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Multiplier Hopf Coquasigroup: Motivation and Biduality

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Abstract: Inspired by the multiplier Hopf algebra theory introduced by A. Van Daele, this paper introduces a new algebraic structure, a multiplier Hopf coquasigroup, by constructing the integral dual of an infinite-dimensional Hopf quasigroup with faithful integrals. Then, it shows that the biduality theorem also holds for Hopf quasigroups and multiplier Hopf coquasigroups of the discrete type.

Keywords: Hopf quasigroup; integral; multiplier Hopf coquasigroup

MSC: 16T05; 17A01; 20N05

1. Introduction

As many Lie groups have an entirely algebraic description, similar to commutative Hopf algebras, J. Kim and S. Majid in [1] developed the corresponding theory of 'algebraic quasigroups', which includes the coordinate algebra $k[S^7]$ of the seven-sphere by using Hopf coquasigroup methods to study differential geometry on $k[S^7]$. The notions of a Hopf quasigroup and a Hopf coquasigroup introduced in [1] generalized the classical notion of an inverse property quasigroup G, expressed as a quasigroup algebra and an algebraic quasigroup, respectively. The authors proved the basic results for Hopf algebras, such as anti-(co)multiplicativity of the antipode and a theory of crossed (co)products. They showed that a theory similar to that of Hopf algebras was possible in this case.

The first author, in his following paper [2], developed the integral theory for Hopf (co)quasigroups and Fourier transform and showed that a finite-dimensional Hopf (co)quasigroup had a unique integration up to the scalar. The dual of a finite-dimensional Hopf quasigroup was a Hopf coquasigroup, as shown in [1]. Then, naturally, there are two motivating questions:

(Q1) Is the integral also unique on an infinite-dimensional Hopf quasigroup?

(Q2) Does the biduality theorem still hold for some infinite-dimensional Hopf quasigroups?

In [3], A. Van Daele developed the theory of multiplier Hopf algebras and algebraic quantum groups, which provided us with the 'integral dual' of an infinite-dimensional Hopf algebra with nondegenerate faithful integrals. Inspired by this, we first consider the properties of the integrals on the infinite-dimensional Hopf quasigroup and then construct the integral dual. We find the integral dual has a similar structure to that of a Hopf coquasigroup, which is a multiplier Hopf coquasigroup of the discrete type.

Furthermore, we show that the duality of a (discrete) multiplier Hopf coquasigroup is exactly a Hopf quasigroup, and the biduality theorem holds for (discrete) multiplier Hopf coquasigroups and Hopf quasigroups. We give positive answers to the above two questions.

This paper is organized as follows. In Section 2, we introduce some basic notions that will be used in the following sections: Hopf (co)quasigroups and multiplier algebras.

In Section 3, we consider the integral on an infinite-dimensional Hopf quasigroup and show that the faithful integrals are unique up to the scalar. For an infinite-dimensional Hopf quasigroup H with a faithful integral φ, we construct the integral dual $\widehat{H} = \varphi(\cdot H)$ and show that $\widehat{H}$ is a discrete multiplier Hopf coquasigroup with a faithful integral.

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In Section 4, we introduce our motivating example, comment on multiplier Hopf coquasigroups, and consider the properties of integrals. We show that a multiplier Hopf coquasigroup has local units, and the integrals are unique up to the scalar.

In Section 5, we construct the dual of a (infinite-dimensional) multiplier Hopf coquasigroup of the discrete type and show that this duality has the Hopf quasigroup structure introduced in [1].

In Section 6, we show that the biduality theorem holds for some infinite-dimensional Hopf quasigroups and multiplier Hopf coquasigroups of the discrete type and, finally, give the concrete isomorphisms in the motivation example.

2. Preliminaries

Throughout this paper, all the linear spaces we considered are over a fixed field k (e.g., the complex number field $\mathbb{C}$).

2.1. Hopf (Co)quasigroups

Recall from [1] that a *Hopf quasigroup* is a possibly nonassociative though unital algebra (H, m, μ) equipped with the algebra homomorphisms $\Delta : H \longrightarrow H \otimes H$, $\varepsilon : H \longrightarrow k$, forming a co-associative co-algebra and a map $S : H \longrightarrow H$ such that

$$m(id \otimes m)(S \otimes id \otimes id)(\Delta \otimes id) = \varepsilon \otimes id = m(id \otimes m)(id \otimes S \otimes id)(\Delta \otimes id),$$
$$m(m \otimes id)(id \otimes S \otimes id)(id \otimes \Delta) = id \otimes \varepsilon = m(m \otimes id)(id \otimes id \otimes S)(id \otimes \Delta).$$

These two equations can be written more explicitly as follows: for all $g, h \in H$,

$$\sum S(h_{(1)})(h_{(2)}g) = \sum h_{(1)}\big(S(h_{(2)})g\big) = \varepsilon(h)g,$$
$$\sum \big(gS(h_{(1)})\big)h_{(2)} = \sum (gh_{(1)})S(h_{(2)}) = \varepsilon(h)g,$$

where we write $\Delta h = \sum h_{(1)} \otimes h_{(2)}$, and, for brevity, we shall omit the summation signs.

The Hopf quasigroup H is called *flexible* if

$$h_{(1)}(gh_{(2)}) = (h_{(1)}g)h_{(2)}, \qquad \forall g, h \in H, \tag{1}$$

and *alternative* if also

$$h_{(1)}(h_{(2)}g) = (h_{(1)}h_{(2)})g, \qquad h(g_{(1)}g_{(2)}) = (hg_{(1)})g_{(2)}, \qquad \forall g, h \in H. \tag{2}$$

H is called *Moufang* if

$$h_{(1)}\big(g(h_{(2)}f)\big) = \big((h_{(1)}g)h_{(2)}\big)f, \qquad \forall h, g, f \in H. \tag{3}$$

It was proved that the antipode S is antimultiplicative and anticomultiplicative, i.e., for all $g, h \in H$,

$$S(gh) = S(h)S(g), \qquad \Delta(Sh) = S(h_{(2)}) \otimes S(h_{(1)}).$$

Moreover, if H is a cocommutative flexible Hopf quasigroup, then $S^2 = id$, and, for all $g, h \in H$,

$$h_{(1)}\big(gS(h_{(2)})\big) = (h_{(1)}g)S(h_{(2)}).$$

Dually, we can obtain a Hopf coquasigroup by reversing the arrows on each map in a Hopf quasigroup. A Hopf coquasigroup is a unital associative algebra A equipped

with a co-unital algebra homomorphism $\Delta : A \longrightarrow A \otimes A$, $\varepsilon : A \longrightarrow k$ and a linear map $S : A \longrightarrow A$ such that, for all $a \in A$,

$$(m \otimes id)(S \otimes id \otimes id)(id \otimes \Delta)\Delta = 1 \otimes id = (m \otimes id)(id \otimes S \otimes id)(id \otimes \Delta)\Delta,$$
$$(id \otimes m)(id \otimes S \otimes id)(\Delta \otimes id)\Delta = id \otimes 1 = (id \otimes m)(id \otimes id \otimes S)(\Delta \otimes id)\Delta.$$

In other words,

$$S(a_{(1)})a_{(2)(1)} \otimes a_{(2)(2)} = 1 \otimes a = a_{(1)}S(a_{(2)(1)}) \otimes a_{(2)(2)},$$
$$a_{(1)(1)} \otimes S(a_{(1)(2)})a_{(2)} = a \otimes 1 = a_{(1)(1)} \otimes a_{(1)(2)}S(a_{(2)}).$$

A Hopf coquasigroup is flexible if

$$a_{(1)}a_{(2)(2)} \otimes a_{(2)(1)} = a_{(1)(1)}a_{(2)} \otimes a_{(1)(2)}, \qquad \forall a \in A, \tag{4}$$

and alternative if also

$$a_{(1)}a_{(2)(1)} \otimes a_{(2)(2)} = a_{(1)(1)}a_{(1)(2)} \otimes a_{(2)}, \tag{5}$$
$$a_{(1)} \otimes a_{(2)(1)}a_{(2)(2)} = a_{(1)(1)} \otimes a_{(1)(2)}a_{(2)}, \qquad \forall a \in A. \tag{6}$$

A is called Moufang if

$$a_{(1)}a_{(2)(2)(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)(2)} = a_{(1)(1)(1)}a_{(1)(2)} \otimes a_{(1)(1)(2)} \otimes a_{(2)}, \quad \forall a \in A, \tag{7}$$

The term 'co-unital' here means

$$(id \otimes \varepsilon)\Delta = id = (\varepsilon \otimes id)\Delta.$$

However, Δ is not assumed to be co-associative.

It was shown in Proposition 5.2 [1] that: Let A be a Hopf coquasigroup, then

(1) $m(S \otimes id)\Delta = \mu\varepsilon = m(id \otimes S)\Delta$;
(2) S is an antimultiplicative $S(ab) = S(b)S(a)$ for all $a, b \in A$;
(3) S is an anticomultiplicative $\Delta S(a) = S(a_{(2)}) \otimes S(a_{(1)})$ for all $a \in A$.

Hence, a Hopf coquasigroup is Hopf algebraic if and only if it is co-associative.

2.2. Multiplier Algebras

Let A be an (associative) algebra. We do not assume that A has a unit, but we do require that the product, seen as a bilinear form, is nondegenerated. This means that whenever $a \in A$ and $ab = 0$ for all $b \in A$ or $ba = 0$ for all $b \in A$, we must have that $a = 0$.

Recall from [3,4] that $M(A)$ is characterized as the largest algebra with an identity containing A as an essential two-sided ideal. In particular, we still have that, whenever $a \in M(A)$ and $ab = 0$ for all $b \in A$ or $ba = 0$ for all $b \in A$, $a = 0$. Furthermore, we consider the tensor algebra $A \otimes A$. It is still nondegenerated, and we have its multiplier algebra $M(A \otimes A)$. There are natural embeddings:

$$A \otimes A \subseteq M(A) \otimes M(A) \subseteq M(A \otimes A).$$

In general, when A has no identity, these two inclusions are strict. If A already has an identity 1_A, the product is obviously nondegenerate, and $M(A) = A$ and $M(A \otimes A) = A \otimes A$.

Let A and B be nondegenerate algebras, if the homomorphism $f : A \longrightarrow M(B)$ is nondegenerated (i.e., $f(A)B = B$ and $Bf(A) = B$), then it has a unique extension to the homomorphism $M(A) \longrightarrow M(B)$, which we also denote as f.

3. Duality on an Infinite-Dimensional Hopf Quasigroup

Let H be a finite-dimensional Hopf quasigroup and $H^* = Hom(H,k)$ be the dual space with natural Hopf coquasigroup structure given by: for $a,b \in H^*$ and $h \in H$

$$\begin{aligned}&\langle ab,h\rangle = \langle a,h_{(1)}\rangle\langle b,h_{(2)}\rangle, \qquad \langle \Delta(a),h\otimes g\rangle = \langle a,hg\rangle,\\ &\langle 1_{H^*},h\rangle = \varepsilon(h), \quad \langle a,1_H\rangle = \varepsilon(a), \quad \langle S(a),h\rangle = \langle a,S(h)\rangle.\end{aligned}$$

Then, there is a natural question: for an infinite-dimensional Hopf quasigroup H, what is its dual?

3.1. Integrals on an Infinite-Dimensional Hopf Quasigroup

Recall from [2] that a left (resp. right) integral on H is a nonzero element $\varphi \in H^*$ (resp. $\psi \in H^*$) such that

$$(id\otimes\varphi)\Delta(h) = \varphi(h)1_H \quad \big(\text{resp.}(\psi\otimes id)\Delta(h) = \psi(h)1_H\big), \qquad \forall h \in H,$$

Moreover, from Lemma 3.3 in [2], we have that $\varphi \circ S$ is a right integral on H.

Lemma 1 ([2])**.** *Let φ (resp. ψ) be a left (resp. right) integral on H, then for $h,g \in H$*

$$h_{(1)}\varphi\big(h_{(2)}S(g)\big) = \varphi\big(hS(g_{(1)})\big)g_{(2)}, \qquad h_{(1)}\varphi(gh_{(2)}) = S(g_{(1)})\varphi(g_{(2)}h). \tag{8}$$

$$\psi\big(S(g)h_{(1)}\big)h_{(2)} = \psi\big(S(g_{(2)})h\big)g_{(1)}, \qquad \psi(g_{(1)}h)g_{(2)} = \psi(gh_{(1)})S(h_{(2)}). \tag{9}$$

Proof. From the proofs of Lemma 3.4 and 3.8 in [2], we can easily check whether the above equations also hold in the infinite-dimensional case. □

In the following, we will construct the 'integral dual' of a class of infinite-dimensional Hopf quasigroups. Let H be an infinite-dimensional Hopf quasigroup with a bijective antipode and a left faithful integral, i.e., $\varphi(gh) = 0, \forall h \in H \Rightarrow g = 0$ and $\varphi(gh) = 0, \forall g \in H \Rightarrow h = 0$.

First, we show that for the infinite-dimensional Hopf quasigroup, the left faithful integral is unique up to the scalar.

Proposition 1. *Let φ' be another left faithful integral on H. Then, $\varphi' = \lambda\varphi$ for some scalar $\lambda \in k$, i.e., the left faithful integral on H is unique up to the scalar.*

Proof. From Lemma 1 , we have $h_{(1)}\varphi(gh_{(2)}) = S(g_{(1)})\varphi(g_{(2)}h)$ for all $h,g \in H$. Apply φ' to both expressions in this equation. Because $\varphi' \circ S$ is a right integral, the right-hand side will give

$$\varphi'(Sg_{(1)})\varphi(g_{(2)}h) = \varphi(\varphi' S(g_{(1)})g_{(2)}h) = \varphi(\varphi' S(g)1_H \cdot h) = \varphi' S(g)\varphi(h).$$

For the left-hand side,

$$\varphi'(h_{(1)}\varphi(gh_{(2)})) = \varphi'(h_{(1)})\varphi(gh_{(2)}) = \varphi(g\varphi'(h_{(1)})h_{(2)}) = \varphi(g\delta_h),$$

where $\delta_h = \varphi'(h_{(1)})h_{(2)}$. Therefore, $\varphi' S(g)\varphi(h) = \varphi(g\delta_h)$ for all $h,g \in H$.

We claim that there is an element $\delta \in H$ such that $\delta_h = \varphi(h)\delta$ for all $h \in H$. Indeed, for any $h' \in H$

$$\begin{aligned}\varphi(g\varphi(h')\delta_h) &= \varphi(h')\varphi(g\delta_h) = \varphi(h')\varphi' S(g)\varphi(h)\\ &= \varphi(h)\varphi' S(g)\varphi(h') = \varphi(h)\varphi(g\delta_{h'})\\ &= \varphi(g\varphi(h)\delta_{h'}),\end{aligned}$$

then $\varphi(h')\delta_h = \varphi(h)\delta_{h'}$ for all $h, h' \in H$, since φ is faithful. Choose an $h' \in H$ such that $\varphi(h') = 1$ and that denote $\delta = \delta_{h'}$. Then, $\delta_h = \varphi(h)\delta$.

If we apply ε, we obtain

$$\begin{aligned} \varphi(h)\varepsilon(\delta) &= \varepsilon(\delta_h) = \varepsilon(\varphi'(h_{(1)})h_{(2)}) \\ &= \varphi'(h_{(1)})\varepsilon(h_{(2)}) = \varphi'(h_{(1)}\varepsilon(h_{(2)})) \\ &= \varphi'(h) \end{aligned}$$

for all $h \in H$ and with $\lambda = \varepsilon(\delta)$, we find the desired result. □

Remark 1. *Similarly, the right faithful integral on H is unique up to the scalar. However, it is a pity that the nonzero faithful integrals do not always exist in the infinite-dimensional case, not even for the special infinite-dimensional Hopf algebra case.*

Proposition 2. *There is a unique group-like element $\delta \in H$ such that, for all $h \in H$,*

(1) $(\varphi \otimes id)\Delta(h) = \varphi(h)\delta$.

(2) $\varphi S(a) = \varphi(a\delta)$.

Furthermore, if the antipode S is bijective, then $(id \otimes \psi)\Delta(h) = \psi(h)\delta^{-1}$.

Proof. From the proof of Proposition 1, $\varphi(h)\delta = \delta_h = \varphi'(h_{(1)})h_{(2)}$ and $\varphi' S(g)\varphi(h) = \varphi(g\delta_h)$, we take $\varphi' = \varphi$ and obtain an element $\delta \in H$ such that $(\varphi \otimes id)\Delta(h) = \varphi(h)\delta$ and $\varphi S(h) = \varphi(h\delta)$. This gives the first part of (1) and (2).

If we apply ε and Δ to the first equation, we find $\varepsilon(\delta) = 1$ and $\Delta(\delta) = \delta \otimes \delta$. According to Proposition 4.2 (1) in [1], if $S(\delta)\delta = 1 = \delta S(\delta)$, then $S(\delta) = \delta^{-1}$. Hence, δ is a group-like element.

S flips the coproduct, so, if we let $\psi = \varphi \circ S$, we obtain

$$\begin{aligned} (id \otimes \psi)\Delta(h) &= S^{-1}(S \otimes \psi)\Delta(h) = S^{-1}(S \otimes \varphi \circ S)\Delta(h) \\ &= S^{-1}(id \otimes \varphi)(S \otimes S)\Delta(h) = S^{-1}(id \otimes \varphi)\Delta^{cop}(S(h)) \\ &= S^{-1}(\varphi \otimes id)\Delta(S(h)) \overset{(1)}{=} S^{-1}(\varphi(S(h))\delta) \\ &= \psi(h)\delta^{-1}. \end{aligned}$$

This completes the proof. □

Remark 2. *(1) The square S^2 leaves the coproduct invariant, therefore, it follows that the composition $\varphi \circ S^2$ of the left faithful integral φ with S^2 will again be a left faithful integral. Due to the uniqueness of left faithful integrals, there is a number $\tau \in k$ such that $\varphi \circ S^2 = \tau\varphi$.*

(2) If we apply (2) to Proposition 2 twice, we obtain

$$\varphi\big(S^2(a)\big) = \varphi\big(S(a)\delta\big) = \varphi\big(S(\delta^{-1}a)\big) = \varphi\big((\delta^{-1}a)\delta\big).$$

So, $\varphi\big((\delta^{-1}a)\delta\big) = \tau\varphi(a)$.

3.2. Integral Dual of an Infinite-Dimensional Hopf Quasigroup

Now, we will construct the dual of an infinite-dimensional Hopf quasigroup. This construction is based on the faithful integrals introduced in the previous subsection. Here, we also start by defining the following subspace of the dual space H^*.

Definition 1. *Let φ be a left faithful integral on a Hopf quasigroup H. We define $\widehat{H}$ as the space of the linear functionals on H of the form $\varphi(\,\cdot\, h)$, where $h \in H$, i.e.,*

$$\widehat{H} = \{\varphi(\,\cdot\, h) \mid h \in H\}.$$

Lemma 2. *Let H be a Hopf quasigroup and φ (resp. ψ) be a left (resp. right) integral on H. If $a \in H$, then there is a $b \in H$ such that $\varphi(ax) = \psi(xb)$ for all $x \in H$. Similarly, given $q \in H$, we have $p \in H$ such that $\varphi(xp) = \psi(qx)$ for all $x \in H$.*

Proof. According to Equations (8) and (9) in Lemma 1, we have for any $h, x \in H$,

$$\begin{aligned}(\psi \otimes \varphi)(xq_{(1)} \otimes pS(q_{(2)})) &= \psi(xq_{(1)})\varphi(pS(q_{(2)})) = \varphi\big(p\underline{\psi(xq_{(1)})S(q_{(2)})}\big)\\ &\overset{(9)}{=} \varphi\big(p\psi(x_{(1)}q)x_{(2)}\big) = \psi(x_{(1)}q)\varphi\big(px_{(2)}\big)\\ &= \psi\big(\underline{x_{(1)}\varphi(px_{(2)})}q\big) \overset{(8)}{=} \psi\big(S(p_{(1)})\varphi(p_{(2)}x)q\big)\\ &= \varphi(p_{(2)}x)\psi\big(S(p_{(1)})q\big) = \varphi\Big(\big(\psi\big(S(p_{(1)})q\big)p_{(2)}\big)x\Big)\\ &= \varphi\Big(\big(\psi \circ S \otimes id\big)\big((S^{-1}(q) \otimes 1)\Delta(p)\big)x\Big).\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}(\psi \otimes \varphi)(xq_{(1)} \otimes pS(q_{(2)})) &= \psi(xq_{(1)})\varphi(pS(q_{(2)})) = \psi\Big(x\big(q_{(1)}\varphi\big(pS(q_{(2)})\big)\big)\Big)\\ &= \psi\Big(x\big(id \otimes \varphi \circ S\big)\big(\Delta(q)(1 \otimes S^{-1}(p))\big)\Big).\end{aligned}$$

According to Theorem 4.5 in [5], the Galois maps $T_1 : a \otimes b \mapsto \Delta(a)(1 \otimes b)$ and $T_2 : a \otimes b \mapsto (a \otimes 1)\Delta(b)$ are bijective, so any element in H has the form $(\psi \circ S \otimes id)\big((S^{-1}(q) \otimes 1)\Delta(p)\big)$. Hence, the above calculation will give us the formula $\varphi(ax) = \psi(xb)$ for all $x \in H$.

Similarly, by computing $(\psi \otimes \varphi)\big(S(q_{(2)})x \otimes q_{(1)}S(p)\big)$, we obtain the second assertion. □

Remark 3. *(1) In the proof of the second part, the Galois maps $T_3 : a \otimes b \mapsto \Delta(a)(b \otimes 1)$ and $T_4 : a \otimes b \mapsto (1 \otimes a)\Delta(b)$ are required to be bijective, which follows due to the fact that the antipode S is bijective, and $T_3^{-1} : a \otimes b \mapsto b_{(2)} \otimes S^{-1}(b_{(1)})a$ and $T_4^{-1} : a \otimes b \mapsto bS^{-1}(a_{(2)}) \otimes a_{(1)}$.*

(2) From Lemma 2, we obtain that

$$\begin{aligned}&\widehat{H} = \{\varphi(\cdot h) \mid h \in H\} = \{\psi(h\cdot) \mid h \in H\}.\\ \text{and}\quad &\{\varphi(h\cdot) \mid h \in H\} = \{\psi(\cdot h) \mid h \in H\}.\end{aligned}$$

We need the following assumption to construct the dual:

Assumption 1.

$$\varphi\big((\,\cdot\, h)h'\big), \varphi\big(h'(h\,\cdot\,)\big) \in \widehat{H}, \quad \forall h, h' \in H. \tag{10}$$

Remark 4. *Following this assumption, Proposition 2 (2), and Lemma 2, we have*

$$\{\varphi(\,\cdot\, h) \mid h \in H\} = \{\varphi(h\,\cdot\,) \mid h \in H\}, \quad \forall h \in H.$$

Therefore,

$$\widehat{H} = \{\varphi(\,\cdot\, h) \mid h \in H\} = \{\psi(h\,\cdot\,) \mid h \in H\} = \{\varphi(h\,\cdot\,) \mid h \in H\} = \{\psi(\,\cdot\, h) \mid h \in H\}$$

and $\psi\big((\,\cdot\, h)h'\big), \psi\big(h'(h\,\cdot\,)\big) \in \widehat{H}$.

We start by making $\widehat{H}$ into an algebra by dualizing the coproduct.

Proposition 3. *For $w, w' \in \widehat{H}$, we can define a linear functional ww' on H with the formula*

$$(ww')(h) = (w \otimes w')\Delta(h), \quad \forall h \in H. \tag{11}$$

Then, $ww' \in \widehat{H}$. This product in $\widehat{H}$ is associative and nondegenerate.

Proof. Let $w, w' \in \widehat{H}$ and assume that $w' = \varphi(\,\cdot\, m)$ with $m \in H$. We have

$$\begin{aligned}(ww')(h) &= (w \otimes \varphi(\,\cdot\, m))\Delta(h) = (w \otimes \varphi)\big(\Delta(h)(1 \otimes m)\big)\\ &= w\big(h_{(1)}\varphi(h_{(2)}m)\big) \overset{(8)}{=} w\Big(S^{-1}\big(m_{(1)}\varphi(hm_{(2)})\big)\Big)\\ &= \varphi\Big(h\big(wS^{-1}(m_{(1)})m_{(2)}\big)\Big)\end{aligned}$$

We see that the product ww' is well-defined as a linear functional on H, and it has the form $\varphi(\,\cdot\, g)$, where $g = wS^{-1}(m_{(1)})m_{(2)}$. Hence, $ww' \in \widehat{H}$. Therefore, we have defined a product in $\widehat{H}$.

The associativity of this product in $\widehat{H}$ is a consequence of the co-associativity of the Δ in H.

To prove that the product is nondegenerate, assume that $ww' = 0$ for all $w \in \widehat{H}$. From the above calculation, for any $h \in H$, $0 = (ww')(h) = \varphi\Big(h\big(wS^{-1}(m_{(1)})m_{(2)}\big)\Big)$, then $wS^{-1}(m_{(1)})m_{(2)} = 0$ because of the faithfulness of φ. This implies $wS^{-1}(m) = 0$ for all $w \in \widehat{H}$, i.e., $\varphi(S^{-1}(m)h) = 0$ for all $h \in H$. We conclude that $S^{-1}(m) = 0$ then $m = 0$, i.e., $w' = 0$. Similarly, $ww' = 0$ for all $w' \in \widehat{H}$ implies $w = 0$. □

Remark 5. *Under Assumption 1, the elements of $\widehat{H}$ can be expressed in four different forms. When we use these different forms in the definition of a product in $\widehat{H}$, we obtain the following useful expressions:*

(1) $w\varphi(\,\cdot\, a) = \varphi(\,\cdot\, b)$ *with* $b = wS^{-1}(a_{(1)})a_{(2)}$; *(2)* $w\varphi(a\,\cdot\,) = \varphi(c\,\cdot\,)$ *with* $c = wS(a_{(1)})a_{(2)}$.

(2) $\psi(\,\cdot\, a)w = \psi(\,\cdot\, d)$ *with* $d = a_{(1)}wS(a_{(2)})$; *(4)* $\psi(a\,\cdot\,)w = \psi(e\,\cdot\,)$ *with* $e = a_{(1)}wS^{-1}(a_{(2)})$.

Moreover, the multiplier algebra $M(\widehat{H})$ of $\widehat{H}$ can be identified with the space H^*. Indeed, for $f \in H^*$ and $w \in \widehat{H}$, $fw, wf \in \widehat{H}$; the co-unit ε, as a linear functional on H, is in fact the unit in the multiplier algebra $M(\widehat{H})$; $fw = 0$ (resp. $wf = 0$) for all $w \in \widehat{H}$ implies $f = 0$.

Let us now define the comultiplication $\widehat{\Delta}$ on $\widehat{H}$. Roughly speaking, the coproduct is dual to the multiplication in H in the sense that

$$\langle \widehat{\Delta}(w), x \otimes y\rangle = \langle w, xy\rangle, \quad \forall x, y \in H.$$

Definition 2. *Let $w_1, w_2 \in \widehat{H}$. Then, we put*

$$\langle (w_1 \otimes 1)\widehat{\Delta}(w_2), x \otimes y\rangle = \langle w_1 \otimes w_2, x_{(1)} \otimes x_{(2)}y\rangle \tag{12}$$

$$\langle \widehat{\Delta}(w_1)(1 \otimes w_2), x \otimes y\rangle = \langle w_1 \otimes w_2, xy_{(1)} \otimes y_{(2)}\rangle \tag{13}$$

for all $x, y \in H$.

We will first show that the functionals in Definition 2 are well-defined and then do so again in $\widehat{H} \otimes \widehat{H}$.

Lemma 3. *$(w_1 \otimes 1)\widehat{\Delta}(w_2), \widehat{\Delta}(w_1)(1 \otimes w_2) \in \widehat{H} \otimes \widehat{H}$. These above two formulas define $\widehat{\Delta}(w)$ as a multiplier in $M(\widehat{H} \otimes \widehat{H})$ for all $w \in \widehat{H}$.*

Proof. Let $w_1 = \psi(a\,\cdot\,)$ and $w_2 = \psi(b\,\cdot\,)$, where $a, b \in H$. For any $x, y \in H$, we have

$$\begin{aligned}\langle (w_1 \otimes 1)\widehat{\Delta}(w_2), x \otimes y\rangle &= \langle w_1 \otimes w_2, x_{(1)} \otimes x_{(2)}y\rangle \\ &= \psi(ax_{(1)})\psi\big(b(x_{(2)}y)\big) = \psi\big(b(\psi(ax_{(1)})x_{(2)}y)\big) \\ &\overset{(9)}{=} \psi\big(b(\psi(a_{(1)}x)S^{-1}(a_{(2)})y)\big) = \psi(a_{(1)}x)\psi\big(b(S^{-1}(a_{(2)})y)\big) \\ &= \Big(\psi(a_{(1)}\,\cdot\,) \otimes \psi\big(b(S^{-1}(a_{(2)})\,\cdot\,)\big)\Big)(x \otimes y).\end{aligned}$$

According to the assumption, we obtain that $(w_1 \otimes 1)\widehat{\Delta}(w_2)$ is a well-defined element in $\widehat{H} \otimes \widehat{H}$. It is similar to $\widehat{\Delta}(w_1)(1 \otimes w_2)$.

Using the fact that the product in $\widehat{H}$ is dual to the coproduct in H and that Δ in H is co-associative, it easily follows that $\big((w_1 \otimes 1)\widehat{\Delta}(w_2)\big)(1 \otimes w_3) = (w_1 \otimes 1)\big(\widehat{\Delta}(w_2)(1 \otimes w_3)\big)$. Therefore, $\widehat{\Delta}(w)$ is defined as a two-sided multiplier in $M(\widehat{H} \otimes \widehat{H})$. □

Proposition 4. *$\widehat{\Delta} : \widehat{H} \longrightarrow M(\widehat{H} \otimes \widehat{H})$ is an algebra homomorphism, as is $(1 \otimes w_1)\widehat{\Delta}(w_2)$, $\widehat{\Delta}(w_1)(w_2 \otimes 1) \in \widehat{H} \otimes \widehat{H}$.*

Proof. It is straightforward that $\widehat{\Delta}$ is an algebra homomorphism, since for all $x, y \in H$

$$\begin{aligned}\langle \widehat{\Delta}(w_1w_2)(1 \otimes w_3), x \otimes y\rangle &= \langle w_1w_2 \otimes w_3, xy_{(1)} \otimes y_{(2)}\rangle \\ &= \langle w_1, x_{(1)}y_{(1)(1)}\rangle\langle w_2, x_{(2)}y_{(1)(2)}\rangle\langle w_3, y_{(2)}\rangle, \\ \langle \widehat{\Delta}(w_1)\widehat{\Delta}(w_2)(1 \otimes w_3), x \otimes y\rangle &= \langle \widehat{\Delta}(w_1)(f \otimes g), x \otimes y\rangle \qquad (\widehat{\Delta}(w_2)(1 \otimes w_3) := f \otimes g\,) \\ &= \langle \widehat{\Delta}(w_1)(1 \otimes g), x_{(1)} \otimes y\rangle\langle f, x_{(2)}\rangle \\ &= \langle w_1 \otimes g, x_{(1)}y_{(1)} \otimes y_{(2)}\rangle\langle f, x_{(2)}\rangle \\ &= \langle w_1, x_{(1)}y_{(1)}\rangle\langle f \otimes g, x_{(2)} \otimes y_{(2)}\rangle \\ &= \langle w_1, x_{(1)}y_{(1)}\rangle\langle \widehat{\Delta}(w_2)(1 \otimes w_3), x_{(2)} \otimes y_{(2)}\rangle \\ &= \langle w_1, x_{(1)}y_{(1)}\rangle\langle w_2 \otimes w_3, x_{(2)}y_{(2)(1)} \otimes y_{(2)(2)}\rangle \\ &= \langle w_1, x_{(1)}y_{(1)}\rangle\langle w_2, x_{(2)}y_{(2)(1)}\rangle\langle w_3, y_{(2)(2)}\rangle.\end{aligned}$$

Due to the co-associativity of the Δ in H, we obtain $\widehat{\Delta}(w_1w_2)(1 \otimes w_3) = \widehat{\Delta}(w_1)\widehat{\Delta}(w_2)(1 \otimes w_3)$ for all $w_3 \in \widehat{H}$. This implies $\widehat{\Delta}(w_1w_2) = \widehat{\Delta}(w_1)\widehat{\Delta}(w_2)$.

With the bijective antipode, the proof of the second assertion is similar to the proof of Lemma 3. □

Let $w \in \widehat{H}$ and assume $w = \varphi(\,\cdot\, a)$ with $a \in H$. Define $\widehat{\varepsilon}(w) = \varphi(a) = w(1_H)$. Then, $\widehat{\varepsilon}$ is a co-unit on $(\widehat{H}, \widehat{\Delta})$ as follows.

Proposition 5. *$\widehat{\varepsilon} : \widehat{H} \longrightarrow k$ is an algebra homomorphism satisfying*

$$(id \otimes \widehat{\varepsilon})\big((w_1 \otimes 1)\widehat{\Delta}(w_2)\big) = w_1w_2 \tag{14}$$

$$(\widehat{\varepsilon} \otimes id)\big(\widehat{\Delta}(w_1)(1 \otimes w_2)\big) = w_1w_2 \tag{15}$$

for all $w_1, w_2 \in \widehat{H}$.

Proof. Firstly, let $w_1 = \varphi(a\,\cdot\,)$ and $w_2 = \varphi(b\,\cdot\,)$. Then, $w_1w_2 = \varphi(c\,\cdot\,)$ with $c = \varphi\big(aS(b_{(1)})\big)b_{(2)}$. Therefore, if $\psi = \varphi \circ S$, we have

$$\begin{aligned}\widehat{\varepsilon}(w_1w_2) &= \varphi(c) = \varphi\big(aS(b_{(1)})\big)\varphi(b_{(2)}) \\ &= \varphi\big(aS(b_{(1)}\varphi(b_{(2)}))\big) = \varphi(a)\varphi(b) \\ &= \widehat{\varepsilon}(w_1)\widehat{\varepsilon}(w_2).\end{aligned}$$

Secondly, let $w_1 = \psi(a\,\cdot\,)$ and $w_2 = \psi(b\,\cdot\,)$. Then, we have

$$\begin{aligned}(w_1 \otimes 1)\widehat{\Delta}(w_2) &= \psi(a_{(1)}\,\cdot\,) \otimes \psi\big(b(S^{-1}(a_{(2)})\,\cdot\,)\big),\\ \psi(a\,\cdot\,)\psi(b\,\cdot\,) &= \psi(a_{(1)}\,\cdot\,)\psi\big(bS^{-1}(a_{(2)})\big).\end{aligned}$$

Hence,

$$\begin{aligned}(id \otimes \widehat{\varepsilon})\big((w_1 \otimes 1)\widehat{\Delta}(w_2)\big) &= \psi(a_{(1)}\,\cdot\,)\psi\big(b(S^{-1}(a_{(2)})1)\big)\\ &= \psi(a_{(1)}\,\cdot\,)\psi\big(bS^{-1}(a_{(2)})\big)\\ &= w_1 w_2.\end{aligned}$$

Finally, the second formula is proven in a similar way, in this case, by letting $w_1 = \varphi(\,\cdot\, a)$ and $w_2 = \varphi(\,\cdot\, b)$. □

Let $\widehat{S} : \widehat{H} \longrightarrow \widehat{H}$ be the dual to the antipode of H, i.e., $\widehat{S}(w) = w \circ S$. Then, it is easy to see that $\widehat{S}(w) \in \widehat{H}$, and we have the following property.

Proposition 6. *$\widehat{S}$ is antimultiplicative and co-antimultiplicative such that*

$$\begin{aligned}w' \otimes w &= (m \otimes id)(id \otimes \widehat{S} \otimes id)\Big((id \otimes \widehat{\Delta})\big((w' \otimes 1)\widehat{\Delta}(w)\big)\Big)\\ &= (m \otimes id)(\widehat{S} \otimes id \otimes id)\Big((id \otimes \widehat{\Delta})\big(\widehat{\Delta}(w)(\widehat{S}^{-1}(w') \otimes 1)\big)\Big)\\ &= (id \otimes m)(id \otimes \widehat{S} \otimes id)\Big((\widehat{\Delta} \otimes id)\big(\widehat{\Delta}(w')(1 \otimes w)\big)\Big)\\ &= (id \otimes m)(id \otimes id \otimes \widehat{S})\Big((\widehat{\Delta} \otimes id)\big((1 \otimes \widehat{S}^{-1}(w))\widehat{\Delta}(w')\big)\Big).\end{aligned}$$

Proof. For $w_1, w_2 \in \widehat{H}$ and any $x \in H$,

$$\begin{aligned}\langle \widehat{S}(w_1 w_2), x\rangle &= \langle w_1 w_2, S(x)\rangle = \langle w_1, S(x_2)\rangle\langle w_2, S(x_1)\rangle\\ &= \langle \widehat{S}(w_1), x_2\rangle\langle \widehat{S}(w_2), x_1\rangle = \langle \widehat{S}(w_2)\widehat{S}(w_1), x\rangle\end{aligned}$$

This implies that $\widehat{S}$ is antimultiplicative.

$$\begin{aligned}\langle \widehat{\Delta}\widehat{S}(w_1)(1 \otimes S(w_2)), x \otimes y\rangle &\overset{(13)}{=} \langle \widehat{S}(w_1) \otimes \widehat{S}(w_2), xy_{(1)} \otimes y_{(2)}\rangle\\ &= \langle w_1, S(xy_{(1)})\rangle\langle w_2, S(y_{(2)})\rangle\\ &= \langle w_1, S(y)_{(2)}S(x)\rangle\langle w_2, S(y)_{(1)}\rangle\\ &= \langle w_2 \otimes w_1, S(y)_{(1)} \otimes S(y)_{(2)}S(x)\rangle\\ &\overset{(12)}{=} \langle (w_2 \otimes 1)\widehat{\Delta}(w_1), S(y) \otimes S(x)\rangle\\ &= \langle (\widehat{S} \otimes \widehat{S})\widehat{\Delta}^{cop}(w_1)(1 \otimes S(w_2)), x \otimes y\rangle,\end{aligned}$$

We conclude that $\widehat{S}$ is co-antimultiplicative.

Finally, we show that $w' \otimes w = (m \otimes id)(id \otimes \widehat{S} \otimes id)\Big((id \otimes \widehat{\Delta})\big((w' \otimes 1)\widehat{\Delta}(w)\big)\Big)$. The other three formulas are similar.

$$\begin{aligned}&\langle (m \otimes id)(id \otimes \widehat{S} \otimes id)\Big((id \otimes \widehat{\Delta})\big((w' \otimes 1)\widehat{\Delta}(w)\big)\Big), x \otimes y\rangle\\ \overset{(11)}{=}\ &\langle (id \otimes \widehat{\Delta})\big((w' \otimes 1)\widehat{\Delta}(w)\big), x_{(1)} \otimes S(x_{(2)}) \otimes y\rangle\\ =\ &\langle \big((w' \otimes 1)\widehat{\Delta}(w)\big), x_{(1)} \otimes S(x_{(2)})y\rangle\\ \overset{(12)}{=}\ &\langle w' \otimes w, x_{(1)(1)} \otimes x_{(1)(2)}(S(x_{(2)})y)\rangle\\ =\ &\langle w' \otimes w, x \otimes y\rangle.\end{aligned}$$

This completes the proof. □

The equation in Proposition 6 can be expressed by generalized Sweedler notation as follows:

$$\begin{aligned} w' \otimes w &= w'\widehat{S}(w_{(1)})w_{(2)(1)} \otimes w_{(2)(2)} = w'w_{(1)}\widehat{S}(w_{(2)(1)}) \otimes w_{(2)(2)} \\ &= w'_{(1)(1)} \otimes \widehat{S}(w'_{(1)(2)})w'_{(2)}w = w'_{(1)(1)} \otimes w'_{(1)(2)}\widehat{S}(w'_{(2)})w. \end{aligned}$$

As a consequence, the antipode $\widehat{S}$ also satisfies

$$\begin{aligned} m(id \otimes \widehat{S})\big((w_1 \otimes 1)\widehat{\Delta}(w_2)\big) &= \widehat{\varepsilon}(w_2)w_1, \\ m(\widehat{S} \otimes id)\big(\widehat{\Delta}(w_1)(1 \otimes w_2)\big) &= \widehat{\varepsilon}(w_1)w_2. \end{aligned}$$

In fact, there is another way to prove this:

$$\begin{aligned} \langle m(id \otimes \widehat{S})\big((w_1 \otimes 1)\widehat{\Delta}(w_2)\big), x\rangle &= \langle (id \otimes \widehat{S})\big((w_1 \otimes 1)\widehat{\Delta}(w_2)\big), x_{(1)} \otimes x_{(2)}\rangle \\ &= \langle \big((w_1 \otimes 1)\widehat{\Delta}(w_2)\big), x_{(1)} \otimes S(x_{(2)})\rangle \\ &\overset{(12)}{=} \langle w_1 \otimes w_2), x_{(1)(1)} \otimes x_{(1)(2)}S(x_{(2)})\rangle \\ &= \langle w_1 \otimes w_2, x \otimes 1\rangle \\ &= \widehat{\varepsilon}(w_2)w_1. \end{aligned}$$

Let ψ be a right faithful integral on H. For $w = \psi(a\,\cdot\,)$, we set $\widehat{\varphi}(w) = \varepsilon(a)$. Then, we have the following result:

Proposition 7. *$\widehat{\varphi}$ Defined above is a left faithful integral on $\widehat{H}$.*

Proof. It is clear that $\widehat{\varphi}$ is nonzero. Assume $w_1 = \psi(a\,\cdot\,)$ and $w_2 = \psi(b\,\cdot\,)$ with $a, b \in H$. Then,

$$(w_1 \otimes 1)\widehat{\Delta}(w_2) = \psi(a_{(1)}\,\cdot\,) \otimes \psi\big(b(S^{-1}(a_{(2)})\,\cdot\,)\big).$$

Therefore, we have

$$\begin{aligned} (id \otimes \widehat{\varphi})\Big((w_1 \otimes 1)\widehat{\Delta}(w_2)\Big) &= \psi(a_{(1)}\,\cdot\,) \otimes \widehat{\varphi}\psi\big(b(S^{-1}(a_{(2)})\,\cdot\,)\big) \\ &= \psi(a\,\cdot\,)\varepsilon(b) = \widehat{\varphi}(w_2)w_1. \end{aligned}$$

Next, we show that $\widehat{\varphi}$ is faithful. If $w_1, w_2 \in \widehat{H}$, and assuming that $w_1 = \psi(a\,\cdot\,)$ with $a \in H$, we have $w_1w_2 = \psi\big(a_{(1)}w_2S^{-1}(a_{(2)})\,\cdot\,\big)$. Therefore, $\widehat{\varphi}(w_1w_2) = w_2S^{-1}(a)$. If this is 0 for all $a \in H$, then $w_2 = 0$, while, if this is 0 for all w_2, then $a = 0$. This proves the faithfulness of $\widehat{\varphi}$. □

Now, we introduce an algebraic structure, a multiplier Hopf coquasigroup, generalizing the ordinary Hopf coquasigroup to a nonunital case. Let A be an (associative) algebra, which may not have a unit, but the product, seen as a bilinear form, is nondegenerated.

Definition 3. *A multiplier Hopf coquasigroup is a nondegenerate associative algebra A equipped with the algebra homomorphisms $\Delta : A \longrightarrow M(A \otimes A)$(coproduct), $\varepsilon : A \longrightarrow k$(co-unit) and a linear map $S : A \longrightarrow A$ (antipode) such that*

(1) $T_1(a \otimes b) = \Delta(a)(1 \otimes b)$ and $T_2(a \otimes b) = (a \otimes 1)\Delta(b)$ belong to $A \otimes A$ for any $a, b \in A$;
(2) The co-unit satisfies $(\varepsilon \otimes id)T_1(a \otimes b) = ab = (id \otimes \varepsilon)T_2(a \otimes b)$;

(3) S is antimultiplicative and anticomultiplicative such that for any $a, b \in A$

$$S(a_{(1)})a_{(2)(1)} \otimes a_{(2)(2)} = 1 \otimes a = a_{(1)}S(a_{(2)(1)}) \otimes a_{(2)(2)}, \tag{16}$$
$$a_{(1)(1)} \otimes S(a_{(1)(2)})a_{(2)} = a \otimes 1 = a_{(1)(1)} \otimes a_{(1)(2)}S(a_{(2)}). \tag{17}$$

A the multiplier Hopf coquasigroup (A, Δ) is called regular *if the antipode S is bijective.*

Remark 6. *(1) In multiplier Hopf coquasigroup (A, Δ), T_1 and T_2 are bijective. If (A, Δ) is regular, then T_3 and T_4 are as well. In fact, from (3) in Definition 3, we can easily obtain*

$$m(id \otimes S)\big((a \otimes 1)\Delta(b)\big) = \varepsilon(b)a,$$
$$m(S \otimes id)\big(\Delta(a)(1 \otimes b)\big) = \varepsilon(a)b.$$

(2) Equations (16) and (17) make sense. Take (16), for example, and (17) is similar.

$$\begin{aligned} b \otimes ac &= ba_{(1)}S(a_{(2)(1)}) \otimes a_{(2)(2)}c \\ &= (m \otimes id)(id \otimes S \otimes id)\Big((id \otimes \Delta)\big((b \otimes 1)\Delta(a)\big)(1 \otimes 1 \otimes c)\Big). \end{aligned}$$

$ba_{(1)} \otimes a_{(2)} = (b \otimes 1)\Delta(a) \in A \otimes A$, and then $a_{(2)(1)} \otimes a_{(2)(2)}c \in A \otimes A$. Therefore, $b \otimes ac = ba_{(1)}S(a_{(2)(1)}) \otimes a_{(2)(2)}c$ holds for all $b, c \in A$. This implies that $a_{(1)}S(a_{(2)(1)}) \otimes a_{(2)(2)} = 1 \otimes a$.

$$\begin{aligned} b \otimes ac &= S(a_{(1)})a_{(2)(1)}b \otimes a_{(2)(2)}c = S(a_{(1)})a_{(2)(1)}x_{(1)} \otimes a_{(2)(2)}x_{(2)}y \\ &= (m \otimes id)(S \otimes id \otimes id)\Big((id \otimes \Delta)\big(\Delta(a)(1 \otimes x)\big)(1 \otimes 1 \otimes y)\Big), \end{aligned}$$

where $b \otimes c = \Delta(x)(1 \otimes y)$. $b \otimes ac = S(a_{(1)})a_{(2)(1)}b \otimes a_{(2)(2)}c$ for all $b, c \in A$ implies that $1 \otimes a = S(a_{(1)})a_{(2)(1)} \otimes a_{(2)(2)}$.

(3) The comultiplication may be not co-associative. The multiplier Hopf coquasigroup weakens the co-associativity of the coproduct in the multiplier Hopf algebra, while the algebraic quantum hypergroup in [6] weakens the homomorphism of the coproduct. This is the main difference.

Because a (infinite-dimensional) Hopf quasigroup (H, Δ) has the unit 1_H, there is a special element $\varphi = \varphi(\,\cdot\, 1_H) \in \widehat{H}$ such that for $w \in \widehat{H}$

$$(w\varphi)(h) = (w \otimes \varphi)\Delta(h) = \varphi(h)w(1) = \widehat{\varepsilon}(w)\varphi(h).$$

This implies that $w\varphi = \widehat{\varepsilon}(w)\varphi$. We call φ a co-integral in $(\widehat{H}, \widehat{\Delta})$.

Analogous to the multiplier Hopf algebra case in [3], we say that a regular multiplier Hopf coquasigroup with a faithful integral (A, Δ) is of the discrete type if there is a nonzero element $\xi \in A$ such that $a\xi = \varepsilon(a)\xi$ for all $a \in A$. This element is called a *left co-integral*. Similarly, a *right co-integral* is a nonzero element $\eta \in A$ such that $\eta a = \varepsilon(a)\eta$. The antipode will turn a left co-integral into a right one and a right one into a left one.

Then, we obtain the main result of this section.

Theorem 1. *Let (H, Δ) be an infinite-dimensional Hopf quasigroup with a faithful integral φ and a bijective antipode S. Then, under Assumption 1, the integral dual $(\widehat{H}, \widehat{\Delta})$ is a discrete multiplier Hopf coquasigroup with a faithful integral.*

4. Multiplier Hopf Coquasigroup

4.1. Motivating Example

In the following, we first introduce the motivating example, where Assumption 1 naturally holds. Then, we comment directly on the multiplier Hopf coquasigroups.

Example 1. *Let G be an infinite (IP) quasigroup with an identity element e, which is by definition $u^{-1}(uv) = v = (vu)u^{-1}$ for all $u, v \in G$. We have that the quasigroup algebra kG is a Hopf quasigroup with the structure shown on the base element $\{u | u \in G\}$*

$$\Delta(u) = u \otimes u, \quad \varepsilon(u) = 1, \quad S(u) = u^{-1}.$$

The function $\delta_u, u \in G$ on kG is given by $\delta_u(v) = \delta_{u,v}$, where $\delta_{u,v}$ is the Kronecker delta. Then, δ_e is the left and right integral on kG.

The integral dual $k(G) = \widehat{kG} = \text{span}\{\delta_e(\cdot\, u) | u \in G\} = \text{span}\{\delta_{u^{-1}} | u \in G\} = \text{span}\{\delta_e(u\,\cdot) | u \in G\}$, where 'span' means the linear span of a set of element. $\delta_e\big((\cdot\, u)v\big) = \delta_{v^{-1}u^{-1}} \in k(G)$ and $\delta_e\big(u(v\,\cdot)\big) = \delta_{v^{-1}u^{-1}} \in k(G)$. Assumption 1

naturally holds. Then, $(k(G), \widehat{\Delta}, \widehat{\varepsilon}, \widehat{S})$ is a multiplier Hopf coquasigroup with the structure that follows.

As an algebra, $k(G)$ is a nondegenerate algebra with the product

$$\delta_u \delta_v = \delta_{u,v} \delta_v,$$

and $1 = \sum_{u \in G} \delta_u$ is the unit in $M(k(G))$. The coproduct, co-unit, and antipode are given by

$$\widehat{\Delta}(\delta_u) = \sum_{v \in G} \delta_v \otimes \delta_{v^{-1}u}, \quad \widehat{\varepsilon}(\delta_u) = \delta_{u,e}, \quad \widehat{S}(\delta_u) = \delta_{u^{-1}}.$$

By the definition of $\widehat{\varphi}$, we obtain the left integral on $k(G)$ is the function that maps every δ_u to 1.

As in the theory of multiplier Hopf algebra in [4], we also can define a multiplier Hopf $*$-coquasigroup (A, Δ) over $\mathbb{C}$, in which (A, Δ) is a regular multiplier Hopf coquasigroup, and the coproduct, co-unit, and antipode are compatible with the involution $*$, i.e.,

(1) The comultiplication Δ is also a $*$-homomorphism (i.e., $\Delta(a^*) = \Delta(a)^*$);
(2) $\varepsilon(a^*) = \overline{\varepsilon(a)}$, where $\overline{(\,\cdot\,)}$ means the conjugation of complex numbers;
(3) $S(S(a)^*)^* = a$.

Example 2. *In Example 1, if $k = \mathbb{C}$, then $\mathbb{C}(G)$ is a multiplier Hopf $*$-coquasigroup.*

Proposition 8. *Let (A, Δ) be a multiplier Hopf ($*$-)coquasigroup. Then, (A, Δ) is the multiplier Hopf ($*$-)algebra introduced in [4], if and only if the comultiplication Δ is co-associative.*

Proposition 9. *If a multiplier Hopf coquasigroup (A, Δ) has the unit 1_A, then (A, Δ) is the usual Hopf coquasigroup.*

Following these two results, the multiplier Hopf coquasigroup can be considered as the generalization of the multiplier Hopf algebra and Hopf coquasigroup. Naturally, we can define a flexible, alternative, and Moufang multiplier Hopf coquasigroup.

A multiplier Hopf coquasigroup (A, Δ) is called *flexible* if

$$a_{(1)} a_{(2)(2)} \otimes a_{(2)(1)} = a_{(1)(1)} a_{(2)} \otimes a_{(1)(2)}, \quad \forall a \in A, \tag{18}$$

and *alternative* if also

$$a_{(1)} a_{(2)(1)} \otimes a_{(2)(2)} = a_{(1)(1)} a_{(1)(2)} \otimes a_{(2)}, \tag{19}$$

$$a_{(1)} \otimes a_{(2)(1)} a_{(2)(2)} = a_{(1)(1)} \otimes a_{(1)(2)} a_{(2)}, \quad \forall a \in A. \tag{20}$$

A is called *Moufang* if

$$a_{(1)} a_{(2)(2)(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)(2)} = a_{(1)(1)(1)} a_{(1)(2)} \otimes a_{(1)(1)(2)} \otimes a_{(2)}, \quad \forall a \in A, \tag{21}$$

Remark 7. *(1) According to the 'cover technique' introduced in [7], these four equations make sense.*

(2) From the dual, we can obtain that the integral dual $(\widehat{H}, \widehat{\Delta})$ of an infinite-dimensional flexible (resp. alternative, Moufang) Hopf quasigroup (H, Δ) is a flexible (resp. alternative, Moufang) multiplier Hopf coquasigroup.

4.2. Integrals on a Multiplier Hopf Coquasigroup

Let (A, Δ) be a regular multiplier Hopf coquasigroup with a faithful integral φ. Just as in the case of an algebraic quantum group (see Proposition 2.6 in [8]) or algebraic quantum hypergroups (see Proposition 1.6 in [6]), we first show that the multiplier Hopf coquasigroup (A, Δ) must have local units in the sense of the following proposition.

Proposition 10. *Let (A, Δ) be a regular multiplier Hopf coquasigroup with a nonzero integral φ. Given finite numbers of the elements $\{a_1, a_2, \cdots, a_n\}$, there exists an element $e \in A$ such that $a_i e = a_i = e a_i$ for all i.*

Proof. It is similar to the proof of Proposition 1.6 in [6]. Set the linear space

$$V = \{(aa_1, aa_2, \cdots, aa_n, a_1a, a_2a, \cdots, a_na) \mid a \in A\} \subseteq A^{2n}.$$

Consider a linear functional on A^{2n} that is zero on V. This means that we have the functionals w_i and ρ_i on A for $i = 1, 2, \cdot s, n$, such that

$$\sum_{i=1}^{n} w_i(aa_i) + \sum_{i=1}^{n} \rho_i(a_i a) = 0 \quad \text{for all } a \in A.$$

Then, for all $x, a \in A$, we have

$$\begin{aligned} & x\Big(\sum_{i=1}^{n}(w_i \otimes id)\big(\Delta(a)(a_i \otimes 1)\big) + \sum_{i=1}^{n}(\rho_i \otimes id)\big((a_i \otimes 1)\Delta(a)\big)\Big) \\ = & \sum_{i=1}^{n}(w_i \otimes id)\big((1 \otimes x)\Delta(a)(a_i \otimes 1)\big) + \sum_{i=1}^{n}(\rho_i \otimes id)\big((a_i \otimes 1)(1 \otimes x)\Delta(a)\big) \\ = & \ 0, \end{aligned}$$

since $(1 \otimes x)\Delta(a) \in A \otimes A$. Because the product in A is nondegenerate, we have for all $a \in A$ that

$$\sum_{i=1}^{n}(w_i \otimes id)\big(\Delta(a)(a_i \otimes 1)\big) + \sum_{i=1}^{n}(\rho_i \otimes id)\big((a_i \otimes 1)\Delta(a)\big) = 0.$$

Now, applying φ to this expression, we obtain

$$\varphi(a)\Big(\sum_{i=1}^{n} w_i(a_i) + \sum_{i=1}^{n} \rho_i(a_i)\Big) = 0 \quad \text{for all } a \in A.$$

As the integral φ is nonzero, we have

$$\sum_{i=1}^{n} w_i(a_i) + \sum_{i=1}^{n} \rho_i(a_i) = 0.$$

So, any linear functional on A^{2n} that is zero on the space V is also zero on the vector $(a_1, a_2, \cdots, a_n, a_1, a_2, \cdots, a_n)$. Therefore, $(a_1, a_2, \cdots, a_n, a_1, a_2, \cdots, a_n) \in V$. This means that there exists an element $e \in A$ such that $a_i e = a_i = e a_i$ for all i. □

Recall from [6] that a linear functional f on A is called faithful if, for $a \in A$, we must have $a = 0$ when either $f(ab) = 0$ for all $b \in A$ or $f(ba) = 0$ for all $b \in A$. Then, under faithfulness, we can obtain the following result.

Lemma 4. *Let (A, Δ) be a multiplier Hopf coquasigroup. If f is a faithful linear functional on A, then for any $a \in A$, there is an element $e \in A$ such that*

$$a = (id \otimes f)\big(\Delta(a)(1 \otimes e)\big).$$

Proof. Take $a \in A$ and set $V = \{(id \otimes f)\big(\Delta(a)(1 \otimes b)\big) \mid b \in A\}$, we need to show $a \in V$. Suppose that $a \notin V$; then, on A, there is a functional $w \in A^*$ such that $w(a) \neq 0$ while $w|_V = 0$, i.e.,

$$0 = w\Big((id \otimes f)\big(\Delta(a)(1 \otimes b)\big)\Big) = f\Big(\big((w \otimes id)\Delta(a)\big)b\Big) \quad \text{for all } b \in A.$$

Observe that $(w \otimes id)\Delta(a) \in M(A)$, and that it does not necessarily belong to A. However, we obtain $f\Big(\big((w \otimes id)\Delta(a)\big)b'b''\Big) = 0$ for all $b', b'' \in A$, and, by the faithfulness of f, we must have $\big((w \otimes id)\Delta(a)\big)b' = 0$ for all $b' \in A$.

If we apply the co-unit, $w(a)\varepsilon(b') = 0$ for all $b' \in A$, and, hence, $w(a) = 0$. This is a contradiction. □

Similarly, we have for a regular multiplier Hopf coquasigroup (A, Δ),

$$\begin{aligned} a &\in \{(id \otimes f)\big((1 \otimes b)\Delta(a)\big) \mid b \in A\} \\ a &\in \{(f \otimes id)\big((b \otimes 1)\Delta(a)\big) \mid b \in A\} \\ a &\in \{(f \otimes id)\big(\Delta(a)(b \otimes 1)\big) \mid b \in A\} \end{aligned}$$

for any faithful $f \in A^*$. In particular, when we assume that a left integral φ is faithful, then

$$\begin{aligned} A &= \text{span}\{(id \otimes \varphi)\big(\Delta(a)(1 \otimes b)\big) \mid a, b \in A\}, \\ A &= \text{span}\{(id \otimes \varphi)\big((1 \otimes a)\Delta(b)\big) \mid a, b \in A\}, \end{aligned}$$

where 'span' means the linear span of a set of an element.

Next, we give some equations on the left and right integrals.

Proposition 11. *Let φ (resp. ψ) be a left (resp. right) integral on A; then, for $a, b \in A$,*

$$a_{(1)}\varphi\big(a_{(2)}S(b)\big) = \varphi\big(aS(b_{(1)})\big)b_{(2)}, \qquad a_{(1)}\varphi(ba_{(2)}) = S(b_{(1)})\varphi(b_{(2)}a). \tag{22}$$

$$\psi\big(S(a)b_{(1)}\big)b_{(2)} = \psi\big(S(a_{(2)})b\big)a_{(1)}, \qquad \psi(a_{(1)}b)a_{(2)} = \psi(ab_{(1)})S(b_{(2)}). \tag{23}$$

Proof. We prove the first two equations on φ, and the others are similar.

$$\begin{aligned} a_{(1)}\varphi\big(a_{(2)}S(b)\big) &\overset{(23)}{=} a_{(1)}\big(S(b_{(1)(2)})b_{(2)}\big)\varphi\big(a_{(2)}S(b_{(1)(2)})\big) \\ &= \big(a_{(1)}S(b_{(1)(2)})\big)b_{(2)}\varphi\big(a_{(2)}S(b_{(1)(2)})\big) \\ &= \big(a_{(1)}S(b_{(1)})_{(1)}\big)b_{(2)}\varphi\big(a_{(2)}S(b_{(1)})_{(2)}\big) \\ &= \big(aS(b_{(1)})\big)_{(1)}b_{(2)}\varphi\big((aS(b_{(1)}))_{(2)}\big) \\ &= \varphi\big(aS(b_{(1)})\big)b_{(2)}, \end{aligned}$$

and

$$\begin{aligned} a_{(1)}\varphi(ba_{(2)}) &\overset{(22)}{=} \big(S(b_{(1)})b_{(2)(1)}\big)a_{(1)}\varphi\big(b_{(2)(2)}a_{(2)}\big) \\ &= S(b_{(1)})\big(b_{(2)(1)}a_{(1)}\big)\varphi\big(b_{(2)(2)}a_{(2)}\big) \\ &= S(b_{(1)})\big(b_{(2)}a\big)_{(1)}\varphi\big((b_{(2)}a)_{(2)}\big) \\ &= S(b_{(1)})\varphi(b_{(2)}a). \end{aligned}$$

This completes the proof. □

Remark 8. *(1) Following Lemma 4, we can easily check that (22) and (23) make sense.*

(2) These formulas are useful in the following part. We take the first one as an example. When the antipode of (A, Δ) is bijective, $a_{(1)}\varphi\big(a_{(2)}S(b)\big) = \varphi\big(aS(b_{(1)})\big)b_{(2)}$ is equivalent to

$$S(id \otimes \varphi)\big(\Delta(a)(1 \otimes b)\big) = (id \otimes \varphi)\big((1 \otimes a)\Delta(b)\big),$$

which is used to define the antipode in the algebraic quantum hypergroup (see Definition 1.9 in [6]).

Set $x = (id \otimes \varphi)\big(\Delta(a)(1 \otimes b)\big)$ and apply ε to the above equation; we have $\varepsilon\big(S(x)\big) = \varphi(ab) = \varepsilon(x)$. According to Lemma 4, we obtain $\varepsilon \circ S = \varepsilon$.

In the following, we will show the uniqueness of the left faithful integrals.

Theorem 2. *Let φ' be another left faithful integral on (A, Δ); then, $\varphi' = \lambda\varphi$ for some scalar $\lambda \in k$, i.e., the left faithful integral on A is unique up to the scalar.*

Proof. From Proposition 11, we have $a_{(1)}\varphi(ba_{(2)}) = S(b_{(1)})\varphi(b_{(2)}a)$ for all $a, b \in A$. Apply φ' to both expressions in this equation. Because $\varphi' \circ S$ is a right integral, the right-hand side will give

$$\varphi' S(b_{(1)})\varphi(b_{(2)}a) = \varphi(\varphi' S(b_{(1)})b_{(2)}a) = \varphi(\varphi' S(b)1_{M(A)} \cdot a) = \varphi' S(b)\varphi(a).$$

For the left-hand side,

$$\varphi'(a_{(1)}\varphi(ba_{(2)})) = \varphi'(a_{(1)})\varphi(ba_{(2)}) = \varphi(b\varphi'(a_{(1)})a_{(2)}) = \varphi(b\delta_a),$$

where $\delta_a = \varphi'(a_{(1)})a_{(2)}$. Therefore, $\varphi' S(b)\varphi(a) = \varphi(b\delta_a)$ for all $a, b \in H$.

We claim that there is an element $\delta \in M(A)$ such that $\delta_a = \varphi(a)\delta$ for all $a \in A$. Indeed, for any $a' \in A$,

$$\begin{aligned} \varphi(b\varphi(a')\delta_a) &= \varphi(a')\varphi(b\delta_a) = \varphi(a')\varphi' S(b)\varphi(a) \\ &= \varphi(a)\varphi' S(b)\varphi(a') = \varphi(a)\varphi(b\delta_{a'}) \\ &= \varphi(b\varphi(a)\delta_{a'}), \end{aligned}$$

then, $\varphi(a')\delta_a = \varphi(a)\delta_{a'}$ for all $a, a' \in A$ since φ is faithful. Choose an $a' \in A$ such that $\varphi(a') = 1$ and denote that $\delta = \delta_{a'}$; then, $\delta_a = \varphi(a)\delta$.

If we apply ε, we obtain

$$\begin{aligned} \varphi(a)\varepsilon(\delta) &= \varepsilon(\delta_a) = \varepsilon(\varphi'(a_{(1)})a_{(2)}) \\ &= \varphi'(a_{(1)})\varepsilon(a_{(2)}) = \varphi'(a_{(1)}\varepsilon(a_{(2)})) \\ &= \varphi'(a) \end{aligned}$$

for all $a \in A$, and, with $\lambda = \varepsilon(\delta)$, we find the desired result. □

Remark 9. *(1) Similarly, the right faithful integral on A is unique up to the scalar. However, as in the special infinite-dimensional Hopf algebra case, the nonzero faithful integrals do not always exist in an infinite-dimensional case.*

(2) The uniqueness of the faithful integral also provides the uniqueness of the antipode, as in [6].

Proposition 12. *There is a unique invertible element $\delta \in M(A)$ such that for all $a \in A$*

(1) $(\varphi \otimes id)\Delta(a) = \varphi(a)\delta$ *and* $(id \otimes \psi)\Delta(a) = \psi(a)\delta^{-1}$.

(2) $\varphi S(a) = \varphi(a\delta)$.

Proof. In the proofs of Theorem 4.4, $\varphi(a)\delta = \delta_a = \varphi'(a_{(1)})a_{(2)}$ and $\varphi' S(b)\varphi(a) = \varphi(b\delta_a)$, we take $\varphi' = \varphi$ and obtain an element $\delta \in M(A)$ such that $(\varphi \otimes id)\Delta(a) = \varphi(a)\delta$ and $\varphi S(a) = \varphi(a\delta)$. This gives the first part of both (1) and (2).

If we apply ε to the first equation, we find that $\varepsilon(\delta) = 1$. Because S flips the coproduct and if we let $\psi = \varphi \circ S$, we obtain

$$\begin{aligned}(id \otimes \psi)\Delta(a) &= S^{-1}(S \otimes \psi)\Delta(a) = S^{-1}(S \otimes \varphi \circ S)\Delta(a)\\ &= S^{-1}(id \otimes \varphi)(S \otimes S)\Delta(a) = S^{-1}(id \otimes \varphi)\Delta^{cop}(S(a))\\ &= S^{-1}(\varphi \otimes id)\Delta(S(a)) \overset{(1)}{=} S^{-1}(\varphi(S(a))\delta)\\ &= \psi(a)S^{-1}(\delta).\end{aligned}$$

It remains to be proven that $S^{-1}(\delta) = \delta^{-1}$.

If we apply φ to Formula (23), $\psi(a_{(1)}b)a_{(2)} = \psi(ab_{(1)})S(b_{(2)})$, we obtain

$$\begin{aligned}\psi(b)\varphi(a) &= \psi(ab_{(1)})\varphi S(b_{(2)}) = \psi(ab_{(1)}\psi(b_{(2)})) = \psi(a\psi(b)S^{-1}(\delta))\\ &= \psi(b)\psi(aS^{-1}(\delta))\end{aligned}$$

for all $a, b \in A$. Then, $\varphi(a) = \psi(aS^{-1}(\delta))$ for all $a \in A$. Therefore, $\varphi(a) = \varphi S(aS^{-1}(\delta)) = \varphi\big(aS^{-1}(\delta)\delta\big)$, and, therefore, $S^{-1}(\delta)\delta = 1_{M(A)}$. On the other hand, $\psi(a) = \varphi S(a) = \varphi(a\delta) = \psi(a\delta S^{-1}(\delta))$, and, therefore, $\delta S^{-1}(\delta) = 1_{M(A)}$. Hence, δ is invertible, and $S^{-1}(\delta) = \delta^{-1}$, which is equivalent to $S(\delta) = \delta^{-1}$. □

Remark 10. *(1) The square S^2 leaves the coproduct invariant, so it follows that the composition $\varphi \circ S^2$ of the left faithful integral φ with S^2 will again be a left faithful integral. According to the uniqueness of left faithful integrals, there is a number $\tau \in k$ such that $\varphi \circ S^2 = \tau\varphi$.*

(2) If we apply (2) to Proposition 8 twice, we obtain

$$\varphi\big(S^2(a)\big) = \varphi\big(S(a)\delta\big) = \varphi\big(S(\delta^{-1}a)\big) = \varphi\big((\delta^{-1}a)\delta\big) = \varphi\big(\delta^{-1}a\delta\big).$$

So, $\varphi\big(\delta^{-1}a\delta\big) = \tau\varphi(a)$.

(3) We call δ the modular element, as in an algebraic quantum group. Here, we cannot conclude that $\Delta(\delta) = \delta \otimes \delta$ due to the lack of the co-associativity of the Δ.

Finally, just as in the algebraic quantum and algebraic quantum hypergroup cases, we will show the existence of a modular automorphism.

Proposition 13. *(1) There is a unique automorphism σ of A such that $\varphi(ab) = \varphi\big(b\sigma(a)\big)$ for all $a, b \in A$. We also have $\varphi\big(\sigma(a)\big) = \varphi(a)$ for all $a \in A$.*

(2) Similarly, there is a unique automorphism σ' of A that satisfies $\psi(ab) = \psi\big(b\sigma'(a)\big)$ for all $a, b \in A$. Moreover, $\psi\big(\sigma'(a)\big) = \psi(a)$ for all $a \in A$.

Proof. (1) For any $p, q, x \in A$,

$$\begin{aligned}(\psi \otimes \varphi)\big(xq_{(1)} \otimes pS(q_{(2)})\big) &= \psi(xq_{(1)})\varphi(pS(q_{(2)})) = \varphi\big(p\underline{\psi(xq_{(1)})S(q_{(2)})}\big)\\ &\overset{(23)}{=} \varphi\big(p\psi(x_{(1)}q)x_{(2)}\big) = \psi(x_{(1)}q)\varphi\big(px_{(2)}\big)\\ &= \psi\big(\underline{x_{(1)}\varphi(px_{(2)})}q\big) \overset{(22)}{=} \psi\big(S(p_{(1)})\varphi(p_{(2)}x)q\big)\\ &= \varphi(p_{(2)}x)\psi\big(S(p_{(1)})q\big) = \varphi\Big(\big(\psi\big(S(p_{(1)})q\big)p_{(2)}\big)x\Big)\\ &= \varphi\Big(\big(\psi \circ S \otimes id\big)\big((S^{-1}(q) \otimes 1)\Delta(p)\big)x\Big).\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}(\psi\otimes\varphi)\big(xq_{(1)}\otimes pS(q_{(2)})\big) &= \psi(xq_{(1)})\varphi(pS(q_{(2)})) = \psi\Big(x(q_{(1)}\varphi\big(pS(q_{(2)})\big))\Big)\\ &= \psi\Big(x(id\otimes\varphi\circ S)\big(\Delta(q)(1\otimes S^{-1}(p))\big)\Big).\end{aligned}$$

Now, assume that $\psi=\varphi\circ S$. Then, we have $\psi\circ S=\tau\varphi$ and $\psi(y)=\varphi(y\delta)$ according to Proposition 12 (2). Then, the above calculation will give us

$$\begin{aligned}\varphi(ax) &= \psi(xb)=\frac{1}{\tau}\varphi\circ S(xb)\\ &= \frac{1}{\tau}\varphi(xb\delta)=\varphi\big(x(\frac{1}{\tau}b\delta)\big)\\ &= \varphi\big(xb\big)\end{aligned}$$

for all $x\in A$, where $a=(\psi\circ S\otimes id)\big((S^{-1}(q)\otimes 1)\Delta(p)\big)$, $b'=(id\otimes\varphi\circ S)\big(\Delta(q)(1\otimes S^{-1}(p))\big)$, and $b=b'\delta$.

Because φ is faithful, the element b is uniquely determined by the element a. Therefore, we can define $\sigma(a)=b$. Moreover, according to Lemma 2 and its remark, all element in A are of the form a above; the map σ is defined on all of A. The map σ is injective due to the faithfulness of φ. It is also surjective because all elements in A are also of the form b above.

Take $a,b,c\in A$, then

$$\begin{aligned}\varphi\big(c\sigma(ab)\big) &= \varphi\big((ab)c\big)=\varphi(abc)=\varphi\big(a(bc)\big)\\ &= \varphi\Big((bc)\sigma(a)\Big)=\varphi\Big(b\big(c\sigma(a)\big)\Big)=\varphi\Big(\big(c\sigma(a)\big)\sigma(b)\Big)\\ &= \varphi\Big(c\big(\sigma(a)\sigma(b)\big)\Big).\end{aligned}$$

It follows from the faithfulness of φ that $\sigma(ab)=\sigma(a)\sigma(b)$. So, $\sigma:A\longrightarrow A$ is an algebraic homomorphism. Applying this result twice, we have

$$\varphi(ab)=\varphi\big(b\sigma(a)\big)=\varphi\big(\sigma(a)\sigma(b)\big)=\varphi\big(\sigma(ab)\big)\quad\text{for all } a,b\in A.$$

According to Proposition 10, A has local units; then, $A^2=A$, so φ is σ-invariant.

(2) Using $\psi=\varphi\circ S$, we can easily obtain the statement for ψ.

$$\begin{aligned}\psi(ab) &= \varphi S(ab)=\varphi\big(S(b)S(a)\big)\\ &= \varphi\Big(\sigma^{-1}S(a)S(b)\Big)=\varphi S\Big(bS^{-1}\sigma^{-1}S(a)\Big)\\ &= \psi\Big(bS^{-1}\sigma^{-1}S(a)\Big).\end{aligned}$$

Therefore, $\sigma'=S^{-1}\sigma^{-1}S$. □

Remark 11. *In the proof of Proposition 13, we have*

$$\varphi\Big((\psi\circ S\otimes id)\big((S^{-1}(q)\otimes 1)\Delta(p)\big)x\Big)=\psi\Big(x(id\otimes\varphi\circ S)\big(\Delta(q)(1\otimes S^{-1}(p))\big)\Big).$$

According to Lemma 4, we have that if $a\in A$, then there is a $b\in A$ such that $\varphi(ax)=\psi(xb)$ for all $x\in A$. This result will be used in the next section.

As in the algebraic quantum group and hypergroup cases, the automorphisms σ and σ' are called the modular automorphisms of A, associated with φ and ψ, respectively. There are some extra properties derived from the above proposition.

Proposition 14. *With the notation above, we have*

(1) $\sigma' = S^{-1}\sigma^{-1}S$ *and* $\sigma'(a) = \delta\sigma(a)\delta^{-1}$;
(2) $\sigma(\delta) = \frac{1}{\tau}\delta$ *and* $\sigma'(\delta) = \frac{1}{\tau}\delta$;
(3) The modular automorphisms σ *and* σ' *commute with each other;*
(4) The modular automorphisms σ *and* σ' *commute with* S^2;
(5) For all $a \in A$, $\Delta\big(\sigma(a)\big) = (S^2 \otimes \sigma)\Delta(a)$ *and* $\Delta\big(\sigma'(a)\big) = (\sigma' \otimes S^{-2})\Delta(a)$.

Compared with algebraic quantum groups, the Δ in the multiplier Hopf coquasigroup is not necessarily co-associative. This is a significant difference between these two objects. Other than that, the proof is similar.

5. Duality of Discrete Multiplier Hopf Coquasigroups

In this section, we will construct the dual of a (infinite-dimensional) multiplier Hopf coquasigroup of the discrete type. The construction is based on the faithful integrals introduced in the last section. Here, we also start by defining the following subspace of the dual space A^*.

Definition 4. *Let* φ *be a left faithful integral on a regular multiplier Hopf coquasigroup* (A, Δ). *We define* $\widehat{A}$ *as the space of the linear functionals on* A *of the form* $\varphi(\,\cdot\, a)$ *where* $a \in A$, *i.e.,*

$$\widehat{A} = \{\varphi(\,\cdot\, a) \mid a \in A\}.$$

Because of Proposition 13 and the following remark, we have

$$\widehat{A} = \{\varphi(\,\cdot\, a) \mid a \in A\} = \{\psi(a\,\cdot\,) \mid a \in A\} = \{\varphi(a\,\cdot\,) \mid a \in A\} = \{\psi(\,\cdot\, a) \mid a \in A\}.$$

Let ξ be a left co-integral. We conclude that $\varphi(\xi) \neq 0$. (If not, $0 = \varepsilon(a)\varphi(\xi) = \varphi(a\xi)$ for all $a \in A$; then, $\xi = 0$ by the faithfulness of φ. This is a contradiction.)

We start by making a discrete multiplier Hopf coquasigroup (A, Δ) into a unital algebra by dualizing the coproduct.

Proposition 15. *For* $w, w' \in \widehat{A}$, *we can define a linear functional* ww' *on* A *by the formula*

$$(ww')(x) = (w \otimes w')\Delta(x), \quad \forall x \in A. \tag{24}$$

Then, $ww' \in \widehat{A}$. *This product in* $\widehat{A}$ *is not necessarily associative, but it does have a unit.*

Proof. Let $w, w' \in \widehat{A}$ and assume that $w' = \varphi(\,\cdot\, a)$ with $a \in A$. We have

$$\begin{aligned}(ww')(x) &= (w \otimes \varphi(\,\cdot\, a))\Delta(x) = (w \otimes \varphi)\big(\Delta(x)(1 \otimes a)\big)\\ &= w\big(x_{(1)}\varphi(x_{(2)}a)\big) \overset{(22)}{=} w\Big(S^{-1}\big(a_{(1)}\varphi(xa_{(2)})\big)\Big)\\ &= \varphi\Big(x\big((wS^{-1} \otimes id)\Delta(a)\big)\Big)\end{aligned}$$

We see that the product ww' is well-defined as a linear functional on A, and it has the form $\varphi(\,\cdot\, b)$, where $b = (wS^{-1} \otimes id)\Delta(a)$. So, $ww' \in \widehat{A}$. Therefore, we have defined a product in $\widehat{A}$.

The associativity of this product in $\widehat{A}$ is a consequence of the co-associativity of the Δ in A, and A is not necessarily co-associative.

To prove that $\widehat{A}$ has a unit, assume that there is a co-integral $\xi \in A$ such that $a\xi = \varepsilon(a)\xi$ for all $a \in A$.

$$\begin{aligned}\varphi\big(\,\cdot\, \frac{1}{\varphi(\xi)}\xi\big)(a) &= \frac{1}{\varphi(\xi)}\varphi\big(a\xi\big) = \frac{1}{\varphi(\xi)}\varphi\big(\varepsilon(a)\xi\big)\\ &= \varepsilon(a),\end{aligned}$$

so $\varepsilon = \varphi\big(\ \cdot\ \frac{1}{\varphi(\xi)}\xi\big) \in \widehat{A}$. $\square$

Remark 12. *(1) Under the assumption, the elements of $\widehat{A}$ can be expressed in four different forms. When we use these different forms in the definition of the product in $\widehat{A}$, we obtain the following useful expressions:*

(1) $w\varphi(\,\cdot\, a) = \varphi(\,\cdot\, b)$ *with* $b = wS^{-1}(a_{(1)})a_{(2)}$; *(2)* $w\varphi(a\,\cdot\,) = \varphi(c\,\cdot\,)$ *with* $c = wS(a_{(1)})a_{(2)}$.

(2) $\psi(\,\cdot\, a)w = \psi(\,\cdot\, d)$ *with* $d = a_{(1)}wS(a_{(2)})$; *(4)* $\psi(a\,\cdot\,)w = \psi(e\,\cdot\,)$ *with* $e = a_{(1)}wS^{-1}(a_{(2)})$.

(2) The reason for being restricted to the discrete case is that there is no definition of multiplier algebra $M(A)$ for a non-associative algebra A.

Let us now define the comultiplication $\widehat{\Delta}$ on the unital algebra $\widehat{A}$. Roughly speaking, the coproduct is dual to the multiplication in H in the sense that

$$\langle\widehat{\Delta}(w), x \otimes y\rangle = \langle w, xy\rangle, \quad \forall x, y \in H.$$

We will first show that the above functional is well-defined and then show this again in $\widehat{H} \otimes \widehat{H}$.

Proposition 16. *Let $w \in \widehat{A}$; then, we have $\widehat{\Delta}(w) \in \widehat{A} \otimes \widehat{A}$, and $\widehat{\Delta}$ is co-associative.*

Proof. The unit $1_{\widehat{A}} = \varepsilon = \psi\Big(\frac{1}{\psi S(\xi)} S(\xi)\ \cdot\ \Big) \in \widehat{A}$, and let $w = \psi(b\,\cdot\,)$. Then,

$$\begin{aligned}
\langle\widehat{\Delta}(w), x \otimes y\rangle &= \langle(\varepsilon \otimes 1_{\widehat{A}})\widehat{\Delta}(w), x \otimes y\rangle = \langle \varepsilon \otimes w, x_{(1)} \otimes x_{(2)}y\rangle \\
&= \langle \frac{1}{\psi S(\xi)}\psi\Big(S(\xi)\ \cdot\ \Big) \otimes \psi(b\,\cdot\,), x_{(1)} \otimes x_{(2)}y\rangle \\
&= \frac{1}{\psi S(\xi)}\psi\Big(S(\xi)x_{(1)}\Big)\psi(bx_{(2)}y) = \frac{1}{\psi S(\xi)}\psi\Big(b\underline{\psi\big(S(\xi)x_{(1)}\big)x_{(2)}}y\Big) \\
&\overset{(23)}{=} \frac{1}{\psi S(\xi)}\psi\Big(b\psi\big(S(\xi_{(2)})x\big)\xi_{(1)}y\Big) = \psi\big(\frac{1}{\psi S(\xi)}S(\xi_{(2)})x\big)\psi\Big(b\xi_{(1)}y\Big) \\
&= \langle\psi\big(\frac{1}{\psi S(\xi)}S(\xi_{(2)})\ \cdot\ \big) \otimes \psi\Big(b\xi_{(1)}\ \cdot\ \Big), x \otimes y\rangle
\end{aligned}$$

Hence, $\widehat{\Delta}(w) = \psi\big(\frac{1}{\psi S(\xi)}S(\xi_{(2)})\ \cdot\ \big) \otimes \psi\Big(b\xi_{(1)}\ \cdot\ \Big) \in \widehat{A} \otimes \widehat{A}$.

The co-associativity is a direct consequence of the product associativity in A. $\square$

Proposition 17. *$\widehat{\Delta} : \widehat{A} \longrightarrow \widehat{A} \otimes \widehat{A}$ is an algebra homomorphism.*

Proof. It is straightforward that $\widehat{\Delta}$ is an algebra homomorphism, since, for all $x, y \in H$,

$$\begin{aligned}
\langle\widehat{\Delta}(w_1w_2), x \otimes y\rangle &= \langle w_1w_2, xy\rangle = \langle w_1 \otimes w_2, \Delta(xy)\rangle \\
&= \langle w_1, x_{(1)}y_{(1)}\rangle\langle w_2, x_{(2)}y_{(2)}\rangle \\
\langle\widehat{\Delta}(w_1)\widehat{\Delta}(w_2), x \otimes y\rangle &= \langle\widehat{\Delta}(w_1) \otimes \widehat{\Delta}(w_2), (x_{(1)} \otimes y_{(1)}) \otimes (x_{(2)} \otimes y_{(2)})\rangle \\
&= \langle\widehat{\Delta}(w_1), x_{(1)} \otimes y_{(1)}\rangle\langle\otimes\widehat{\Delta}(w_2), x_{(2)} \otimes y_{(2)}\rangle \\
&= \langle w_1, x_{(1)}y_{(1)}\rangle\langle w_2, x_{(2)}y_{(2)}\rangle
\end{aligned}$$

This completes the proof. $\square$

Let $w \in \widehat{A}$ and assume $w = \varphi(\,\cdot\, a)$ with $a \in A$. Define $\widehat{\varepsilon}(w) = \varphi(a) = w(1_{M(A)})$. Then, $\widehat{\varepsilon}$ is a co-unit on $(\widehat{A}, \widehat{\Delta})$, as follows.

Proposition 18. *$\widehat{\varepsilon}: \widehat{A} \longrightarrow k$ is an algebra homomorphism satisfying*

$$(id \otimes \widehat{\varepsilon})\widehat{\Delta}(w) = w = (\widehat{\varepsilon} \otimes id)\widehat{\Delta}(w) \tag{25}$$

for all $w \in \widehat{A}$.

Proof. Firstly, let $w_1 = \varphi(a\,\cdot\,)$ and $w_2 = \varphi(b\,\cdot\,)$. Then, $w_1 w_2 = \varphi(c\,\cdot\,)$ with $c = \varphi\big(aS(b_{(1)})\big)b_{(2)}$. Therefore, if $\psi = \varphi \circ S$, we have

$$\begin{aligned}\widehat{\varepsilon}(w_1 w_2) &= \varphi(c) = \varphi\big(aS(b_{(1)})\big)\varphi(b_{(2)}) \\ &= \varphi\big(aS(b_{(1)}\varphi(b_{(2)}))\big) = \varphi(a)\varphi(b) \\ &= \widehat{\varepsilon}(w_1)\widehat{\varepsilon}(w_2).\end{aligned}$$

Secondly, let $w = \varphi(\,\cdot\, a)$. Then, we have

$$\begin{aligned}\langle\widehat{\Delta}(w), x \otimes y\rangle &= \langle\widehat{\Delta}(w)(1_{\widehat{A}} \otimes \varepsilon), x \otimes y\rangle = \langle w \otimes \varepsilon, xy_{(1)} \otimes y_{(2)}\rangle \\ &= \langle\varphi(\,\cdot\, a) \otimes \varphi\Big(\,\cdot\, \frac{1}{\varphi(\xi)}\xi\Big), xy_{(1)} \otimes y_{(2)}\rangle \\ &= \frac{1}{\varphi(\xi)}\varphi(xy_{(1)}a)\varphi\big(y_{(2)}\xi\big) = \frac{1}{\varphi(\xi)}\varphi\Big(x\underline{y_{(1)}\varphi\big(y_{(2)}\xi\big)}a\Big) \\ &\overset{(22)}{=} \varphi\Big(xS^{-1}(\xi_{(1)})\varphi\big(y\xi_{(2)}\big)a\Big) \\ &= \langle\varphi\Big(\,\cdot\, \frac{1}{\varphi(\xi)}S^{-1}(\xi_{(1)})a\Big) \otimes \varphi\Big(\,\cdot\, \xi_{(2)}\Big), x \otimes y\rangle.\end{aligned}$$

Hence, $\widehat{\Delta}(w) = \varphi\Big(\,\cdot\, \frac{1}{\varphi(\xi)}S^{-1}(\xi_{(1)})a\Big) \otimes \varphi\Big(\,\cdot\, \xi_{(2)}\Big)$. Therefore,

$$\begin{aligned}(id \otimes \widehat{\varepsilon})\widehat{\Delta}(w) &= \varphi\Big(\,\cdot\, \frac{1}{\varphi(\xi)}S^{-1}(\xi_{(1)})a\Big)\varphi\big(\xi_{(2)}\big) \\ &= \varphi\Big(\,\cdot\, \frac{1}{\varphi(\xi)}S^{-1}\big(\xi_{(1)}\varphi(\xi_{(2)})\big)a\Big) = \varphi\Big(\,\cdot\, a\Big) \\ &= w.\end{aligned}$$

Finally, from Proposition 16, $\widehat{\Delta}(\psi(b\,\cdot\,)) = \psi\big(\frac{1}{\psi S(\xi)}S(\xi_{(2)})\,\cdot\,\big) \otimes \psi\Big(b\xi_{(1)}\,\cdot\,\Big)$. Then, $(\widehat{\varepsilon} \otimes id)\widehat{\Delta}(w) = \psi\big(\frac{1}{\psi S(\xi)}S(\xi_{(2)})\big)\psi\Big(b\xi_{(1)}\,\cdot\,\Big) = \frac{1}{\psi S(\xi)}\psi\Big(b\xi_{(1)}\psi\big(S(\xi_{(2)})\big)\,\cdot\,\Big) = \psi(b\,\cdot\,)$. This completes the proof. □

Let $\widehat{S}: \widehat{A} \longrightarrow \widehat{A}$ be the dual to the antipode of A, i.e., $\widehat{S}(w) = w \circ S$. Then, it is easy to see that $\widehat{S}(w) \in \widehat{A}$, and we have the following property:

Proposition 19. *$\widehat{S}$ is antimultiplicative and co-antimultiplicative such that*

$$\begin{aligned}m(id \otimes m)(\widehat{S} \otimes id \otimes id)(\widehat{\Delta} \otimes id) &= \widehat{\varepsilon} \otimes id = m(id \otimes m)(id \otimes \widehat{S} \otimes id)(\widehat{\Delta} \otimes id), \\ m(m \otimes id)(id \otimes \widehat{S} \otimes id)(id \otimes \widehat{\Delta}) &= id \otimes \widehat{\varepsilon} = m(m \otimes id)(id \otimes id \otimes \widehat{S})(id \otimes \widehat{\Delta}).\end{aligned}$$

Proof. For $w_1, w_2 \in \widehat{H}$ and any $x \in H$,

$$\begin{aligned}\langle\widehat{S}(w_1 w_2), x\rangle &= \langle w_1 w_2, S(x)\rangle = \langle w_1, S(x_{(2)})\rangle\langle w_2, S(x_{(1)})\rangle \\ &= \langle\widehat{S}(w_1), x_{(2)}\rangle\langle\widehat{S}(w_2), x_{(1)}\rangle = \langle\widehat{S}(w_2)\widehat{S}(w_1), x\rangle\end{aligned}$$

This implies that $\widehat{S}$ is antimultiplicative.

$$\begin{aligned}\langle\widehat{\Delta}\widehat{S}(w), x\otimes y\rangle &= \langle\widehat{S}(w), xy\rangle = \langle w, S(xy)\rangle = \langle w, S(y)S(x)\rangle\\ &= \langle\widehat{\Delta}(w), S(y)\otimes S(x)\rangle = \langle\widehat{\Delta}^{cop}(w), S(x)\otimes S(y)\rangle\\ &= \langle(\widehat{S}\otimes\widehat{S})\widehat{\Delta}^{cop}(w), x\otimes y\rangle,\end{aligned}$$

We conclude that $\widehat{S}$ is co-antimultiplicative.

Finally, we show that $m(id\otimes m)(\widehat{S}\otimes id\otimes id)(\widehat{\Delta}\otimes id) = \widehat{\varepsilon}\otimes id$. The other three formulas are similar.

$$\begin{aligned}&\langle m(id\otimes m)(\widehat{S}\otimes id\otimes id)(\widehat{\Delta}\otimes id)\big(w\otimes w'\big), x\rangle\\ =\ &\langle(\widehat{S}\otimes id\otimes id)(\widehat{\Delta}\otimes id)\big(w\otimes w'\big), (id\otimes\Delta)\Delta(x)\rangle\\ =\ &\langle(\widehat{\Delta}\otimes id)\big(w\otimes w'\big), (S\otimes id\otimes id)(id\otimes\Delta)\Delta(x)\rangle\\ =\ &\langle w\otimes w', (m\otimes id)(S\otimes id\otimes id)(id\otimes\Delta)\Delta(x)\rangle\\ =\ &\langle w\otimes w', 1_{M(A)}\otimes x\rangle\\ =\ &\widehat{\varepsilon}(w)w'(x).\end{aligned}$$

This completes the proof. □

From Propositions 15–19, we obtain the first main result of this section.

Theorem 3. *Let (A,Δ) be a regular multiplier Hopf coquasigroup of the discrete type with a left faithful integral φ. Then, $(\widehat{A},\widehat{\Delta})$ is the Hopf quasigroup introduced in [1].*

Let ψ be a right faithful integral on A. For $w = \psi(a\,\cdot\,)$, we set $\widehat{\varphi}(w) = \varepsilon(a)$. Then, we have the following result.

Proposition 20. *The functional $\widehat{\varphi}$ defined above is a left faithful integral on the Hopf quasigroup $(\widehat{A},\widehat{\Delta})$.*

Proof. It is clear that $\widehat{\varphi}$ is nonzero. Assume $w = \psi(b\,\cdot\,)$; then, according to Proposition 16

$$\widehat{\Delta}(w) = \psi\big(\frac{1}{\psi S(\xi)}S(\xi_{(2)})\ \cdot\ \big)\otimes\psi\Big(b\xi_{(1)}\ \cdot\ \Big).$$

Therefore, we have

$$\begin{aligned}(id\otimes\widehat{\varphi})\widehat{\Delta}(w) &= \psi\big(\frac{1}{\psi S(\xi)}S(\xi_{(2)})\ \cdot\ \big)\varepsilon\Big(b\xi_{(1)}\Big)\\ &= \psi\big(\frac{1}{\psi S(\xi)}S(\xi)\ \cdot\ \big)\varepsilon(b) = \widehat{\varphi}(w)1_{\widehat{A}}.\end{aligned}$$

Next, we show that $\widehat{\varphi}$ is faithful. If $w_1, w_2\in\widehat{A}$ and assuming that $w_1 = \psi(a\,\cdot\,)$ with $a\in A$, we have $w_1w_2 = \psi\big(a_{(1)}w_2S^{-1}(a_{(2)})\ \cdot\ \big)$. Therefore, $\widehat{\varphi}(w_1w_2) = w_2S^{-1}(a)$. If this is 0 for all $a\in H$, then $w_2 = 0$, while if this is 0 for all w_2, then $a = 0$. This proves the faithfulness of $\widehat{\varphi}$. □

If we set $\widehat{\psi} = \widehat{\varphi}\circ\widehat{S}$ as we do for the multiplier Hopf coquasigroup (A,Δ), we find that, when $w = \varphi(\,\cdot\,a)$,

$$\widehat{\psi}(w) = \widehat{\varphi}\circ\widehat{S}(w) = \widehat{\varphi}(w\circ S) = \widehat{\varphi}\big(\varphi S(S^{-1}(a)\,\cdot\,)\big) = \varepsilon\big(S^{-1}(a)\big) = \varepsilon(a).$$

6. Biduality

Following Section 3, the integral dual $\widehat{H}$ of an infinite-dimensional Hopf quasigroup H is a regular multiplier Hopf coquasigroup of the discrete type. Specifically, let H be

an infinite-dimensional Hopf quasigroup with a faithful left integral φ, and $\widehat{H} = \varphi(\,\cdot\, H)$. If $\varphi(\,\cdot\, H) = \varphi(H\,\cdot\,)$ and $\varphi\big((\,\cdot\, h)h'\big), \varphi\big(h'(h\,\cdot\,)\big) \in \widehat{H}$ for all $h, h' \in H$, then $\widehat{H}$ is a regular multiplier Hopf coquasigroup of the discrete type.

Moreover, according to Theorem 3, the integral dual $\widehat{\widehat{H}}$ of the regular multiplier Hopf coquasigroup of the discrete type $\widehat{H}$ is a Hopf quasigroup. Then, what is the relation between H and $\widehat{\widehat{H}}$? Similarly, for a discrete multiplier Hopf coquasigroup A, $\widehat{A}$ is a Hopf quasigroup, and the relation of A and $\widehat{\widehat{A}}$ is what we care about. This is the content of the following theorem (a biduality theorem).

Theorem 4. *Let (H, Δ) be a Hopf quasigroup with a faithful integral and $(\widehat{H}, \widehat{\Delta})$ be the dual-multiplier Hopf coquasigroup of the discrete type. For $h \in H$ and $f \in \widehat{H}$, we set $\Gamma(h)(f) = f(h)$. Then, $\Gamma(h) \in \widehat{\widehat{H}}$ for all $h \in H$. Moreover, Γ is an isomorphism between the Hopf quasigroups (H, Δ) and $(\widehat{\widehat{H}}, \widehat{\widehat{\Delta}})$.*

Proof. For $h \in H$, first, we show that $\Gamma(h)$, as a linear functional on $\widehat{H}$, is in $\widehat{\widehat{H}}$. Indeed, let $f = \varphi\big(\,\cdot\, S(h)\big)$ and take any $f' \in \widehat{H}$. According to Proposition 3, $f'f = \varphi(\,\cdot\, h')$ where $h' = f'(h_{(2)})S(h_{(1)})$. Therefore,

$$\widehat{\psi}(f'f) = \varepsilon(h') = f'(h) = \Gamma(h)(f').$$

So, $\Gamma(h) = \widehat{\psi}(\,\cdot\, f)$, and $\Gamma(h) \in \widehat{\widehat{H}}$.

It is clear that Γ is bijective between the linear space H and $\widehat{\widehat{H}}$ because of the bijection of the antipode. Γ respects that the multiplication and comultiplication are straightforward because, in both cases, the products are dual to the coproducts and vice versa. For details,

$$\begin{aligned} \langle \Gamma(hh'), f\rangle &= \langle f, hh'\rangle = \langle \widehat{\Delta}(f), h \otimes h'\rangle \\ &= \langle (\Gamma \otimes \Gamma)(h \otimes h'), \widehat{\Delta}(f)\rangle = \langle \Gamma(h)\Gamma(h'), \widehat{\Delta}(f)\rangle, \\ \langle \widehat{\widehat{\Delta}}\Gamma(h), f \otimes f'\rangle &= \langle \Gamma(h), ff'\rangle = \langle ff', h\rangle \\ &= \langle f \otimes f', \Delta(h)\rangle = \langle (\Gamma \otimes \Gamma)\Delta(h), f \otimes f'\rangle. \end{aligned}$$

Hence, Γ is an isomorphism between H and $\widehat{\widehat{H}}$. □

Similarly, we can obtain another isomorphism.

Theorem 5. *Let (A, Δ) be a discrete multiplier Hopf coquasigroup and $(\widehat{A}, \widehat{\Delta})$ be the dual Hopf quasigroup. For $a \in A$ and $w \in \widehat{A}$, we set $\Gamma(a)(w) = w(a)$. Then, $\Gamma(a) \in \widehat{\widehat{A}}$ for all $a \in A$. Moreover, Γ is an isomorphism between the multiplier Hopf coquasigroup (A, Δ) and $(\widehat{\widehat{A}}, \widehat{\widehat{\Delta}})$.*

As in the cases of an algebraic quantum group and an algebraic quantum group hypergroup, all of the results also hold for the (flexible (resp. alternative, Moufang)) multiplier Hopf ($*$-) coquasigroups. At the end of this section, we return to our motivating example of the multiplier Hopf coquasigroups.

Example 3. *Let G be an infinite (IP) quasigroup with an identity element e, which is, by definition, $u^{-1}(uv) = v = (vu)u^{-1}$ for all $u, v \in G$. The quasigroup algebra kG has a natural Hopf quasigroup structure. δ_e represents the left and right integrals on kG. The integral dual $k(G)$, introduced in Example 1, is a multiplier Hopf coquasigroup of the discrete type.*

Now, we construct the dual of $k(G)$, as introduced in Section 4. Then,

$$\widehat{k(G)} = \{\widehat{\varphi}(\,\cdot\, \delta_u) \mid u \in G\}.$$

The element $\widehat{\varphi}(\,\cdot\,\delta_u) = \widehat{\psi}(\,\cdot\,\delta_u)$ maps δ_u to 1 and maps $\delta_v (v \neq u)$ to 0.

According to Theorem 4, $kG \cong \widehat{k(G)}$ as Hopf quasigroups. The isomorphism $\Gamma : kG \longrightarrow \widehat{k(G)}$ is given by

$$\begin{aligned}\Gamma(u) &= \widehat{\psi}\Big(\,\cdot\,\varphi(\,\cdot\,S(u))\Big) = \widehat{\psi}\Big(\,\cdot\,\delta_e(\,\cdot\,u^{-1})\Big)\\ &= \widehat{\psi}\Big(\,\cdot\,\delta_u\Big).\end{aligned}$$

So, if we identify $\widehat{\psi}(\,\cdot\,\delta_u)$ with u, then $\widehat{k(G)} = kG$.

According Theorem 5, $k(G) \cong \widehat{kG}$ as multiplier Hopf coquasigroups. The isomorphism $\Gamma : k(G) \longrightarrow \widehat{kG}$ is given by

$$\begin{aligned}\Gamma(\delta_u) &= \widehat{\widehat{\psi}}\Big(\,\cdot\,\widehat{\varphi}(\,\cdot\,S(\delta_u))\Big) = \delta_e\Big(\,\cdot\,\widehat{\varphi}(\,\cdot\,\delta_{u^{-1}})\Big)\\ &= \delta_e\Big(\,\cdot\,u^{-1}\Big) = u.\end{aligned}$$

So, $\widehat{kG} = k(G)$.

7. Conclusions

Following A. Van Daele's idea, this paper gives an answer to a class of infinite-dimensional Hopf quasigroups and shows that the biduality theorem holds for Hopf quasigroups and discrete multiplier Hopf coquasigroups. Furthermore, inspired by [9,10], a possible topic for further research is the biduality theorem for weak Hopf quasigroups and weak multiplier Hopf coquasigroups, which I believe to be correct.

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