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Trigonometric Polynomial Solutions of Bernoulli Trigonometric Polynomial Differential Equations

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Abstract: We consider real trigonometric polynomial Bernoulli equations of the form $A(\theta)y' = B_1(\theta) + B_n(\theta)y^n$ where $n \geq 2$, with A, B_1, B_n being trigonometric polynomials of degree at most $\mu \geq 1$ in variables θ and $B_n(\theta) \neq 0$. We also consider trigonometric polynomials of the form $A(\theta)y^{n-1}y' = B_0(\theta) + B_n(\theta)y^n$ where $n \geq 2$, being A, B_0, B_n trigonometric polynomials of degree at most $\mu \geq 1$ in the variable θ and $B_n(\theta) \neq 0$. For the first equation, we show that when $n \geq 4$, it has at most 3 real trigonometric polynomial solutions when n is even and 5 real trigonometric polynomial solutions when n is odd. For the second equation, we show that when $n \geq 3$, it has at most 3 real trigonometric polynomial solutions when n is odd and 5 real trigonometric polynomial solutions when n is even. We also provide trigonometric polynomial equations of the two types mentioned above where the maximum number of trigonometric polynomial solutions is achieved. The proof method will be to apply extended Fermat problems to polynomial equations.

Keywords: trigonometric polynomial; Bernoulli equations; trigonometric solutions

MSC: Primary 34A05; Secondary 37C10



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1. Introduction and Statement of the Main Results

Generalized Bernoulli polynomial equations are equations of the form

$$a(x)x' = b_n(x)x^n + b_1(x)x \quad (1)$$

where a, b_n, b_1 are polynomials in real variables and $b_n(x) \neq 0$. It goes without saying the importance of such equations, which appear in all textbooks of ordinary differential equations as the first examples of nonlinear equations that can be solved. The study of given polynomial solutions of a differential equation is of main interest for understanding the set of solutions of a differential system.

There are several previous works asking for polynomial solutions of the above or more general equations for some values of n , see, for instance, [1–6] and the references therein. When $n = 2$ and $a(x)$ are constant, it is the well-known trigonometric Riccati equation, and when $n = 3$ and $a(x)$ are constant, it is the well-known Abel equation. The question of asking for polynomial solutions of such equations is reminiscent of a similar one proposed by Poincaré on the number and degree of algebraic solutions of planar autonomous polynomial differential systems in terms of their degrees. The characterization of the number of polynomial solutions of Equation (1) was completely solved in [7] where the authors determine the maximum number of polynomial solutions of Bernoulli equations of the form in (1), and this maximum is achieved.

In this paper, we go a step further and consider Equation (1) where the functions that appear are real trigonometric polynomials of a certain degree μ . In this case, we will rename them as *trigonometric polynomial Bernoulli differential equations of degree μ* . We note that the degree of a trigonometric polynomial is well defined in terms of its Fourier series, since it is the degree of its corresponding Fourier series. The ring of trigonometric polynomials

in the variables $\cos \theta, \sin \theta$ with coefficients in $\mathbb{R}$ will be denoted as $\mathbb{R}_t[\theta] := \mathbb{R}[\cos \theta, \sin \theta]$. We note that these equations (especially when $n = 2$ or $n = 3$ and $a(x)$ is constant) appear in the study of the number of limit cycles of planar polynomial differential equations with homogeneous nonlinearities (see, for example, [8,9] and the references therein). In particular, periodic orbits surrounding the origin of these planar systems correspond to 2π -periodic solutions of the corresponding Riccati or Abel equations. When $a(x)$ is not constant, these real trigonometric equations are singular at the zeros of $a(x)$, which implies that the Cauchy problem has no unique solutions on the zero set of $a(x)$, and so the behavior of the solutions becomes more and more complex. For example, when $n = 2$, there are a large number of trigonometric polynomial solutions of the equation, and the number of solutions is bounded from above in terms of the degrees of the corresponding trigonometric polynomials defining the equation. More concretely, we will work with real trigonometric Bernoulli differential equations of the form

$$A(\theta)Y' = B_1(\theta)Y + B_n(\theta)Y^n, \quad n \geq 2 \quad (2)$$

where the prime denotes the derivative with respect to θ , $A(\theta), B_i(\theta) \in \mathbb{R}_t[\theta]$, for $i \in \{1, n\}$ and $B_n(x) \not\equiv 0$. We also have $\mu = \max\{\mu_1, \mu_2, \mu_3\}$ where μ_1, μ_2, μ_3 are the degrees of $A(\theta)$, $B_1(\theta)$ and $B_n(\theta)$, respectively. In this case, we say that the real trigonometric Bernoulli equation has degree μ .

Our first result completely solves the problem of Bernoulli trigonometric polynomials (2). The case $n = 2$ was proved in [10] and the case $n = 3$ was proved in [11]. We include them in the next theorem for the sake of completeness.

Theorem 1. *Consider real trigonometric Bernoulli differential equations*

$$A(\theta)Y' = B_1(\theta)Y + B_n(\theta)Y^n, \quad n \geq 2 \quad (3)$$

with $A, B_1, B_n \in \mathbb{R}_t[\theta]$ and $B_n \not\equiv 0$ of degree μ . Then

1. *For $n = 2$, Equation (3) has at most 2μ real trigonometric polynomial solutions if $\mu \geq 2$ and 3 trigonometric polynomial solutions if $\mu = 1$ and there are trigonometric differential equations of this type that have exactly this maximum number of trigonometric polynomial solutions.*
2. *For $n = 3$, Equation (3) has at most 7 real trigonometric polynomial solutions, and there are real trigonometric differential equations of this type having exactly these maximum number of trigonometric polynomial solutions.*
3. *For $n \geq 4$, Equation (3) has at most 3 real trigonometric polynomial solutions when n is even and 5 real trigonometric polynomial solutions when n is odd, and there are real trigonometric differential equations of this type having exactly these maximum number of trigonometric polynomial solutions.*

Theorem 1 is proved in Section 2. As we shall see, the case of trigonometric polynomials is much more involved than the case of polynomials because the proofs of the maximum number of solutions are based on divisibility arguments. In contrast to the ring of polynomials, which is a Unique Factorization Domain, the ring of trigonometric polynomials is not a Unique Factorization Domain, as we will see in the next section. This causes the difficulty in proving that such results are much higher than in the case of polynomials.

Notice that the question we are interested in is also reminiscent of a similar one proposed by Poincaré about the number and degree of algebraic solutions of planar autonomous polynomial differential systems in terms of their degrees.

Our second objective in the paper is on equations of the form

$$A(\theta)Y^{n-1}Y' = B_0(\theta)Y + B_n(\theta)Y^n, \quad n \geq 2,$$

where again the prime denotes derivative in the variable θ , $A, B_i \in \mathbb{R}_t[\theta]$ for $i \in \{0, n\}$ with $B_n \neq 0$. These types of equations will be called *real trigonometric Bernoulli equations of the first kind*. In this case neither the case $A(\theta)$ not constant nor the case $A(\theta)$ constant have been studied anywhere for $n \geq 3$. The case $n = 2$ has been studied in [11], but we introduce it here for the sake of completeness.

Theorem 2. Consider real trigonometric Bernoulli differential equations of the first kind

$$A(\theta)Y^{n-1}Y' = B_0(\theta) + B_n(\theta)Y^n, \quad n \geq 2 \quad (4)$$

with $A, B_0, B_n \in \mathbb{R}_t[\theta]$ and $B_n \neq 0$ of degree μ . Then

1. For $n = 2$, Equation (4) has at most 7 real trigonometric polynomial solutions and there are trigonometric differential equations of this type that have exactly this maximum number of trigonometric polynomial solutions.
2. For $n \geq 3$, Equation (4) has at most 5 real trigonometric polynomial solutions when n is even and 3 real trigonometric polynomial solutions when n is odd, and there are real trigonometric differential equations of this type having exactly the maximum number of trigonometric polynomial solutions.

Theorem 2 is proved in Section 2.

2. Preliminaries

We first observe that the proofs of Theorems 1 and 2 are based on divisibility arguments and that the ring of polynomials is not a Unique Factorization Domain. This can be seen for instance using the identity $\cos^2 \theta + \sin^2 \theta = (1 - \sin \theta)(1 + \sin \theta)$. So, $\cos \theta$ divides the right-hand side expression but does not divide the left-hand side. This difficulty is overcome using the isomorphism $\Phi: \mathbb{R}_t[\theta] \rightarrow \mathbb{R}(x)$ given by

$$(\cos \theta, \sin \theta) \mapsto \left(\frac{1 - x^2}{1 + x^2}, \frac{2x}{1 + x^2} \right)$$

between the two fields $\mathbb{R}_t(\theta) = \mathbb{R}(\cos \theta, \sin \theta)$ and $\mathbb{R}(x)$, being $\mathbb{R}(x)$ the ring of rational functions with coefficients in $\mathbb{R}$.

The following result is well-known

Lemma 1. Let $P(\theta) \in \mathbb{R}_t[\theta]$ with degree μ . Then

$$\Phi(P(\theta)) = \frac{p(x)}{(1 + x^2)^\mu},$$

where $\gcd(p(x), 1 + x^2) = 1$ and $\deg(p(x)) \leq 2\mu$. Conversely, any rational function $g(x)/(1 + x^2)^\mu$ with $g(x)$ an arbitrary polynomial of degree at most 2μ can be written as a trigonometric polynomial through the inverse change Φ^{-1} .

The following result is an extension of Fermat's theorem for polynomials. For a proof of it, we refer the reader to [12].

Theorem 3. Equation $p^k + q^k = r^k$ where $p, q, r \in \mathbb{C}[x]$ has polynomial solutions not pairwise similar if and only if $k = 1, 2$.

In particular when $k \geq 3$ there are no solutions of the equation $p^k + q^k = r^k$. Now we write (3) in terms of $a(x)$, $b_1(x)$ and $b_n(x)$ with $a, b_1, b_n \in \mathbb{R}[x]$.

Lemma 2. Let $Y(\theta)$ be a nonconstant real trigonometric polynomial solution of Equation (3) and set

$$Y(\theta) = \frac{y(x)}{(1+x^2)^{\mu_0}}, \quad A(\theta) = \frac{a(x)}{(1+x^2)^{\mu_1}},$$

$$B_1(\theta) = \frac{b_1(x)}{(1+x^2)^{\mu_2}}, \quad B_n(\theta) = \frac{b_n(x)}{(1+x^2)^{\mu_3}}$$

with $\deg(y) \leq 2\mu_0$, $\deg(a) \leq 2\mu_1$, $\deg(b_1) \leq 2\mu_2$, $\deg(b_n) \leq 2\mu_3$ and $\gcd(y, 1+x^2) = 1$, $\gcd(a, 1+x^2) = 1$, $\gcd(b_1, 1+x^2) = 1$ and $\gcd(b_n, 1+x^2) = 1$. Then Equation (3) becomes

$$\begin{aligned} & \frac{a(x)}{2(1+x^2)^{\mu_1}} (\dot{y}(x)(1+x^2) - 2\mu_0 xy(x)) \\ &= \frac{b_1(x)}{(1+x^2)^{\mu_2}} y(x) + \frac{b_n(x)}{(1+x^2)^{\mu_3+(n-1)\mu_0}} y^n(x), \end{aligned} \quad (5)$$

where the dot denotes the derivative in the new variable x .

Proof. It follows from Φ that

$$x' = \frac{dx}{d\theta} = \frac{1+x^2}{2} \quad \text{yielding} \quad Y(\theta)' = \frac{\dot{y}(x)(1+x^2) - 2\mu_0 xy(x)}{2(1+x^2)^{\mu_0}}, \quad (6)$$

where the dot denotes the derivative in the variable x and the prime denotes the derivative in the variable μ . Using this, it is clear that Equation (3) becomes (5). $\square$

In the same manner, using (6) we have the following lemma, which we state without proof.

Lemma 3. Let $Y(\theta)$ be a nonconstant real trigonometric polynomial solution of Equation (4) and set

$$Y(\theta) = \frac{y(x)}{(1+x^2)^{\mu_0}}, \quad A(\theta) = \frac{a(x)}{(1+x^2)^{\mu_1}},$$

$$B_0(\theta) = \frac{b_0(x)}{(1+x^2)^{\mu_2}}, \quad B_n(\theta) = \frac{b_n(x)}{(1+x^2)^{\mu_3}}$$

with $\deg(y) \leq 2\mu_0$, $\deg(a) \leq 2\mu_1$, $\deg(b_0) \leq 2\mu_2$, $\deg(b_n) \leq 2\mu_3$ and $\gcd(y, 1+x^2) = 1$, $\gcd(a, 1+x^2) = 1$, $\gcd(b_0, 1+x^2) = 1$ and $\gcd(b_n, 1+x^2) = 1$. Then Equation (3) becomes

$$\begin{aligned} & \frac{a(x)}{2(1+x^2)^{\mu_1}} y^{n-1}(x) (\dot{y}(x)(1+x^2) - 2\mu_0 xy(x)) \\ &= \frac{b_0(x)}{(1+x^2)^{\mu_2-n\mu_0}} y(x) + \frac{b_n(x)}{(1+x^2)^{\mu_3}} y^n(x), \end{aligned}$$

where the dot denotes the derivative in the new variable x .

Proof of Theorem 1.

Let $Y_0(\theta) \neq 0$ be a solution of (3). First we note that if $Y_0(\theta)$ is solution of (3) then $\alpha Y_0(\theta)$ is also a solution for all $\alpha \in \mathbb{C}$ such that $\alpha^n = 1$. It follows from the fact that when n is even the identity $\alpha^{n-1} = 1$ has a unique real solution $\alpha = 1$ while it has two real solutions $\alpha = \pm 1$ when n is odd, that if $Y_0(\theta)$ is a solution of Equation (3) then $-Y_0(\theta)$ is also a solution of (3) which is different from $Y_0(\theta)$ when n is odd.

Now we claim the following: Let $Y_0(\theta)$, $Y_1(\theta)$, $Y_2(\theta)$ be different polynomial solutions of Equation (3) with $Y_0(\theta) \neq 0$, $Y_1(\theta) \neq 0$, $Y_2(\theta) \neq 0$ and $Y_2(\theta) \neq -Y_1(\theta)$ (when n is odd). Set $Y_i(\theta) = y_i(x)/(1+x^2)^{\mu_i}$ for $i = 0, 1, 2$ where $\mu_i = \deg(Y_i)$, $\deg(y_i) \leq 2\mu_i$, $\mu_1 \leq \mu_2$ and $\gcd(y_i, 1+x^2) = 1$ for $i = 0, 1, 2$. Write $y_i(x) = g(x)\tilde{y}_i(x)$ for $i = 1, 2$ where $g = \gcd(y_1, y_2)$.

Then except the solution $Y(\theta) \equiv 0$, all other polynomial solutions of Equation (3), if they exist, can be expressed as $Y(\theta) = y(x)/(1+x^2)^{\mu_0}$ where $\mu_0 = \deg(Y_0)$, $\deg(y_0) \leq 2\mu_0$,

$$y(x; c) = (\pm 1)^n \frac{\tilde{y}_1(x)\tilde{y}_2(x)g(x)}{(c\tilde{y}_1^{n-1}(x)(1+x^2)^{(n-1)(\mu_2-\mu_1)} + (1-c)\tilde{y}_2^{n-1}(x))^{1/(n-1)}}, \quad (7)$$

being c a constant with $c \notin \{0, 1\}$ and

$$(c\tilde{y}_1^{n-1}(x)(1+x^2)^{(n-1)(\mu_2-\mu_1)} + (1-c)\tilde{y}_2^{n-1}(x))^{1/(n-1)}$$

is a polynomial.

Now we prove the claim. Let $Y(\theta)$ be a nonzero real trigonometric polynomial solution of Equation (3). The functions

$$Z_0(\theta) = 1/Y_0^{n-1}(\theta), \quad Z_1(\theta) = 1/Y_1^{n-1}(\theta), \quad \text{and} \quad Z_2(\theta) = 1/Y_2^{n-1}(\theta)$$

are solutions of a linear differential equation and satisfy

$$-A(\theta)Z_i'(\theta) = (n-1)B_1(\theta)Z_i(\theta) + (n-1)B_n(\theta), \quad i = 0, 1, 2.$$

Therefore, we have

$$\frac{Z'(\theta) - Z_1'(\theta)}{Z(\theta) - Z_1(\theta)} = \frac{Z_2'(\theta) - Z_1'(\theta)}{Z_2(\theta) - Z_1(\theta)}.$$

Integrating the above equality, we obtain

$$Z(\theta) = Z_1(\theta) + c(Z_2(\theta) - Z_1(\theta)),$$

where c is an arbitrary constant. So, the general solution of Equation (3) is

$$\begin{aligned} Y^{n-1}(\theta) &= \frac{1}{Z(\theta)} = \frac{1}{Z_1(\theta) + c(Z_2(\theta) - Z_1(\theta))} \\ &= \frac{Y_1^{n-1}(\theta)Y_2^{n-1}(\theta)}{cY_1^{n-1}(\theta) + (1-c)Y_2^{n-1}(\theta)} \end{aligned}$$

or equivalently

$$\frac{y^{n-1}(x)}{(1+x^2)^{(n-1)(\mu_0-\mu_1)}} = \frac{g^{n-1}(x)\tilde{y}_1^{n-1}(x)\tilde{y}_2^{n-1}(x)}{c(1+x^2)^{(n-1)(\mu_2-\mu_1)}\tilde{y}_1^{n-1}(x) + (1-c)\tilde{y}_2^{n-1}(x)}. \quad (8)$$

Since the right-hand side of Equation (8) is not divisible by $1+x^2$ we must have $\mu_0 \geq \mu_1$, but if $\mu_0 > \mu_1$ then taking into account that neither $\tilde{y}_2$ nor y_0 (and when $\mu_1 = \mu_2$ also $\tilde{y}_1$) do not divide $1+x^2$, we get a contradiction with (8). In summary, we have $\mu_0 = \mu_1$ and Equation (8) becomes

$$y^{n-1}(x) = \frac{g^{n-1}(x)\tilde{y}_1^{n-1}(x)\tilde{y}_2^{n-1}(x)}{c(1+x^2)^{(n-1)(\mu_2-\mu_1)}\tilde{y}_1^{n-1}(x) + (1-c)\tilde{y}_2^{n-1}(x)},$$

being c an arbitrary constant not equal $c = 0$ (otherwise $y(x) = y_1(x)$) and not equal to $c = 1$ (in which case $y(x) = y_2(x)$). This concludes the proof of the claim.

In view of the claim, if $Y_1(\theta), Y_2(\theta)$ are real trigonometric polynomial solutions of (3) such that $Y_i(\theta) \not\equiv 0$ for $i = 1, 2$, $Y_1(\theta) \neq (\pm 1)^n Y_2(\theta)$, then any other real trigonometric

polynomial solution of Equation (3), say $Y_3(\theta)$, different from them is of the form given in (7) for some appropriate constant c with $c \neq 0$ and $c \neq 1$. In particular

$$c\tilde{y}_1^{n-1}(x)(1+x^2)^{(n-1)(\mu_2-\mu_1)} + (1-c)\tilde{y}_2^{n-1}(x) = c\tilde{y}_3^{n-1}(x) + (1-c)\tilde{y}_2^{n-1},$$

where $\tilde{y}_3 = y_1(1+x^2)^{\mu_2-\mu_1}$ is the $(n-1)$ -th root of a polynomial q and q divides g . This implies that

$$(\sqrt[n-1]{c}\tilde{y}_3(x))^{n-1} + (\sqrt[n-1]{1-c}\tilde{y}_2(x))^{n-1} = q(x)^{n-1}$$

for some polynomial $q(x)$. It follows from Theorem 3 that this last equation has no non-similar polynomial solutions for $n \geq 4$, and as a consequence, such polynomial q does not exist, implying that such real trigonometric polynomial $Y_3(\theta)$ does not exist unless the polynomial solutions are similar. However, in this last case taking into account that $B_n(\theta) \neq 0$, we conclude that it is also not possible.

In short, among the solution $Y_0(\theta) \equiv 0$, there can only be two sets of solutions $Y_1(\theta), Y_2(\theta)$ if n is even and $Y_1(\theta), -Y_1(\theta), Y_2(\theta), -Y_2(\theta)$ if n is odd.

Now we show that there are equations of the form (3) with this number of real trigonometric polynomial solutions. Note that $y_1(x) = (\pm 1)^n x$ and $y_2(x) = (\pm 1)^n x^2$ are polynomial solutions of equation

$$a(x)y' = b_1(x)y + b_n(x)y^n$$

with

$$a(x) = x^{2n-1} - x^n, \quad b_1(x) = x^{2n-2} - 2x^{n-1}, \quad b_n(x) = 1.$$

Therefore, in view of Lemmas 1 and 2 if we set

$$\begin{aligned} A(\theta) &= \frac{2a(x)}{(1+x^2)^n} = \frac{x^{2n-1} - x^n}{(1+x^2)^n}, \\ B_1(\theta) &= \frac{b_1(x)(1+x^2) - 2xa(x)}{(1+x^2)^n} = \frac{x^{2n-2} - 2x^{n-1} - x^{2n}}{(1+x^2)^n}, \\ B_n(\theta) &= b_n(x) = 1, \end{aligned}$$

then the solutions

$$Y_1(\theta) = (\pm 1)^n \frac{x}{1+x^2}, \quad Y_2(\theta) = (\pm 1)^n \frac{x^2}{1+x^2}, \quad Y_0(\theta) \equiv 0$$

are real trigonometric polynomial solutions of Equation (3). This concludes the proof of the theorem. $\square$

Proof of Theorem 2.

Let $Y_0(\theta) \neq 0$ be a solution of (4). First we note that if $Y_0(\theta)$ is a solution of (4) then $\alpha Y_0(\theta)$ is also a solution for all $\alpha \in \mathbb{C}$ such that $\alpha^n = 1$. It follows from the fact that when n is odd the identity $\alpha^n = 1$ has a unique real solution $\alpha = 1$ while it has two real solutions $\alpha = \pm 1$ when n is even, that if $Y_0(\theta)$ is a solution of Equation (4) then $-Y_0(\theta)$ is also a solution of (4) which is different from $Y_0(\theta)$ when n is even.

Now we claim that if $Y_0(\theta), Y_1(\theta), Y_2(\theta)$ are different polynomial solutions of Equation (3) with $Y_0(\theta) \neq 0, Y_1(\theta) \neq 0, Y_2(\theta) \neq 0$ and $Y_2(\theta) \neq -Y_1(\theta)$ (when n is odd) then setting $Y_i(\theta) = y_i(x)/(1+x^2)^{\mu_i}$ for $i = 0, 1, 2$ where $\mu_i = \deg(Y_i)$ and $\deg(y_i) \leq 2\mu_i$, $\mu_1 \leq \mu_2$ and $\gcd(y_i, 1+x^2) = 1$ for $i = 0, 1, 2$ and writing $y_i(x) = g(x)\tilde{y}_i(x)$ for $i = 1, 2$ where $g = \gcd(y_1, y_2)$ then except the solution $Y(\theta) \equiv 0$, all the other polynomial solutions of Equation (3), if they exist, can be expressed as $Y(\theta) = y(x)/(1+x^2)^{\mu_0}$ where $\mu_0 = \deg(Y_0), \deg(y_0) \leq 2\mu_0$ and

$$y(x; c) = (\pm 1)^{n-1} g(x) ((1-c)(1+x^2)^{(n-1)(\mu_2-\mu_1)} \tilde{y}_1^{n-1}(x) + c\tilde{y}_2^{n-1}(x))^{1/(n-1)} \quad (9)$$

being c a constant with $c \notin \{0, 1\}$ and

$$((1-c)(1+x^2)^{(n-1)(\mu_2-\mu_1)}\tilde{y}_1^{n-1}(x) + c\tilde{y}_2^{n-1}(x))^{1/(n-1)}$$

a polynomial.

Now we prove the claim. Let $Y(\theta)$ be a nonzero real trigonometric polynomial solution of Equation (3). The functions

$$Z_0(\theta) = Y_0^{n-1}(\theta), \quad Z_1(\theta) = Y_1^{n-1}(\theta), \quad \text{and} \quad Z_2(\theta) = Y_2^{n-1}(\theta)$$

are solutions of a linear differential equation and satisfy

$$-A(\theta)Z_i'(\theta) = (n-1)B_0(\theta) + (n-1)B_n(\theta)Z_i(\theta), \quad i = 0, 1, 2.$$

Therefore, we have

$$\frac{Z'(\theta) - Z_1'(\theta)}{Z(\theta) - Z_1(\theta)} = \frac{Z_2'(\theta) - Z_1'(\theta)}{Z_2(\theta) - Z_1(\theta)}.$$

Integrating the above equality, we obtain

$$Z(\theta) = Z_1(\theta) + c(Z_2(\theta) - Z_1(\theta)),$$

where c is an arbitrary constant. So, the general solution of Equation (3) is

$$Y^{n-1}(\theta) = Z(\theta) = Z_1(\theta) + c(Z_2(\theta) - Z_1(\theta)) = (1-c)Y_1^{n-1}(\theta) + cY_2^{n-1}(\theta)$$

where c is an arbitrary constant. Therefore,

$$\begin{aligned} \frac{y^{n-1}(x)}{(1+x^2)^{(n-1)\mu_0}} &= \frac{(1-c)y_1^{n-1}(x)}{(1+x^2)^{(n-1)\mu_1}} + \frac{cy_2^{n-1}(x)}{(1+x^2)^{(n-1)\mu_2}} \\ &= \frac{(1-c)(1+x^2)^{(n-1)(\mu_2-\mu_1)}y_1^{n-1}(x) + cy_2^{n-1}(x)}{(1+x^2)^{(n-1)\mu_2}}, \end{aligned}$$

or equivalently

$$\frac{y^{n-1}(x)}{(1+x^2)^{(n-1)(\mu_0-\mu_1)}} = g(x)^2((1-c)(1+x^2)^{(n-1)(\mu_2-\mu_1)}\tilde{y}_1^{n-1}(x) + c\tilde{y}_2^{n-1}(x)). \quad (10)$$

Since the right-hand side of Equation (10) is a polynomial that is not divisible by $1+x^2$, we must have $\mu_0 \leq \mu_2$, but if $\mu_0 < \mu_2$ then, taking into account that $\tilde{y}_2$ does not divide $1+x^2$ we get a contradiction to (10). In summary, we have $\mu_0 = \mu_2$ and Equation (10) becomes

$$y^{n-1}(x) = g^{n-1}(x)((1-c)(1+x^2)^{(n-1)(\mu_2-\mu_1)}\tilde{y}_1^{n-1}(x) + c\tilde{y}_2^{n-1}(x)),$$

being c an arbitrary constant not equal $c = 0$ (otherwise $y(x) = y_1(x)$) and not equal to $c = 1$ (in which case $y(x) = y_2(x)$). This concludes the proof of the claim.

In view of the claim, if $Y_1(\theta), Y_2(\theta)$ are real trigonometric polynomial solutions of (4) such that $Y_i(\theta) \neq 0$ for $i = 1, 2$, $Y_1(\theta) \neq (\pm 1)^{n-1}Y_2(\theta)$, then any other real trigonometric polynomial solution of Equation (4), say $Y_3(\theta)$, different from them is of the form given in (9) for some appropriate constant c with $c \neq 0$ and $c \neq 1$. In particular

$$(1-c)\tilde{y}_1^{n-1}(x)(1+x^2)^{(n-1)(\mu_2-\mu_1)} + c\tilde{y}_2^{n-1}(x) = (1-c)\tilde{y}_3^{n-1}(x) + c\tilde{y}_2^{n-1},$$

where $\tilde{y}_3 = y_1(1+x^2)^{\mu_2-\mu_1}$ is the $(n-1)$ -th root of a polynomial q . This implies that

$$(\sqrt[n-1]{1-c\tilde{y}_3(x)})^{n-1} + (\sqrt[n-1]{c\tilde{y}_2(x)})^{n-1} = q(x)^{n-1}$$

for some polynomial $q(x)$. It follows from Theorem 3 that this last equation has no non-similar polynomial solutions for $n \geq 4$, and as a consequence, such polynomial q does not exist, implying that such real trigonometric polynomial $Y_3(\theta)$ does not exist unless the polynomial solutions are similar. However, in this last case taking into account that $B_n(\theta) \neq 0$, we conclude that it is also not possible.

In short, among the solution $Y_0(\theta) \equiv 0$, there can only be two sets of solutions $Y_1(\theta), Y_2(\theta)$ if n is odd and $Y_1(\theta), -Y_1(\theta), Y_2(\theta), -Y_2(\theta)$ if n is even.

Now we show that there are equations of the form (4) with this number of real trigonometric polynomial solutions. Note that $y_1(x) = (\pm 1)^{n+1}x$ and $y_2(x) = (\pm 1)^{n+1}x^2$ are polynomial solutions of equation

$$a(x)y^{n-1}y' = b_0(x) + b_n(x)y^n$$

with

$$a(x) = x^n - x^{2n}, \quad b_0(x) = x^{3n-1}, \quad b_n(x) = x^{n-1} - 2x^{2n-1}$$

Therefore, in view of Lemmas 1 and 2 if we set

$$\begin{aligned} A(\theta) &= \frac{2a(x)}{(1+x^2)^n} = \frac{x^n - x^{2n}}{(1+x^2)^{2n-1}}, \\ B_0(\theta) &= \frac{b_0(x)}{(1+x^2)^{2n-1}} = \frac{x^{3n-1}}{(1+x^2)^{2n-1}}, \\ B_n(\theta) &= \frac{b_n(x)(1+x^2) - 2ax}{(1+x^2)^{2n}} = \frac{x^{n-1} - x^{n+1} - 2x^{2n-1}}{(1+x^2)^{2n}}, \end{aligned}$$

then the solutions

$$Y_1(\theta) = (\pm 1)^{n+1} \frac{x}{1+x^2}, \quad Y_2(\theta) = (\pm 1)^{n+1} \frac{x^2}{1+x^2}, \quad Y_0(\theta) \equiv 0$$

are real trigonometric polynomial solutions of Equation (4). This concludes the proof of the theorem. $\square$

3. Conclusions

In this paper, we consider real trigonometric polynomial equations of a certain degree μ that are called *trigonometric polynomial Bernoulli differential equations of degree μ* . The form of the real trigonometric polynomial Bernoulli equation is $A(\theta)y' = B_1(\theta) + B_n(\theta)y^n$ with $n \geq 2$, $B_n(\theta) \neq 0$, $A(\theta), B_1(\theta), B(\theta)$ being trigonometric polynomials of degree at most μ . With the same conditions but with $B_0(\theta)$ instead of $B_1(\theta)$ is considered also the trigonometric polynomials having the form $A(\theta)y^{n-1}y' = B_0(\theta) + B_n(\theta)y^n$. For the first equation, it is shown that for $n \geq 4$, it has at most 3 real trigonometric polynomial solutions when n is even, at most 5 real trigonometric polynomial solutions when n is odd and these upper bounds are reached. For the second equation, it is proved that for $n \geq 3$, it has at most 3 real trigonometric polynomial solutions when n is odd, at most 5 real trigonometric polynomial solutions when n is even and these upper bounds are reached. This paper solves completely for these trigonometric polynomial Bernoulli differential equations the question about the number and degree of trigonometric solutions of planar trigonometric polynomial differential systems in terms of their degrees (similar to the one given by Poincaré for polynomial differential systems).

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References

1. Abul-Dahab, M.; Abul-Ez, M.; Kishka, Z.; Constaes, D. Reverse generalized Bessel matrix differential equation, polynomial solutions, and their properties. *Math. Methods Appl. Sci.* **2015**, *38*, 1005–1013. [\[CrossRef\]](#)
2. Campbell, J.G.; Golomb, M. On the polynomial solutions of a Riccati equation. *Amer. Math. Monthly* **1954**, *61*, 402–404. [\[CrossRef\]](#)
3. Ferragut, T.; Llibre, J. On the polynomial solutions of the polynomial differential equations $yy' = a_0(x) + a_1(x)y + a_2(x)y^2 + \dots + a_n(x)y^n$. *Indian J. Pure Appl. Math.* **2020**, *51*, 217–232. [\[CrossRef\]](#)
4. Giné, J.; Grau, M.; Llibre, J. On the polynomial limit cycles of polynomial differential equations. *Isr. J. Math.* **2011**, *181*, 461–475. [\[CrossRef\]](#)
5. Rainville, E.D. Necessary conditions for polynomial solutions of certain Riccati equations. *Amer. Math. Mon.* **1936**, *43*, 473–476. [\[CrossRef\]](#)
6. Saad, N.; Hall, R.L.; Trenton, V.A. Polynomial solutions for a class of second-order linear differential equations. *Appl. Math. Comput.* **2014**, *226*, 615–634. [\[CrossRef\]](#)
7. Cima, A.; Gasull, A.; Mañosas, F. On the number of polynomial solutions of Bernoulli and Abel polynomial differential equations. *Differ. Equ.* **2017**, *263*, 7099–7122. [\[CrossRef\]](#)
8. Gasull, A.; Llibre, J. Limit cycles for a class of Abel equations. *SIAM J. Math. Anal.* **1990**, *21*, 1235–1244. [\[CrossRef\]](#)
9. Neto, A.L. On the number of solutions of the equation $dx/dt = \sum_{j=0}^n a_j(t)x^j$, $0 \leq t \leq 1$ for which $x(0) = x(1)$. *Invent. Math.* **1980**, *59*, 67–76. [\[CrossRef\]](#)
10. Gasull, A.; Torregrossa, J.; Zhang, X. The number of polynomial solutions of polynomial Riccati equations. *Differ. Equ.* **2016**, *261*, 5071–5093. [\[CrossRef\]](#)
11. Valls, C. Trigonometric polynomial solutions of equivariant trigonometric polynomial Abel differential equations. *Electron. Differ. Equ.* **2017**, *261*, 1–9.
12. de Borat, M. Another generalization of Mason’s ABC Theorem. *arXiv* **2009**, arXiv:0707.043402.2009.