in [5,7], we conclude that $N(p,n,s,t)$, in this case, is

$$\sum_{m|z} \frac{\phi(m)}{\tau(m)}.$$

□

Next, we give full classification of local rings with singleton basis and associated polynomial $x^2 - p^d u$.

Corollary 2. *Assume that R is a local ring with singleton basis and associated polynomial $x^2 - p^d u$. Then,*

(i) If $p = 2$, then there is only one ring up to ismororphism, i.e.,

$$N(2,n,s,t) = 1.$$

(ii) If $p \neq 2$, then $k = 2$, and thus $z = (p^{s'} - 1, 2) = 2$. Which implies

$$\sum_{m|2} \frac{\phi(m)}{\tau(m)} = 2.$$

This means there are only two isomorphism classes of such rings.

If the associated polynomial is $x^2 - p^d u\lambda$, then by Theorem 2, we have the following remark.

Remark 3. *Note that*

(i) *If p is even, then $u = 1$ and $\lambda \in 1 + pS_0 \backslash \{1 + p\, c^{p^i} : 1 \leq i \leq m\}$, where $S_0 = Z_{p^{n-d}}[c] = GR(p^{n-d}, m)$ is a Galois subring of S^σ and c is an element of S_0 of order $p^m - 1$ for all divisors m of s'.*

(ii) *If p is odd and R is commutative, then $\lambda = 1$ and $u = 1$ or $u = a$.*

(iii) *If p is odd and R is noncommutative, then $\lambda = 1$ and u is equal to one of the k-roots in $K = GF(p^s)$ of the polynomial $x^k - u^{p^i} v^{-1}$ over $F = GF(p^{s'})$.*

In case when $g(x) = \pi^2 = p^d u\lambda + p^e u_1 \lambda_1 \pi$, we have the following notes.

Remark 4. *Taking into consideration that $J = pS \oplus S\pi$ and $d + t \geq n$, then we distinguish the followings cases:*

(i) *If π is element of $J \backslash J^2$ and p is an element of $J \setminus J^2$. Then, $\pi^2 = p^2 u\lambda + p^e u_1 \lambda_1 \pi$ with $e \geq 1$ and $t \geq n - 2$ or $\pi^2 = p^d u\lambda + p u_1 \lambda_1 \pi$ with $d \geq 2$.*

(ii) *If π is element of $J \setminus J^2$ and p is an element of $J^i \setminus J^{i+1}$ with $i \geq 2$. Then, $\pi^2 = p^d u\lambda + p^e u_1 \lambda_1 \pi$ implies that $p^d \in J^2$. But $p \notin J$. Thus, $p \in J^2$, and hence $\pi^2 = pu\lambda + p^e u_1 \lambda_1 \pi$ with $e \geq 1$ and $t \geq n - 1$.*

(iii) *If π is element of $J^i \setminus J^{i+1}$ with $i \geq 2$. Then, p is an element of $J \setminus J^2$ and hence $\pi^2 = p^{2i} u\lambda + p^e u_1 \lambda_1 \pi$ with $e \geq i$ and $t \geq n - 2i$ or $\pi^2 = p^d u\lambda + p^i u_1 \lambda_1 \pi$ with $d \geq 2i$.*

The following remark links local rings with singleton basis to chain rings.

Remark 5. *Let R be a local ring with singleton basis given in case (ii) of the last remark. Then, $p = \pi^2 (u\lambda + p^{e-1} u_1 \lambda_1 \pi)^{-1}$, and subsequently $pR = \pi^2 R$. By using R is a local ring, then it is obvious that R is a commutative chain ring with invariants $p, n, s, k = 2, t \geq n - 1$ with $n > 1$. The classification up to isomorphism of such chain rings with the same invariants p, n, s, k, t is already known (cf. [7]).*

Next, we classify all commutative local rings with singleton basis and associated polynomial $\pi^2 = p^d u\lambda + p^e u_1 \lambda_1 \pi$.

Remark 6. *Let R be a commutative local ring with singleton basis with invariants p, n, s, t with $n > 1$ and $\pi^2 = p^d u\lambda + p^e u_1 \lambda_1 \pi$. If $d \leq e$, then $\pi^2 = p^d u(\lambda + p^{e-d}\, u^{-1} u_1 \lambda_1 \pi)$. If $\mu = u(\lambda + p^{e-d}\, u^{-1} u_1 \lambda_1 \pi)$, then $\pi^2 = p^d\, u\mu$. In the case that $p \neq 2$, then there exists $\xi \in 1 + J(R)$ such that $\xi^2 = \mu$. Now, put $\theta = \xi^{-1}\pi$ then $\theta^2 = (\xi^{-1}\pi)^2 = (\xi^{-1})^2 \pi^2 = p^d u$. Then, obviously $(S, \theta) \in T_R$ with $\theta^2 = p^d\, u$. Thus, we can assmue that R in such case is a commutative local ring with singleton basis with invariants p, n, s, t with $n > 1$ and $\pi^2 = p^d u$ and the classification up to isomorphism of such local rings with singleton basis with the same invariants p, n, s, t with $n > 1$ is already known according to the last theorem.*

5. The Enumeration of Artinian Local Rings with Singleton Basis and with Absolutely Algebraic Residue Field

First we introduce the following important result.

Theorem 3. *Let R and T be Artinian local rings with singleton basis and with the same invariants p, n, s, t, and R', T' be their associated subrings respectlively with K, K' be their absolutely algebraic residue fields, respectively. Then, $R \cong T$ if and only if $R' \cong T'$ and $K \cong K'$.*

Proof. By using Hensel Lemma, the isomorphism $F \cong F'$ can be lifted to an isomorphism ψ between coefficient subrings of R and T. Let us define the map ς from R to T as follows:

$$\varsigma\ (s_0 + s_1\pi) = \ \psi(s_0) + \psi(s_1)\varphi(\pi),$$

where φ is the isomorphism between R' and T'. Now, it is easy to see that ς is an isomorphism. The converse is direct. □

Note that the previous proof can be generalized to prove the theorem under general condition, i.e., not necessarily singleton.

Theorem 4. *Artinian local rings with singleton basis, with isomorphic absolutely algebraic residue fields, with the same invariants p, n, s, t and with same associated polynomial x^2 are uniquely determined by p, n, s, t.*

Proof. Since every Artinian ring R with singleton basis has an associated subring R' which is finite local with singleton basis and invarinats p, n, s, t. Thus, by Theorem 3, the enumeration is reduced to that of finite local rings with singleton basis. By Theorem 2, these rings are uniquely determined by their invariants. □

Theorem 5. *The number $N(\ p,\ n,\ s,\ t)$ of mutually non- isomorphic Artinian local rings with singleton basis, with isomorphic absolutely algebraic residue fields, with the same invariants p, n, s, t, with $n \neq 1$ and with the same associated polynomial $x^2 - p^d u\lambda$ is*

(i) *If $p = 2$,*

$$N(2, n, s, t) = \frac{1}{s'} \sum_{i=0}^{s'} p^{(i,s')n-d-1}.$$

(ii) *If $p \neq 2$, then*

$$N(p, n, s, t) = \sum_{m|z} \frac{\phi(m)}{\tau(m)}.$$

Proof. Let $E(p, n, s, t)$ be the number of non-ismomorphic classes of Artinian local rings with invariants p, n, s, t and associated polynomial $x^2 - p^d u\lambda$. From Theorem 3,

$$E(p, n, s, t) = N(p, n, s, t).$$

and Theorem 2 concludes the results. □

Corollary 3. *Let R be an Artinian local ring with singleton basis with absolutely algebraic residue field and with the same associated polynomial $x^2 - p^d u$.*

(i) *If R is commutative and p is even, $N(2, n, s, t) = 1$, and then $u = 1$.*
(ii) *If R is commutative and p is odd, $N(p, n, s, t) = 2$, and thus $u = 1$ or $u = a$.*
(iii) *If R is non-commutative, then $u = 1, a, a^2, \ldots, a^{\,p^{s/2}}$.*

Remark 7.

(1) *As a particular case if the ring R is isomorphic to T_3 given by construction A, let us assume that $pu \neq 0$. Then, $p = \pi u^{-1}(\pi - pu^{1/2})$ and hence p is an element of πR. As R is local, then R is a commutative chain ring of characteristic p^2 (cf. [7]).*
(2) *If the ring R is finite and isomorphic to T_2 given by construction A, we note the following:*
    (i) *Suppose $d = n - 1$. Then, R is of order $p^{(n+1)r}$, $R/J(R) \cong GF(p^r)$ and of characteristic p^n. Such ring is called near Galois ring (cf. [9]).*

   (ii) *If u is a unit in S. Then, $\pi^2 = pu$ implies $p = \pi^2 u^{-1}$ is an element of $\pi^2 R$. Thus, $J(R) = R\pi$. But R is local, and therefore, R is a chain ring of characteristic p^2 (not necessarily commutative) (cf. [5]).*

(3) *If R is finite and isomorphic to T_1, note the following:*
   (i) *If $n = 1$, then R is a ring with few zero divisors (cf. [13]).*
   (ii) *Assume that $n = 1$ or $n = 2$, then R is a particular case of a finite ring in which the multiplication of any two zero divisors is zero.*

6. Conclusions

In this article, we have fully determined the structure of local rings with singleton basis. Moreover, we investigated enumeration of local rings with singleton basis with fixed invariants p, n, s, t. Under certain conditions on Eisenstein polynomials, we classified these rings up to isomorphism.

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