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Point Processes in a Metric Space and Their Applications

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Abstract: Point processes are important in extreme value theory due to their equivalent formulations of two popular models in various applications: the block maxima models and the peak-over-threshold model. Point processes in a metric space provide tools to analyze heavy-tailed phenomena that appear in the research of extremal behaviors of functional data. To facilitate these applications of point processes, the equivalence between the weak convergence of point processes and the $\mathbb{M}_{\mathbb{O}}$-convergence is established in the paper.

Keywords: point processes; functional data; the $\mathbb{M}_{\mathbb{O}}$-convergence

MSC: 60G70; 60G55; 62G32; 62R10

1. Introduction

A point process on a space S is a stochastic process composed of a time series of indications of a specific event, which provides an elegant formulation of extremal behaviors of a stochastic process. The *block maxima* model and the *peak-over-threshold* model as two important approaches in extreme value theory can be formulated as applications of point processes. The important role of point processes has been widely discussed in various monographs such as [1–4], to name a few.

Let $(X_t)_{t\in\mathbb{N}}$ be an iid sequence of random variables. Define the partial maxima $M_n = \max\{X_1, \ldots, X_n\}$. According to the Fisher–Tippett theorem (see for example Theorem 3.2.3 in [2]), if there exist constants $a_n > 0$, $b_n \in \mathbb{R}$ and a non-degenerate distribution H such that

$$a_n^{-1}(M_n - b_n) \xrightarrow{d} H, \tag{1}$$

then H is of the same type as one of the three distributions: Fréchet, Weibull, or Gumbel. By introducing the parameter ξ, the *generalized extreme value distribution* (GEV) has the distribution function

$$G_\xi(x) = \exp\Big\{-\Big[1 + \xi\Big(\frac{x-u}{\sigma}\Big)\Big]^{-1/\xi}\Big\}, \tag{2}$$

where u and σ are two parameters, and x_- and x_+ are the lower and upper endpoints of G, respectively. When $\xi > 0$, G_ξ corresponds to the Fréchet distribution; when $\xi = 0$, G_ξ corresponds to the Gumbel distribution; and when $\xi < 0$, G_ξ corresponds to the Weibull distribution. Define the marked point process

$$N_n = \sum_{i=1}^{n} \mathbf{1}_{(i/n,(X_i - b_n)/a_n)}, \tag{3}$$

where $\mathbf{1}_x(A)$ is the indicator function such that

$$\mathbf{1}_x(A) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

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According to Theorem 7.1 in [3], we have

$$N_n \xrightarrow{d} N\,, \quad n \to \infty\,. \tag{4}$$

The process N is a Poisson process with intensity measure $|\cdot| \times \mu$, where $|\cdot|$ is the Lebesgue measure on $\mathbb{R}$ and $\mu\big([x, x_+)\big) = \big[1 + \xi(x-u)/\sigma\big]^{-1/\xi}$. Moreover, Theorem 5.2.4 in [2] ensures that (4) holds if and only if

$$nP\big(a_n^{-1}(X_n - b_n) \in \cdot\big) \xrightarrow{v} \mu(\cdot)\,, \tag{5}$$

where $\xrightarrow{v}$ stands for the vague convergence; please refer to [1] for more details on the vague convergence.

The event $\{(M_n - b_n)/a_n \le x\}$ is equivalent to the event $N_n\big((0,1] \times (x, x_+)\big) = 0$, which is the connection to the block maxima model. The peak-over-threshold model considers the probability of the form

$$P\big((X_i - b_n)/a_n > x_1 \mid (X_i - b_n)/a_n > x_2\big)\,, \quad (x_1, x_2) \in \mathbb{R}^2\,,$$

which is equivalent to

$$\frac{\mu([x_1, x_+))}{\mu([x_2, x_+))} = \frac{(1+\xi(x_1 - u)/\sigma)^{-1/\xi}}{(1+\xi(x_2 - u)/\sigma)^{-1/\xi}} = \Big[1 + \xi\Big(\frac{x_1 - x_2}{\sigma + \xi(x_2 - u)}\Big)\Big]^{-1/\xi}.$$

For more details on the connections between the two models and point processes, please also refer to Section 7.4, [3].

Extreme value theory takes heavy-tailed phenomena as the main objects to study, and it focuses on events that are bounded away from the origin. The M_0-convergence introduced in [5] considers the convergence of measures on the space without the origin point, which uses a similar idea to the $w^\#$-convergence studied in the monographs [6,7] on point processes. A gap between the M_0-convergence and the weak convergence of point processes exists, and one goal of this paper is to fill this gap. In the meantime, point processes provide a tool to analyze functional data, which might not be from a Hilbert space. As the review paper [8] explains, the Karhunen–Loeve expansion is the main tool to convert an infinite-dimensional curve into a finite-dimensional vector, which plays an important role in functional data analysis and requires the curves defined in a Hilbert space. The objects exhibit heavy-tailed features such as extreme temperature curves, and high-frequency stock prices, to name a few. They are more reasonable to be viewed as a random element from a Banach space than from a Hilbert space. Thus, the framework of functional data analysis described in [8] is not feasible to analyze these objects. Alternatively, point processes can be used for analyzing heavy-tailed functional data due to their connection to extreme value theory.

The paper will be organized as follows. In Section 2, we introduce the space of measures $\mathbb{M}_O$ and the corresponding convergence, the $\mathbb{M}_\mathbb{O}$-convergence. The properties of point processes are studied in Section 3, in which the equivalence between the weak convergence of point processes and the $\mathbb{M}_\mathbb{O}$-convergence is proved. Some interesting examples of metric spaces allow for the construction of the space $\mathbb{M}_\mathbb{O}$.

2. The Space of Measures $\mathbb{M}_\mathbb{O}$ and the $\mathbb{M}_\mathbb{O}$-Convergence

The M_0-convergence is a special case of the $\mathbb{M}_\mathbb{O}$-convergence when the set $\mathbb{O}$ is a singleton. The space $\mathbb{M}_\mathbb{O}$ and the $\mathbb{M}_\mathbb{O}$-convergence have a natural connection with the concept of *hidden regular variation*, and they are introduced and studied in [9]. Here, we will briefly introduce some key results of the $\mathbb{M}_\mathbb{O}$-convergence.

2.1. The Metric Space $(\mathbb{S}, d, \mathbb{O})$

Let $(\mathbb{S}, d)$ be a complete and separable metric space, and the Borel σ-field on $\mathbb{S}$ is denoted $\mathcal{S}$, which is generated by open balls $B_r(x) = \{y \in \mathbb{S} : d(x, y) < r\}$ for $x \in \mathbb{S}$. Let $\mathbb{O}$ be a closed subset of $\mathbb{S}$ and let $\mathbb{C} = \mathbb{S} \setminus \mathbb{O}$. We further assume that $\mathbb{S}$ is equipped with a scalar multiplication; see Section 3.1 [9].

Definition 1. *A scalar multiplication on $\mathbb{S}$ is a map $[0, \infty) \times \mathbb{S} \to \mathbb{S} : (\lambda, x) \to \lambda x$ satisfying the following properties:*

1. *$\lambda_1(\lambda_2 x) = (\lambda_1\lambda_2)x$ for all $\lambda_1, \lambda_2 \in [0, \infty)$ and $x \in \mathbb{S}$;*
2. *$1x = x$ for $x \in \mathbb{S}$;*
3. *The map is continuous with respect to the product topology;*
4. *if $x \in \mathbb{C}$ and if $0 \le \lambda_1 < \lambda_2$, then $d(\lambda_1 x, \mathbb{O}) < d(\lambda_2 x, \mathbb{O})$.*

The set $\mathbb{O}$ is assumed to be a cone, i.e., $\lambda\mathbb{O} = \mathbb{O}$ for $\lambda \in [0, +\infty)$. We shall prove that for all $x \in \mathbb{S}$, the point $0x \in \mathbb{O}$. Trivially, $0x \in \mathbb{O}$ for $x \in \mathbb{O}$. Choose an arbitrary $x \in \mathbb{C}$. We have $\lambda(0x) = (\lambda 0)x = 0x$ for $\lambda \ge 0$ and, moreover, $d(\lambda_1(0x), \mathbb{O}) = d(0x, \mathbb{O}) = d(\lambda_2(0x), \mathbb{O})$, which leads to a contradiction to (iv) in Definition 1 when $0x \in \mathbb{C}$ and $\lambda_1 \ne \lambda_2$. Therefore, $0x \in \mathbb{O}$ for all $x \in \mathbb{S}$. The underlying space we consider in the paper is the complete and separable metric space $(\mathbb{S}, d, \mathbb{O})$, which is equipped with scalar multiplication and $\mathbb{O}$ as a closed cone. Later in the paper, we will write $(\mathbb{S}, d)$ or $\mathbb{S}$ instead of $(\mathbb{S}, d, \mathbb{O})$ for convenience.

Remark 1. *It is possible that $0x \ne 0y$ when $x \ne y$ and $\mathbb{O}$ is not a singleton. Here is an example. Consider a space $(\mathbb{S}, d, \mathbb{O})$ and a real line $\mathbb{R}$. Let $\widetilde{\mathbb{S}} = \mathbb{R} \times \mathbb{S}$. Take two elements from $\widetilde{\mathbb{S}}$, $\widetilde{x} = (a, x)$ and $\widetilde{y} = (b, y)$, where $a, b \in \mathbb{R}$ and $x, y \in \mathbb{S}$. Define $\widetilde{d}(\widetilde{x}, \widetilde{y}) = |a - b| + d(x, y)$ and $\lambda\widetilde{x} = (a, \lambda x)$ for $\lambda \in \mathbb{R}$. It is easy to verify that $(\widetilde{\mathbb{S}}, \widetilde{d}, \mathbb{R} \times \mathbb{O})$ is complete and separately equipped with a scalar multiplication and $\mathbb{R} \times \mathbb{O}$ as a closed cone . If $a \ne b \in \mathbb{R}$, $0\widetilde{x} = (a, 0x) \ne (b, 0y) = 0\widetilde{y}$ even when $0x = 0y$.*

2.2. The Space of Measures on $(\mathbb{S}, d, \mathbb{O})$ and Their Convergences

Let $\mathcal{S}_{\mathbb{C}} = \{A \in \mathcal{S} : A \subset \mathbb{C}\}$ be the σ-algebra, and let $\mathcal{C}_{\mathbb{C}} = \mathcal{C}_{\mathbb{C}}(\mathbb{S})$ be a collection of real-valued, non-negative, bounded continuous functions f on $\mathbb{C}$ vanishing on $\mathbb{O}^r = \{x \in \mathbb{S} : d(x, \mathbb{O}) = \inf_{y \in \mathbb{O}} d(x, y) < r\}$ for some $r > 0$. We say a set $A \in \mathcal{S}_{\mathbb{C}}$ is *bounded away from* $\mathbb{O}$ if $A \subset \mathbb{S} \setminus \mathbb{O}^r$ for some $r > 0$.

Let $\mathbb{M}_{\mathbb{O}} = \mathbb{M}(\mathbb{S} \setminus \mathbb{O})$ be the space of Borel measures on $\mathcal{S}_{\mathbb{C}}$ that are bounded on the complements of $\mathbb{O}^r$, $r > 0$. As discussed in [9], the space $(\mathbb{M}_{\mathbb{O}}, d_{\mathbb{M}_{\mathbb{O}}})$ is complete and separable with a proper choice of the metric $d_{\mathbb{M}_{\mathbb{O}}}$. For a measure $\mu \in \mathbb{M}_{\mathbb{O}}$, we must have $\mu(\mathbb{S} \setminus \mathbb{O}^r) < \infty$ for all $r > 0$, and thus there is, at most, countable $r > 0$ such that $\mu(\partial\mathbb{O}^r) > 0$. Here, ∂A stands for the boundary of a set A. Moreover, for any $r > 0$, we can find $\widetilde{r} < r$ such that $\mu(\partial\mathbb{O}^{\widetilde{r}}) = 0$. The set $A \in \mathcal{S}_{\mathbb{C}}$ is said to be *μ-smooth* if $\mu(\partial A) = 0$.

The $\mathbb{M}_{\mathbb{O}}$-convergence is characterized by functions in $\mathcal{C}_{\mathbb{C}}$. Suppose that there is a sequence of measures (μ_n) with $\mu_n \in \mathbb{M}_{\mathbb{O}}$ and a measure $\mu \in \mathbb{M}_{\mathbb{O}}$. We say that $\mu_n \to \mu$ in $\mathbb{M}_{\mathbb{O}}$ or $\mu_n \xrightarrow{M} \mu$ if $\int f\,\mathrm{d}\mu_n \to \int f\,\mathrm{d}\mu$ as $n \to \infty$ for all $f \in \mathcal{C}_{\mathbb{C}}$. A Portmanteau theorem for the $\mathbb{M}_{\mathbb{O}}$-convergence is given as Theorem 2.1 in [9], and we will present useful parts of this theorem for the paper.

Theorem 1 (Portmanteau theorem)**.** *Let $\mu, \mu_n \in \mathbb{M}_{\mathbb{O}}$. The following statements are equivalent:*

1. *$\mu_n \to \mu$ in $\mathbb{M}_{\mathbb{O}}$ as $n \to \infty$.*
2. *$\int f\,\mathrm{d}\mu_n \to \int f\,\mathrm{d}\mu$ for all $f \in \mathcal{C}_{\mathbb{C}}$.*
3. *$\lim_{n\to\infty} \mu_n(A) = \mu(A)$ for all $A \in \mathcal{S}_{\mathbb{C}}$ with $\mu(\partial A) = 0$.*

2.3. The Counting Measures $\mathbb{N}_{\mathbb{O}}$

Let $\mathbb{N}_{\mathbb{O}} = \mathbb{N}(\mathbb{S} \setminus \mathbb{O})$ be the space of all measures $N \in \mathbb{M}_{\mathbb{O}}$ such that $N(A \setminus \mathbb{O}^r)$ is a non-negative integer for all $A \in \mathcal{S}$ and $r > 0$. A measure N is a counting measure if $N \in \mathbb{N}_{\mathbb{O}}$.

Proposition 1. *The space $\mathbb{N}_{\mathbb{O}}$ is a closed subset of $\mathbb{M}_{\mathbb{O}}$.*

Proof. It is enough to show that the limit of a converging sequence in $\mathbb{N}_{\mathbb{O}}$ is still in $\mathbb{N}_{\mathbb{O}}$. Let $(N_n)_{n\in\mathbb{N}}$ be a sequence of counting measures and $N_n \to N$ in $\mathbb{M}_{\mathbb{O}}$. Let y be an arbitrary point in $\mathbb{C}$. Since $N \in \mathbb{M}_{\mathbb{O}}$, for all but a countable set of values of $r \in (0, d(y, \mathbb{O}))$, $N(\partial B_r(y)) = 0$. We can find a decreasing sequence $(r_j)_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} r_j = 0$ and $N(\partial B_{r_j}(y)) = 0$, $j \geq 1$. By Portmanteau theorem, we have that for $j \geq 1$,

$$N_n(B_{r_j}(y)) \to N(B_{r_j}(y)), \quad n \to \infty.$$

Since $N_n(B_{r_j}(y))$ are non-negative integers, $N(B_{r_j}(y))$ are also non-negative integers, and thus N is a counting measure by Lemma A1 in Appendix A. □

3. Properties of Point Processes

We are interested in showing the equivalence between the weak convergence of point processes and the $\mathbb{M}_{\mathbb{O}}$-convergence as the equivalence between (4) and (5).

3.1. Random Measures and Point Processes

Definition 2. *1. A random measure ξ with state space $\mathbb{C}$ is a measurable mapping from a probability space $(\Omega, \mathcal{E}, p)$ into $(\mathbb{M}_{\mathbb{O}}, d_{\mathbb{M}_{\mathbb{O}}})$.*
2. A point process on $\mathbb{C}$ is a measurable mapping from $(\Omega, \mathcal{E}, p)$ into $(\mathbb{N}_{\mathbb{O}}, d_{\mathbb{M}_{\mathbb{O}}})$.

A realization of a random measure ξ has the value $\xi(A, \omega)$ on the Borel set $A \in \mathcal{S}_{\mathbb{C}}$. For each fixed A, $\xi_A = \xi(A, \cdot)$ is a function mapping $(\Omega, \mathcal{E}, p)$ into $\mathbb{R}^+ = [0, \infty]$. The following theorem provides a convenient way to examine whether a mapping is a random measure or a process is a point process.

Theorem 2. *Let ξ be a mapping from a probability space $(\Omega, \mathcal{E}, p)$ into $\mathbb{M}_{\mathbb{O}}$. Then, ξ is a random measure if and only if ξ_A is a random variable taking values from $\mathbb{R}^+$ for each $A \in \mathcal{S}_{\mathbb{C}}$. Similarly, N is a point process if and only if $N(A)$ is a random variable taking values from non-negative integers for each $A \in \mathcal{S}_{\mathbb{C}}$.*

Proof. Let $\mathcal{M}_{\mathbb{O}}$ be the σ-algebra of $(\mathbb{M}_{\mathbb{O}}, d_{\mathbb{M}_{\mathbb{O}}})$. Let $\mathcal{U}$ be the σ-algebra of subsets of $\mathbb{M}_{\mathbb{O}}$ whose inverse images under ξ are events, and let Φ_A denote the mapping taking a measure $\mu \in \mathbb{M}_{\mathbb{O}}$ into $\mu(A)$. Because $\xi_A(\omega) = \xi(A, \omega) = \Phi_A(\xi(\cdot, \omega))$,

$$\xi^{-1}(\Phi_A^{-1}(B)) = (\xi_A)^{-1}(B), \quad B \in \mathcal{B}(\mathbb{R}^+).$$

If ξ_A is a random variable, $(\xi_A)^{-1}(B) \in \mathcal{E}$ and we have $\Phi_A^{-1}(B) \in \mathcal{U}$ by definition. This implies that $\mathcal{M}_{\mathbb{O}} \subset \mathcal{U}$ and thus ξ is a random measure. Conversely, if ξ is a random measure, $\Phi_A^{-1}(B) \in \mathcal{M}_{\mathbb{O}}$ for $B \in \mathcal{B}(\mathbb{R}^+)$ and hence $\xi^{-1}(\Phi_A^{-1}(B)) \in \mathcal{E}$. This shows that ξ_A is a random variable.

Similarly, we can prove the property for N; thus, the details are omitted. □

3.2. Laplace Functionals

Let $f \in \mathcal{C}_{\mathbb{C}}$ and let ξ be a random measure. The Laplace functional of ξ is given by

$$L_\xi[f] = \mathbb{E}\left[\exp(-\int_{\mathbb{C}} f(x)\xi(\mathrm{d}x))\right].$$

We are interested in two important properties of Laplace functionals, which are listed in the two propositions.

Proposition 2. *The Laplace functions $\{L_\xi[f] : f \in \mathcal{C}_{\mathbb{C}}\}$ uniquely determine the distribution of a random measure ξ.*

To prove Proposition 2, we need the following lemma, which is derived directly from Theorem 3.3 [10].

Lemma 1. *The distribution of a random measure is completely determined by the fidi distributions*

$$p(\xi(A_1) \le x_1, \ldots, \xi(A_k) \le x_k),$$

for all finite families $\{A_1, \ldots, A_k\}$ of disjoint sets generating $\mathcal{S}_{\mathbb{O}}$.

A point process N is a random measure, and if Proposition 2 holds, $\{L_N(f) : f \in \mathcal{C}_{\mathbb{C}}\}$ uniquely determines the distribution of N.

Proof of Proposition 2. For $k \ge 1$ and Borel sets $A_1, \ldots, A_k \in \mathcal{S}_{\mathbb{O}}$ bounded away from $\mathbb{C}$ and $\lambda_i > 0$, $i = 1 \ldots, k$, the function $f : \mathbb{O} \to [0, \infty)$ is given by

$$f(x) = \sum_{i=1}^{k} \lambda_i \mathbf{1}_{A_i}(x), \quad x \in \mathbb{O}.$$

Then for each realization $\omega \in \Omega$,

$$\xi(\omega, f) = \int_{\mathbb{S}} f(x)\xi(\omega, dx) = \sum_{i=1}^{k} \lambda_i \xi(\omega, A_i),$$

and

$$L_\xi[f] = E \exp\Big(- \sum_{i=1}^{k} \lambda_i \xi(A_i) \Big),$$

which is the joint Laplace transform of the random vector $(\xi(A_i))_{i=1,\ldots,k}$. The uniqueness of Laplace transform for random vectors yields that L_ξ uniquely determines the law of $(\xi(A_i))_{i=1,\ldots,k}$ and Lemma 1 completes the proof. □

The following proposition shows that the convergence of Laplace functionals is equivalent to the convergence of random measures, which will be useful in the proofs.

Proposition 3. *Let (ξ_n) and ξ be random measures defined in Definition 2. The Laplace functional $L_{\xi_n}[f] \to L_\xi[f]$ as $n \to \infty$ for all $f \in \mathbb{C}_{\mathbb{C}}$ if and only if $\xi_n \xrightarrow{d} \xi$ as $n \to \infty$.*

Proof. Think of the simple functions of the form $f = \sum_{i=1}^{k} c_i \mathbf{1}_{A_i}$, where k is a finite positive integer, $\sum_i |c_i| < \infty$ and $(A_i)_{i \ge 1}$ are a family of Borel sets with $A_i \in \mathcal{S}_{\mathbb{C}}$. The convergence of distributions of the integrals $\int_{\mathbb{C}} f \, d\xi$ is equivalent to the finite-dimensional convergence for every finite k. Following a classical argument, we can find $h_l^+, h_l^- \in \mathcal{C}_{\mathbb{C}}$ satisfying that $0 < h_l^-(x) \uparrow f(x)$ and $h_l^+(x) \downarrow f(x)$ holds uniformly for every $x \in \mathbb{S}$ as $l \to \infty$. Proposition A1 implies finite-dimensional convergence and hence weak convergence. □

Since point processes (N_n) and N are also random measures, the convergence of $N_n \xrightarrow{d} N$ as $n \to \infty$ is equivalent to the convergence $L_{N_n}[f] \to L_N[f]$ as $n \to \infty$ for all $f \in \mathcal{C}_{\mathbb{C}}$.

3.3. Poisson Processes and Marked Point Processes

As an important example of point processes, we shall give a definition of Poisson processes.

Definition 3. *Given a random measure $\mu \in M_{\mathbb{O}}$, a point process N is called a* Poisson process *or* Poisson random measure *(PRM) with mean measure μ if it satisfies the following conditions:*

1. *For $A \in \mathcal{S}_{\mathbb{C}}$ and a non-negative integer k,*

$$P\big(N(A) = k\big) = \begin{cases} \exp\big(\mu(A)\big)\big(\mu(A)\big)^k/k!\,, & \mu(A) < \infty\,, \\ 0\,, & \mu(A) = \infty\,. \end{cases}$$

2. *For $k \geq 1$, if $A_1, \ldots, A_k$ are mutually disjoint Borel sets in $\mathcal{S}_{\mathbb{C}}$, then $N(A_i)$, $i = 1, \ldots, k$ are independent random variables.*

We will write a Poisson process with mean measure μ as $PRM(\mu)$.

Proposition 4. *For a measure $\mu \in M_{\mathbb{O}}$, $PRM(\mu)$ exists and its law is determined by two conditions in Definition 3. Moreover, the Laplace functional of $PRM(\mu)$ is given by*

$$L[f] = \exp\Big(- \int_{\mathbb{C}} (1 - e^{-f(x)})\mu(\mathrm{d}x)\Big)\,, \quad f \in \mathcal{C}_{\mathbb{C}}\,, \tag{6}$$

and conversely, a point process N with Laplace functional of the form (6) must be $PRM(\mu)$.

Proof. Following the lines in the proof of Proposition 3.6, [1] and choosing $f = c\mathbf{1}_A$ for $c > 0$ and $A \in \mathcal{S}_{\mathbb{C}}$, the Laplace functional $L_N[f]$ has the form (6). Let

$$f = \sum_{i=1}^{k} c_i \mathbf{1}_{A_i}\,, \tag{7}$$

where $k > 0$, $c_i \geq 0$ and $A_1, \ldots, A_k$ are disjoint sets bounded away from $\mathbb{O}$. Similarly, it can be shown that the Laplace function $L_N[f]$ has the form (6). Then, for any $f \in \mathcal{C}_{\mathbb{C}}$, there are simple functions f_n of the form (7) such that $f_n \uparrow f$ with $\sup_{x \in \mathbb{C}} |f_n(x) - f(x)| \to 0$ as $n \to \infty$. By Proposition 3.6 in [1], we have that the Laplace function $L_N[f]$ has the form (6). Moreover, it is easy to prove by following the lines in the proof of Proposition 3.6 [1].

The proof of the existence of PRM(μ) is through construction. We will use the same trick as in the proof of Lemma A1 to divide $\mathbb{C}$ into countable disjoint subspaces $\mathbb{C}^{(r_j)}$, $j = 1, 2, \ldots$. Then, let $\mu_j(\cdot) = \mu(\cdot \cap \mathbb{C}^{(r_j)})$ for $\mu \in \mathbb{M}_{\mathbb{O}}$. Using the arguments in the proof of Proposition 3.6, [1], it is easy to construct PRM(μ_i) for $i \geq 1$, named N_i. Let $N = \sum_{i=1}^{\infty} N_i$. For $f \in \mathcal{C}_{\mathbb{C}}$,

$$\begin{aligned} L_N[f] &= E\exp\big(-\sum_{i=1}^{\infty} N_i(f)\big) = \lim_{n\to\infty} E\exp\Big(- \sum_{i=1}^{n} N_i(f)\Big) \\ &= \lim_{n\to\infty} \prod_{i=1}^{n} E\exp(-N_i(f)) \\ &= \lim_{n\to\infty} \prod_{i=1}^{n} E\exp\Big(- \int_{\mathbb{C}} (1 - e^{-f(x)})\mu_i(dx)\Big) \\ &= \exp\Big(- \int_{\mathbb{C}} (1 - e^{-f(x)}) \sum_{i=1}^{\infty} \mu_i(dx)\Big) \\ &= \exp\Big(- \int_{\mathbb{C}} (1 - e^{-f(x)})\mu(dx)\Big)\,. \end{aligned}$$

This shows that PRM(μ) exists. □

As discussed before, the block maxima and peak-over-threshold models have close relations to marked point processes.

Theorem 3. *Let $(X_{n,i})$ be iid copies of a random variable X taking values from $(\mathbb{S}, d)$ and a measure $\mu \in \mathbb{M}_{\mathbb{O}}$. Suppose (N_n) is a sequence of point processes*

$$N_n = \sum_{i=1}^{n} \mathbf{1}_{(n^{-1}i, X_{n,i})} \tag{8}$$

with state space $[0, \infty) \times \mathbb{C}$ and N is PRM$(|\cdot| \times \mu)$. Then,

$$N_n \xrightarrow{d} N\,, \quad n \to \infty\,, \tag{9}$$

in $\mathbb{N}\big([0, \infty) \times \mathbb{C}\big)$ if and only if

$$nP(X_{n,1} \in \cdot) \xrightarrow{M} \mu(\cdot)\,, \quad n \to \infty\,, \tag{10}$$

holds.

The proof is technical and is in Appendix B.

4. Examples

According to Theorem 3, the applications of marked point processes to the block maxima and peak-over-threshold models require that the underlying metric space is complete, separable, and equipped with scalar multiplication. In this section, we will provide some interesting examples.

4.1. The Space $C[0, 1]$ as a Complete and Separable Space

The space $C = C[0, 1]$ consists of continuous functions on the unit interval $[0, 1]$, and the distance between $\mathbf{x}, \mathbf{y} \in C[0, 1]$ is given by

$$d(\mathbf{x}, \mathbf{y}) = \sup_{t \in [0,1]} |\mathbf{x}(t) - \mathbf{y}(t)|\,.$$

We choose the zero function $\mathbf{x} = 0$ as 0_C. As shown in Chapter 2, [11], the space (C, d) is complete and separable. Moreover, with natural scalar multiplication, the space $(C, d, \{0_C\})$ satisfies the conditions in Section 2. Therefore, the results in Section 3 are applicable to the space $(C, d, \{0_C\})$.

In Chapter 9 of [12], extreme value theory in $C[0, 1]$ is studied. Some of the results therein can be easily shown by an application of Theorem 3. We take Theorem 9.3.1 of [12], for example. Given that $\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_n$ are iid stochastic processes in $C^+[0, 1]$, where $C^+[0, 1] = \{f \in C[0, 1] : f > 0\,, \|f\|_\infty = 1\}$. Theorem 9.3.1 of [12] states that if

$$\frac{1}{n} \bigvee_{i=1}^{n} \mathbf{x}_i \xrightarrow{d} \mathbf{y}\,, \quad \text{in } C^+[0, 1]\,, \tag{11}$$

then

$$v_n \xrightarrow{v} v\,, \tag{12}$$

with $v_n(A) = nP(n^{-1}\mathbf{x} \in A)$. Define the marked point process as $N_n = \sum_{i=1}^{n} \mathbf{1}_{(n^{-1}i, \mathbf{x}_i)}$. The event $\{n^{-1} \vee_{i=1}^{n} \mathbf{x}_i \in A\}$ is equivalent to the event $\{N_n\big((0, 1) \times A^c\big) = 0\}$ with A^c as the complement of the set A. According to Theorem 3, the weak convergence $N_n \xrightarrow{d} N$ leads to $v_n \xrightarrow{M} v$. In this example, the $\mathbb{M}_{\mathbb{O}}$-convergence is equivalent to the vague convergence, i.e, $v_n \xrightarrow{v} v$. Furthermore, two convergences (11) and (12) are equivalent due to Theorem 3, which extend the results in [12].

4.2. The Space of Sequences in $(\mathbb{S}, d)$

The M_0-convergence is a special case of the $\mathbb{M}_{\mathbb{O}}$-convergence when $\mathbb{O}$ is a singleton with exactly the zero point, which has been used to study the regular variation of sequences. Suppose that $(\mathbb{S}, d, \{0_S\})$ is the underlying space satisfying the conditions in Section 2. Let $\mathbb{S}^{\mathbb{Z}}$ be the space of all sequences $\mathbf{x} = (x_t)_{t\in\mathbb{Z}}$ with elements in $\mathbb{S}$, and the corresponding metric $d_{\mathbb{Z}}$ is given by

$$d_{\mathbb{Z}}(\mathbf{x}, \mathbf{y}) = \sum_{t\in\mathbb{Z}} 2^{-|t|} \frac{d(x_t, y_t)}{1 + d(x_t, y_t)}, \quad \mathbf{x}, \mathbf{y} \in S^{\mathbb{Z}}.$$

The space $(S^{\mathbb{Z}}, d_{\mathbb{Z}})$ is complete and separable. By choosing $\mathbb{O} = \{(0_S, \ldots, 0_S, \ldots)\}$, scalar multiplication is defined componentwise.

Let B be a Banach space with a countable base $(b_i)_{i\in\mathbb{Z}}$. For $\mathbf{x} \in B$, we have $\mathbf{x} = \sum_{i=1}^{\infty} x_i b_i$ with $(x_i)_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$. We further assume that $0_B = 0\mathbf{x}$ for $\mathbf{x} \in B$. Assume that the norm in B, $\|\cdot\|$ is a Hilbert-Schimdt norm

$$\|\mathbf{x}\|^2 = \sum_{i=1}^{\infty} 2^{-i} x_i^2.$$

Then, the space $(B, \|\cdot\|, \{0_B\})$ is the underlying space satisfying the conditions in Section 2. It allows for the study of functional data from a Banach space instead of a Hilbert space by using point processes, which might be of interest in the context of extreme value theory.

5. Discussion

There were some efforts to study functional data in the context of extreme value theory—take [13] for example. It remains challenging to work with functional data with heavy-tailed features, for which a more natural assumption is that data are from a Banach space instead of a Hilbert space. Point processes provide useful tools to study the extremal behaviors of functional data, especially from a Banach space. In the future, a lot of work needs to be done to construct a framework to analyze functional data with heavy-tailed features.

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Appendix A. Preliminaries on Random Measures and Laplace Functionals

Appendix A.1. Necessary and Sufficient Conditions of the Counting Measures

We say that the measure $N \in \mathbb{N}_{\mathbb{O}}$ is a counting measure. For $x \in \mathbb{C}$ and $\mu \in \mathbb{M}_{\mathbb{O}}$, we say that the measure μ has an atom x if $\mu(\{x\}) > 0$. A measure with only atoms is *purely atomic*, while a *diffuse* measure has no atom. The following lemma shows the necessary and sufficient conditions of a counting measure in $\mathbb{M}_{\mathbb{O}}$.

Lemma A1. *Assume that the measure $\mu \in \mathbb{M}_{\mathbb{O}}$.*

1. *The measure μ is uniquely decomposable as $\mu = \mu_a + \mu_d$, where*

$$\mu_a = \sum_{i=1}^{\infty} \kappa_i \mathbf{1}_{x_i} \tag{A1}$$

is a purely atomic measure, uniquely determined by a countable set $\{(x_i, \kappa_i)\} \subset \mathbb{O} \times (0, \infty)$, and μ_d ia a diffuse measure.

2. *A measure $N \in \mathbb{M}_{\mathbb{O}}$ is a counting measure if and only if (1) its diffuse component is null; (2) all κ_i in (A1) are positive integers; and (3) the set $\{x_i\}$ defined in (A1) is a countable set with, at most, many finite x_i in any set $A \in \mathbb{S} \setminus \mathbb{O}^r$ with $r > 0$.*

Proof. Let $r_j = 1/j$, $j = 1, 2, \ldots$. Let $\mathbb{C}^{(1)} = \mathbb{S} \setminus \mathbb{O}^{r_1}$ and $\mathbb{C}^{(j+1)} = \mathbb{O}^{r_j} \setminus \mathbb{O}^{r_{j+1}}$, $j = 1, 2, \ldots$. Then, $\mathbb{C} = \cup_{j=1}^{\infty} \mathbb{C}^{(j)}$. By definiton of $\mathbb{M}_{\mathbb{O}}$, if $\mu \in \mathbb{M}_{\mathbb{O}}$, the measure $\mu_j(\cdot) = \mu(\cdot \cap \mathbb{O}^{(r_j)})$ and hence μ is σ-finite. Part (i) is a property of σ-finite measures; see Appendix A1.6 [6] for details.

Since μ_j is finite, Proposition 9.1.III in [7] implies that μ_j is a counting measure if and only if all the three conditions in (ii) are satisfied. Moreover, if μ_j is a counting measure, all of its atoms must lie in $\mathbb{C}^{(r_j)}$. Because $\mathbb{C}^{(r_j)}$ are disjoint sets and $\mu = \sum_{j=1}^{\infty} \mu_j$, the measure μ is a counting measure if and only if all the three conditions in (ii) are satisfied. □

Appendix A.2. Convergence of Laplace Functionals

The first property of Laplace functional is that the uniform convergence of functions in $\mathcal{C}_{\mathbb{C}}$ implies the convergence of Laplace functionals.

Proposition A1. *Let ξ be a random measure. For a sequence of functions $(f_n)_{n\in\mathbb{N}}$ with $f_n \in \mathcal{C}_{\mathbb{C}}$ and a function $f \in \mathcal{C}_{\mathbb{C}}$, the convergence $L_\xi[f_n] \to L_\xi[f]$ as $\sup_{x\in\mathbb{C}} |f_n(x) - f(x)| \to 0$ if one of three conditions holds: (i) $\xi(\mathbb{C}) < \infty$; (ii) the pointwise convergence $f_n \to f$ is monotonic; and (iii) there exists $r > 0$ such that for each $n \geq 1$, f_n vanishes on $\mathbb{O}^r$.*

Proof. If condition (i) holds,

$$|L_\xi[f_n] - L_\xi[f]| \leq \xi(\mathbb{C}) \sup_{x\in\mathbb{C}} |f_n(x) - f(x)| \to 0\,.$$

If condition (iii) holds, since $\xi(\mathbb{S} \setminus \mathbb{C}^r) < \infty$,

$$|L_\xi[f_n] - L_\xi[f]| \leq \xi(\mathbb{S} \setminus \mathbb{O}^r) \sup_{x\in\mathbb{C}} |f_n(x) - f(x)| \to 0\,.$$

Suppose that condition (ii) holds. If $f_n(x) \downarrow f(x)$ for each $x \in \mathbb{C}$, this implies that $r > 0$ exists such that f_n vanishes on $\mathbb{O}^r$ because $f_1 \in \mathcal{C}_{\mathbb{C}}$ and condition (iii) is satisfied. If $f_n(x) \uparrow f(x)$ for each $x \in \mathbb{C}$, the dominated convergence theorem ensures that $L_\xi[f_n] \to L_\xi[f]$. □

Appendix B. Proof of Theorem 3

The proof of Theorem 3 follows the general idea of the proof of Proposition 3.21, [1], which needs the properties of the Laplace functionals. Let $\widetilde{S} = [0, \infty) \times S$ and $\widetilde{\mathbb{O}} = [0, \infty) \times \mathbb{O}$. Define $\widetilde{d}((t_1, x_1), (t_2, x_2)) = |t_1 - t_2| + d(x_1, x_2)$ for $(t_i, x_i) \in \widetilde{\mathbb{S}}$. By defining $\lambda(t, x) = (t, \lambda x)$ for $(t, x) \in \widetilde{\mathbb{S}}$ as scalar multiplication, the metric space $(\widetilde{\mathbb{S}}, \widetilde{d})$ is complete and separable, and the set $\widetilde{\mathbb{O}}$ is a cone. Now, we can define the corresponding $\mathbb{M}_{\mathbb{O}}$-convergence as above.

Assume that $\mathcal{C}_{\widetilde{\mathbb{C}}}$ is a collection of real-valued, non-negative, bounded continuous functions f on $\widetilde{\mathbb{C}}$ vanishing on the set $\widetilde{\mathbb{O}}^r = \{(t, x) \in \widetilde{\mathbb{S}} : \widetilde{d}((t, x), \widetilde{\mathbb{O}}) < r\}$ for some $r > 0$. The Laplace functional of f is given by

$$\begin{aligned} L_{N_n}[f] &= \mathbb{E}[\exp(-N_n(f))] = \mathbb{E} \exp\Big\{ -\sum_{i=1}^{n} f(n^{-1}i, X_{n,i}) \Big\} \\ &= \prod_{i=1}^{n} \Big(1 - \int_{\mathbb{C}} (1 - e^{-f(n^{-1}i, x)}) P(X_{n,1} \in \mathrm{d}x) \Big)\,. \end{aligned}$$

Equivalently, $L_{N_n}[f] \to L_N[f]$ if and only if

$$-\log L_{N_n}[f] = -\sum_{i=1}^{n} \log \Big(1 - \int_{\mathbb{C}} (1 - e^{-f(n^{-1}i, x)}) P(X_{n,1} \in \mathrm{d}x) \Big) \to -\log L_N[f]\,.$$

Moreover, according to Proposition 2, $L_{N_n}[f] \to L_N[f]$ for all $f \in \mathcal{C}_{\widetilde{\mathbb{C}}}$ if and only if $N_n \xrightarrow{d} N$ as $n \to \infty$.

Suppose that (10) holds. Let $\lambda_n([t_1, t_2) \times A) = |t_1 - t_2| p(X_{n,1} \in A)$ with $0 \le t_1 < t_2$ and $A \in \mathcal{S}_{\mathbb{C}}$ satisfying $\mu(\partial A) = 0$. The limit (10) implies that

$$\lim_{n\to\infty} \lambda_n([t_1, t_2) \times A) = |t_1 - t_2| \mu(A) .$$

Therefore, we have

$$\begin{aligned}
&\sum_{i=1}^{n} \int_{\mathbb{C}} (1 - e^{-f(n^{-1}i,x)}) P(X_{n,1} \in \mathrm{d}x) \\
&= \int \int_{[0,\infty)\times\mathbb{C}} (1 - e^{-f(s,x)})\, \mathrm{d}\lambda_n(s,x) \\
&\to \int \int (1 - e^{-f(s,x)})\, \mathrm{d}s \mu(\mathrm{d}x) , \quad n \to \infty .
\end{aligned}$$

Suppose that f vanishes on $[0,\infty) \times \widetilde{O}^r$ for a real number $r > 0$ and $\mu(\partial \mathbb{O}^r) = 0$. Then, we have

$$\sup_{i\ge 1} \int_{\mathbb{C}} (1 - e^{-f(n^{-1}i,x)}) P(X_{n,1} \in \mathrm{d}x) \le P(X_{n,1} \in \mathbb{O}^r) \to 0 , \quad n \to \infty , \tag{A2}$$

by (10). Notice that $\log(1 + y) = y - y^2/2 + o(y^2)$. We have

$$\begin{aligned}
&\Big| - \log L_{N_n}[f] - \sum_{i=1}^{n} \int_{\mathbb{C}} (1 - e^{-f(n^{-1}i,x)}) P(X_{n,1} \in \mathrm{d}x) \Big| \\
&\le \sum_{i=1}^{n} \Big(\int_{\mathbb{C}} (1 - e^{-f(n^{-1}i,x)}) P(X_{n,1} \in \mathrm{d}x) \Big)^2 \\
&\le \Big(\sup_{i\ge 1} \int_{\mathbb{O}^r} (1 - e^{-f(n^{-1}i,x)}) P(X_{n,1} \in \mathrm{d}x) \Big) \sum_{i=1}^{\infty} \int_{\mathbb{O}^r} (1 - e^{-f(n^{-1}i,x)}) P(X_{n,1} \in \mathrm{d}x) \\
&\to 0 , \quad n \to \infty .
\end{aligned}$$

This shows that (10) implies that $L_{N_n}[f] \to L_N[f]$.

Suppose that $L_{N_n}[f] \to L_N[f]$ as $n \to \infty$ for $f \in \mathcal{C}_{\widetilde{\mathbb{C}}}$. Take $f(s,x) = 1_{[0,1]}(s) g(x)$ with $g \in \mathcal{C}_{\mathbb{C}}$. Then,

$$\begin{aligned}
L_{N_n}[f] &= \mathbb{E} \exp \Big(- \sum_{i=1}^{n} g(X_{n,i}) \Big) = \big(\mathbb{E} \exp(-g(X_{n,1})) \big)^n \\
&= \Big(1 - \frac{\int_{\mathbb{C}} (1 - e^{-g(x)}) n P(X_{n,1} \in \mathrm{d}x)}{n} \Big)^n \\
&\to \exp \Big(- \int_{\mathbb{C}} (1 - g(x)) \mu(\mathrm{d}x) \Big) , \quad n \to \infty ,
\end{aligned}$$

which is the Laplace functional of PRM(μ). This relation implies that (9) holds.

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