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Closed-Form Solutions to a Forced Damped Rotational Pendulum Oscillator

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Abstract: In this investigation, some analytical solutions to both conserved and non-conserved rotational pendulum systems are reported. The exact solution to the conserved oscillator (unforced, undamped rotational pendulum oscillator), is derived in the form of a Jacobi elliptical function. Moreover, an approximate solution for the conserved case is obtained in the form of a trigonometric function. A comparison between both exact and approximate solutions to the conserved oscillator is examined. Moreover, the analytical approximations to the non-conserved oscillators including the unforced, damped rotational pendulum oscillator and forced, damped rotational pendulum oscillator are obtained. Furthermore, all mentioned oscillators (conserved and non-conserved oscillators) are linearized, and their exact solutions are derived. In addition, all obtained approximations are compared with the four-order Runge–Kutta (RK4) numerical approximations and with the exact solutions to the linearized oscillators. The obtained results can help several authors for discussing and interpreting their results.

Keywords: rotational pendulum system; damped rotational pendulum oscillator; forced damped rotational pendulum oscillator; Jacobian elliptic functions; trigonometric function

MSC: 70K40; 70K60; 70K75; 34B15; 34C15

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1. Introduction

The nonlinear differential equations (NDEs) including all their types, such as ordinary, partial, linear, and nonlinear types have a huge impact on scientific research because they can model a wide range of real-life phenomena, engineering, and physical problems [1–10]. For instance, Wazwaz [1] discussed a huge number of (non)linear partial differential equations (PDEs), (in)homogeneous PDEs, some systems of (non)linear PDEs, and one-dimensional and multidimensional PDEs by using many analytical and numerical methods, such as the Adomian decomposition method (ADM), modified ADM, the variational iteration method (VIM), the tanh method, the tanh-coth method, the sine–cosine method, Hirota's bilinear method, etc. In addition, Wazwaz [1] and many authors [2–5] discussed many applications of physical equations in different fields of science, including the physics of plasmas, fluids, and optical fibers, such as one-dimensional and multi-dimensional (in)homogeneous heat equations, one-dimensional and multi-dimensional

(in)homogeneous wave equations, the (non)linear Klein–Gordon equation (KGE), Burgers equation, the telegraph equation, the (non)linear Schrödinger equation, the family of Korteweg–deVries (KdV) equations, etc. El-Dib [6] used the linearizing method for determining the displacement amplitude and approximate frequency to the third-order ordinary differential equations (ODEs). He [7] applied some asymptotic techniques such as parameter-expanding methods, variational approaches, the homotopy perturbation method (HPM), the parameterized perturbation technique (PT), ancient Chinese methods, iteration PT for analyzing both weakly and strongly NDEs. He and El-Dib [8] reduced the damped KGE to the Duffing equation (DE) by using suitable transformation and solved the DE by using the hybrid reducing rank method with the HPM. All these models could be solved analytically or numerically to obtain results that help to demonstrate the researcher's hypothesis and to achieve new innovative applications.

The model of nonlinear oscillators that describes the motion of different types of pendulum oscillators with different rotations and directions is one of the most successful models for describing and modeling many physical and engineering applications such as the wind vibration [11–20]. To find an exact solution to the dynamical system of the pendulum oscillation is not an easy task, and sometimes it can be impossible due to its nonlinearity. However, some numerical methods or analytical approximation methods could be more suitable to get some approximate solutions that may be close enough to the exact solutions [21]. Many authors derived in details the rotating pendulum equation of motion and solved this equation by using different approaches [11–20]. For instance, He et al. [11] used the HPM to derive an analytic solution to the rotating pendulum oscillator. Hieu et al. [12] applied both an equivalent linearization approach with a weighted averaging for studying different types of undamped nonlinear oscillators such as the cubic nonlinearity Duffing oscillator (DO), DO with discontinuity, rotating pendulum oscillator, and many others. Moreover, the authors [12] made a comparison with many other methods (e.g., HPM, VIM, energy balance method (BM), harmonic BM, and many other methods) for checking the accuracy of the obtained solutions. González–Gaxiola et al. [15] studied the system of the following rotational pendulum model by proposing a new numerical approach, known as the Rach–Adomian–Meyers decomposition method (MDM)

$$\begin{cases} R_1 \equiv \ddot{\theta} + \frac{g}{l}\sin(\theta) - \frac{1}{2}\Omega^2\sin(2\theta) = 0, \\ \theta(0) = \theta_0 \text{ and } \dot{\theta}(0) = \dot{\theta}_0. \end{cases} \tag{1}$$

The authors [15], solved the initial value problem (i.v.p.) (1) for $\dot{\theta}_0 = 0$. Moreover, they calculated the relative error compared to the exact solution to verify the accuracy of the proposed method. They mentioned that they reach a solution to the given system with a high degree of accuracy. However, in Ref. [16], the following nonlinear pendulum system with arbitrary amplitude was investigated by using an analytical approximation method based on Maclurin series and Chebyshev polynomials

$$\begin{cases} R_2 \equiv \ddot{\theta} + \omega_0^2(1 - \Lambda\cos(\theta))\sin(\theta) = 0, \\ \theta(0) = \theta_0 \text{ and } \dot{\theta}(0) = 0. \end{cases} \tag{2}$$

Here, $\omega_0^2 = g/l$ where l and g represent the pendulum length and the gravitational acceleration. The authors derived a new analytical approximation for the i.v.p. (2) by transforming it to the cubic-quintic DO. Furthermore, the conservative rotational pendulum system was solved by applying the Hamiltonian approach [17]. The authors applied the Hamiltonian approach for deriving natural frequency to the rotating pendulum. In this technique, the problem solved without series approximation and without more restrictive assumptions.

For all these previous studies [15–17], the authors described the different pendulum oscillators without considering both forcing and damping terms, which they only investigated in the conserved case. There are many applications to the rotating pendulum oscillators such as fly-ball governors, vibration absorbers, and breaking symmetry in quantum mechanics [13]. Moreover, the rotational pendulum oscillator can be used for modeling

mechanical and civil structure problems [22,23] and the wind-excited vibration [24]. Motivated by these applications and many others, the current paper is devoted to studying the rotational pendulum problem taking the impact of both the damping term and the excited periodic force, in this case the rotational pendulum problem called the forced, damped rotational pendulum model

$$\begin{cases} R_4 \equiv \ddot{\theta} + 2\varepsilon\dot{\theta} + \frac{g}{l}\sin(\theta) - \frac{1}{2}\Omega^2\sin(2\theta) - \gamma\cos(\omega t) = 0, \\ \theta(0) = \theta_0 \text{ and } \dot{\theta}(0) = \dot{\theta}_0. \end{cases} \tag{3}$$

In addition, this paper presents different analytical approaches to get the exact solution of the conserved case, i.e., for $\varepsilon = \gamma = 0$. Remember that $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$.

Many analytical and approximate methods have been developed for solving different types of nonlinear oscillators, including the harmonic balance method (HBM) [25,26], the parameter-expansion method (PEM) [27], the energy balance method (EBM) [28,29], the use of special functions [30–32], the Hamiltonian approach [33], HPM [34,35], the max–min approach [36], the VIM [37,38], and many other analytical and numerical techniques. Thus, in order to understand the mechanism of many nonlinear oscillators, it is important to find their analytical solutions. However, most dynamical system problems can't be solved analytically. Therefore, many researchers resort to solving such problems numerically or by using some approximate analytical methods. However, the numerical approaches are frequently costly and time consuming when acquring highly accurate approximations to the complicated nonlinear problems under consideration. Accordingly, in the present study, we will try to find some exact and analytical approximations to the rotational pendulum-type oscillators by using the ansatz method. We will prove that all approximations that will be obtained are distinguished by high accuracy and are more stable for a long-time.

2. Analytical Solutions to the Rotational Pendulum-Type Oscillators

We proceed to find some exact and approximations to the undamped and damped rotational pendulum oscillators and for the forced, damped rotating pendulum oscillator by using the ansatz method.

2.1. Conserved Rotational Pendulum Oscillator

The solution of the i.v.p. (1) is supposed to have the following form:

$$\theta = 2\arctan(u), \tag{4}$$

where $u \equiv u(t)$ and $\theta \equiv \theta(t)$ for simplicity only.

Inserting Equation (4) into Equation (1) yields

$$R_1 = \frac{2}{l(u^2+1)^2}S_0, \tag{5}$$

with

$$S_0 = gu^3 + gu - 2l(\dot{u})^2u + l\ddot{u}u^2 + l\Omega^2u^3 - l\Omega^2u + l\ddot{u}. \tag{6}$$

In order to solve the ode $S_0 = 0$, the following ansatz is assumed,

$$u = A\,\mathrm{cn}\left(\sqrt{\omega}t + B, m\right), \tag{7}$$

where (A, B, ω, m) are undermined parameters.

Before inserting ansatz (7) into $S_0 = 0$, the following identities are introduced,

$$\begin{aligned}
\frac{d}{dt}\mathrm{cn} &= -\sqrt{\omega}\mathrm{dn} \times \mathrm{sn},\\
\frac{d^2}{dt^2}\mathrm{cn} &= -\omega\mathrm{cn} \times \left(\mathrm{dn}^2 - m\mathrm{sn}^2\right),\\
\mathrm{dn}^2 &= 1 - m(1 - \mathrm{cn}^2),\\
\mathrm{sn}^2 &= 1 - \mathrm{cn}^2,
\end{aligned} \tag{8}$$

where $\mathrm{cn} \equiv \mathrm{cn}(\sqrt{\omega}t + B, m)$ and likewise with respect to the rest elliptic functions.

By using the above identities in $S_0 = 0$, we get

$$\begin{aligned}
&S_0 = A\left(A^2 g - 2A^2 lm\omega + A^2 l\omega + A^2 l\Omega^2 - 2lm\omega\right)\mathrm{cn}^3 +\\
&A\left(2A^2 lm\omega - 2A^2 l\omega + g + 2lm\omega - l\omega - l\Omega^2\right)\mathrm{cn}.
\end{aligned} \tag{9}$$

Equating to zero the coefficients of cn^3 and cn, solving the obtained system, we finally get

$$\begin{aligned}
m &= \frac{A^2\left(A^2 g + A^2 l\Omega^2 + g\right)}{\left(A^4 - 1\right)l\Omega^2 + \left(A^2 + 1\right)^2 g},\\
\omega &= \frac{g}{l} + \frac{\left(A^2 - 1\right)\Omega^2}{A^2 + 1}.
\end{aligned} \tag{10}$$

For the following initial conditions (ICs), the constants A and B can be determined:

$$\begin{aligned}
u(0) &= \tan\left(\frac{\theta_0}{2}\right) := u_0,\\
\dot{u}(0) &= \frac{\dot{\theta}_0}{1 + \cos\theta_0} := \dot{u}_0.
\end{aligned} \tag{11}$$

By using of the following formula—

$$\mathrm{cn}(f + g, m) = \frac{\mathrm{cn}(f, m)\mathrm{cn}(g, m) - \mathrm{sn}(f, m)\mathrm{dn}(f, m)\mathrm{sn}(g, m)\mathrm{dn}(g, m)}{1 - m \cdot \mathrm{sn}^2(f, m)\mathrm{sn}^2(g, m)}, \tag{12}$$

the closed form solution to the i.v.p. (1) is finally obtained,

$$\theta_{\mathrm{cn}} = 2\arctan\left(\frac{\tan\left(\frac{\theta_0}{2}\right)\mathrm{cn}\left(\sqrt{\omega}t, m\right) + b_1\mathrm{sn}\left(\sqrt{\omega}t, m\right)\mathrm{dn}\left(\sqrt{\omega}t, m\right)}{1 + b_2\mathrm{sn}^2\left(\sqrt{\omega}t, m\right)}\right), \tag{13}$$

with

$$\begin{aligned}
b_1 &= \frac{\dot{\theta}_0}{\left(\cos(\theta_0) + 1\right)\sqrt{\frac{g}{l} - \Omega^2\cos(\theta_0)}},\\
b_2 &= \frac{\dot{\theta}_0^2 l\sec^2\left(\frac{\theta_0}{2}\right)}{4l\Omega^2\cos(\theta_0) - 4g}.
\end{aligned}$$

The exact period of oscillations is calculated as

$$T = \frac{4K(m)}{\sqrt{\omega}},$$

where $K(m)$ represents the first kind of elliptic function.

We also may obtain an approximate trigonometric solution to the ode (6) via introducing the following ansatz

$$u = u_0 \cos\left(\sqrt{w}t\right) + C \sin\left(\sqrt{w}t\right). \tag{14}$$

Inserting the ansatz (14) into $S_0 = 0$, yields

$$S_0 = \frac{1}{4} u_0 \cos\left(t\sqrt{w}\right) \left(\begin{array}{c} 3C^2 g - 5C^2 l w + 3C^2 l \Omega^2 + 3 g u_0^2 \\ +4g - 5 l u_0^2 w + 3 l u_0^2 \Omega^2 - 4 l w - 4 l \Omega^2 \end{array} \right) + \cdots . \tag{15}$$

Equating to zero the coefficient of $\cos \sqrt{w}t$ in expression (15) gives

$$w = \frac{1}{5l} \left[\frac{8\left(g - 4l\Omega^2\right)}{5C^2 + 5u_0^2 + 4} + 3\left(g + l\Omega^2\right) \right]. \tag{16}$$

By using the ICs (11), we final get C as

$$C = \pm \frac{1}{\sqrt{6}} \sqrt{ \begin{array}{c} \frac{l\left(5\dot{u}_0^2 + 8\Omega^2\right)}{g + l\Omega^2} - 3u_0^2 - 4 \\ \pm \frac{\sqrt{12l\left(5u_0^2+4\right)\dot{u}_0^2 (g + l\Omega^2) + \left(3u_0^2 (g + l\Omega^2) + 4g - 5l\dot{u}_0^2 - 4l\Omega^2\right)^2}}{g + l\Omega^2} \end{array} }. \tag{17}$$

Inserting Equation (14) into Equation (4), we finally get the solution to the i.v.p. (1) in terms of trigonometric function

$$\theta_{\text{Trig}} = 2 \arctan\left(u_0 \cos\left(\sqrt{w}t\right) + C \sin\left(\sqrt{w}t\right)\right). \tag{18}$$

Furthermore, by linearizing Equation (1), we get

$$\begin{cases} R_1 \equiv \ddot{\theta} + Q^2 \theta = 0, \\ \theta(0) = \theta_0 \text{ and } \dot{\theta}(0) = \dot{\theta}_0. \end{cases} \tag{19}$$

Solving the conservative i.v.p. (19), the following exact solution is obtained,

$$\theta_{\text{lin}} = \theta_0 \cos(Qt) + \frac{\dot{\theta}_0}{Q} \sin(Qt), \tag{20}$$

where $Q = \sqrt{g/l - \Omega^2}$.

The analytical solution (13) and the approximate solution (16) to the i.v.p. (1) and the exact solution (20) to the i.v.p. (19) (linearized form to the i.v.p. (1)) are compared with the RK4 numerical approximation to the i.v.p. (1) for $\left(g, l, \Omega, \theta_0, \dot{\theta}_0\right) = \left(9.81, 10, \sqrt{1/120}, \pi/8, 0\right)$. In addition, the phase portrait to the dynamical system of the i.v.p. (1) near the equilibrium point $(0, 0)$ is presented in Figure 1. Moreover, for checking the accuracy of the obtained solutions to the conserved case, the global maximum distance error (GMDE) L_d as compared to RK4 numerical approximation to the i.v.p. (1) in the whole time domain is estimated:

$$\begin{aligned} L_d &= \max_{t_0 < t < T} \left| \text{RK4} - \theta_{\text{Trig}} \right| = 0.0029364, \\ L_d &= \max_{t_0 < t < T} \left| \text{RK4} - \theta_{\text{lin}} \right| = 0.108229. \end{aligned}$$

Moreover, the distance error L_d as compared to the exact solution (13) to the original problem (1) is estimated:

$$\begin{aligned} L_d &= \max_{t_0 < t < T} \left| \theta_{\text{cn}} - \theta_{\text{Trig}} \right| = 0.00174605, \\ L_d &= \max_{t_0 < t < T} \left| \theta_{\text{cn}} - \theta_{\text{lin}} \right| = 0.0181119. \end{aligned}$$

In spite of this, solution (20) is exact solution to the i.v.p. (19) but by comparing this solution with the analytical solution (13) and the approximate solution (16) to the i.v.p. (1), we can observe that the linearized form (we mean here the i.v.p. (19)) to the i.v.p. (1) is not a good idea for getting a solution to original problem (1) as shown in Figure 2. On the other side, the approximate solution (18) is distinguished by high accuracy and higher stability against the corresponding time. Moreover, there is a good agreement between this approximation and the RK4 numerical approximation.

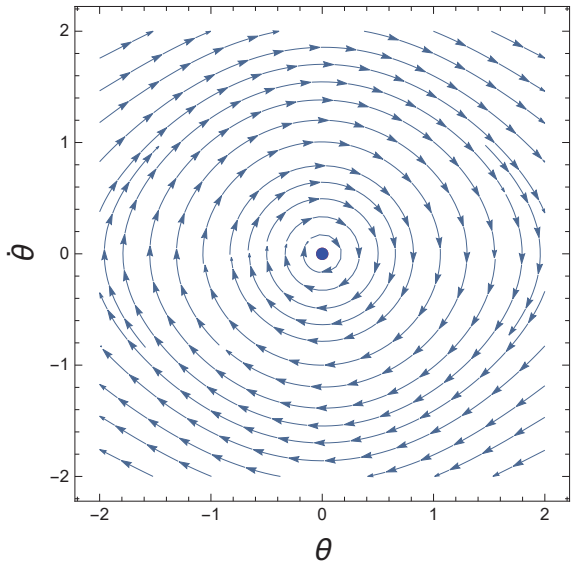


Figure 1. The phase portrait to the dynamical system of the i.v.p. (1) near the equilibrium point $(0, 0)$ is considered for $(g, l, \Omega, \theta_0, \dot{\theta}_0) = (9.81, 10, \sqrt{1/120}, \pi/8, 0)$.

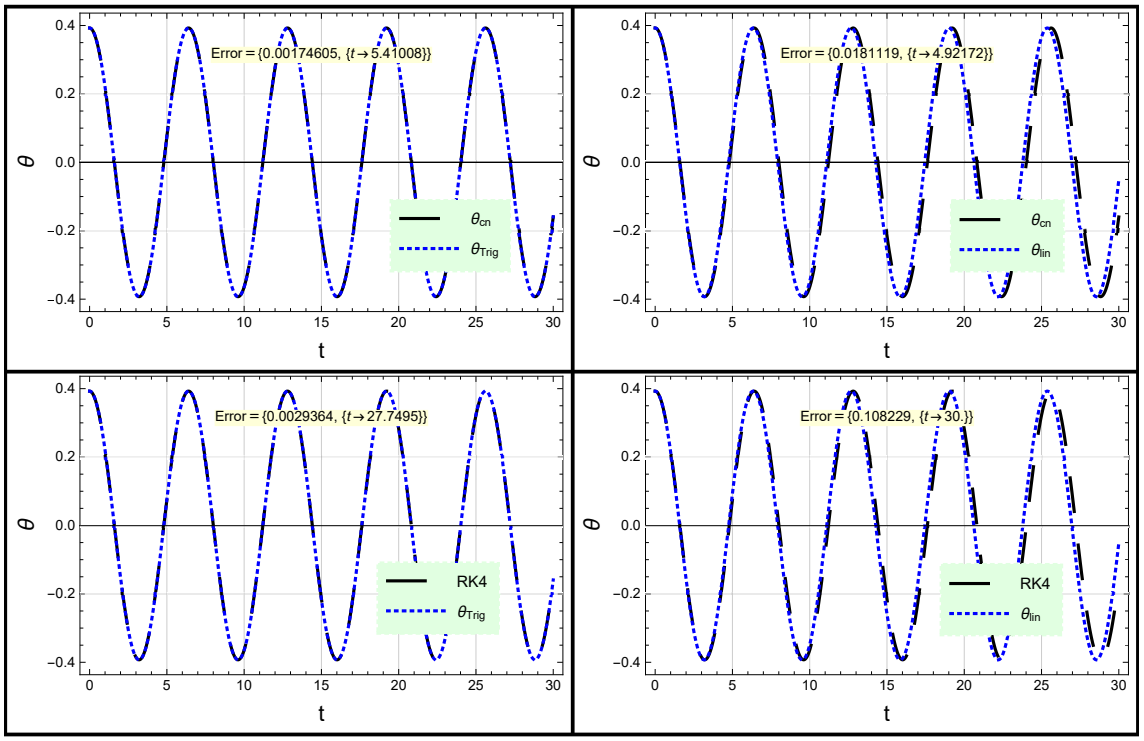


Figure 2. The analytical solution (13) and the approximation (16) to the conserved i.v.p. (1) as well as the exact solution (20) to the i.v.p. (19) (linearized form to the i.v.p. (1)) are compared to the RK4 numerical approximation for $(g, l, \Omega, \theta_0, \dot{\theta}_0) = (9.81, 10, \sqrt{1/120}, \pi/8, 0)$.

2.2. Unforced Damped Rotational Pendulum Oscillator

Let us consider the following damped i.v.p.:

$$\begin{cases} R_5 \equiv \ddot{\theta} + 2\varepsilon\dot{\theta} + \frac{g}{l}\sin(\theta) - \frac{1}{2}\Omega^2\sin(2\theta) = 0, \\ \theta(0) = \theta_0 \text{ and } \theta'(0) = \dot{\theta}_0. \end{cases} \tag{21}$$

The solution of the i.v.p. (21) can be introduced in the form

$$\theta_{\text{Trig}} = 2\arctan(u), \tag{22}$$

where $u \equiv u(t)$ and $\theta = \theta(t)$ for simplicity only.

Inserting the ansatz (22) into R_5, we get

$$R_5 = \frac{2}{l(u^2+1)^2} G_1, \tag{23}$$

with

$$G_1 = \begin{array}{c} gu^3 + gu + \varepsilon l u^4 u' + 2\varepsilon l u^2 u' - 2lu(u')^2 \\ +lu^2u'' + l\Omega^2u^3 - l\Omega^2u + \varepsilon l u' + lu'' \end{array}. \tag{24}$$

Next, assume the following ansatz

$$u = e^{-\varepsilon t}[c_0\cos(w(t)) + c_1\sin(w(t))], \tag{25}$$

where $c_{0,1}$ are undetermined parameters.

Inserting ansatz (25) into Equation (24), we obtain

$$G_1 = S_0\cos(w(t)) + S_1\sin(w(t)) + \cdots, \tag{26}$$

where the coefficients $S_{0,1}$ are given in Appendix A. Equating to zero the coefficients S_0 and S_1 and eliminating the second derivative $w''(t)$ from the resulting system gives an ode for determining the unknown function $w(t)$. Solving this ode yields

$$w(t) = \int_0^t \sqrt{\frac{8(g - l\Omega^2)e^{4\varepsilon\tau} + 6(g + l(-3\varepsilon^2 + \Omega^2))(c_0^2 + c_1^2)e^{2\varepsilon\tau} - 5l\varepsilon^2(c_0^2 + c_1^2)^2}{8e^{4\varepsilon\tau}l + 10e^{2\varepsilon\tau}l(c_0^2 + c_1^2)}}\, d\tau, \tag{27}$$

where the constants c_0 and c_1 can be determined from the ICs.

Moreover, by linearizing the i.v.p. (21), we get

$$\begin{cases} R_5 \equiv \ddot{\theta} + 2\varepsilon\dot{\theta} + Q^2\theta = 0, \\ \theta(0) = \theta_0 \text{ and } \dot{\theta}(0) = \dot{\theta}_0. \end{cases} \tag{28}$$

By solving the non-conservative i.v.p. (28), the following exact solution is obtained:

$$\theta_{\text{lin}} = e^{-\varepsilon t}\left(\theta_0\cos\left(t\tilde{Q}\right) + \frac{(\varepsilon\theta_0 + \dot{\theta}_0)}{\tilde{Q}}\sin\left(t\tilde{Q}\right)\right), \tag{29}$$

where $\tilde{Q} = \sqrt{Q^2 - \varepsilon^2}$.

Both approximation (22) to the i.v.p. (21) and the exact solution (29) to the linearized i.v.p. (28) are compared with the RK4 numerical approximation to the i.v.p. (21) as illustrated in Figure 3a,b, respectively, for $(g, l, \Omega, \varepsilon) = \left(9.81, 10, \sqrt{0.5}, 0.025\right)$ and $\left(\theta_0, \dot{\theta}_0\right) = (0, 0)$. Moreover, the phase portrait to the dynamical system of the i.v.p. (21)

near the equilibrium point $(0,0)$ is displayed in Figure 4. In addition, the GMDE L_d to the approximation (22) and the solution (29) is estimated as follows

$$L_d = \max_{t_0<t<170} \left|\text{RK4} - \theta_{\text{Sol. (22)}}\right| = 0.00095286,$$
$$L_d = \max_{t_0<t<170} \left|\text{RK4} - \theta_{\text{Sol. (29)}}\right| = 0.00210528.$$

As is clear, the analytical approximation (22) to the original problem (21) is more accurate than the exact solution (29) to the linearized problem (28).

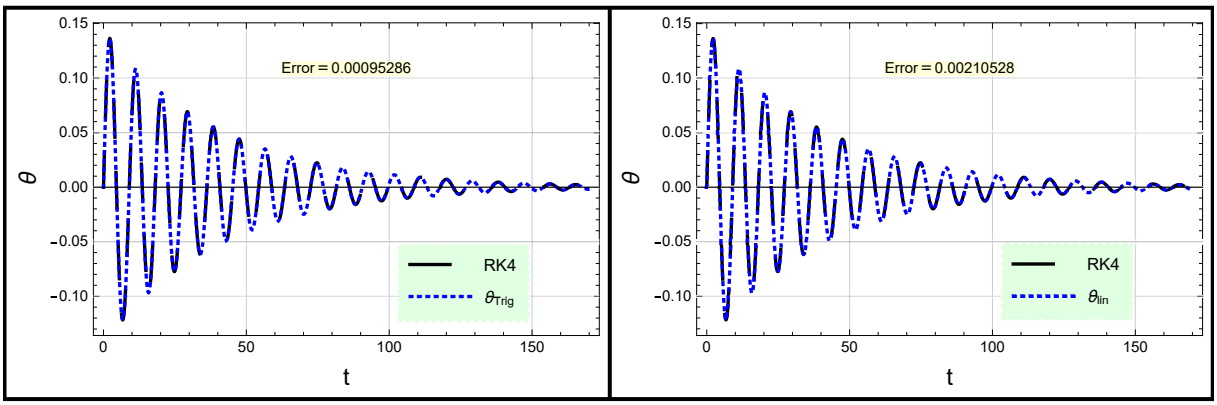


Figure 3. The approximation (22) to the non-conserved i.v.p. (21) and the exact solution (29) to the linearized i.v.p. (28) are compared with the RK4 numerical approximation to the i.v.p. (21) for $(g, l, \Omega, \varepsilon) = \left(9.81, 10, \sqrt{0.5}, 0.025\right)$ and $(\theta_0, \dot{\theta}_0) = (0,0)$.

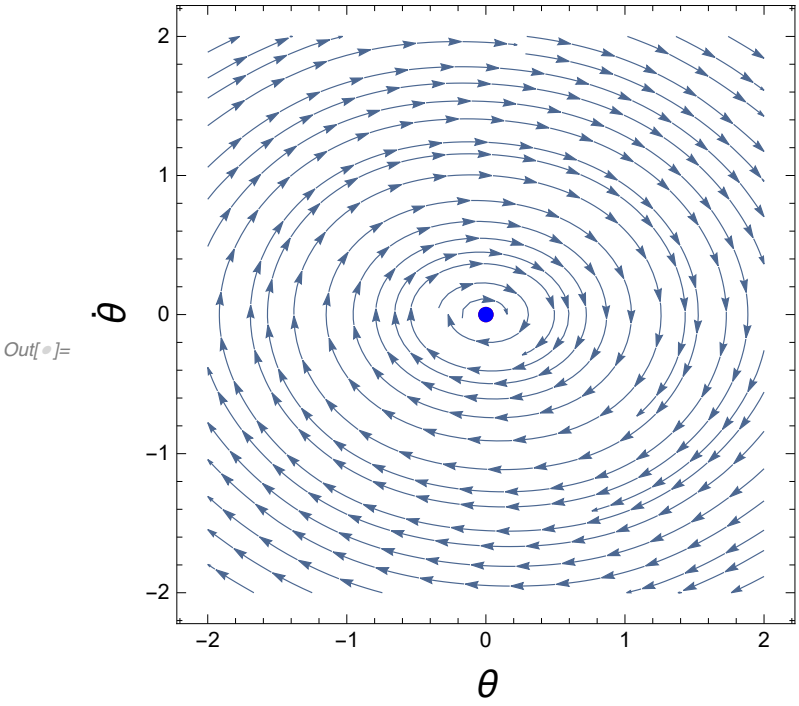


Figure 4. The phase portrait to the dynamical system of the i.v.p. (21) near the equilibrium point $(0,0)$ is considered for $(g, l, \Omega, \varepsilon) = \left(9.81, 10, \sqrt{0.5}, 0.025\right)$ and $(\theta_0, \dot{\theta}_0) = (0,0)$.

3. Forced Damped Rotational Pendulum Oscillator

Now, to find an approximation to the i.v.p. (3), the following ansatz is introduced,

$$\theta(t) = \varphi(t) + c_1 \cos(\omega t) + c_2 \sin(\omega t), \tag{30}$$

where $\varphi \equiv \varphi(t)$ indicates a solution to the following i.v.p.:

$$\begin{cases} \ddot{\varphi} + 2\varepsilon\dot{\varphi} + \frac{g}{l}\sin(\varphi) - \frac{1}{2}\Omega^2\sin(2\varphi) = 0, \\ \varphi(0) = \theta_0 - c_1 \text{ and } \varphi'(0) = \dot{\theta}_0 - \omega c_2. \end{cases} \tag{31}$$

The constants c_1 and c_2 can be determined later.

For $\sin(\theta) \approx \left(\theta - \frac{1}{6}\theta^3\right)$ and $\sin(2\theta) \approx \left(2\theta - \frac{1}{6}(2\theta)^3\right)$, we have

$$\begin{aligned} R_4 &\approx \ddot{\theta} + 2\varepsilon\dot{\theta} + \tfrac{g}{l}\left(\theta - \tfrac{1}{6}\theta^3\right) - \tfrac{1}{2}\Omega^2\left(2\theta - \tfrac{1}{6}(2\theta)^3\right) - \gamma\cos(\omega t) \\ &= W_0\cos(\omega t) + W_1\sin(\omega t) + \cdots, \end{aligned} \tag{32}$$

where the coefficients $W_{0,1}$ are given in Appendix B. For $W_0 = W_1 = 0$, at $\varphi(t) = 0$, the values of the constants (c_1, c_2), can be obtained as

$$\left(g - 4l\Omega^2\right)c_1^3 + 8l(\gamma - 2\varepsilon\omega c_2) + c_1\left(-8g + 8l\left(\omega^2 + \Omega^2\right) + \left(g - 4l\Omega^2\right)c_2^2\right) = 0, \tag{33}$$

and

$$-2\varepsilon\omega c_1 - \omega^2 c_2 - \frac{gc_2\left(-8 + c_1^2 + c_2^2\right)}{8l} + \frac{1}{2}\Omega^2 c_2\left(-2 + c_1^2 + c_2^2\right) = 0. \tag{34}$$

Eliminating c_2 form Equations (33) and (34), the following cubic equation is obtained:

$$\begin{aligned} &8l\gamma\left(g\gamma^2 - 32g\varepsilon^2\omega^2 + 32l\varepsilon^2\omega^4 - 4l\gamma^2\Omega^2 + 32l\varepsilon^2\omega^2\Omega^2\right) \\ &-8\begin{pmatrix} g^2\gamma^2 - gl\gamma^2\omega^2 - 32g^2\varepsilon^2\omega^2 + 64gl\varepsilon^2\omega^4 - 128l^2\varepsilon^4\omega^4 \\ -32l^2\varepsilon^2\omega^6 - 5gl\gamma^2\Omega^2 + 4l^2\gamma^2\omega^2\Omega^2 + 64gl\varepsilon^2\omega^2\Omega^2 \\ -64l^2\varepsilon^2\omega^4\Omega^2 + 4l^2\gamma^2\Omega^4 - 32l^2\varepsilon^2\omega^2\Omega^4 \end{pmatrix}c_1 \\ &-64l\gamma\varepsilon^2\omega^2\left(g - 4l\Omega^2\right)c_1^2 + \gamma^2\left(g - 4l\Omega^2\right)^2 c_1^3 = 0. \end{aligned} \tag{35}$$

Eliminating c_1 form Equations (33) and (34), we get another cubic

$$\begin{aligned} &256\varepsilon^2\omega^2\left(g^2 - 2gl\omega^2 + 4l^2\varepsilon^2\omega^2 + l^2\omega^4 - 2gl\Omega^2 + 2l^2\omega^2\Omega^2 + l^2\Omega^4\right)c_2 \\ &-32\gamma\varepsilon\omega\left(g - 4l\Omega^2\right)\left(g - l\omega^2 - l\Omega^2\right)c_2^2 + \gamma^2\left(g - 4l\Omega^2\right)^2 c_2^3 = 0. \end{aligned} \tag{36}$$

The value of c_1 is defined to be the least in magnitude real root to the cubic Equation (35) and the value of c_2 equals the least in magnitude real root to the cubic Equation (36).

Furthermore, the linearization of the i.v.p. (3) gives us

$$\begin{cases} R_5 \equiv \ddot{\theta} + 2\varepsilon\dot{\theta} + Q^2\theta = f(t), \\ \theta(0) = \theta_0 \text{ and } \dot{\theta}(0) = \dot{\theta}_0. \end{cases} \tag{37}$$

Solving the forced non-conservative i.v.p. (28), the following exact solution is obtained

$$\theta_{\text{lin}} = \frac{e^{-\varepsilon t}}{\tilde{Q}Q_\varepsilon}\left\{\begin{array}{c} \sin\left(t\tilde{Q}\right)\left[Q_\varepsilon(\varepsilon\theta_0 + \dot{\theta}_0) - \gamma\varepsilon\left(Q^2 + \omega^2\right)\right] \\ +\tilde{Q}\left[\begin{array}{c} \cos\left(t\tilde{Q}\right)(\gamma Q_\omega + \theta_0 Q_\varepsilon) \\ +\gamma e^{\varepsilon t}(-Q_\omega\cos(t\omega) + 2\varepsilon\omega\sin(t\omega)) \end{array}\right] \end{array}\right\}, \tag{38}$$

where $Q_\omega = \left(\omega^2 - Q^2\right)$ and $Q_\varepsilon = \left(4\omega^2\varepsilon^2 + Q_\omega^2\right)$.

Both approximation (30) to the i.v.p. (3) and the exact solution to the i.v.p. (37) are compared with the RK4 numerical approximation to the i.v.p. (3) as illustrated in Figure 5a,b, respectively, for $(g, l, \omega_0, \omega, \Omega, \varepsilon, \gamma) = \left(9.81, 10, \sqrt{g/l}, 0.1, \omega_0/2, 0.1, 0.1\right)$ and $\left(\theta_0, \dot{\theta}_0\right) = (0, 0)$. Moreover, the GMDE L_d to the approximation (30) is estimated:

$$\begin{aligned} L_d &= \max_{t_0 < t < 30}\left|\text{RK4} - \theta_{\text{Sol. (30)}}\right| = 0.00147272, \\ L_d &= \max_{t_0 < t < 30}\left|\text{RK4} - \theta_{\text{Sol. (38)}}\right| = 0.0000218495. \end{aligned}$$

In addition, for $(g, l, \omega, \Omega, \varepsilon, \gamma) = (9.8, 1, 0.5, 0.1, 0.02, 0.2)$ and $(\theta_0, \dot{\theta}_0) = (0, 0)$, both approximation (30) to the i.v.p. (3) and the exact solution to the i.v.p. (37) are compared with the RK4 numerical approximation to the i.v.p. (3) as illustrated in Figure 6a,b, respectively. The GMDE L_d is also estimated as follows

$$L_d = \max_{t_0<t<50} \left|\text{RK4} - \theta_{\text{Sol. (30)}}\right| = 0.0000921758,$$
$$L_d = \max_{t_0<t<50} \left|\text{RK4} - \theta_{\text{Sol. (38)}}\right| = 0.000082668.$$

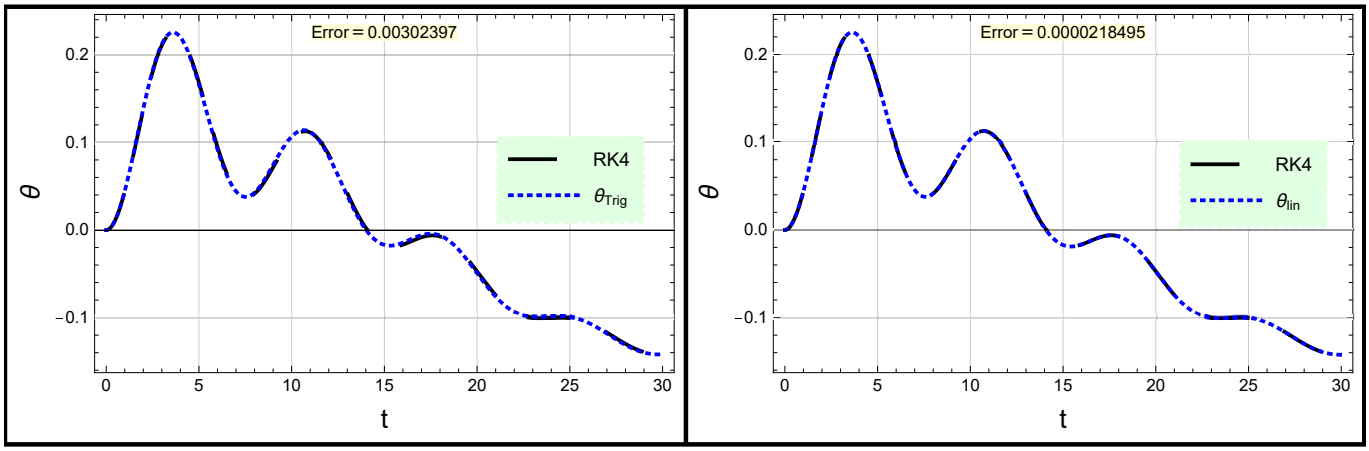


Figure 5. The approximation (30) to the i.v.p. (3) and the exact solution to the i.v.p. (37) are compared with the RK4 numerical approximation to the i.v.p. (3) for $(g, l, \omega_0, \omega, \Omega, \varepsilon, \gamma) = (9.81, 10, \sqrt{g/l}, 0.1, \omega_0/2, 0.1, 0.1)$ and $(\theta_0, \dot{\theta}_0) = (0, 0)$.

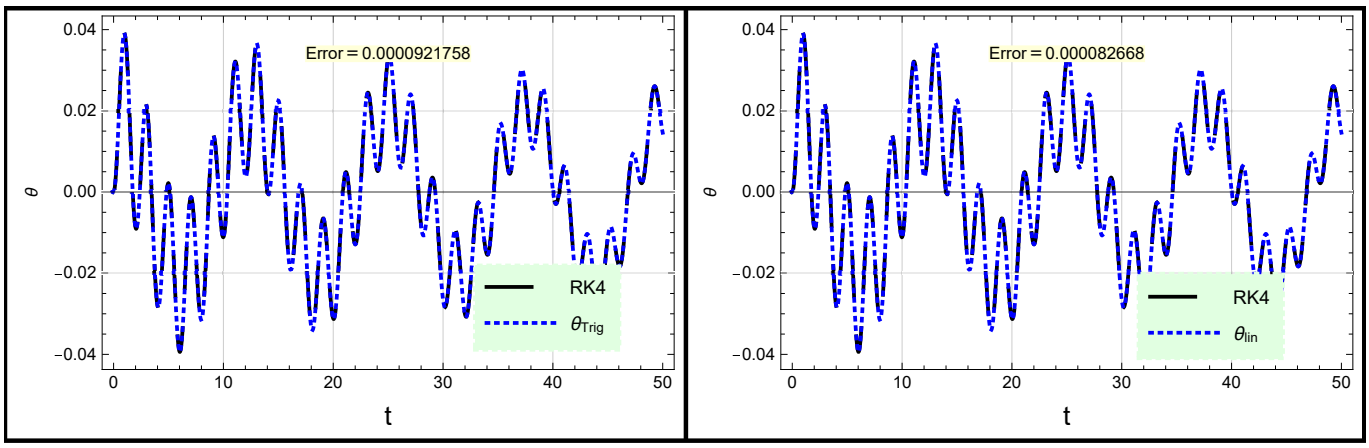


Figure 6. The approximation (30) to the i.v.p. (3) and the exact solution to the i.v.p. (37) are compared with the RK4 numerical approximation to the i.v.p. (3) for $(g, l, \omega, \Omega, \varepsilon, \gamma) = (9.8, 1, 0.5, 0.1, 0.02, 0.2)$ and $(\theta_0, \dot{\theta}_0) = (0, 0)$.

4. Conclusions

Some exact and analytical approximations to the rotational pendulum oscillators including (un)damped and forced, damped oscillators have been derived. In the first, an exact analytical solution to the unforced, undamped rotational pendulum oscillator in the form of elliptic function was derived in detail. In addition, an analytical approximation to the unforced, undamped rotating pendulum oscillator was obtained in the form of trigonometric function. Furthermore, an exact solution to the linearized the unforced, undamped rotational pendulum oscillator was derived and compared with all other obtained solutions. Moreover, some approximations to both the unforced, damped rotational pendulum oscillator and the forced, damped rotational pendulum oscillator have been derived in

detail. We also linearized both the unforced, damped rotational pendulum oscillator and the forced, damped rotational pendulum oscillator and derived their exact solutions. All obtained solutions for the conserved and non-conserved cases have been discussed and compared with the numerical approximation. The obtained solutions exhibit a high degree of accuracy and have all benefits of numerical methods for being applied directly to many nonlinear and complicated problems that come from vibratory systems. They also have the versatility, elegance, and other benefits of most analytical techniques.

In the present investigation, we focused our efforts to find some approximate solutions with high accuracy to the forced, damped rotational pendulum oscillator. However, studying the stability analysis to damped rotational pendulum oscillator is an important research point, but it will be addressed in future work.

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Conflicts of Interest: The authors declare that they have no conflict of interest.

Appendix A

The coefficients of $S_{0,1}$ Equation (26)

$$S_0 = \frac{1}{8}e^{-5\varepsilon t}\begin{bmatrix} c_0\begin{pmatrix} 6c_0^2 g e^{2\varepsilon t} + 6c_1^2 g e^{2\varepsilon t} - 5c_0^4\varepsilon^2 l - 10c_1^2c_0^2\varepsilon^2 l \\ -5c_1^4\varepsilon^2 l - 18c_0^2\varepsilon^2 l e^{2\varepsilon t} - 18c_1^2\varepsilon^2 l e^{2\varepsilon t} \\ +6c_0^2 l\Omega^2 e^{2\varepsilon t} + 6c_1^2 l\Omega^2 e^{2\varepsilon t} + 8g e^{4\varepsilon t} - 8l\Omega^2 e^{4\varepsilon t} \end{pmatrix} \\ -2c_0 l e^{2\varepsilon t} w'(t)(t)^2\left(5c_0^2 + 5c_1^2 + 4e^{2\varepsilon t}\right) \\ -c_1\varepsilon l w'(t)(t)\left(-8c_0^2 e^{2\varepsilon t} - 8c_1^2 e^{2\varepsilon t} - c_0^4 - 2c_1^2c_0^2 - c_1^4 + 8e^{4\varepsilon t}\right) \\ +2c_1 l e^{2\varepsilon t} w''(t)(t)\left(c_0^2 + c_1^2 + 4e^{2\varepsilon t}\right) \end{bmatrix},$$

and

$$S_1 = \frac{1}{8}e^{-5\varepsilon t}\begin{pmatrix} c_1\begin{pmatrix} 6c_0^2 g e^{2\varepsilon t} + 6c_1^2 g e^{2\varepsilon t} - 5c_0^4\varepsilon^2 l - 10c_1^2c_0^2\varepsilon^2 l - 5c_1^4\varepsilon^2 l \\ -18c_0^2\varepsilon^2 l e^{2\varepsilon t} - 18c_1^2\varepsilon^2 l e^{2\varepsilon t} + 6c_0^2 l\Omega^2 e^{2\varepsilon t} \\ +6c_1^2 l\Omega^2 e^{2\varepsilon t} + 8g e^{4\varepsilon t} - 8l\Omega^2 e^{4\varepsilon t} \end{pmatrix} \\ +c_0\varepsilon l w'(t)(t)\left(-8c_0^2 e^{2\varepsilon t} - 8c_1^2 e^{2\varepsilon t} - c_0^4 - 2c_1^2c_0^2 - c_1^4 + 8e^{4\varepsilon t}\right) \\ -2c_1 l e^{2\varepsilon t} w'(t)(t)^2\left(5c_0^2 + 5c_1^2 + 4e^{2\varepsilon t}\right) \\ +2c_0 l e^{2\varepsilon t} w''(t)(t)\left(c_0^2 + c_1^2 + 4e^{2\varepsilon t}\right) \end{pmatrix}.$$

Appendix B

The coefficients of $W_{0,1}$ Equation (32)

$$W_0 = \begin{bmatrix} -\frac{\left(g-4l\Omega^2\right)c_1^3 + 8l(\gamma - 2\varepsilon\omega c_2) + c_1\left(-8g + 8l\left(\omega^2+\Omega^2\right) + \left(g-4l\Omega^2\right)c_2^2\right)}{8l} \\ -\frac{\left(g-4l\Omega^2\right)c_1}{2l}\varphi(t)^2 \end{bmatrix},$$

and

$$W_1 = \begin{bmatrix} -2\varepsilon\omega c_1 - \omega^2 c_2 - \frac{g c_2\left(-8 + c_1^2 + c_2^2\right)}{8l} + \frac{1}{2}\Omega^2 c_2\left(-2 + c_1^2 + c_2^2\right) \\ -\frac{\left(g-4l\Omega^2\right)c_2}{2l}\varphi(t)^2 \end{bmatrix}.$$

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