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# Construction of Dual Optimal Bidirectional Double-Loop Networks for Optimal Routing †

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**Abstract:** Bidirectional double-loop networks (BDLNs) are widely used in computer networks for their simplicity, symmetry and scalability. One common way to improve their performance is to decrease the diameter and average distance. Attempts have been made to find BDLNs with minimal diameters; however, such BDLNs will not necessarily have the minimum average distance. In this paper, we construct dual optimal BDLNs with minimum diameters and average distances using an efficient method based on coordinate embedding and transforming. First, we get the lower bounds of both the diameter and average distance by embedding a BDLN into Cartesian coordinates. Then, we construct tight optimal BDLNs that provide the aforementioned lower bounds based on an embedding graph. On the basis of node distribution regularity in tight optimal BDLNs, we construct dual optimal BDLNs with minimum diameters and average distances for any number of nodes. Finally, we present on-demand optimal message routing algorithms for the dual optimal BDLNs that we have constructed. The presented algorithms do not require routing tables and are efficient, requiring little computation.

**Keywords:** bidirectional double-loop networks; diameter; average distance; lower bound; optimal routing

**MSC:** 94C15



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## 1. Introduction

In recent years, data centers [1,2], cloud computing [3,4], edge computing [5,6], and the Internet of Things [7] have become research hotspots. The enhancement of computing power no longer simply relies on improving the processing capacity of a single computer, but rather, on increasing the number of computing devices to form a large-scale processing system. For systems employing a large number of computing devices, the question of what kind of interconnection structure is used to obtain the best performance is a research field of mathematics and computer science.

With the further expansion of network scale, to improve network performance, we not only need to consider the communication between nodes, but also the cost of the construction of the network. For networks with fixed nodes, the construction of connections among nodes is an important research topic; the goal is to achieve efficient communication and adequate fault tolerance.

When analyzing the network topology, components in the system are abstracted as vertexes, and the communication channels between two components as edges. In this way, the topology of a network is abstracted into a graph, which can be used as a mathematical model to study the topology of the network. Thus, the topology properties of networks can be studied using graph theory. For example, network information transmission delay is related to the diameter and average distance of the graph, while the network fault tolerance is attributed to the connectivity of the graph [8–10].

The study of topology is closely related to the development of computer networks. Early network topologies only included some simple topological structures [11], such as line arrays, single-loop networks (circles), star graphs, trees and complete graphs. Later, the topology transitioned to double-loop networks, Petersen graphs and hypercubes [12].

A double-loop network (DLN) is a widely used, interconnection topology in local area networks, as well as in parallel and distributed computing [13]. Because of its simplicity, symmetry and scalability, it is also applied in neural networks [14,15], the Internet of Things [16] and data center networks [17].

There are two types of DLNs: unidirectional double-loop networks (UDLNs) and bidirectional double-loop networks (BDLNs) [13]. As shown in Figure 1, A UDLN has N vertices $0, 1, \ldots, N-1$, for each vertex i ($0 \le i < N$), and there are two unidirectional edges: $i \to i + r(\mathrm{mod}N)$, $i \to i + s(\mathrm{mod}N)$, $1 \le r < s < N$. It is strongly connected if and only if $N$, $r$ and $s$ are relatively prime.

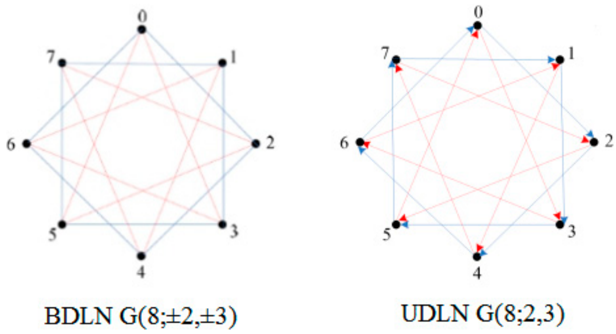


**Figure 1.** BDLN G(8;±2,±3) & UDLN G(8;2,3).

**Definition 1.** *Unidirectional Double loop networks (UDLN)* $G = G(N;r,s)$, *with* $1 \le r < s < N$ *and* $gcd(N,r,s) = 1$ *have vertex set* $V = \{0, 1, \ldots, N-1\}$, *and the adjacencies are defined by* $i \to i + r(\mathrm{mod}N)$, $i \to i + s(\mathrm{mod}N)$ *for* $i \in V$.

**Definition 2.** *Bidirectional Double loop networks (BDLN)* $G = G(N;\pm r,\pm s)$, *with* $1 \le r < s < N$ *and* $gcd(N,r,s) = 1$ *have vertex set* $V = \{0, 1, \ldots, N-1\}$, *and the adjacencies are defined by* $i \to i + r(\mathrm{mod}N)$, $i \to i - r(\mathrm{mod}N)$, $i \to i + s(\mathrm{mod}N)$, $i \to i - s(\mathrm{mod}N)$ *for* $i \in V$.

Unlike a UDLN, for each vertex $i$ in a BDLN, there are four unidirectional edges: $i \to i + r(\mathrm{mod}N)$, $i \to i + s(\mathrm{mod}N)$, $i \to i - r(\mathrm{mod}N)$, $i \to i - s(\mathrm{mod}N)$, represented as $[+r]$, $[+s]$, $[-r]$, $[-s]$ edge, respectively. Compared with a unidirectional double-loop network (UDLN), a BDLN can use both forward and backward edges to send messages. Hence, with a higher bandwidth and a smaller diameter, a BDLN is more resilient to node or link failures.

In network communication, information is transmitted from the source node to the destination node through storage and forwarding via several intermediate devices. If the distance between two nodes is too great, the information transmission will take more time; additionally, the longer the transmission path, the weaker the anti-interference ability. Thus, the distance among nodes is an important factor for the communication efficiency of the whole network. When analyzing the performance and efficiency of computer networks, two factors related to distance are important: maximum network transmission delay and average transmission delay. In graph theory, these factors are usually abstracted into two parameters: diameter and average distance [18].

Diameter, the maximum distance between any two nodes in the graph, has been the object of considerable attention in the context of BDLNs, as a small-diameter network can ensure the efficiency of communications. The lower bound of the diameter of a DLN is given in [19]; those authors also found many optimal DLNs whose diameters reach the lower bound. Great importance has been placed on other DLN network properties, such as fault tolerance [20] and message routing [21,22]; however, little work has focused on the average distance of BDLNs.

The diameter reflects the worst-case delay of communication between two nodes, while the average distance represents the average delay of communications. The average distance refers to the average of the distances between all pairs of points on the graph, which is used to indicate the centrality of the graph [18]. This concept was first used to evaluate building floor designs [23] and to study chemical molecular structures [24]; later, it was used extensively in mathematics and the analysis and design of computer systems and communication networks [25–27].

These parameters are important for BDLNs to measure message transmission delay. Taking an optimal BDLN as an example, the average distance is not necessarily the minimum. Some optimal BDLNs may have bigger average distances than non-optimal ones; when this happens, the overall efficiency of transmission is reduced.

Concerning the method to research double-loop networks, the L-shaped title (the Minimum Distance Diagram of UDLN is like an L-shaped title) is widely used in UDLN; it requires just four parameters [28]. This method, however, is not wholly applicable to BDLNs. Other structures, such as spiral rings [29] or trees [30], have been proposed to calculate the diameter of DLNs. Compared with the simplicity and intuition of the L-shape title method, spiral rings and trees are difficult to construct.

Naturally, we associate the graph with a coordinate system which represents a bridge between algebra and geometry, so that geometric issues can be tackled by algebraic methods [31]. In this paper, we propose an efficient method based on coordinate embedding and transforming to construct dual BDLNs for optimal routing.

Our main contributions are as follows:

1. To the best of our knowledge, this is the first work proposing the construction of dual optimal BDLNs for any number of nodes in which both the diameter and average distance are at their minima. We expand tight optimal BDLNs [22] to more general situations.
2. Embedding a BDLN into a coordinate system is useful for research of BDLNs, especially with regard to network properties such as the diameter and average distance. By means of coordinate transforming, we find node distribution regularity on the embedding graph of tight optimal BDLNs that achieves the lower bounds of both the diameter and average distance.
3. We present an on-demand optimal message routing algorithm for the dual optimal BDLN that we have constructed. The presented algorithms are efficient, requiring little computational overhead.

The rest of this paper is organized as follows: In Section 2, we provide a brief description of the embedding of BDLNs in Cartesian coordinates. In Section 3, we get the lower bounds of the diameter and average distance based on the embedding graph. In Section 4, we construct a kind of tight optimal BDLN that achieves the lower bounds of the diameter and average distance; we also find node distribution regularity on the embedding graph. In Section 5, we construct dual optimal BDLNs with minimum diameters and average distances for any number of nodes on the basis of node distribution regularity. We present on-demand optimal message routing algorithms for dual optimal BDLNs. Finally, we summarize our results and make some concluding remarks.

## 2. Embedding the BDLN into Cartesian Coordinates

### *2.1. Coordinate Embedding of BDLN*

Once a message has been routed, it traverses a series of $r$ and $s$ edges to reach its destination. Supposing a message traverses from the source to the destination in a BDLN $\mathrm{G(N;\pm r,\pm s)}$, the routing may be represented as $x_1[+r]+x_2[-r]+y_1[+s]+y_2[-s]$, where $x_1$, $x_2$, $y_1$, $y_2$, are the number of [+r] edges, [−r] edges, [+s] edges, [−s] edges, respectively.

**Lemma 1.** *If $x_1[+r] + x_2[-r] + y_1[+s] + y_2[-s]$ is the optimal routing from node i to node j, then $x_1x_2 = 0$, $y_1y_2 = 0$.*

**Proof.** Since [+r] is in the opposite direction to [−r], intuitively, if both edges exist simultaneously, the source node does not go any forward while the routing length adds two more. Then, either $x_1$ or $x_2$ is 0, as is $y_1$ and $y_2$. □

From this lemma, the optimal routing between $x_1$ and $x_2$ includes four cases: [+*r*], [+*s*]; [−*r*], [+*s*]; [+*r*], [−*s*]; and [−*r*], [−*s*]. Thus, a BDLN has a tabular representation where the *X* axis is represented by "r links" and the *Y* axis is represented by "s links".

Without loss of generality, source node 0 is considered to be the origin because of the high degree of symmetry of the BDLN.

**Definition 3.** *Visiting nodes of $G(N;\pm r,\pm s)$ in Cartesian coordinates in the order of coordinates: (0,0), (0,−1), (−1,0), (0,1), (1,0), (0,−2), (−1,−1), (−2,0), (−1,1), (0,2), (1,−1), (2,0), (1,1). Location (x,y) on the plane is occupied by a node of k, where $k \in \{0, 1, \cdots, N-1\}$ Location (x,y) also represents a coordinate number $k'$, $k' = xr + ys(mod N)$. If $k' > 0$, then $k = k'$. If $k' \le 0$, then $k = N + k'$. Each k, represented by a corresponding grid in Cartesian coordinates, fills the tabular cell by turns if the number has not been visited previously. Such an embedding graph of $G(N;\pm r,\pm s)$ is called $CG(N;\pm r,\pm s)$ in this paper.*

Figure 1 shows BDLN *G*(8;±2,±3). Figure 2 shows the coordinate embedding graph of *G*(25;±1,±7) in Cartesian coordinates, i.e., *CG*(25;±1,±7).

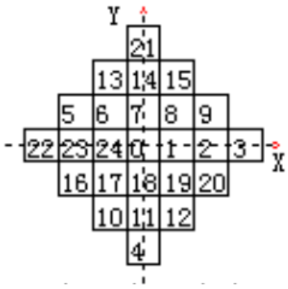


**Figure 2.** *CG*(25;±1,±7).

**Example 1.** *For the second layer, the nodes are visited in the order: (0,−2), (−1,−1), (−2,0), (−1,1), (0,2), (1,−1), (2,0), (1,1).*

From $CG(N;\pm r,\pm s)$, it is clear that the distance between node 0 and *k* is $|x| + |y|$ if node *k* is at $(x, y)$.

**Definition 4.** *Node-Set in layer i is defined as $U_i (1 \le i \le c)$ and the number of nodes in $U_i$ is $u_i (1 \le i \le c)$ when the maximum layer is C. From the above definition, $u_1 = 4$, $U_1 = \{(0,-1), (-1,0), (0,1), (1,0)\}$; here, the node is represented with its coordinates. In particular, $u_0 = 1$, $U_0 = \{(0,0)\}$. Location (x,y) on the plane is occupied by node $V_{(x,y)}(-c \le x \le c, -c \le y \le c)$, $V_{(x,y)} = xr + ys(mod N)(xr + ys(mod N) > 0)$ or $V_{(x,y)} = N + xr + ys(mod N)(xr + ys(mod N) < 0)$.*

**Example 2.** *In Figure 2, for CG(25;±1,±7), $U_1 = \{(0,-1), (-1,0), (0,1), (1,0)\}$, $V_{(0,-1)} = 18$, $V_{(-1,0)} = 24$, $V_{(0,1)} = 7$, $V_{(1,0)} = 1$.*

## *2.2. Properties of the Embedding Graph*

From Definitions 3, 4, we can get some interesting properties of $CG(N;\pm r,\pm s)$.

**Property 1.** *If* $(x, y) \in U_i$, *then* $i = |x| + |y|$.

**Property 2.** *If* $(x, y) \in U_i$, *then* $V(x, y) + V(-x, -y) = N$.

**Property 3.** *In* $CG(N; \pm r, \pm s)$, $1 \le u_i \le 4i$ $(1 \le i \le c)$, *where* $u_i$ *is the number of nodes.*

**Lemma 2.** *In* $CG(N; \pm r, \pm s)$, $2c \le N \le 2c^2 + 2c + 1$, *where c is the maximum layer.*

**Proof.** We know that $N = 1 + \sum_{i=1}^{c} u_i$. From Property 3, it is clear that $N \le 1 + 4\sum_{i=1}^{c} i$. Therefore, we can conclude that $N \le 2c^2 + 2c + 1$. □

The worst condition is that all the nodes are on the $X$ or $Y$ axis. In thas case, if $N$ is even, then $N = 2c$; whereas if $N$ odd, then $N = 2c + 1$ under the condition $gcd(N, r, s) = 1$ and $1 \le r < s < N$.

Notably, when $N = 2c$ or $N = 2c + 1$, $G(N; \pm r, \pm s)$ degenerates to a ring network. While $N = 2c^2 + 2c + 1$, the vertices are fully located on the embedding graph. As shown in Figure 2, we call such a graph $FCG(N; \pm r, \pm s)$ (Full Coordinate Embedding Graph). $FCG(N; \pm r, \pm s)$ has more special properties than other $CG(N; \pm r, \pm s)$ due to its high degree of symmetry, as we will demonstrate in the following sections.

**Lemma 3.** *Step i is the shortest distance from node 0 to all other nodes in* $U_i$.

**Proof.** Assuming $k \in U_i$, then $k \equiv x + ys(\mathrm{mod} N)$ $(x + ys(\mathrm{mod} N) > 0)$ or $k \equiv N + x + ys(\mathrm{mod} N)$ $(x + ys(\mathrm{mod} N) < 0)$, $|x| + |y| = i \ge 1, -i \le x \le i, -i \le y \le i$

a. Assuming $x_1, y_1$, where $k \equiv x_1 + y_1 s(\mathrm{mod} N)$ $(x_1 + y_1 s(\mathrm{mod} N) > 0)$ or $k \equiv N + x_1 + y_1 s(\mathrm{mod} N)$ $(x_1 + y_1 s(\mathrm{mod} N) < 0)$, $|x_1| + |y_1| = l < i, -l < x_1 < l$, $-l < y_1 < l$. According to Definition 3, $k$ should be put in node set $U_h$ in front of $U_i$, which contradicts $k \in U_i$.

b. Supposing that another $x_2, y_2$ exists, where $k \equiv x_2 + y_2 s(\mathrm{mod} N)$ $(x_2 + y_2 s(\mathrm{mod} N) > 0)$ or $k \equiv N + x_2 + y_2 s(\mathrm{mod} N)$ $(x_2 + y_2 s(\mathrm{mod} N) < 0)$, $|x_2| + |y_2| = l > i$, $-l < x_2 < l$, $-l < y_2 < l$; According to Definition 3, $k$ should be put in node set $U_j$ followed by node set $U_i$, which contradicts $k \in U_i$.

Therefore, step $i$ is the shortest distance from node 0 to all other nodes in $U_i$. □

Combining Lemma 3 with Property 1, we have:

**Theorem 1.** *In*$CG(N; \pm r, \pm s)$, *the shortest distance from node 0 to any other node* $(x, y)$ *in* $U_i$ *is* $|x| + |y|$, *where* $(x, y) \in U_i$.

### 3. Diameter and Average Distance of BDLN in Cartesian Coordinates

Embedding a BDLN into a coordinates system is useful for research of BDLNs. In this section, we get the lower bounds of the diameter and average distance of a BDLN based on the embedding graph.

#### *3.1. Diameter*

In this paper, $d$ denotes the diameter of $G(N; \pm r, \pm s)$.

**Lemma 4.** *Maximum layer c is the diameter of double-loop network* $G(N; \pm r, \pm s)$.

**Proof.** Diameter $d$ of the double-loop network $G(N; \pm r, \pm s)$ is defined as $d$ = *max*{*d(i,j)*}, where *d(i,j)* is the shortest distance from node $i$ to node $j$. Since BDLN is highly symmetric, its diameter can also be calculated from $d$ = *max*{*d(0,j)*}. From Lemma 3, nodes with the shortest distance from 0 to other nodes fall into node-set {1, 2, … c}. It is clear that *max*{1, 2, … *c*} = $u$. Therefore, $d$ = $c$. □

Combining Lemma 2 with Lemma 4, we derive the following lemma:

**Lemma 5.** $2d \leq N \leq 2d^2 + 2d + 1$.

From Lemma 5, we know the lower bound diameter of $G(N; \pm r, \pm s)$ is $lbd(N) = \lceil (\sqrt{2N-1} - 1)/2 \rceil$; here, $\lceil X \rceil$ is the minimum integer, which is greater than $X$.

*3.2. Average Distance*

**Definition 5.** *The average distance of BDLN $G(N; \pm r, \pm s)$ is:*

$$avG(N; \pm r, \pm s) = \frac{1}{N} \sum_{i=0}^{N-1} \left[ \frac{1}{(N-1)} \sum_{j=0}^{N-1} d_{ij} \right] \tag{1}$$

$d_{ij} = 0 (i = j)$, where $d_{ij}$ is the shortest distance from node $i$ to node $j$. Equally, $avgd(N; \pm r, \pm s) = \frac{1}{N-1} \sum_{i=1}^{N-1} d_i$, and $d_i$ is the shortest distance from node 0 to node $i$.

From Definition 5 and Theorem 1, we derive the following lemma:

**Lemma 6.** *To all nodes $(x_i, y_i)$ in the embedding graph, $(x_i, y_i) \in U_i$, $1 \leq i \leq d$, the average distance of BDLN is:*

$$avG(N; \pm r, \pm s) = \frac{1}{N-1} \sum (|x_i| + |y_i|) \tag{2}$$

**Lemma 7.**

$$avG(N; \pm r, \pm s) \geq 2d(d+1)(2d+1)/3(N-1) \tag{3}$$

**Proof.** From Lemma 6 and Property 1, 3, we have

$$avG(N; \pm r, \pm s) \geq \frac{4}{N-1} \sum_{i=1}^{d} i^2 = 2d(d+1)(2d+1)/3(N-1)$$

□

We can now summarize and state the lower bound of the average distance in the form of the following lemma:

**Lemma 8.** *In $CG(N; \pm r, \pm s)$, the average distance $avgd(N; \pm r, \pm s) \geq \sqrt{2N-1}/3$, if and only if $N = 2d^2 + 2d + 1$, then $avG(N; \pm r, \pm s) = \sqrt{2N-1}/3$.*

The lower bound of the average distance of $G(N; \pm r, \pm s)$ can be determined by

$$\mathrm{lba}(N) = \sqrt{2N-1}/3 \tag{4}$$

## 4. Tight optimal BDLN

From the embedding graph of BDLN in the Cartesian coordinates, we know that in $FCG(N; \pm r, \pm s)$, both the diameter and average distance have reached their lower bounds.

*4.1. Construction of Tight Optimal BDLN*

**Definition 6.** *A $G(N; \pm r, \pm s)$ is a tight optimal BDLN when both the diameter and average distance reach their lower bounds, i.e., $BestG(N; \pm r, \pm s)$.*

**Lemma 9.** *For a tight optimal BDLN, $G(N;\pm r,\pm s)$. When its diameter is d, then $N = 2d^2 + 2d + 1$ and the average distance is $(2d+1)/3$.*

**Proof.** For a tight optimal BDLN $G(N;\pm r,\pm s)$, $N = 1 + 4\sum_{i=1}^{d} i = 2d^2 + 2d + 1$. Putting this into Lemma 7, we get an average distance of a tight optimal BDLN of $(2d+1)/3$. □

**Property 4.** *Based on Definition 3, 4, the four top nodes in $FullCG(N;\pm r,\pm s)$ are $U_d = \{(d,0),(0,d),(-d,0),(0,-d)\}$, $V_{(d,0)} = rd$, $V_{(0,d)} \equiv Sd(modN)$, $V_{(-d,0)} = N - rd$, and $V_{(0,-d)} = rd + r$.*

**Theorem 2.** *In $BestG(N;\pm r,\pm s)$, for a given r, the corresponding s is:*

$$s \equiv (2d+1)r(modN) \tag{5}$$

**Proof.** From Property 4 and Property 2, we have $sd(modN) + r(d+1) = N = 2d^2 + 2d + 1$. So, $s \equiv (2d+1)r(modN)$. □

From Theorem 2, we can construct $BestG(N;\pm r,\pm s)$ as $G(2d^2+2d+1;\pm r,\pm(2d+1)r(modN))$.

Notably, when $r = 1$, $s = 2d + 1$, the corresponding dual optimal BDLN is $G(2d^2 + 2d + 1;\pm 1,\pm(2d+1))$; while when $r = d$, $s = d + 1$, the correlative dual optimal BDLN is $G(2d^2 + 2d + 1;\pm d,\pm(d+1))$.

Since network structures of this kind are relatively more stable and well-regulated, we will focus on them in the following sections. Without loss of generality, a tight optimal BDLN mainly refers to this kind of BDLN and is represented by $G_d$ with diameter $d$, i.e., $G_d = G(2d^2 + 2d + 1;\pm d,\pm(d+1))$.

Table 1 shows the value of $r$ and the related $s$ of the tight optimal diameter for $d$ = 1~2. Note that $r < s$, and for each $s$, there is another $s$: $N$ -$s$.

**Table 1.** $BestG(N;\pm r,\pm s)$ for diameter = 1, 2.

| Diameter | Average Distance | N | r | S |
|---|---|---|---|---|
| 1 | 0.800 | 5 | 1<br>2<br>3 | 2<br>4<br>4 |
| 2 | 1.538 | 13 | 1<br>2<br>3<br>4<br>5<br>6 | 5<br>3<br>11<br>7<br>12<br>9 |

*4.2. Coordinates Transforming of BDLN*

Since a BDLN is vertex transitive, any node can be relocated in the coordinates. Taking node 0 for an example, it can be relocated in the first ($d$,$d$ + 1) and fourth quadrants ($d$ + 1,$-d$). As shown in Figure 3, $\vec{\rho}$ and $\vec{\sigma}$ are defined as vectors from the origin to the copies of node 0 at ($d$,$d$ + 1) and ($d$ + 1,$-d$), respectively. That is, $\vec{\rho} = d\vec{x} + (d+1)\vec{y}$ and $\vec{\sigma} = (d+1)\vec{x} - d\vec{y}$, respectively.

By means of coordinate transformations, any node can be relocated with $\vec{\rho}$ or $\vec{\sigma}$ or a combination thereof. Letting $u'$ and $u''$ be two copies of vertex $u(x_0,y_0)$, $u'$ and $u''$ can be reached from u by placing vector $\vec{\rho}$ and $\vec{\sigma}$ at $u$, respectively. Thus, the coordinates of $u'$ and $u''$ are ($x_0 + d + 1$, $y_0 - d$) and ($x_0 + d + 1$, $y_0 - d$), respectively.

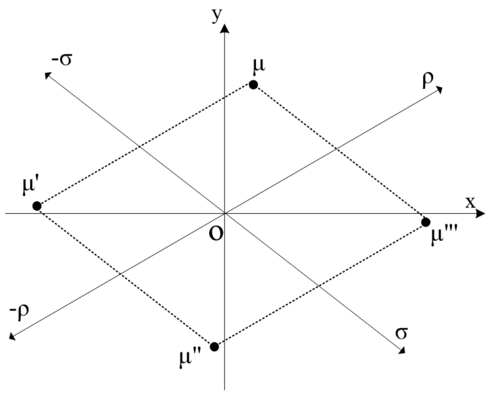


**Figure 3.** Vectors of copies of node 0.

In particular, vertices are located according to the order of the number sequence on line $y = -x + \text{a}$ on the embedding graph (see Figure 4). Without loss of generality, Theorem 3 is compliant with the embedding graph of $G_d$.

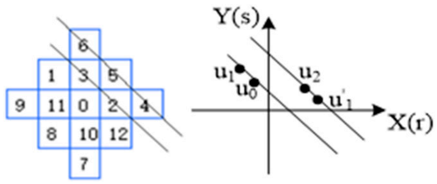

**Figure 4.** Number sequence on line $y = -x + \text{a}$.

**Theorem 3** [22]**.** *The nodes are distributed successively on the line* $y = -x + a$ ($-d \le a \le d$) *in the embedding graph of* $G_d$ *according to their number order.*

## 5. Dual Optimal BDLN

### *5.1. Construction of Dual Optimal BDLN*

If $N \neq 2d^2 + 2d + 1$, although the diameter or average distance of BDLN $G(N; \pm r, \pm s)$ cannot reach their lower bounds, they can achieve the minimum value. For example, when $N = 10$, according to Lemma 8 (4), the lower bound average distance of $\text{G}(10; \pm\text{r}, \pm\text{s})$ is 1.308 and the corresponding lower bound of diameter is 2. In this case, $10 \neq 2d^2 + 2d + 1 = 13$ ($d = 2$), which means that the average distance of $\text{G}(10; \pm\text{r}, \pm\text{s})$ cannot reach 1.308; however, as shown in Table 2, we still find that the average distance of $\text{G}(10; \pm 2, \pm 3)$ and $\text{G}(10; \pm 2, \pm 7)$ are the minimum among $\text{G}(10; \pm\text{r}, \pm\text{s})$. Note that $\text{G}(10; \pm 2, \pm 3)$ and $\text{G}(10; \pm 2, \pm 7)$ are isomorphic BDLNs. We also find that the coordinate embedding graphs of $\text{G}(10; \pm 2, \pm 3)$ and $\text{G}(10; \pm 2, \pm 7)$ are more compact than the others in Figure 5.

**Table 2.** Diameter and Average Distance of $G(10; \pm 2, \pm s)$.

| $N$ | $r$ | $S$ | Diameter | Average Distance |
|---|---|---|---|---|
| 10 | 2 | 3 | 2 | 1.4 |
| | | 5 | 3 | 1.7 |
| | | 7 | 2 | 1.4 |
| | | 9 | 3 | 1.5 |

**Definition 7.** *Given the number of nodes N,* $G(N; \pm r, \pm s)$ *is called a dual optimal BDLN when both the diameter and average distance are the minimum among those BDLNs with the same number of nodes N.*

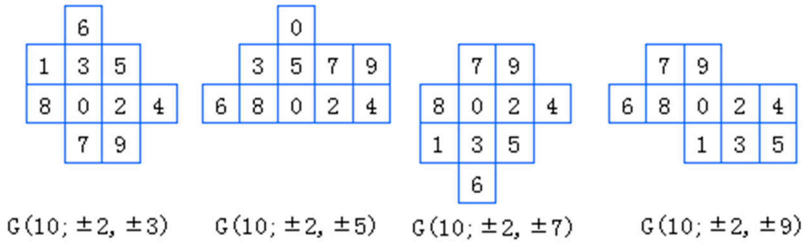


**Figure 5.** $CG(10; \pm 2, \pm s)$.

Comparing Definition 7 with Definition 6, we see that a tight optimal BDLN is a special kind of dual optimal BDLN. Given the number of nodes $N$, we need to determine how to construct a dual optimal BDLN whose diameter and average distance are the minimum.

From Theorem 3, we know that the nodes are distributed on line $y = -x + \text{a}$ by their number order. Further, they can be divided into two groups according to their coordinate numbers.

Case 1: The coordinate numbers of the positive integers are $1, 2, 3, \ldots, d(d+1)$ and the associated vertices are the same number, i.e., $1, 2, 3, \ldots, d(d+1)$.

Case 2: The coordinate numbers of the other group are $0, -1, -2, -3, \ldots, -d(d+1)$ and the associated vertices are $N, N-1, N-2, N-3, \ldots, N-d(d+1)$.

In Case 2, for vertices whose coordinate number is not greater than 0, their corresponding number will change with $N$. Let us consider two cases, i.e., adding or deleting $x$ vertices to $G_d$, to construct a dual optimal BDLN. To simplify the problem, we first consider $1 \leq x \leq d+1$.

(1) Adding vertices to $G_d$

If $N$ is 1, the number series in Case 2 is $N+1, N, N-1, N-2, \ldots, N-d(d+1)+1$. Then, $N-d(d+1)$ should be removed from the series. It is easy to find that if N adds $x_1$ vertices, the number series $N-d(d+1), N-d(d+1)+1, \ldots, N-d(d+1)+x_1$ should be removed, appearing at $(0,-d), (-1,-d+1) \ldots$ on line $y = -x-d$.

Since the nodes can be relocated someplace in the embedding graph of $G_d$, as shown in Figure 6, the nodes on line $y = -x-d$ can be relocated through coordinate transforming.

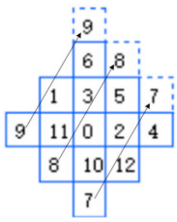

**Figure 6.** Coordinate transforming of BDLN $G_2$.

$$(0, -d) \xrightarrow{\vec{\rho}} (d, 1)$$

$$(-1, -d+1) \xrightarrow{\vec{\rho}} (-1+d, 2)$$

$$\cdots\cdots$$

$$(-d, 0) \xrightarrow{\vec{\rho}} (0, d+1)$$

By placing vector $\vec{\rho}$ at $(0,-d), (-1,-d+1), \ldots, (-d,0)$, we get copies at $(d,1), (-1+d,2), \ldots, (0,d+1)$.

In this case, the tile is obtained by adding $1 \sim d+1$ squares along the direction at $(d,1)$ to $(0,d+1)$, as illustrated in Figure 7, until, at most, $d+1$ vertices are added to the embedding graph.

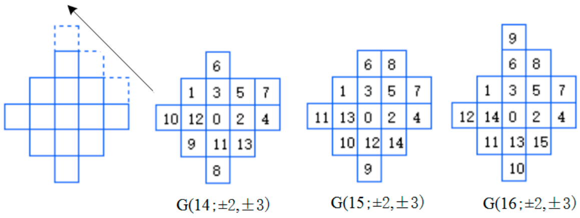


**Figure 7.** Adding vertices to $G_d$.

**Lemma 10.** *When adding x* $(1 \leq x \leq d+1)$ *vertices to* $G_d$*, if all the x vertices have a distance from the origin of d* + 1*, then* $G(N+x; \pm d, \pm(d+1))$ *is a dual optimal BDLN.*

**Proof.** Suppose the diameter of $G(N+x; \pm d, \pm(d+1))$ is not the minimum. In this case, there should be a vertex $u$ whose distance from the origin is less than $d + 1$. Since the vertices are all located on $G_d$, vertex $u$ meeting such conditions is an impossibility. Therefore, the diameter of $G(N+x; \pm d, \pm(d+1))$ is the minimum, as is the average distance.

As such, $G(N+x; \pm d, \pm(d+1))$ is a dual optimal BDLN with a minimum diameter and average distance. □

(2) Deleting vertices from $G_d$

Similar to the case of adding vertices, if we subtract 1 from $N$, the vertex at $(0,-d)$ should be removed in the embedding graph, while if we subtract $x$ from $N$, the vertices at $(0,-d)$,$(-1,-d+1)$ ... on line $y = -x - d$ in the direction of the upper right until, at most, $d$ + 1 should be removed from the embedding graph (see Figure 8).

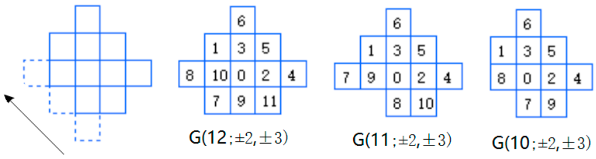


**Figure 8.** Deleting vertices from $G_d$.

For given node number $N$, the algorithm to construct the relevant dual optimal BDLN is presented schematically in Figure 9.

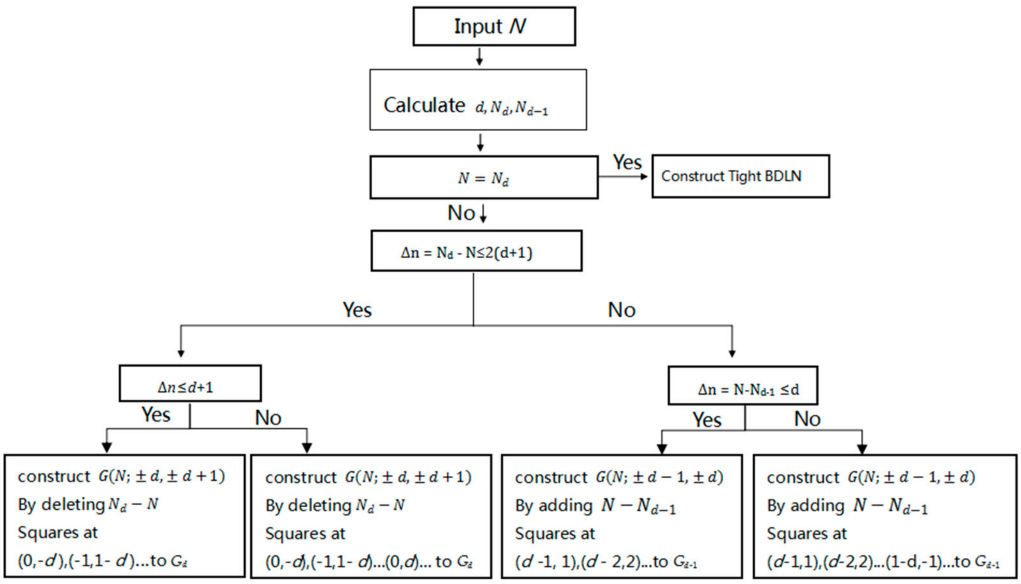


**Figure 9.** Algorithm for constructing a dual optimal BDLN.

Step 1: Calculating a feasible diameter for the BDLN

$$d = \left\lceil \sqrt{2N-1} - 1)/2 \right\rceil$$

From Lemma 5 and Lemma 8, we can determine that $N \leq 2d^2 + 2d + 1$.

Step 2: If $2d^2 + 2d + 1 = N$, then the tight optimal BDLN $G\left(2d^2 + 2d + 1; \pm d, \pm(d+1)\right)$ is the relevant dual optimal BDLN. Note that:

$$N_d = 2d^2 + 2d + 1,\ G_d = G\left(2d^2 + 2d + 1; \pm d, \pm(d+1)\right)$$

Step 3: If $0 < N_d - N \leq 2(d+1)$, we construct the relevant dual optimal BDLN $G(N; \pm d, \pm d + 1)$ by adding vertices to $G_d$ in two cases:

① $0 < N_d - N \leq d + 1$

In this case, dual optimal BDLN $G(N; \pm d, \pm(d+1))$ is constructed by deleting $N_d - N$ squares at (0,$-d$),($-1$,$-d+1$) ... to $G_d$, as shown in Figure 8.

② $d + 1 < N_d - N \leq 2(d+1)$

Except for the $d$ + 1 vertices deleted along the direction (0,$-d$), ($-1$,$-d$ + 1) ... ($-d$,0) in case ①, we construct a dual optimal BDLN $G(N; \pm d, \pm(d+1))$ by deleting $N_d - N - (d+1)$ squares at (0,$d$), (1,$d$ − 1) ... to $G_d$, as illustrated in Figure 10.

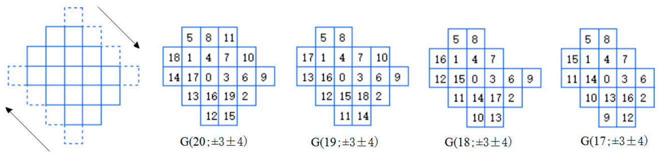


**Figure 10.** Deleting the other $d$ + 1 layer vertices from $G_d$.

Step 4: If $N_d - N > 2(d+1)$, we construct the relevant dual optimal BDLN $G(N; \pm(d-1), \pm d)$ by adding vertices to $G\left(2d^2 - 2d + 1; \pm(d-1), \pm d\right)$ in two cases.

Note that:

$$N_{d-1} = 2(\mathrm{d}-1)^2 + 2(d-1) + 1 = 2d^2 - 2d + 1$$

$$G_{d-1} = G\left(2d^2 - 2d + 1; \pm(d-1), \pm d\right)$$

③ $0 < N - N_{d-1} \leq d$

We construct a dual optimal BDLN $G(N; \pm(d-1), \pm d)$ by adding $N - N_{d-1}$ squares at ($-1$ + $d$,1), ($-2$ + $d$,2) ... to $G_{d-1}$, as shown in Figure 7.

④ $d < N - N_{d-1} \leq 2d$

Except for the incremental $d$ vertices added along direction ($-1$ + $d$,1), ($-2$ + $d$,2) ... (0,$d$) in case ③, we construct dual optimal BDLN $G(N; \pm(d-1), \pm d)$ by adding surplus $N - N_{d-1} - d$ squares at (1 − $d$, −1),(2 − $d$, −2) ... to $G_{d-1}$, as shown in Figure 11.

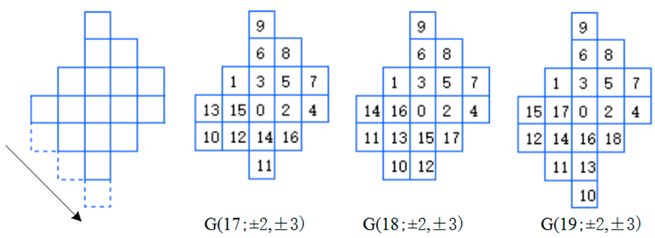


**Figure 11.** Adding the other $d$ + 1 layer vertices to $G_{d-1}$.

(3) For some special cases, adding vertices to $G_{d-1}$ or deleting vertices to $G_d$ can yield the same diameter and average distance in the dual optimal BDLN.

Suppose the number of nodes of BDLN is N.
① adding $x_1$ vertices to $G_{d-1}$
$N = N_{d-1} + x_1 = 2(d-1)^2 + 2(d-1) + 1 + x_1$
② deleting $x_2$ vertices to $G_d$
$N = N_d - x_2 = 2d^2 + 2d + 1 + x_2$
So, we get: $x_1 + x_2 = 4d$

If $x_1 + x_2 = 4d$, it is not difficult to certify that adding $x_1$ vertices to $G_{d-1}$ or deleting $x_2$ vertices to $G_d$ can afford the same diameter and average distance.

By combining $x_1 + x_2 = 4d$ with restrictive conditions $1 \leq x_1 \leq 2d$, $1 \leq x_2 \leq 2(d+1)$, we have:

$$x_1 = 2(d-1),\ x_2 = 2(d+1)$$

$$x_1 = 2d-1,\ x_2 = 2d+1$$

$$x_1 = 2d,\ x_2 = 2d$$

Therefore, there are three pairs of dual optimal BDLNs with the same diameter and average distance:

① $G\left(2d^2-1; \pm(d-1), \pm d\right)$ and $G\left(2d^2-1; \pm d, \pm(d+1)\right)$
② $G\left(2d^2; \pm(d-1), \pm d\right)$ and $G\left(2d^2; \pm d, \pm(d+1)\right)$
③ $G\left(2d^2+1; \pm(d-1), \pm d\right)$ and $G\left(2d^2+1; \pm d, \pm(d+1)\right)$

### *5.2. Optimal Routing of Dual Optimal BDLN*

Once a message has been routed in a BDLN, it traverses a series of $r$ and $s$ edges to reach its destination. Suppose $x$ and $y$ are the number of $r$ and $s$ edges in the route, respectively, then the length of the route is $x + y$. When the length of the route is minimum, it is defined as the optimal route. As illustrated in Theorem 1, the shortest distance from source vertex 0 to destination vertex u is $|x| + |y|$, where $x$ and $y$ are coordinate numbers of vertex $u$ in the embedding graph of BDLN. As shown in Figure 11, the coordinates of destination vertex 14 in the embedding graph of $G(27; \pm 3, \pm 4)$ are (2,2), which means that a message can be transferred from source vertex 0 to destination vertex 14 by taking 2[+$r$] edges and 2[+$s$] edges. Therefore, the optimal route between the vertexes in the embedding graph can be represented with coordinates of vertexes.

The optimal routing of a dual optimal BDLN is presented schematically in Algorithm 1. Since the optimal route can be expressed with coordinates of a vertex, now, we introduce the algorithm to calculate the coordinates ($x$,$y$) of vertex $u$ for a given dual optimal BDLN $G(N; \pm d, \pm d+1)$:

(1) $N = N_d = 2d^2 + 2d + 1$

In this case, the dual optimal BDLN $G(N; \pm d, \pm d+1)$ is tight optimal BDLN $G\left(2d^2 + 2d + 1; \pm d, \pm(d+1)\right)$, represented as $G_d$. The optimal routing of the tight optimal BDLN is given in [22].

(2) $N > N_d$
In this case, the given dual optimal BDLN is constructed by adding vertices to $G_d$.
Note that $\Delta n = N - N_d$
① $0 < u \leq d(d+1)$
$a = \lfloor (2u+d)/(2d+1) \rfloor$, $x = a(d+1) - u$, $y = -x + a$
② $d(d+1) < u \leq d(d+1) + \Delta n'$
Here, if $1 \leq \Delta n \leq d+1$, then $\Delta n' = \Delta n$, whereas if $\Delta n > d+1$, then $\Delta n' = d+1$, $x = (d+1)^2 - u$, $y = u - d(d+1)$

③ Other cases

When $u' = N - u$, we first calculate the coordinate $(x',y')$ of $u'$; then, the coordinates of $u$ are $x = -x'$, $y = -y'$.

(3) $N < N_d$ In this case, the given dual optimal BDLN is constructed by deleting vertices to $G_d$. Note that $\Delta n = N_d - N$.

① $0 < u \leq d(d+1) - \Delta n'$

Here, if $1 \leq \Delta n \leq d+1$, then $\Delta n' = 0$, whereas if $\Delta n > d+1$, then $\Delta n' = \Delta n - (d+1)$. $a = \lfloor (2u+d)/(2d+1) \rfloor$, $x = \mathrm{a}(d+1) - u$, $y = -x + \mathrm{a}$

② Other cases

When $u' = N - u$, we first calculate the coordinate $(x',y')$ of $u'$; then, the coordinates of $u$ are $x = -x'$, $y = -y'$.

---

**Algorithm 1:** Optimal routing of dual optimal BDLN $G(N; \pm d, \pm d+1)$

---

**input:** $G(N; \pm d, \pm d+1)$, source vertex 0, destination vertex $u$
**output:** coordinates $(x,y)$ of vertex $u$
Calculate $N_d = 2d^2 + 2d + 1$, $\Delta n = N_d - N$
**If** $N = N_d$ **then**
  Tight optimal routing of $G_d$
**Else if** $N > N_d$
  **If** $0 < u \leq d(d+1)$ **then**
    Calculate $a = \lfloor (2u+d)/(2d+1) \rfloor$
    $\mathrm{x} = \mathrm{a}(d+1) - u$, $y = -x + \mathrm{a}$
  **Else**
    if $\Delta n > d+1$ **then** $\Delta n = d+1$
    **if** $d(d+1) < u \leq d(d+1) + \Delta n$ then
      $\mathrm{x} = (d+1)^2 - u$, $y = u - d(d+1)$
    **Else**
      $u' = N - u$
      Calculate coordinate $(x',y')$ of $u'$
      $x = -x'$, $y = -y'$
    **Endif**
  **Endif**
**Else**
  $\Delta n = N_d - N$
  **if** $\Delta n > d+1$ **then**
    $\Delta n = \Delta n - (d+1)$
  **Else**
    $\Delta n = 0$
  **Endif**
  **if** $0 < u \leq d(d+1) - \Delta n$ **then**
    Calculate $a = \lfloor (2u+d)/(2d+1) \rfloor$
    $x = \mathrm{a}(d+1) - u$, $y = -x + \mathrm{a}$
  **Else**
    $u' = N - u$, calculate coordinate $(x',y')$ of $u'$
    $x = -x'$, $y = -y'$
  **Endif**
**Endif**
**return** $x$, $y$

---

We have implemented an optimal routing demo system in the .NET C# language. The system was created on a PC with 18 cores (3 GHz Intel Xeon Platinum, 8124 M). As shown in Figure 12, the system could show the embedding graph and optimal routing of the BDLN, which is useful for research of dual optimal BDLN.

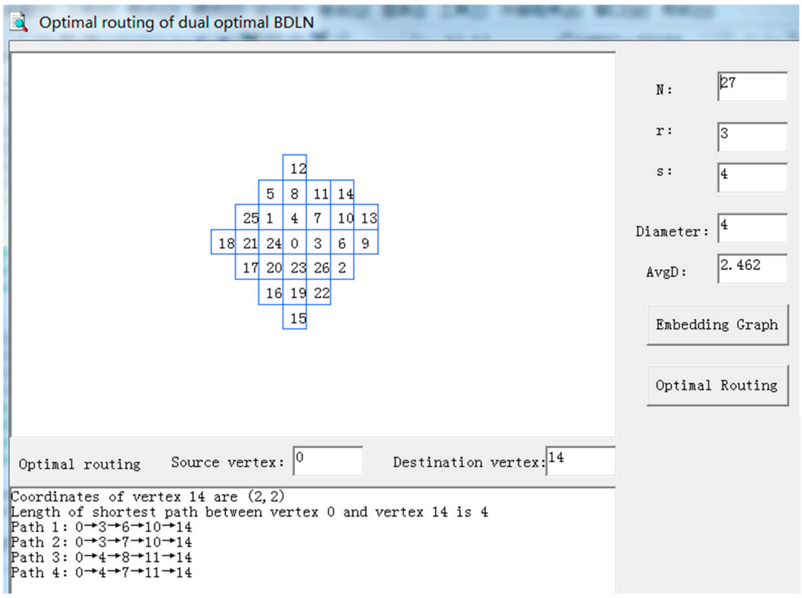


**Figure 12.** Optimal routing of dual optimal BDLN.

To test the running speed of the optimal routing, we noted the running time results of the optimal routing between any two nodes in the dual optimal BDLN with different numbers of nodes. In order to test the efficiency of the routing in the system, we compared the computation time to determine the optimal routing with those of two other algorithms. The first algorithm was Breadth-first search (BFS); this algorithm is used to search graph data to quickly visit each node in a graph. The second algorithm is called HP [13]; it is based on four DLN parameters. As shown in Figure 13, with a growing number of nodes, the computation time required to determine the optimal routing increases significantly when using the BFS and HP algorithms; in contrast, using our proposed algorithm, the computation time increased only minimally. Therefore, our algorithm can reduce the running time costs when dealing with large numbers of nodes.

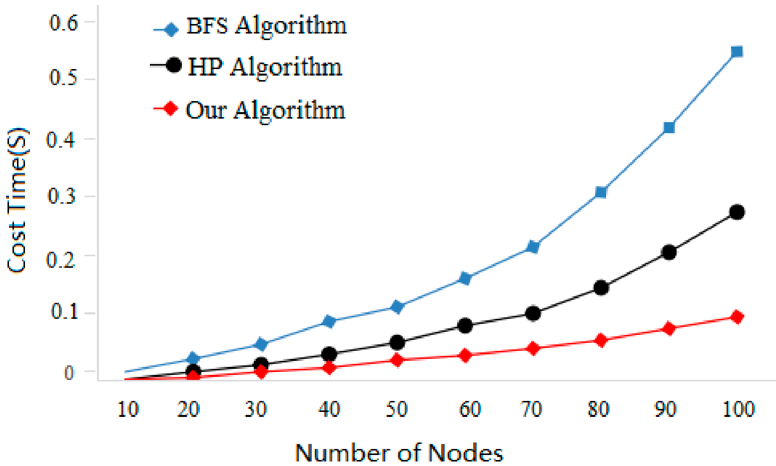


**Figure 13.** Computational time to determine the optimal routing of dual optimal BDLNs.

By comparing the running speed of the optimal routing between dual optimal BDLN and other BDLNs (non-dual optimal BDLNs), as shown in Figure 14, with an increasing number of nodes, the computational time of the latter increased more than that of the former. Thus, our dual optimal BDLN can effectively improve the running speed of the optimal routing.

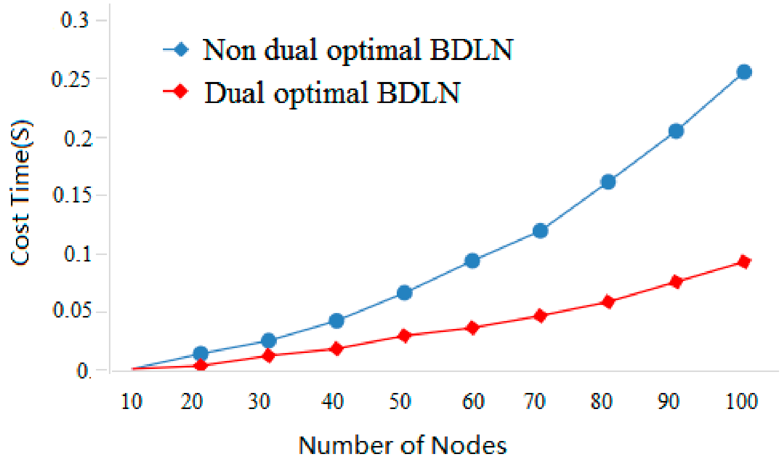


**Figure 14.** Cost time of optimal routing of dual optimal BDLN and other BDLNs.

### 5.3. Average Distance of the Dual Optimal BDLN

(1) When adding $x_1$ vertices to $G_d$, the average distance of the newly constructed BDLN $G(N+x_1;\pm d,\pm(d+1))$ is:

$$avG(N+x_1;\pm d,\pm(d+1)) = (avG_d(N-1)+x_1(d+1))/(N+x_1-1) \quad (6)$$

Here, $N = 2d^2+2d+1$, $avG_d = 2d(2d+1)(d+1)/3(N-1)$, $1 \le x_1 \le 2(d+1)$.

From the above equations, it is not difficult to see that average distance of BDLN $G(N+x_1;\pm d,\pm(d+1))$ increases as $x_1$ increases.

(2) When deleting $x_2$ vertices from $G_d$, the average distance of the newly constructed BDLN $G(N-x_2;\pm d,\pm(d+1))$ is:

$$avG(N-x_2;\pm d,\pm(d+1)) = (avG_d(N-1)-x_2(d+1))/(N-x_2-1) \quad (7)$$

Here, $N = 2d^2+2d+1$, $avG_d = 2d(2d+1)(d+1)/3(N-1)$, $1 \le x_2 \le 2(d+1)$.

Similar to the above, the average distance of BDLN $G(N+x_1;\pm d,\pm(d+1))$ decreases as $x_2$ decreases.

Combining these two cases, we can state that average distance increases with the number of nodes in the dual optimal BDLN, as shown in Figure 15. The $X$ axis represents the number of nodes and the $Y$ axis the diameter or average distance.

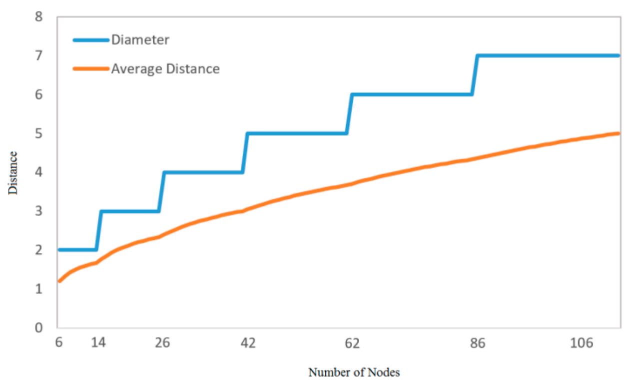


**Figure 15.** Diameter and average distance of the dual BDLN.

## 6. Conclusions

We made a thorough inquiry of the diameter and average distance of a BDLN using an efficient method based on coordinate embedding and transforming. Our method can also be used for large-scaled graphs. Once the large-scaled graph data has been successfully embedded in the coordinate system, the query speed of the graph was greatly improved, as each node could be located according to its coordinates.

For a tight optimal BDLN [22], the number of nodes should follow the equation $N = 2d^2 + 2d + 1$, where d is the diameter of BDLN. If $N \neq 2d^2 + 2d + 1$, although diameter or average distance cannot reach their lower bounds, they can achieve the minimum values. To address this issue, we propose a dual optimal BDLN for any given number of nodes. We expanded a tight optimal BDLN to suit more general situations.

Because of the minimum of both diameter and average distance, it is evident that the dual optimal BDLN constructed in this paper yielded better performance compared with non-dual optimal ones, as verified by experiments. By means of coordinate embedding and transforming in the coordinate system, each node of the BDLN could be located, thereby simplifying the message routing. The on-demand optimal routing algorithms presented in this paper do not require routing tables and are efficient, requiring little computational overhead.

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**Data Availability Statement:** The data used to support the findings of this study are available from the authors upon request.

**Conflicts of Interest:** The authors declare no conflict of interest.

## Abbreviations

| | |
|---|---|
| DLN | Double-loop networks |
| BDLN | Bidirectional Double-loop networks |
| UDLN | Unidirectional Double-loop network |
| BFS | Breadth First Search |

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