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Application of Generalized Logistic Function to Travelling Wave Solutions for a Class of Nonlinear Evolution Equations

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Abstract: The generalized Logistic function that solves a first-order nonlinear ODE with an arbitrary positive power term of the dependent variable is introduced in this paper, by means of which the traveling wave solutions of a class of nonlinear evolution equations, including the generalized Fisher equation, the generalized Nagumo equation, the generalized Burgers-Fisher equation, the generalized Gardner equation, the generalized KdV-Burgers equation, and the generalized Benney equation, are obtained successfully. In these particular cases, traveling wave solutions of several important model PDEs in mathematical physics are also discovered.

Keywords: generalized Logistic function; traveling wave solution; generalized Fisher equation; generalized Nagumo equation; generalized Burgers-Fisher equation; generalized Gardner equation; generalized KdV-Burgers equation; generalized Benney equation

MSC: 35Q51



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1. Introduction

In the present paper, we investigate a class of nonlinear evolution equations with a nonlinear term of any order in the following:

The generalized Fisher equation [1] (G-Fisher for short)

$$u_t - \alpha u_{xx} = \beta u(1 - u^n). \quad (1)$$

The generalized Fitzhugh-Nagumo equation [2] (G-Fitzhugh-Nagumo):

$$u_t - \alpha u_{xx} = \beta u(1 - u^n)(u^n - a). \quad (2)$$

The generalized Burgers-Fisher equation [3] (G-Burgers-Fisher):

$$u_t - \alpha u_{xx} + (n + 1)u^n u_x = \beta u(1 - u^n). \quad (3)$$

The generalized Gardner equation (G-Gardner):

$$u_t + (\alpha u^n + \beta u^{2n})u_x - \delta u_{xxx} = 0. \quad (4)$$

The generalized KdV-Burgers equation [4,5] (G-KdV-Burgers):

$$u_t - \gamma u_{xx} + \delta u_{xxx} = (n + 1)u^n u_x. \quad (5)$$

The generalized Kuramoto-Sivashinsky equation [6] (G-KS) or the generalized Benney equation (G-Benney):

$$u_t + pu_{xx} + qu_{xxx} + ru_{xxxx} + (n + 1)u^n u_x = 0. \quad (6)$$

For Equations (2)–(6) above, n is an arbitrary positive integer. In particular, when $n = 1$, Equations (1)–(6) become the important nonlinear model equations in mathematical physics. To be more precise, Equations (1)–(6) become the famous Fisher equation [7], Fitzhugh-Nagumo equation [8], Burgers-Fisher equation [9], Gardner equation [10], KdV-Burgers equation [11–13] and Benney equation [14], respectively. Since Equations (1)–(6) are the generalized forms of the important model equations in mathematical physics, their investigation is significant both in mathematics and physics.

In recent years, various methods to derive exact solutions of nonlinear PDEs have been presented and developed, such as Hirota's bilinear operators [15], the tanh-function expansion and its extension [16,17], the Jacobi elliptic function method [18], the Exp-function method [19], the F-expansion method [20,21], the (G'/G) -expansion method [22–24], the simplest equation method [6,25], sub-ODE method [26], and so on. As far as the Sub-ODE method is concerned, its core idea is to construct the traveling wave solutions of complex partial differential equations with simple and solvable ordinary differential equations. In dealing with a nonlinear differential equation, especially a partial differential equation with nonlinear terms of any order, finding a suitable Sub-ODE is still the most critical step in the PDE solution process.

In the present paper, we introduce a first-order nonlinear ODE as follows:

$$F'(\xi) = -F^{n+1}(\xi) + F(\xi), n = 1, 2, \dots, \quad (7)$$

which admits solution:

$$F(\xi) = \left(\frac{1}{1 + e^{-n\xi}} \right)^{\frac{1}{n}}, \quad (8)$$

This can be proved directly. For convenience sake, hereafter, function (8) is called the generalized Logistic function, and Equation (7) is the generalized Logistic equation.

In this particular case, when $n = 1$, function (8) becomes the standard Logistic function, and Equation (7) is the Logistic equation. It is well-known that the Logistic function has applications in many fields, including artificial neural networks, bio-mathematics, ecology, mathematical psychology, chemistry, demography, probability and statistics, and so on.

The main objective of the paper is to look for traveling wave solutions of Equations (1)–(6) using the generalized Logistic function (8) and Equation (7). The results obtained by the proposed method will be described in detail in Sections 2–7. In Section 8, the conclusion is made briefly.

2. Traveling Wave Solution of G-Fisher Equation (1)

Let us begin with Equation (1) and look for its traveling wave solution in the form

$$u(x, t) = u(\xi), \xi = k(x - Vt), \quad (9)$$

where k and V are constants to be determined. Substituting (9) into Equation (1) yields an ODE for $u = u(\xi)$,

$$-kVu' - \alpha k^2 u'' + \beta u^{n+1} - \beta u = 0. \quad (10)$$

Considering the homogeneous balance between the highest order term u'' and the nonhomogeneous term u^{n+1} via the generalized Logistic Equation (7) ($m + 2n = m(n + 1) \Rightarrow m = 2$), we can suppose that the solution of Equation (10) is of the form:

$$u = AF^2, \quad (11)$$

where A be a constant to be determined later, $F = F(\xi)$ is given by (8), which solves Equation (7).

From (11) and using Equation (7), it is derived that:

$$u' = -2AF^{n+2} + 2AF^2, \quad (12)$$

$$u'' = 2A(n+2)F^{2n+2} - 2A(n+4)F^{n+2} + 4AF^2. \quad (13)$$

Substituting (11–13) into the left-hand side of Equation (10), collecting all terms with $F^i (i = 2n+2, n+2, 2)$ together, and equating the coefficients of $F^i (i = 2n+2, n+2, 2)$ to zero, yields a set of algebraic equations for A , k and V as follows:

$$\begin{aligned} F^{2n+2}: & -2(n+2)A\alpha k^2 + \beta A^{n+1} = 0, \\ F^{n+2}: & 2AkV + 2(n+4)A\alpha k^2 = 0, \\ F^2: & -2AkV - 4A\alpha k^2 - A\beta = 0. \end{aligned}$$

Solving the algebraic equations above yields:

$$A = \left(\frac{2(n+2)\alpha}{\beta} \right)^{\frac{1}{n}}, k = \pm \left(\frac{\beta}{2(n+2)\alpha} \right)^{\frac{1}{2}}, V = \mp(n+4) \sqrt{\frac{\alpha\beta}{2(n+2)}}. \quad (14)$$

Substituting (8) and (14) with (9) into (11), we have traveling wave solutions of Equation (1) as follows

$$u(x, t) = \left(\frac{2(n+2)\alpha}{\beta} \right)^{\frac{1}{n}} \left(\frac{1}{1 + e^{-n\xi}} \right)^{\frac{2}{n}}, \quad (15a)$$

where

$$\xi = \pm \left(\frac{\beta}{2(n+2)\alpha} \right)^{\frac{1}{2}} \left(x \pm (n+4)t \sqrt{\frac{\alpha\beta}{2(n+2)}} \right). \quad (15b)$$

In particular, when $n = 1$, (15) becomes:

$$u(x, t) = \frac{6\alpha}{\beta} \left(\frac{1}{1 + e^{-\xi}} \right)^2, \xi = \pm \left(\frac{\beta}{6\alpha} \right)^{\frac{1}{2}} \left(x \pm 5\sqrt{\frac{\alpha\beta}{6}}t \right), \quad (16)$$

which is the traveling wave solution of Fisher's equation:

$$u_t - \alpha u_{xx} = \beta u(1 - u). \quad (17)$$

When $n = 2$, (15) becomes:

$$u(x, t) = \pm \sqrt{\frac{8\alpha}{\beta}} \left(\frac{1}{1 + e^{-2\xi}} \right), \xi = \pm \left(\frac{\beta}{8\alpha} \right)^{\frac{1}{2}} \left(x \pm 6\sqrt{\frac{\alpha\beta}{8}}t \right), \quad (18)$$

which is the traveling wave solution of Huxley's equation:

$$u_t - \alpha u_{xx} = \beta u(1 - u^2). \quad (19)$$

For the travelling wave solutions of Equation (17), when $n = 1$, $\alpha = -1.8$, the corresponding graphs of $\beta = -4.4$, -2 and -0.5 are given as follows (Figure 1):

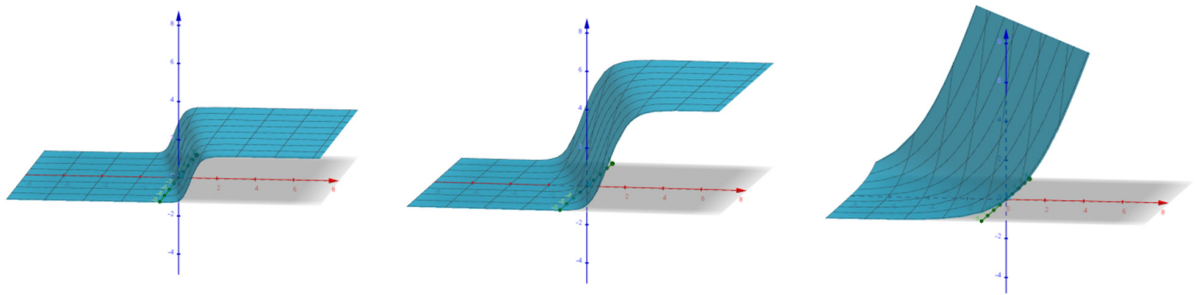


Figure 1. When $n = 1$, fix $\alpha = -1.8$ and take β as -4.4 , -2 and -0.5 respectively.

For the travelling wave solutions of Equation (17), when $n = 1$, $\beta = -1.5$, the corresponding graphs of $\alpha = -4.2$, -1.5 and -0.5 are given as follows (Figure 2):

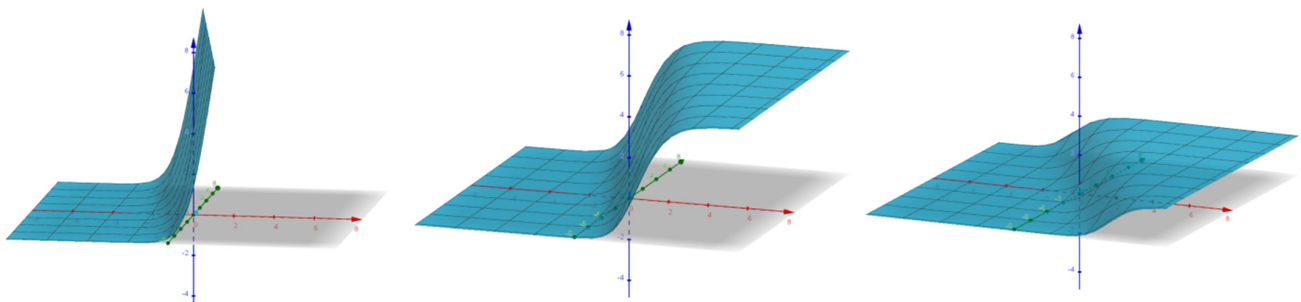


Figure 2. When $n = 1$, fix $\beta = -1.5$ and take α as -4.2 , -1.5 and -0.5 respectively.

For the travelling wave solutions of Equation (19), when $n = 2$, $\alpha = -3$, the corresponding graphs of $\beta = -4.6$, -1 and -0.3 are given as follows (Figure 3):

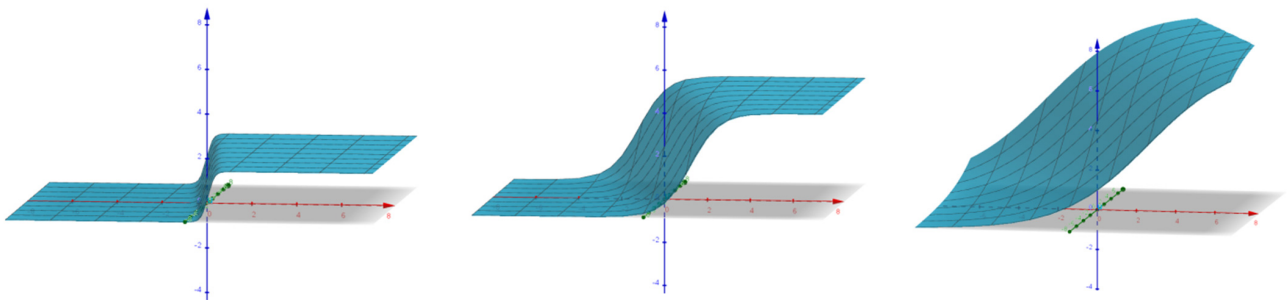


Figure 3. When $n = 2$, fix $\alpha = -3$ and take β as -4.6 , -1 and -0.3 respectively.

For the travelling wave solutions of Equation (19), when $n = 2$, $\beta = -2$, the corresponding graphs of $\alpha = -5$, -1.5 and -0.3 are given as follows (Figure 4):

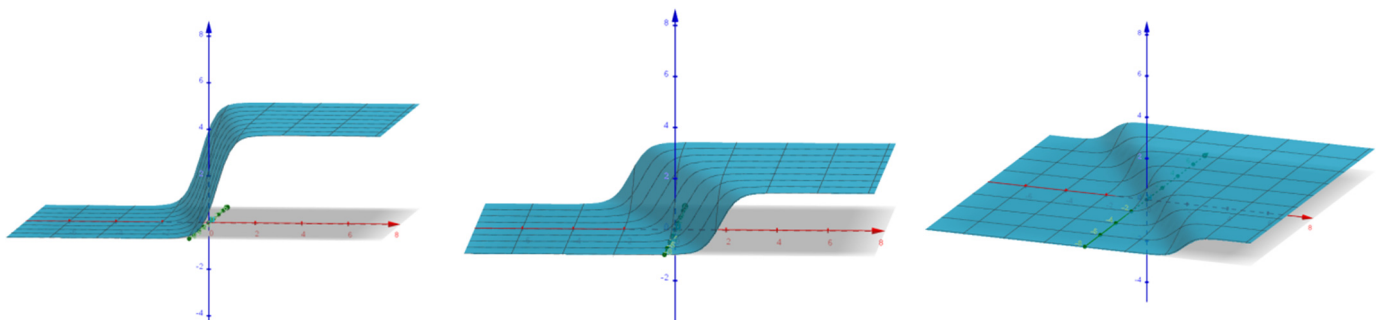


Figure 4. When $n = 2$, fix $\beta = -2$ and take α as -5 , -1.5 and -0.3 respectively.

3. Traveling Wave Solution of G-Fitzhugh-Nagumo Equation (2)

To look for the traveling wave solution of Equation (2), we suppose that:

$$u(x, t) = u(\xi), \quad \xi = k(x - Vt), \quad (20)$$

where k and V are constants to be determined.

The traveling wave variables (20) permit us to reduce Equation (2) into an ODE for $u = u(\xi)$:

$$-kVu' - \alpha k^2 u'' + \beta [u^{2n+1} - (a+1)u^{n+1} + au] = 0. \quad (21)$$

Considering the homogeneous balance between the highest order term u'' and the nonhomogeneous term u^{2n+1} via Equation (7) ($m + 2n = m(2n + 1) \Rightarrow m = 1$), we can suppose that the solution of Equation (21) is of the form:

$$u = AF, \quad (22)$$

where A be a constant to be determined later, $F = F(\xi)$ is given by (8), which solves Equation (7).

From (22) and using Equation (7), it is derived that:

$$u' = -AF^{n+1} + AF, \quad (23)$$

$$u'' = A(n+1)F^{2n+1} - A(n+2)F^{n+1} + AF. \quad (24)$$

Substituting (22)–(24) into the left-hand side of Equation (21), collecting all terms with F^i ($i = 2n + 1, n + 1, 1$) together, and equating the coefficients of F^i ($i = 2n + 1, n + 1, 1$) to zero, yields a set of algebraic equations for A, k and V as follows:

$$\begin{aligned} F^{2n+1}: & \quad -\alpha k^2(n+1)A + \beta A^{2n+1} = 0, \\ F^{n+1}: & \quad kVA + (n+2)\alpha k^2 A - \beta(a+1)A^{n+1} = 0, \\ F: & \quad -kVA - \alpha k^2 A + \beta aA = 0. \end{aligned}$$

Solving the algebraic equations above, we have four solutions:

$$A_1 = 1, k_1 = \sqrt{\frac{\beta}{(n+1)\alpha}}, V_1 = \left(\frac{a}{2} - \frac{n+3}{2(n+1)} \pm \frac{\beta(a+1)}{2}\right) \sqrt{(n+1)\alpha\beta}, \quad (25a)$$

$$A_2 = a^{\frac{1}{n}}, k_2 = a \sqrt{\frac{\beta}{(n+1)\alpha}}, V_2 = \left(\frac{1}{2} - \frac{n+3}{2(n+1)} \pm \frac{\beta(a+1)}{2}\right) \sqrt{(n+1)\alpha\beta}, \quad (25b)$$

$$A_3 = a^{\frac{1}{n}}, k_3 = -a \sqrt{\frac{\beta}{(n+1)\alpha}}, V_3 = \left(-\frac{1}{2} + \frac{n+3}{2(n+1)} \pm \frac{\beta(a+1)}{2}\right) \sqrt{(n+1)\alpha\beta}, \quad (25c)$$

$$A_4 = 1, k_4 = -\sqrt{\frac{\beta}{(n+1)\alpha}}, V_4 = \left(-\frac{a}{2} + \frac{n+3}{2(n+1)} \pm \frac{\beta(a+1)}{2}\right) \sqrt{(n+1)\alpha\beta}. \quad (25d)$$

Substituting (8) and (25) with (20) into (22), we have traveling wave solutions of Equation (2) as follows:

$$u(x, t) = \left(\frac{1}{1+e^{-n\xi}}\right)^{\frac{1}{n}}, \quad (26a)$$

$$\xi = \sqrt{\frac{\beta}{(n+1)\alpha}} \left[x - \left(\frac{a}{2} - \frac{n+3}{2(n+1)} \pm \frac{\beta(a+1)}{2}\right) \sqrt{(n+1)\alpha\beta} t \right].$$

$$u(x, t) = a^{\frac{1}{n}} \left(\frac{1}{1 + e^{-n\zeta}} \right)^{\frac{1}{n}}, \quad (26b)$$

$$\zeta = a \sqrt{\frac{\beta}{(n+1)\alpha}} \left[x - \left(\frac{1}{2} - \frac{n+3}{2(n+1)} \pm \frac{\beta(a+1)}{2} \right) \sqrt{(n+1)\alpha\beta t} \right].$$

$$u(x, t) = a^{\frac{1}{n}} \left(\frac{1}{1 + e^{-n\zeta}} \right)^{\frac{1}{n}}, \quad (26c)$$

$$\zeta = -a \sqrt{\frac{\beta}{(n+1)\alpha}} \left[x - \left(\frac{n+3}{2(n+1)} - \frac{1}{2} \pm \frac{\beta(a+1)}{2} \right) \sqrt{(n+1)\alpha\beta t} \right].$$

$$u(x, t) = \left(\frac{1}{1 + e^{-n\zeta}} \right)^{\frac{1}{n}}, \quad (26d)$$

$$\zeta = -\sqrt{\frac{\beta}{(n+1)\alpha}} \left[x - \left(\frac{n+3}{2(n+1)} - \frac{a}{2} \pm \frac{\beta(a+1)}{2} \right) \sqrt{(n+1)\alpha\beta t} \right].$$

In particular, when $n = 1$, (26) becomes:

$$u(x, t) = \frac{1}{1 + e^{-\zeta}}, \quad \zeta = \sqrt{\frac{\beta}{2\alpha}} \left[x - \left(\frac{a}{2} - 1 \pm \frac{\beta(a+1)}{2} \right) \sqrt{2\alpha\beta t} \right]. \quad (27a)$$

$$u(x, t) = \frac{a}{1 + e^{-\zeta}}, \quad \zeta = a\sqrt{2} \left[x - \left(\frac{1}{2} - a \pm \frac{\beta(a+1)}{2} \right) \sqrt{(n+1)\alpha\beta t} \right]. \quad (27b)$$

$$u(x, t) = \frac{a}{1 + e^{-n\zeta}}, \quad \zeta = -a\sqrt{\frac{\beta}{2\alpha}} \left[x - \left(a - \frac{1}{2} \pm \frac{\beta(a+1)}{2} \right) \sqrt{(n+1)\alpha\beta t} \right]. \quad (27c)$$

$$u(x, t) = \frac{1}{1 + e^{-\zeta}}, \quad \zeta = -\sqrt{\frac{\beta}{2\alpha}} \left[x - \left(1 - \frac{a}{2} \pm \frac{\beta(a+1)}{2} \right) \sqrt{2\alpha\beta t} \right]. \quad (27d)$$

Each one of the expressions (27) is the traveling wave solution of the Nagumo equation:

$$u_t - \alpha u_{xx} = \beta u(1 - u)(u - a). \quad (28)$$

4. Traveling Wave Solution of G-Burgers-Fisher Equation (3)

To look for the traveling wave solution of Equation (3), we suppose that:

$$u(x, t) = u(\zeta), \quad \zeta = k(x - Vt), \quad (29)$$

where k and V are constants to be determined.

The traveling wave variables (29) permit us to reduce Equation (3) into an ODE for $u = u(\zeta)$:

$$-kVu' - \alpha k^2 u'' - (n+1)ku^n u' - \beta u^{n+1} + \beta u = 0. \quad (30)$$

Considering the homogeneous balance between the highest order term u'' and the nonhomogeneous term $u^n u'$ via the generalized Logistic Equation (7) ($m + 2n = m(n+1) + n \Rightarrow m = 1$), we can suppose that the solution of Equation (30) is of the form:

$$u = AF, \quad (31)$$

where A be a constant to be determined later, $F = F(\zeta)$ is given by (8), which solves Equation (7).

From (31) and using Equation (7), it is derived that:

$$\begin{aligned} u' &= -AF^{n+1} + AF, \\ u'' &= A(n+1)F^{2n+1} - A(n+2)F^{n+1} + AF, \\ u^n u' &= -A^{n+1}F^{2n+1} + A^{n+1}F^{n+1}, \\ u^{n+1} &= A^{n+1}F^{n+1}. \end{aligned} \quad (32)$$

Substituting (31) and (32) into the left-hand side of Equation (30), collecting all terms with $F^i (i = 2n + 1, n + 1, 1)$ together, and equating the coefficients of $F^i (i = 2n + 1, n + 1, 1)$ to zero, yields a set of algebraic equations for A , k and V as follows:

$$\begin{aligned} F^{2n+1}: & -\alpha k^2(n+1)A + k(n+1)A^{n+1} = 0, \\ F^{n+1}: & kVA + (n+2)\alpha k^2A - (n+1)kA^{n+1} - \beta A^{n+1} = 0, \\ F: & -kVA - \alpha k^2A + \beta A = 0. \end{aligned}$$

Solving the algebraic equations above yields:

$$A = 1, k = \frac{1}{\alpha}, V_4 = \alpha\beta - 1. \quad (33)$$

Substituting (8) and (33) with (29) into (31), we have a traveling wave solution of Equation (3) as follows:

$$u(x, t) = \left(\frac{1}{1 + e^{-n\xi}} \right)^{\frac{1}{n}}, \xi = \frac{1}{\alpha}[x - (\alpha\beta - 1)t]. \quad (34)$$

In particular, when $n = 1$, (34) becomes:

$$u(x, t) = \frac{1}{1 + e^{-\xi}}, \xi = \frac{1}{\alpha}[x - (\alpha\beta - 1)t], \quad (35)$$

which is the traveling wave solution of Burgers-Fisher's equation:

$$u_t - \alpha u_{xx} + 2uu_x = \beta u(1 - u). \quad (36)$$

When $n = 2$, (34) becomes:

$$u(x, t) = \frac{1}{\sqrt{1 + e^{-2\xi}}}, \xi = \frac{1}{\alpha}[x - (\alpha\beta - 1)t], \quad (37)$$

which is the traveling wave solutions of modified Burgers-Huxley's equation:

$$u_t - \alpha u_{xx} + 3u^2u_x = \beta u(1 - u^2). \quad (38)$$

5. Traveling Wave Solution of G-Gardner Equation (4)

To look for the traveling wave solution of Equation (4), we suppose that

$$u(x, t) = u(\xi), \xi = k(x - Vt), \quad (39)$$

where k and V are constants to be determined.

The traveling wave variables (39) change Equation (4) into an ODE for $u = u(\xi)$:

$$-kVu' + k(\alpha u^n + \beta u^{2n})u' - \delta k^3 u''' = 0.$$

Integrating it with respect to ξ once and taking the constant of integration to zero, and dividing by k , yields:

$$-Vu + \frac{\alpha u^{n+1}}{n+1} + \frac{\beta u^{2n+1}}{2n+1} - \delta k^2 u'' = 0. \quad (40)$$

Considering the homogeneous balance between the highest order term u'' and the nonhomogeneous term u^{2n+1} via Equation (7) ($m + 2n = m(2n + 1) \Rightarrow m = 1$), we can suppose that the solution of Equation (40) is of the form:

$$u = AF(\xi), \quad (41)$$

where A be a constant to be determined later, $F = F(\xi)$ is given by function (8), which solves Equation (7).

From (41) and using Equation (7), it is derived that:

$$\begin{aligned} u' &= -AF^{n+1} + AF, \\ u'' &= A(n+1)F^{2n+1} - A(n+2)F^{n+1} + AF, \\ u^{n+1} &= A^{n+1}F^{n+1}, \quad u^{2n+1} = A^{2n+1}F^{2n+1}. \end{aligned} \quad (42)$$

Substituting (41) and (42) into the left-hand side of Equation (40), collecting all terms with $F^i (i = 2n+1, n+1, 1)$ together, and equating the coefficients of $F^i (i = 2n+1, n+1, 1)$ to zero, yields a set of algebraic equations for A, k and V as follows:

$$\begin{aligned} F^{2n+1} : \quad & \frac{\beta A^{2n+1}}{2n+1} - \delta k^2(n+1)A = 0, \\ F^{n+1} : \quad & \frac{\alpha A^{n+1}}{n+1} + \delta k^2(n+2)\alpha k^2 A = 0, \\ F : \quad & -VA - \delta k^2 A = 0. \end{aligned}$$

Solving the algebraic equations above yields:

$$A_1 = \left[\frac{(2n+1)^2 \alpha^2}{(n+2)^2 \beta^2} \right]^{\frac{1}{2n}}, \quad k = \pm \frac{\alpha}{n+2} \sqrt{\frac{2n+1}{\beta \delta (n+1)}}, \quad V = -\frac{\alpha^2}{\beta} \cdot \frac{2n+1}{(n+2)^2 (n+1)}. \quad (43)$$

Substituting (8) and (43) with (39) into (41), we have traveling wave solutions of Equation (4) as follows:

$$u(x, t) = \left[\frac{(2n+1)^2 \alpha^2}{(n+2)^2 \beta^2} \right]^{\frac{1}{2n}} \left(\frac{1}{1 + e^{-n\xi}} \right)^{\frac{1}{n}}, \quad (44a)$$

$$\xi = \pm \frac{\alpha}{n+2} \sqrt{\frac{2n+1}{\beta \delta (n+1)}} \left[x + \frac{\alpha^2 (2n+1)}{\beta (n+2)^2 (n+1)} t \right]. \quad (44b)$$

In particular, when $n = 1$, (44) becomes:

$$u(x, t) = \pm \frac{\alpha}{\beta} \left(\frac{1}{1 + e^{-n\xi}} \right), \quad \xi = \pm \frac{\alpha}{3} \sqrt{\frac{3}{2\beta\delta}} \left(x + \frac{\alpha^2}{6\beta} t \right), \quad (45)$$

which is the traveling wave solution of Gardner's Equation (10):

$$u_t + (\alpha u + \beta u^2) - \delta u_{xxx} = 0.$$

6. Traveling Wave Solution of G-KdV-Burgers Equation (5)

To look for the travelling wave solution of Equation (5), we suppose that:

$$u(x, t) = u(\xi), \quad \xi = k(x - Vt), \quad (46)$$

where k and V are constants to be determined.

The traveling wave variables (46) permit us to reduce Equation (5) into an ODE for $u = u(\xi)$:

$$-kVu' - \gamma k^2 u'' + \delta k^3 u''' - (n+1)ku^n u' = 0.$$

Integrating it with respect to ξ once, taking the constant of integration to zero, and dividing by k , yields:

$$-Vu - \gamma ku' + \delta k^2 u'' - u^{n+1} = 0. \quad (47)$$

Considering the homogeneous balance between the highest order term u'' and the non-homogeneous term u^{n+1} via Equation (7) ($m + 2n = m(n + 1) \Rightarrow m = 2$), we can suppose that the solution of Equation (47) is of the form:

$$u = AF^2(\xi), \quad (48)$$

where A be a constant to be determined later, $F = F(\xi)$ is given by function (8), which solves Equation (7).

From (48) and using Equation (7), it is derived that:

$$u' = -2AF^{n+2} + 2AF^2, \quad (49)$$

$$u'' = 2A(n+2)F^{2n+2} - 2A(n+4)F^{n+2} + 4AF^2. \quad (50)$$

Substituting (48)–(50) into the left-hand side of Equation (47), collecting all terms with F^i ($i = 2n + 2, n + 2, 2$) together, and equating the coefficients of F^i ($i = 2n + 2, n + 2, 2$) to zero, yields a set of algebraic equations for A, k and V as follows:

$$\begin{aligned} F^{2n+2}: \quad & 2\delta k^2(n+2)A - A^{n+1} = 0, \\ F^{n+2}: \quad & 2k\gamma A - 2(n+4)\delta k^2 A = 0, \\ F^2: \quad & -VA - 2k\gamma A + 4\delta k^2 A = 0. \end{aligned}$$

Solving the algebraic equations above yields:

$$A = \left[\frac{2(n+2)\gamma^2}{(n+4)^2\delta} \right]^{\frac{1}{n}}, \quad k = \frac{\gamma}{(n+4)\delta}, \quad V = -\frac{2(n+2)\gamma^2}{(n+4)^2\delta}. \quad (51)$$

Substituting (8) and (51) with (46) into (48), we have a traveling wave solution of Equation (5) as follows:

$$u(x, t) = \left[\frac{2(n+2)\gamma^2}{(n+4)^2\delta} \right]^{\frac{1}{n}} \left(\frac{1}{1 + e^{-n\xi}} \right)^{\frac{2}{n}}, \quad (52a)$$

$$\xi = \frac{\gamma}{(n+4)\delta} \left[x + \frac{2(n+2)\gamma^2}{(n+4)^2\delta} t \right]. \quad (52b)$$

In particular, when $n = 1$, (52) becomes:

$$u(x, t) = \frac{6\gamma^2}{25\delta} \left(\frac{1}{1 + e^{-\xi}} \right)^2, \quad \xi = \frac{\gamma}{5\delta} \left(x + \frac{6\gamma^2}{25\delta} t \right), \quad (53)$$

which is the traveling wave solution of the KdV-Burgers equation:

$$u_t + \gamma u_{xx} + \delta u_{xxx} = 2uu_x. \quad (54)$$

In particular, when $n = 2$, (52) becomes:

$$u(x, t) = \sqrt{\frac{2\gamma^2}{9\delta}} \frac{1}{1 + e^{-2\xi}}, \quad \xi = \frac{\gamma}{6\delta} \left(x + \frac{2\gamma^2}{9\delta} t \right), \quad (55)$$

which is the traveling wave solution of the modified KdV-Burgers equation:

$$u_t + \gamma u_{xx} + \delta u_{xxx} = 3u^2 u_x. \quad (56)$$

7. Travelling Wave Solution of G-KS Equation (6)

In order to look for the traveling wave solution of Equation (6), we suppose that:

$$u(x, t) = u(\xi), \quad \xi = k(x - Vt), \quad (57)$$

where k and V are constants to be determined.

The traveling wave variables (57) permit us to reduce Equation (6) into an ODE for $u = u(\xi)$:

$$-kVu' + pk^2u'' + qk^3u''' + \gamma k^4u^{(4)} + (n+1)ku''u' = 0.$$

Integrating it with respect to ξ once, taking the constant of integration to zero, and dividing by k , yields:

$$-Vu + pku' + qk^2u'' + \gamma k^3u''' + u^{n+1} = 0. \quad (58)$$

Considering the homogeneous balance between the highest order term u''' and the non-homogeneous term u^{n+1} via Equation (7) ($m + 3n = m(n+1) \Rightarrow m = 3$), we can suppose that the solution of Equation (58) is of the form:

$$u = AF^3(\xi), \quad (59)$$

where A be a constant to be determined later, $F = F(\xi)$ is given by function (8), which solves Equation (7).

From (59) and using Equation (7), it is derived that:

$$u' = -3AF^{n+3} + 3AF^3, \quad (60)$$

$$u'' = 3(n+3)AF^{2n+3} - 3(n+6)AF^{n+3} + 9AF^3, \quad (61)$$

$$u''' = -3(n+3)(2n+3)AF^{3n+3} + 9(n+3)^2AF^{2n+3} - 3[(n+3)(n+6) + 9]AF^{n+3} + 27AF^3. \quad (62)$$

Substituting (59)–(62) into the left-hand side of Equation (58), collecting all terms with F^i ($i = 3n+3, 2n+3, n+3, 3$) together and equating the coefficients of F^i ($i = 3n+3, 2n+3, n+3, 3$) to zero, yields a set of algebraic equations for A , k and V as follows:

$$\begin{aligned} F^{3n+3}: & \quad A^{n+1} - 3(n+3)(2n+3)\gamma k^3 A = 0, \\ F^{2n+3}: & \quad 3(n+3)qk^2 A + 9(n+3)^2\gamma k^3 = 0, \\ F^{n+3}: & \quad -3pA - 3(n+6)qk^2 A - 3[(n+3)(n+6) + 9]\gamma k^3 A = 0, \\ F^3: & \quad -VA + 3pkA + 9qk^2 A + 27\gamma k^3 A = 0. \end{aligned}$$

Solving the algebraic equations above yields:

$$A = \left[\frac{(2n+3)q^3}{9(n+3)^2\gamma^2} \right]^{\frac{1}{n}}, \quad k = -\frac{q}{3(n+3)\gamma}, \quad V = \frac{[9 - 2(n+6)]q^3}{9(n+3)^2\gamma^2}, \quad (63)$$

provided that p, q and γ satisfy an additional condition:

$$9(n+3)^2p\gamma + [9 - 2(n+3)(n+6)]q^2 = 0. \quad (64)$$

Substituting (8) and (63) with (57) into (59), we have a traveling wave solution of Equation (6) as follows:

$$u(x, t) = \left[\frac{(2n+3)q^3}{9(n+3)^2\gamma^2} \right]^{\frac{1}{n}} \left(\frac{1}{1 + e^{-n\xi}} \right)^{\frac{3}{n}}, \quad (65a)$$

$$\xi = -\frac{q}{3(n+3)\gamma} \left\{ x - \frac{[9-2(n+6)]q^3}{9(n+3)^2\gamma^2} t \right\}, \quad (65b)$$

provided that p, q and γ satisfy the additional condition (64).

In particular, when $n = 1$, (65) becomes:

$$u(x, t) = \frac{5q^3}{144\gamma^2} \left(\frac{1}{1 + e^{-\xi}} \right)^3, \quad \xi = -\frac{q}{12\gamma} \left(x + \frac{5q^3}{144\gamma^2} t \right), \quad (66)$$

which is the traveling wave solution of the Benney equation (14):

$$u_t + pu_{xx} + qu_{xxx} + \gamma u_{xxxx} + 2uu_x = 0, \quad (67)$$

provided that p, q and γ satisfy the additional condition:

$$144p\gamma = 47q^2, \quad (68)$$

which is obtained from (64) when $n = 1$.

When $n = 2$, (65) becomes:

$$u(x, t) = \sqrt{\frac{7q^3}{215\gamma^2}} \left(\frac{1}{1 + e^{-2\xi}} \right)^{\frac{3}{2}}, \quad \xi = -\frac{q}{15\gamma} \left(x + \frac{7q^3}{215\gamma^2} t \right), \quad (69)$$

which is the traveling wave solution of the modified Benney equation:

$$u_t + pu_{xx} + qu_{xxx} + \gamma u_{xxxx} + 3u^2u_x = 0, \quad (70)$$

provided that p, q and γ satisfy the additional condition:

$$225p\gamma = 71q^2, \quad (71)$$

which is obtained from (64) when $n = 2$.

When $n = 3$, (65) becomes:

$$u(x, t) = \frac{q}{\sqrt[3]{36\gamma^2}} \left(\frac{1}{1 + e^{-3\xi}} \right), \quad \xi = -\frac{q}{18\gamma} \left(x + \frac{q^3}{36\gamma^2} t \right), \quad (72)$$

which is the traveling wave solution of the G-Benney equation:

$$u_t + pu_{xx} + qu_{xxx} + \gamma u_{xxxx} + 4u^3u_x = 0, \quad (73)$$

provided that p, q and γ satisfy the additional condition:

$$334p\gamma = 99q^2, \quad (74)$$

which is obtained from (64) when $n = 3$.

8. Conclusions

We have seen that the traveling wave solutions of six nonlinear evolution equations with nonlinear terms of any order are expressed by the generalized Logistic function introduced in this paper. To our knowledge, the results expressed by the generalized Logistic function and the method used in this paper have not been seen before in the literature.

The procedure for finding traveling wave solutions of mentioned nonlinear equations is simple and routine:

First, carrying on the traveling wave reduction of PDE to ODE, suppose that:

$$u(x, t) = u(\xi), \xi = k(x - Vt),$$

Second, considering the homogeneous balance between the highest order derivatives and highest order nonlinear term appearing in the reduction ODE to get integers m , then suppose $u = AF^m$, where $F = F(\xi)$ is the generalized Logistic function (8);

Third, substituting $u = AF^m$ into the reduction ODE and using the generalized Logistic Equation (7), the reduction ODE is converted into a polynomial in $F(\xi)$, setting the coefficients of the polynomial to zero, a set of algebraic equations for A, k and V can be derived;

Fourth, after solving the algebraic equations to get A, k and V , then the traveling wave solutions of mentioned nonlinear equations can be easily obtained.

Since the generalized Logistic function is related to the hyperbolic tangent function, that is:

$$F(\xi) = \left(\frac{1}{1 + e^{-n\xi}} \right)^{\frac{1}{n}} = \left(\frac{1}{2} + \frac{1}{2} \tanh \frac{n\xi}{2} \right)^{\frac{1}{n}}, \quad (75)$$

by which the traveling wave solutions obtained in the paper can be expressed by the hyperbolic tangent function.

In this particular case, when $n = 1$, Equation (7) and function (8) become the Logistic equation:

$$F' = -F^2 + F, \quad (76)$$

and standard Logistic function:

$$F(\xi) = \frac{1}{1 + e^{-\xi}}, \quad (77)$$

respectively. Utilizing Equation (76) and its solution (77), traveling wave solutions of many nonlinear evolution equations such as the Burgers, KdV, KP, Boussinesq, Klein-Gordon, and so on, can be expressed by a polynomial in Logistic function (77) and the researching results of these equations will appear elsewhere.

It is worth mentioning that in Reference [27], a fourth-order nonlinear differential equation is solved by using the Logistic function (77). The analytic solution in exponential form is obtained successfully. But these solutions are obtained when m and n take different specific parameters respectively. In this paper, the traveling wave solutions of a class of nonlinear evolution equations are obtained using the generalized Logistic function (75). Relatively speaking, the traveling wave solutions of the latter with a nonlinear term of any order are more abundant and general. Moreover, more nonlinear evolution equations with two nonlinear terms of any order can be obtained using the generalized Logistic function.

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