

## Article

# On Novel Fractional Operators Involving the Multivariate Mittag–Leffler Function

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**Abstract:** The multivariate Mittag–Leffler function is introduced and used to establish fractional calculus operators. It is shown that the fractional derivative and integral operators are bounded. Some fundamental characteristics of the new fractional operators, such as the semi-group and inverse characteristics, are studied. As special cases of these novel fractional operators, several fractional operators that are already well known in the literature are acquired. The generalized Laplace transform of these operators is evaluated. By involving the explored fractional operators, a kinetic differintegral equation is introduced, and its solution is obtained by using the Laplace transform. As a real-life problem, a growth model is developed and its graph is sketched.

**Keywords:** multivariate Mittag–Leffler; fractional integral; fractional derivative; generalized Laplace transform

**MSC:** 26A33; 26D10; 26D53; 05A30



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## 1. Introduction

Fractional calculus is a natural generalization of classical calculus and plays a significant role in mathematics, biology, chemistry, economics, engineering, and physics. Abel was the first to use fractional calculus by solving the Tautocrone problem [1]. Many books [2–4] and articles [5–7] on hypotheses and developments of fractional calculus that have studied the subject in depth have been published.

In fractional calculus, differentiation and integration are generalized to non-integer orders. Most fractional derivatives are represented in terms of integration [6,7]. Cauchy problems for a differential equation with a Caputo Hadamard fractional derivative in spaces of continuously differentiable functions were explored in [8]. In [9,10], the fractional calculus of several known types of fractional operators was extensively investigated, along with the applications thereof.

Generalized fractional operators recover the classical ones as special cases. A new definition and properties of a fractional operator without a singular kernel were studied in [11,12]. Yang et al. defined a new fractional derivative without a singular kernel and described a potential application for modeling the steady heat-conduction problem [13]. Agarwal and Nieto discussed the Marichev–Saigo–Maeda fractional integral operators by involving the Mittag–Leffler type function with four parameters [14]. Some fractional integrals and derivatives with general analytic kernels were briefly studied in [15,16]. The  $(k, s)$ -fractional calculus of the generalized Mittag–Leffler function was discussed by Nisar et al. [17]. For further details and applications of fractional operators, we refer the readers to [18–23].

The classical Mittag–Leffler function was proposed by Gösta [24] in 1903. It has been generalized in various ways [25–27] by introducing the two parameters or three parameters. Further, Dorrego and Cerutti [28] represented the  $k$  Mittag–Leffler function in the following form:

$$\mathcal{E}_{k,\rho,\eta}^{\gamma}(\theta) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} \theta^n}{\Gamma_k(\rho n + \eta) n!},$$

where  $n$  is a natural number,  $k$  is a positive real number, and  $\eta, \rho, \gamma \in \mathbb{C}$  with  $\Re(\rho) > 0$  and  $\Re(\eta) > 0$ .

Bivariate and multivariate generalizations have recently been proposed as new types of generalizations [29,30]. Instead of a single power series in a single variable, Mittag–Leffler functions are defined by two or more variables as a power series. The multivariate Mittag–Leffler function defined by Sexana et al. in [30] is given by the following notation.

$$\mathcal{E}_{\alpha,(\beta_i)}^{(\delta_i)}(z_1, z_2, \dots, z_n) = \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i}}{\Gamma(\alpha + \sum_{i=1}^n \beta_i \kappa_i) (\kappa_1)! (\kappa_2)! \dots (\kappa_n)!} (z_1)^{\kappa_1} (z_2)^{\kappa_2} \dots (z_n)^{\kappa_n},$$

where  $\delta_i, \beta_i \in \mathbb{C}$  with  $\Re(\beta_i) > 0$  for all  $i = 1, 2, \dots, n$ , and the notation  $(\delta_i)_{\kappa_i}$  is used for the Pochhammer symbol with a relation to the gamma function  $(\delta_i)_{\kappa_i} = \frac{\Gamma(\delta_i + \kappa_i)}{\Gamma(\delta_i)}$ .

The special  $k$ -gamma and  $k$ -beta functions were studied by Diaz et al. in [31] with the following definitions.

**Definition 1.** The extended  $\Gamma$  function, which is called the  $k$ -gamma function, is characterized by the relation

$$\Gamma_k(\zeta) = \int_0^{\infty} \alpha^{\zeta-1} e^{-\frac{\alpha^k}{k}} d\alpha, \quad \Re(\zeta), k > 0.$$

Obviously,  $\Gamma(\zeta) = \lim_{k \rightarrow 1} \Gamma_k(\zeta)$  and  $\Gamma_k(\zeta) = k^{\frac{\zeta}{k}-1} \Gamma(\frac{\zeta}{k})$ .

**Definition 2.** For  $\Re(\zeta), \Re(\eta) > 0$ , and  $k > 0$ , the  $k$ -beta function is defined as

$$B_k(\zeta, \eta) = \frac{1}{k} \int_0^1 \tau^{\frac{\zeta}{k}-1} (1-\tau)^{\frac{\eta}{k}-1} d\tau.$$

It is notable that the  $\Gamma_k$  and  $B_k$  functions are related to an identity

$$B_k(\zeta, \eta) = \frac{\Gamma_k(\zeta) \Gamma_k(\eta)}{\Gamma_k(\zeta + \eta)}.$$

**Definition 3** ([32]). Let  $\alpha > 0$  and  $k > 0$ ; then, the  $k$ -Riemann–Liouville (R-L) fractional integral  $I_{a,k}^{\alpha} f$  of order  $\alpha$  is given by

$$I_{a,k}^{\alpha} f(\theta) = \frac{1}{k \Gamma_k(\alpha)} \int_a^{\theta} (\theta - t)^{\frac{\alpha}{k}-1} f(t) dt, \quad t \in (a, b). \quad (1)$$

**Definition 4** ([33]). Let  $f \in L^1[r, q]$ ,  $s$  be a real number, but let  $-1, k$  be a positive real number and  $\eta \in \mathbb{C}$  with  $\Re(\eta) > 0$ ; then, the  $(k, s)$  fractional integral of the Riemann type is defined by

$$\left({}_k^s J_r^{\eta} f\right)(\theta) = \frac{(s+1)^{1-\frac{\eta}{k}}}{k \Gamma_k(\eta)} \int_r^{\theta} (\theta^{s+1} - \kappa^{s+1})^{\frac{\eta}{k}-1} \kappa^s f(\kappa) d\kappa. \quad (2)$$

**Definition 5** ([33]). The  $(k, s)$ -fractional derivative of the Riemann-type be defined by

$$\left({}_k^s D_r^\mu f\right)(\theta) = \frac{(s+1)^{1-\frac{nk-\mu}{k}}}{\Gamma_k(nk-\mu)} \left(\frac{1}{\theta^s} \frac{d}{d\theta}\right)^n k^{n-1} \int_r^\theta (\theta^{s+1} - \kappa^{s+1})^{\frac{nk-\mu}{k}-1} \kappa^s f(\kappa) d\kappa, \quad (3)$$

where  $f \in L[r, q]$ ,  $s$  is a real number, except for  $-1$ ,  $k$  is a positive real number, and  $\mu \in \mathbb{C}$  with  $\Re(\mu) > 0$  and  $n = \Re(\mu) + 1$ .

**Definition 6** ([34]). The Prabhakar integral operator for the choice of  $\eta, \rho, \zeta, \gamma \in \mathbb{C}$ ,  $\Re(\rho) > 0$ ,  $\Re(\eta) > 0$ , and  $f \in L[r, q]$  is defined by

$$\left(\mathbf{P}_{\rho, \eta, \zeta}^\gamma f\right)(\theta) = \int_0^\theta (\theta - \kappa)^{\eta-1} \mathcal{E}_{k, \rho, \eta}^\gamma(\zeta(\theta - \kappa)^\rho) f(\kappa) d\kappa. \quad (4)$$

**Definition 7** ([34]). Let  $\eta, \rho, \zeta, \gamma \in \mathbb{C}$ ,  $\Re(\rho) > 0$ ,  $\Re(\eta) > 0$ ,  $n = [\eta]$ , and  $f \in L[0, q]$ ; then, the Prabhakar derivative operator is given by

$$\left(\mathfrak{D}_{\rho, \eta, \zeta}^\gamma f\right)(\theta) = \left(\mathbf{P}_{\rho, n-\eta, \zeta}^{-\gamma} f\right)(\theta). \quad (5)$$

Dorrego [35] presented the  $k$ -Prabhakar integral operator in the following definition.

**Definition 8.** For all positive real numbers  $k$  and  $\eta, \rho, \zeta, \gamma \in \mathbb{C}$ ,  $\Re(\rho) > 0$ ,  $\Re(\eta) > 0$ , and  $f \in L[r, q]$ , the  $k$ -Prabhakar integral operator is defined by the equation

$$\left({}_k \mathbf{P}_{\rho, \eta, \zeta}^\gamma f\right)(\theta) = \frac{1}{k} \int_0^\theta (\theta - \kappa)^{\frac{\eta}{k}-1} \mathcal{E}_{k, \rho, \eta}^\gamma(\zeta(\theta - \kappa)^{\frac{\rho}{k}}) f(\kappa) d\kappa. \quad (6)$$

In [30], the authors presented the  $k$ -Prabhakar derivative operator as follows.

**Definition 9.** For all positive real numbers  $k$  and  $\mu, \rho, \zeta, \gamma_1 \in \mathbb{C}$  with  $\Re(\rho_1) > 0$ ,  $\Re(\mu) > 0$ , and  $n = [\frac{\mu}{k}] + 1$ ,  $f \in L[r, q]$ , the extended Prabhakar fractional derivative operator is given by

$$\left({}_k \mathbf{D}_{\rho, \mu, \zeta}^{\gamma_1} f\right)(\theta) = \left(\frac{d}{d\theta}\right)^n k^{n-1} \int_0^\theta (\theta - \kappa)^{\frac{nk-\mu}{k}-1} \mathcal{E}_{k, \rho_1, nk-\mu}^{-\gamma_1}(\zeta(\theta - \kappa)^{\frac{\rho_1}{k}}) f(\kappa) d\kappa. \quad (7)$$

The modified fractional operator explored by Samraiz et al. [6] is defined as follows.

**Definition 10.** Let  $s$  be real numbers, but let  $-1, k$  be positive real numbers, with  $\eta_1, \rho, \zeta, \gamma_1 \in \mathbb{C}$ ,  $\Re(\rho) > 0$ ,  $\Re(\eta_1) > 0$ ,  $n = [\frac{\eta_1}{k}] + 1$ , and  $f \in L[0, \alpha_1]$ ; then,

$$\left({}_k^s \mathfrak{J}_{0^+, \rho, \eta_1}^{\zeta, \gamma_1} f\right)(\theta) = \frac{(p+1)^{1-\frac{\eta_1}{k}} k}{k} \int_0^\theta (\theta^{s+1} - t^{s+1})^{\frac{\eta_1}{k}-1} t^s E_{k, \rho, \eta_1}^{\gamma_1}(\zeta(\theta^{s+1} - t^{s+1})^{\frac{\rho_1}{k}}) f(t) dt.$$

The reverse fractional derivative operator is defined by the following definition.

**Definition 11.** Let  $s$  be real numbers, but let  $-1, k$  be positive real numbers, with  $\eta_1, \rho, \zeta, \gamma_1 \in \mathbb{C}$ ,  $\Re(\rho) > 0$ ,  $\Re(\eta_1) > 0$ ,  $n = [\frac{\eta_1}{k}] + 1$ , and  $f \in L[0, \alpha_1]$ ; then,

$$\begin{aligned} \left({}_k^s \mathcal{D}_{0^+, \rho, \eta_1}^{\zeta, \gamma_1} f\right)(\theta) &= \left(\frac{1}{\theta^s} \frac{d}{d\theta}\right)^n \left({}_k^s \mathfrak{J}_{0^+, \rho, nk-\eta_1}^{\zeta, -\gamma_1} f\right)(\theta) \\ &= (s+1)^{1-\frac{nk-\eta_1}{k}} \left(\frac{1}{\theta^s} \frac{d}{d\theta}\right)^n k^{n-1} \int_0^\theta (\theta^{s+1} - t^{s+1})^{\frac{nk-\eta_1}{k}-1} \\ &\quad \times t^s E_{k, \rho, nk-\eta_1}^{-\gamma_1}(\zeta(\theta^{s+1} - t^{s+1})^{\frac{\rho}{k}}) f(t) dt. \end{aligned}$$

**Theorem 1** ([17]). Let  $s$  be real numbers, but let  $-1, k$  be positive real numbers, with  $\eta, \nu, \rho, \zeta, \gamma \in \mathbb{C}, \Re(\rho) > 0, \Re(\eta) > 0$ , and  $\Re(\nu) > 0$ ; then,

$${}_k J_{r+}^{\eta} \left[ (\kappa^{s+1} - r^{s+1})^{\frac{\nu}{k}-1} \mathcal{E}_{k,\rho,\nu}^{\gamma} (\zeta (\kappa^{s+1} - r^{s+1})^{\frac{\rho}{k}}) \right] = \frac{(\theta^{s+1} - r^{s+1})^{\frac{\eta+\nu}{k}-1}}{(s+1)^{\frac{\eta}{k}}} \mathcal{E}_{k,\rho,\eta+\nu}^{\gamma} (\zeta (\theta^{s+1} - r^{s+1})^{\frac{\rho}{k}}).$$

In this paper, we will generalize the multivariate Mittag–Leffler function with the following definition, which was introduced by Sexana et al. in [30].

**Definition 12.** Let  $\alpha, \beta_i, \delta_i \in \mathbb{C} \operatorname{Re}(\beta_i) > 0$ ;

$$\mathcal{E}_{k,\alpha,(\beta_i)}^{(\delta_i)}(z_1, z_2, \dots, z_n) = \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i, k}}{\Gamma_k(\alpha + \sum_{i=1}^n \beta_i \kappa_i) (\kappa_1)! (\kappa_2)! \dots (\kappa_n)!} (z_1)^{\kappa_1} (z_2)^{\kappa_2} \dots (z_n)^{\kappa_n},$$

where  $k > 0$  and the notation  $(\delta_i)_{\kappa_i, k}$  is used for the  $k$ -Pochhammer symbol.

The structure of this article is designed as follows: In Section 2, we introduce a novel fractional integral operator and go over its properties. In Section 3, we explore the inverse derivative operator of the novel integral operator. In Section 4, we study the properties of integer-order integral and derivative operators and evaluate the Laplace transform of the novel operators. Finally, we outline the conclusions in Section 6.

## 2. Generalized Fractional Integral Operators Involving the Multivariate Mittag–Leffler Function

The generalized form of fractional integral operator containing multivariate Mittag–Leffler function as part of its kernel is defined as follows:

**Definition 13.** Let  $s$  be a real number, except for  $-1$ , and let the parameters  $\mu, \beta_i, \delta_i, \zeta_i$  be complex numbers with  $\Re(\beta_i) > 0$  for all  $i = 1, 2, \dots, n; \Re(\mu) > 0$ ,  $f$  is in  $L[a, b]$ , and the function  $\Theta(t)$  is strictly increasing and differentiable. Then,

$$\begin{aligned} & {}_k \mathfrak{J}_{s, \Theta a^+, \mu, (\beta_i)}^{(\delta_i), (\zeta_i)} f(x) \\ &= \frac{(s+1)^{1-\frac{\mu}{k}}}{k} \int_a^x \left( \Theta^{s+1}(x) - \Theta^{s+1}(t) \right)^{\frac{\mu}{k}-1} \\ & \quad \times \mathcal{E}_{k, \mu, (\beta_i)}^{(\delta_i)} \left( \zeta_1 \left( \Theta^{s+1}(x) - \Theta^{s+1}(t) \right)^{\frac{\beta_1}{k}}, \zeta_2 \left( \Theta^{s+1}(x) - \Theta^{s+1}(t) \right)^{\frac{\beta_2}{k}}, \dots, \right. \\ & \quad \left. \zeta_n \left( \Theta^{s+1}(x) - \Theta^{s+1}(t) \right)^{\frac{\beta_n}{k}} \right) \Theta'(t) \Theta^s(t) f(t) dt \\ &= \frac{(s+1)^{1-\frac{\mu}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i, k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n \beta_i \kappa_i) (\kappa_1)! \dots (\kappa_n)!} \\ & \quad \times \int_a^x \left( \Theta^{s+1}(x) - \Theta^{s+1}(t) \right)^{\frac{\mu + \sum_{i=1}^n \beta_i \kappa_i}{k}-1} \Theta'(t) \Theta^s(t) f(t) dt. \end{aligned} \quad (8)$$

The following are special cases of the introduced operator (8) that are fractional integral operators from the literature.

- (i) Substituting  $\Theta(x) = x, n = 1$ , into (8), we get modified  $(k, s)$ -fractional operator introduced by Samraiz et al. in [6] and given by Definition 10.
- (ii) Substituting  $\Theta(x) = x, n = 1$ , and  $\delta = 0$  into (8) results in the  $(k, s)$ -R-L fractional integral introduced by Sarikaya et al. in [33].
- (iii) Substituting  $\Theta(x) = x, s = 0$ , and  $n = 1$ , we obtain the generalized Prabhakar integral operator introduced by Gustavo A. Dorrego in [35] and given in Definition 8.

- (iv) Substituting  $\Theta(x) = x, s = 0, n = 1$ , and  $k = 1$  into (8), we get the fractional operator (4) introduced by Garra et al. in [34].
- (v) Substituting  $\Theta(x) = x, \delta = 0$ , and  $s = 0$  into (8), it reduces to the  $k$ -(R-L) fractional integral (1) defined by Mubeen et al. in [32].
- (vi) Substituting  $\Theta(x) = x, \delta = 0, s = 0$ , and  $k = 1$  into (8), it reduces to the classical (R-L) fractional integral.

Next, we prove the boundedness of the operator by using the definition of the norm presented in [36].

**Definition 14.** Let  $\varphi$  and  $W \neq 0$  be functions defined on  $[a, b]$ , where  $W$  is differentiable and strictly increasing. The space  $X^p(a, b)$ ,  $1 \leq p \leq \infty$  for all Lebesgue measurable functions  $\varphi$  for which  $\|\varphi\|_{X^p} < \infty$ , i.e.,

$$\|\varphi\|_{X^p} = \left[ (s+1) \int_a^b |\varphi(t)|^p W^s(t) W'(t) dt \right]^{\frac{1}{p}}, \quad 1 < p < \infty,$$

where  $s$  is a real number, except for  $-1$ , and

$$\|\varphi\|_{X^\infty} = \operatorname{ess\,sup}_{a \leq t \leq b} |\varphi(t)| < \infty.$$

**Theorem 2.** Let  $s$  be a real number, except for  $-1$ , and let the parameters  $\alpha, \beta_i, \delta_i, \zeta_i$  be complex numbers with  $\Re(\beta_i) > 0$  for all  $i = 1, 2, \dots, n$ ; let  $\Re(\mu) > 0$  and the function  $f$  be in  $L[a, b]$ ; then, the result

$$\begin{aligned} & \left\| {}_{s, \Theta}^k \mathfrak{J}_{a^+, \mu, (\beta_i)}^{(\delta_i), (\zeta_i)} f(x) \right\|_{X^p} \\ & \leq (s+1)^{-\frac{\mu}{k}} \left( \Theta^{s+1}(q) - \Theta^{s+1}(a) \right)^{\frac{\mu}{k}} \mathcal{E}_{k, \mu, (\beta_i)}^{(\delta_i)} \left( \zeta_1 \left( \Theta^{s+1}(q) - \Theta^{s+1}(a) \right)^{\frac{\beta_1}{k}}, \right. \end{aligned} \quad (9)$$

$$\left. \zeta_2 \left( \Theta^{s+1}(q) - \Theta^{s+1}(a) \right)^{\frac{\beta_2}{k}}, \dots, \zeta_n \left( \Theta^{s+1}(q) - \Theta^{s+1}(a) \right)^{\frac{\beta_n}{k}} \right) \|f\|_{X^p} \quad (10)$$

holds.

**Proof.** By using Definition 13, we have

$$\begin{aligned} & \left\| {}_{s, \Theta}^k \mathfrak{J}_{a^+, \mu, (\beta_i)}^{(\delta_i), (\zeta_i)} f(x) \right\|_{X^p} \\ & = \frac{(s+1)^{1-\frac{\mu}{k}}}{k} \left( \int_a^q (s+1) \left| \int_a^\Theta \left( \Theta^{s+1}(x) - \Theta^{s+1}(t) \right)^{\frac{\mu}{k}-1} \right. \right. \\ & \quad \times \mathcal{E}_{k, \mu, (\beta_i)}^{(\delta_i)} \left( \zeta_1 \left( \Theta^{s+1}(x) - \Theta^{s+1}(t) \right)^{\frac{\beta_1}{k}}, \zeta_2 \left( \Theta^{s+1}(x) - \Theta^{s+1}(t) \right)^{\frac{\beta_2}{k}}, \dots, \right. \\ & \quad \left. \left. \zeta_n \left( \Theta^{s+1}(x) - \Theta^{s+1}(t) \right)^{\frac{\beta_n}{k}} \right) \Theta'(t) \Theta^s(t) f(t) dt \right|^p \Theta^s(x) \Theta'(x) dx \right)^{\frac{1}{p}}. \end{aligned}$$

Substituting  $u = \Theta^{s+1}(t)$  and  $v = \Theta^{s+1}(x)$ , we get

$$\begin{aligned} & \left\| {}^k_{s,\Theta} \mathfrak{J}_{a^+,\mu,(\beta_i)}^{(\delta_i),(\zeta_i)} f(x) \right\|_{X^p} \\ &= \frac{(s+1)^{-\frac{\mu}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i,k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\ & \quad \times \left( \int_{\Theta^{s+1}(a)}^{\Theta^{s+1}(q)} \left| \int_{\Theta^{s+1}(a)}^{\Theta^{s+1}(\theta)} (v-u)^{\frac{\mu+\sum_{i=1}^n \beta_i \kappa_i}{k}-1} f(h^{-1}(u^{\frac{1}{s+1}})) du \right|^p dv \right)^{\frac{1}{p}}, \end{aligned}$$

where  $\Theta^{-1}$  is the inverse function of  $\Theta$ . Now, using the generalized Minkowski inequality, we can write

$$\begin{aligned} & \left\| {}^k_{s,\Theta} \mathfrak{J}_{a^+,\mu,(\beta_i)}^{(\delta_i),(\zeta_i)} f(x) \right\|_{X^p} \\ & \leq \frac{(s+1)^{1-\frac{\mu}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i,k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\ & \quad \times \int_{\Theta^{s+1}(a)}^{\Theta^{s+1}(q)} \left( \left| f(\Theta^{-1}(u^{\frac{1}{s+1}})) \right|^p \int_u^{\Theta^{s+1}(q)} (v-u)^{(\frac{\mu+\sum_{i=1}^n \beta_i \kappa_i}{k}-1)p} dv \right)^{\frac{1}{p}} du \\ &= \frac{(s+1)^{-\frac{\mu}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i,k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\ & \quad \times \int_{\Theta^{s+1}(a)}^{\Theta^{s+1}(q)} \left| f(\Theta^{-1}(u^{\frac{1}{s+1}})) \right| \left( \frac{(\Theta^{s+1}(q) - u)^{(\frac{\mu+\sum_{i=1}^n \beta_i \kappa_i}{k}-1)p+1}}{(\frac{\mu+\sum_{i=1}^n \beta_i \kappa_i}{k}-1)p+1} \right)^{\frac{1}{p}} du. \end{aligned}$$

Using Hölder's inequality and through simple calculations, we get

$$\begin{aligned} & \left\| {}^k_{s,\Theta} \mathfrak{J}_{a^+,\mu,(\beta_i)}^{(\delta_i),(\zeta_i)} f(x) \right\|_{X^p} \\ & \leq \frac{(s+1)^{-\frac{\mu}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i,k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\ & \quad \times \left( \int_{\Theta^{s+1}(a)}^{\Theta^{s+1}(q)} \left| f(\Theta^{-1}(u^{\frac{1}{s+1}})) \right|^p du \right)^{\frac{1}{p}} \\ & \quad \times \left[ \int_{\Theta^{s+1}(a)}^{\Theta^{s+1}(q)} \left( \frac{(\Theta^{s+1}(q) - u)^{(\frac{\mu+\sum_{i=1}^n \beta_i \kappa_i}{k}-1)p+1}}{(\frac{\mu+\sum_{i=1}^n \beta_i \kappa_i}{k}-1)p+1} \right)^{\frac{q}{p}} du \right]^{\frac{1}{q}} \\ & \leq \frac{(s+1)^{-\frac{\mu}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i,k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\ & \quad \times \left( \int_{\Theta^{s+1}(a)}^{\Theta^{s+1}(q)} \left| f(\Theta^{-1}(u^{\frac{1}{s+1}})) \right|^p du \right)^{\frac{1}{p}} \frac{(\Theta^{s+1}(q) - \Theta^{s+1}(a))^{\frac{\mu+\sum_{i=1}^n \beta_i \kappa_i}{k}}}{\frac{\mu+\sum_{i=1}^n \beta_i \kappa_i}{k}}, \end{aligned}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ . Through back substitution, we obtain

$$\begin{aligned}
& \left\| {}^k_{s,\Theta} \mathfrak{J}_{a^+,\mu,(\beta_i)}^{(\delta_i),(\zeta_i)} f(x) \right\|_{X^p} \\
& \leq \frac{(s+1)^{-\frac{\mu}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i,k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\
& \quad \times \left( (s+1) \int_a^q |f(t)|^p \Theta^s(t) \Theta'(t) dt \right)^{\frac{1}{p}} \frac{(\Theta^{s+1}(q) - \Theta^{s+1}(a))^{\frac{\mu + \sum_{i=1}^n \beta_i \kappa_i}{k}}}{\frac{\mu + \sum_{i=1}^n \beta_i \kappa_i}{k}} \\
& = (s+1)^{-\frac{\mu}{k}} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i,k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\
& \quad \times \frac{(\Theta^{s+1}(q) - \Theta^{s+1}(a))^{\frac{\mu + \sum_{i=1}^n \beta_i \kappa_i}{k}}}{\mu + \sum_{i=1}^n \beta_i \kappa_i} \|f\|_{X_w^p} \\
& = (s+1)^{-\frac{\mu}{k}} \left( \Theta^{s+1}(q) - \Theta^{s+1}(a) \right)^{\frac{\mu}{k}} \mathcal{E}_{k,\mu,(\beta_i)}^{(\delta_i)} \left( \zeta_1 \left( \Theta^{s+1}(q) - \Theta^{s+1}(a) \right)^{\frac{\beta_1}{k}}, \right. \\
& \quad \left. \zeta_2 \left( \Theta^{s+1}(q) - \Theta^{s+1}(a) \right)^{\frac{\beta_2}{k}}, \dots, \zeta_n \left( \Theta^{s+1}(q) - \Theta^{s+1}(a) \right)^{\frac{\beta_n}{k}} \right) \|f\|_{X_1^p}.
\end{aligned}$$

This completes the proof.  $\square$

**Theorem 3.** Let  $s$  be a real number, except for  $-1$ , and let the parameters  $\alpha, \beta_i, \delta_i, \zeta_i$  be complex numbers with  $\Re(\beta_i) > 0$  for all  $i = 1, 2, \dots, n$ ,  $\Re(\mu) > 0$ . Moreover, the function  $f$  is in  $L[a, b]$ ; then, the following equation holds.

$$\begin{aligned}
& {}^k_{s,\Theta} \mathfrak{J}_{a^+,\mu,(\beta_i)}^{(\delta_i),(\zeta_i)} \left( \Theta^{s+1}(t) - \Theta^{s+1}(a) \right)^{\frac{\eta}{k}-1} \\
& = \frac{(s+1)^{-\frac{\mu}{k}}}{k} \Gamma_k(\eta) (\Theta^{s+1}(x) - \Theta^{s+1}(a))^{\frac{\eta+\mu}{k}-1} \\
& \quad \mathcal{E}_{\mu+\eta,(\beta_i)}^{(\delta_i)} (\zeta_1 (\Theta^{s+1}(x) - \Theta^{s+1}(a))^{\beta_1}, \zeta_2 (\Theta^{s+1}(x) - \Theta^{s+1}(a))^{\beta_2}, \dots, \\
& \quad \zeta_n (\Theta^{s+1}(x) - \Theta^{s+1}(a))^{\beta_n}).
\end{aligned}$$

**Proof.** By using Definition 13, we have

$$\begin{aligned}
& {}^k_{s,\Theta} \mathfrak{J}_{a^+,\mu,(\beta_i)}^{(\delta_i),(\zeta_i)} (\Theta^{s+1}(t) - \Theta^{s+1}(a))^{\frac{\eta}{k}-1} \\
& = \frac{(s+1)^{1-\frac{\mu}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i,k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\
& \quad \times \int_a^x \left( \Theta^{s+1}(x) - \Theta^{s+1}(t) \right)^{\frac{\mu + \sum_{i=1}^n \beta_i \kappa_i}{k}-1} \left( \Theta^{s+1}(t) - \Theta^{s+1}(a) \right)^{\frac{\eta}{k}-1} \Theta^s(t) \Theta'(t) dt \\
& = \frac{(s+1)^{1-\frac{\mu}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i,k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\
& \quad \times \left( \Theta^{s+1}(x) - \Theta^{s+1}(a) \right)^{\frac{\mu + \sum_{i=1}^n \beta_i \kappa_i + \eta}{k}-2} \\
& \quad \times \int_a^x \left( \frac{\Theta^{s+1}(x) - \Theta^{s+1}(t)}{\Theta^{s+1}(x) - \Theta^{s+1}(a)} \right)^{\frac{\mu + \sum_{i=1}^n \beta_i \kappa_i}{k}-1} \left( \frac{\Theta^{s+1}(t) - \Theta^{s+1}(a)}{\Theta^{s+1}(x) - \Theta^{s+1}(a)} \right)^{\frac{\eta}{k}-1} \Theta^s(t) \Theta'(t) dt.
\end{aligned}$$

By substituting  $u = \frac{\Theta^{s+1}(x) - \Theta^{s+1}(t)}{\Theta^{s+1}(x) - \Theta^{s+1}(a)}$ , we obtain

$$\begin{aligned}
 & {}^k_{s,\Theta} \mathfrak{J}_{a^+, \mu, (\beta_i)}^{(\delta_i), (\zeta_i)} (\Theta^{s+1}(t) - \Theta^{s+1}(a))^{\frac{\eta}{k}-1} \\
 &= \frac{(s+1)^{-\frac{\mu}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i, k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\
 &\quad \times \left( \Theta^{s+1}(x) - \Theta^{s+1}(a) \right)^{\frac{\mu + \sum_{i=1}^n \beta_i \kappa_i + \eta}{k} - 1} \int_0^1 u^{\frac{\mu + \sum_{i=1}^n \beta_i \kappa_i}{k} - 1} (1-u)^{\frac{\eta}{k} - 1} du \\
 &= \frac{(s+1)^{-\frac{\mu}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i, k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\
 &\quad \times \left( \Theta^{s+1}(x) - \Theta^{s+1}(a) \right)^{\frac{\mu + \sum_{i=1}^n \beta_i \kappa_i + \eta}{k} - 1} B_k\left(\mu + \sum_{i=1}^n \beta_i \kappa_i, \eta\right) \\
 &= \frac{(s+1)^{-\frac{\mu}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i, k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\
 &\quad \times \left( \Theta^{s+1}(x) - \Theta^{s+1}(a) \right)^{\frac{\mu + \sum_{i=1}^n \beta_i \kappa_i + \eta}{k} - 1} \times \frac{\Gamma_k(\mu + \sum_{i=1}^n \beta_i \kappa_i) \Gamma_k(\eta)}{\Gamma_k(\mu + \eta + \sum_{i=1}^n \beta_i \kappa_i)}.
 \end{aligned}$$

This simple calculation completes the proof.  $\square$

**Theorem 4.** Let  $s$  be a real number, except for  $-1$ , and let the parameters  $\mu, \beta_i, \delta_i, \zeta_i$  be complex numbers with  $\Re(\beta_i) > 0$  for all  $i = 1, 2, \dots, n$ ,  $\Re(\mu)$ , and let  $\Re(\eta) > 0$  and the function  $f$  be in  $L[a, b]$ ; then, we obtain the result

$$\begin{aligned}
 & {}^k_{s,\Theta} \mathfrak{J}_{a^+, \mu, (\beta_i)}^{(\delta_i), (\zeta_i)} (\Theta^{s+1}(t) - \Theta^{s+1}(a))^{\frac{\eta}{k}-1} \mathcal{E}_{k, \beta, \eta}^{\delta} (w(\Theta^{s+1}(t) - \Theta^{s+1}(a))^{\frac{\beta}{k}}) \\
 &= \frac{(s+1)^{-\frac{\mu}{k}}}{k} (\Theta^{s+1}(x) - \Theta^{s+1}(a))^{\frac{\mu+\eta}{k}-1} \\
 &\quad \times \mathcal{E}_{\mu+\eta, (\beta_i), \beta}^{(\delta_i), \delta} (\zeta_1 (\Theta^{s+1}(x) - \Theta^{s+1}(a))^{\beta_1}, \zeta_2 (\Theta^{s+1}(x) - \Theta^{s+1}(a))^{\beta_2}, \dots, \\
 &\quad \zeta_n (\Theta^{s+1}(x) - \Theta^{s+1}(a))^{\beta_n}, (\Theta^{s+1}(x) - \Theta^{s+1}(a))^{\beta}).
 \end{aligned}$$

**Proof.** By using Definition 13, we can write

$$\begin{aligned}
 & {}^k_{s,\Theta} \mathfrak{J}_{a^+, \mu, (\beta_i)}^{(\delta_i), (\zeta_i)} (\Theta^{s+1}(t) - \Theta^{s+1}(a))^{\frac{\eta}{k}-1} \mathcal{E}_{k, \beta, \eta}^{\delta} (w(\Theta^{s+1}(t) - \Theta^{s+1}(a))^{\frac{\beta}{k}}) \\
 &= \frac{(s+1)^{1-\frac{\mu}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i, k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\
 &\quad \times \int_a^x \left( \Theta^{s+1}(x) - \Theta^{s+1}(t) \right)^{\frac{\mu + \sum_{i=1}^n \beta_i \kappa_i}{k} - 1} \left( \Theta^{s+1}(t) - \Theta^{s+1}(a) \right)^{\frac{\eta}{k} - 1} \\
 &\quad \times \mathcal{E}_{k, \beta, \eta}^{\delta} \left( w(\Theta^{s+1}(t) - \Theta^{s+1}(a))^{\frac{\beta}{k}} \right) \Theta^s(t) \Theta'(t) dt \\
 &= \frac{(s+1)^{1-\frac{\mu}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n, n_1}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i, k} (\zeta_i)^{\kappa_i} (w)^{n_1}}{\Gamma_k(\mu + \sum_{i=1}^n \beta_i \kappa_i) \Gamma_k(\beta n_1 + \alpha) (\kappa_1)! (\kappa_2)! \dots (\kappa_n)! (n_1)!} \\
 &\quad \times \int_a^x (\Theta^{s+1}(x) - \Theta^{s+1}(t))^{\frac{\mu + \sum_{i=1}^n \beta_i \kappa_i}{k} - 1} (\Theta^{s+1}(t) - \Theta^{s+1}(a))^{\frac{\eta + \beta n_1}{k} - 1} \Theta^s(t) \Theta'(t) dt. \quad (11)
 \end{aligned}$$

Consider

$$\begin{aligned}
 \mathfrak{J} &= \int_a^x (\Theta^{s+1}(x) - \Theta^{s+1}(t))^{\frac{\mu + \sum_{i=1}^n \beta_i \kappa_i}{k} - 1} (\Theta^{s+1}(t) - \Theta^{s+1}(a))^{\frac{\eta + \beta n_1}{k} - 1} \Theta^s(t) \Theta'(t) dt \\
 &= (\Theta^{s+1}(x) - \Theta^{s+1}(a))^{\frac{\mu + \eta + \sum_{i=1}^n \beta_i \kappa_i + \beta n_1}{k} - 2} \int_a^x \left( \frac{\Theta^{s+1}(x) - \Theta^{s+1}(t)}{\Theta^{s+1}(t) - \Theta^{s+1}(a)} \right)^{\frac{\mu + \sum_{i=1}^n \beta_i \kappa_i}{k} - 1} \\
 &\quad \times \left( \frac{\Theta^{s+1}(t) - \Theta^{s+1}(a)}{h(x)s + 1 - h(a)s + 1} \right)^{\frac{\eta + \beta n_1}{k} - 1} \Theta^s(t) \Theta'(t) dt \\
 &= \frac{(\Theta^{s+1}(x) - \Theta^{s+1}(a))^{\frac{\mu + \eta + \sum_{i=1}^n \beta_i \kappa_i + \beta n_1}{k} - 1}}{s + 1} \int_0^1 u^{\frac{\mu + \sum_{i=1}^n \beta_i \kappa_i}{k} - 1} (1 - u)^{\frac{\eta + \beta n_1}{k} - 1} du \\
 &= \frac{(\Theta^{s+1}(x) - \Theta^{s+1}(a))^{\frac{\mu + \eta + \sum_{i=1}^n \beta_i \kappa_i + \beta n_1}{k} - 1}}{s + 1} B_k(\mu + \eta + \sum_{i=1}^n \beta_i \kappa_i, \eta + \beta n_1) \\
 &= \frac{(\Theta^{s+1}(x) - \Theta^{s+1}(a))^{\frac{\mu + \eta + \sum_{i=1}^n \beta_i \kappa_i + \beta n_1}{k} - 1}}{s + 1} \frac{\Gamma_k(\mu + \sum_{i=1}^n \beta_i \kappa_i) \Gamma_k(\eta + \beta n_1)}{\Gamma_k(\mu + \eta + \sum_{i=1}^n \beta_i \kappa_i + \beta n_1)}. \quad (12)
 \end{aligned}$$

Substituting (12) into (11), we obtain the required result.  $\square$

Now, we present the semi-group property of the new fractional integral operator.

**Theorem 5.**  $s$  is a real number except for  $-1$ , and the parameters  $\mu, \nu, \sigma_i, \beta_i, \delta_i, \zeta_i$  are complex numbers with  $\Re(\beta_i) > 0$  for all  $i = 1, 2, \dots, n$ ;  $\Re(\nu) > 0$ ,  $\Re(\mu) > 0$ , and the function  $f$  is in  $L[a, b]$ ; then, we have the semi-group property

$$\begin{aligned}
 & {}^k_{s, \Theta} \mathfrak{J}_{a^+, \mu, (\beta_i)}^{(\delta_i), (\zeta_i)} \left( {}^k_{s, \Theta} \mathfrak{J}_{a^+, \nu, (\beta_i)}^{(\sigma_i), (\zeta_i)} \right) f(x) \\
 &= {}^k_{s, \Theta} \mathfrak{J}_{a^+, \mu + \nu, \beta_i}^{(\delta_i + \sigma_i), (\zeta_i)} f(x).
 \end{aligned}$$

**Proof.** By using Definition 13 and the Dirichlet formula, we obtain

$$\begin{aligned}
 & {}^k_{s, \Theta} \mathfrak{J}_{a^+, \mu, (\beta_i)}^{(\delta_i), (\zeta_i)} \left( {}^k_{s, \Theta} \mathfrak{J}_{a^+, \nu, (\beta_i)}^{(\sigma_i), (\zeta_i)} \right) f(x) \\
 &= \frac{(s + 1)^{1 - \frac{\mu}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n = 0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i, k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\
 &\quad \times \int_a^x (\Theta^{s+1}(x) - \Theta^{s+1}(t))^{\frac{\mu + \sum_{i=1}^n \beta_i \kappa_i}{k} - 1} h^s(t) \Theta'(t) \\
 &\quad \times \left( {}^k_{s, \Theta} \mathfrak{J}_{a^+, \nu, (\beta_i)}^{(\sigma_i), (\zeta_i)} \right) f(t) dt \\
 &= \frac{(s + 1)^{2 - \frac{(\mu + \nu)}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n} \sum_{l_1, \dots, l_n = 0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i, k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! (\kappa_2)! \dots (\kappa_n)!} \\
 &\quad \times \frac{\prod_{i=1}^n (\sigma_i)_{l_i, k} (\zeta_i)^{l_i}}{\Gamma_k(\nu + \sum_{i=1}^n \beta_i l_i) (l_1)! (l_2)! \dots (l_n)!} \\
 &\quad \times \int_a^x \int_a^t (\Theta^{s+1}(x) - \Theta^{s+1}(t))^{\frac{\mu + \sum_{i=1}^n \beta_i \kappa_i}{k} - 1} \Theta^s(t) \Theta'(t) \\
 &\quad \times \left( \Theta^{s+1}(t) - \Theta^{s+1}(\tau) \right)^{\frac{\nu + \sum_{i=1}^n \beta_i l_i}{k} - 1} \Theta^s(\tau) \Theta'(\tau) f(\tau) d\tau dt
 \end{aligned}$$

$$\begin{aligned}
&= \frac{(s+1)^{2-\frac{(\mu+\nu)}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n} \sum_{l_1, \dots, l_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i, k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\
&\quad \times \frac{\prod_{i=1}^n (\sigma_i)_{l_i, k} (\zeta_i)^{l_i}}{\Gamma_k(\nu + \sum_{i=1}^n \beta_i l_i) (l_1)! (l_2)! \dots (l_n)!} \int_a^x \Theta^s(\tau) \Theta'(\tau) f(\tau) \\
&\quad \times \int_{\tau}^x (\Theta^{s+1}(x) - \Theta^{s+1}(t))^{\frac{\mu + \sum_{i=1}^n \beta_i \kappa_i}{k} - 1} \\
&\quad \times (\Theta^{s+1}(t) - \Theta^{s+1}(\tau))^{\frac{\nu + \sum_{i=1}^n \beta_i l_i}{k} - 1} \Theta^s(t) \Theta'(t) dt d\tau \\
&= \frac{(s+1)^{2-\frac{(\mu+\nu)}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \sum_{l_1, \dots, l_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i, k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\
&\quad \times \frac{\prod_{i=1}^n (\sigma_i)_{l_i, k} (\zeta_i)^{l_i}}{\Gamma_k(\nu + \sum_{i=1}^n \beta_i l_i) (l_1)! (l_2)! \dots (l_n)!} \\
&\quad \times \int_a^x \Theta^s(\tau) \Theta'(\tau) f(\tau) (\Theta^{s+1}(x) - \Theta^{s+1}(\tau))^{\frac{\mu + \nu + \sum_{i=1}^n (\beta_i \kappa_i + \beta_i l_i)}{k} - 2} \\
&\quad \times \int_{\tau}^x \left( \frac{\Theta^{s+1}(x) - \Theta^{s+1}(t)}{\Theta^{s+1}(x) - \Theta^{s+1}(\tau)} \right)^{\frac{\mu + \sum_{i=1}^n \beta_i \kappa_i}{k} - 1} \\
&\quad \times \left( \frac{\Theta^{s+1}(t) - \Theta^{s+1}(\tau)}{\Theta^{s+1}(x) - \Theta^{s+1}(\tau)} \right)^{\frac{\nu + \sum_{i=1}^n \beta_i l_i}{k} - 1} \Theta^s(t) \Theta'(t) dt d\tau.
\end{aligned}$$

Substituting  $u = \frac{\Theta^{s+1}(x) - \Theta^{s+1}(t)}{\Theta^{s+1}(x) - \Theta^{s+1}(\tau)}$ , then,  $du = \frac{-(s+1)\Theta^s(t)\Theta'(t)dt}{\Theta^{s+1}(x) - \Theta^{s+1}(\tau)}$ , and we obtain

$$\begin{aligned}
& {}_{s, \Theta}^k \mathfrak{J}_{a^+, \mu, (\beta_i)}^{(\delta_i), (\zeta_i)} \left( {}_{s, \Theta}^k \mathfrak{J}_{a^+, \nu, (\beta_i)}^{(\sigma_i), (\zeta_i)} \right) f(x) \\
&= \frac{(s+1)^{1-\frac{(\mu+\nu)}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \sum_{l_1, \dots, l_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i, k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\
&\quad \times \frac{\prod_{i=1}^n (\sigma_i)_{l_i, k} (\zeta_i)^{l_i}}{\Gamma_k(\nu + \sum_{i=1}^n \beta_i l_i) (l_1)! (l_2)! \dots (l_n)!} \\
&\quad \times \int_a^x \Theta^s(\tau) \Theta'(\tau) f(\tau) (\Theta^{s+1}(x) - \Theta^{s+1}(\tau))^{\frac{\mu + \nu + \sum_{i=1}^n (\beta_i \kappa_i + \beta_i l_i)}{k} - 1} \\
&\quad \times \int_0^1 u^{\frac{\mu + \sum_{i=1}^n \beta_i \kappa_i}{k} - 1} (1-u)^{\frac{\nu + \sum_{i=1}^n \beta_i l_i}{k} - 1} du d\tau \\
&= \frac{(s+1)^{1-\frac{(\mu+\nu)}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \sum_{l_1, \dots, l_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i, k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\
&\quad \times \frac{\prod_{i=1}^n (\sigma_i)_{l_i, k} (\zeta_i)^{l_i}}{\Gamma_k(\nu + \sum_{i=1}^n \beta_i l_i) (l_1)! (l_2)! \dots (l_n)!} \\
&\quad \times \int_a^x \Theta^s(\tau) \Theta'(\tau) f(\tau) (\Theta^{s+1}(x) - \Theta^{s+1}(\tau))^{\frac{\mu + \nu + \sum_{i=1}^n (\beta_i \kappa_i + \beta_i l_i)}{k} - 1} \\
&\quad \times B_k(\mu + \nu + \sum_{i=1}^n (\beta_i \kappa_i + \beta_i l_i)) d\tau
\end{aligned}$$

$$\begin{aligned}
&= \frac{(s+1)^{1-\frac{(\mu+\nu)}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n} \sum_{l_1, \dots, l_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i, k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)! (\kappa_1)! (\kappa_2)! \dots (\kappa_n)!} \\
&\quad \times \frac{\prod_{i=1}^n (\sigma_i)_{l_i, k} (\zeta_i)^{l_i}}{\Gamma_k(\nu + \sum_{i=1}^n \beta_i l_i) (l_1)! (l_2)! \dots (l_n)!} \\
&\quad \times \int_a^x \Theta^s(\tau) \Theta'(\tau) f(\tau) (\Theta^{s+1}(x) - \Theta^{s+1}(\tau))^{\frac{\mu+\nu+\sum_{i=1}^n (\beta_i \kappa_i + \beta_i l_i)}{k} - 1} \\
&\quad \times \frac{\Gamma_k(\mu + \sum_{i=1}^n \beta_i \kappa_i) \Gamma_k(\nu + \sum_{i=1}^n \beta_i l_i)}{\Gamma_k(\mu + \nu + \sum_{i=1}^n (\beta_i \kappa_i + \beta_i l_i))} d\tau \\
&= {}^k_{s, \Theta} \mathfrak{J}_{a^+, \mu+\nu, (\beta_i)}^{(\delta_i+1), (\zeta_i+\sigma_i)} f(x).
\end{aligned}$$

Hence, the proof is completed.  $\square$

### 3. Generalized Fractional Derivative Involving the Multivariate Mittag–Leffler Function

In this section, we define the inverse operator for the newly defined integral given by (8). Some basic connections of both fractional operators are also part of this section. The derivative operator is defined as follows.

**Definition 15.** Let  $s$  be a real number, except for  $-1$ , let the parameters  $\mu, \beta_i, \delta_i, \zeta_i$  be complex numbers with  $\Re(\beta_i) > 0$  for all  $i = 1, 2, \dots, n$ , and let  $\Re(\mu) > 0, f \in L[a, b]$ , and the function  $\Theta$  be strictly increasing.

$$\begin{aligned}
& {}^k_{s, \Theta} \mathfrak{D}_{a^+, \mu, (\beta_i)}^{(\delta_i), (\zeta_i)} f(x) \\
&= \left( \frac{k}{\Theta^s(x) \Theta'(x)} \frac{d}{dx} \right)^n {}^k_{s, \Theta} \mathfrak{J}_{a^+, nk-\mu, (\beta_i)}^{(-\delta_i), (\zeta_i)} f(x) \\
&= \frac{(s+1)^{1-\frac{nk-\mu}{k}}}{k} \left( \frac{k}{\Theta^s(x) \Theta'(x)} \frac{d}{dx} \right)^n \int_0^x (\Theta^{s+1}(x) - \Theta^{s+1}(t))^{\frac{nk-\mu}{k} - 1} \\
&\quad \times \mathcal{E}_{k, nk-\mu, (\beta_i)}^{(-\delta_i)} \left( \zeta_1 \left( \Theta^{s+1}(x) - \Theta^{s+1}(t) \right)^{\frac{\beta_1}{k}}, \zeta_2 \left( \Theta^{s+1}(x) - \Theta^{s+1}(t) \right)^{\frac{\beta_2}{k}}, \dots, \right. \\
&\quad \left. \zeta_n \left( \Theta^{s+1}(x) - \Theta^{s+1}(t) \right)^{\frac{\beta_n}{k}} \right) \Theta'(t) \Theta^s(t) f(t) dt, \quad s \in \mathbb{R} \setminus \{-1\}.
\end{aligned} \tag{13}$$

The following are some cases of the defined fractional derivative operator that represent operators that exist in the literature.

- (i) Substituting  $\Theta(x) = x, n = 1$  into (13), we get the modified fractional operator given in (11).
- (ii) By setting  $\Theta(x) = x, s = 0$ , and  $n = 1$  in (13), we get the  $k$ -Parabhakar fractional operator (7) given in [30].
- (iii) Corresponding to the choice  $\Theta(x) = x, s = 0, n = 1$ , and  $k = 1$  in (13), we get the Parabhakar fractional operator given in (5).
- (iv) Substituting  $\Theta(x) = x, n = 1$ , and  $\delta_1 = 0$  into (13), it represents the  $(k, s)$ -(R-L) fractional derivative (3) introduced by Nisar et al. [33].
- (v) Substituting  $\Theta(x) = x, n = 1, \delta_1 = 0, s = 0$ , and  $k = 1$  in (13), it reduces to the classical (R-L) fractional derivative.

In the next theorem, we prove the inverse property of the operators that we defined and presented in (8) and (13).

**Theorem 6.** Let  $s$  be a real number, except for  $-1$ , and let the parameters  $\mu, \beta_i, \delta_i, \zeta_i$  be complex numbers with  $\Re(\beta_i) > 0$  for all  $i = 1, 2, \dots, n$ ,  $\Re(\mu) > 0$ . Furthermore,  $f$  is in  $L[a, b]$ , and the function  $\Theta$  is strictly increasing; then, we have the following result:

$${}_{s,\Theta}^k \mathcal{D}_{a^+,\mu,(\beta_i)}^{(\delta_i),(\zeta_i)} \left( {}_{s,\Theta}^k \mathcal{J}_{a^+,\mu,(\beta_i)}^{(\delta_i),(\zeta_i)} f(x) \right) = f(x).$$

The semi-group property of the new fractional derivative operator is given in the following theorem.

**Theorem 7.** Let  $s$  be a real number, except for  $-1$ , and let the parameters  $\mu, \beta_i, \delta_i, \zeta_i, \sigma_i$  be complex numbers with  $\Re(\beta_i) > 0$  for all  $i = 1, 2, \dots, n$ ,  $\Re(\mu) > 0$ ,  $\Re(\nu) > 0$ . Furthermore,  $f$  is in  $L[a, b]$ , and the function  $\Theta$  is strictly increasing; then, we have the following result:

$$\begin{aligned} & {}_{s,\Theta}^k \mathcal{D}_{a^+,\mu,(\beta_i)}^{(\delta_i),(\zeta_i)} \left( {}_{s,\Theta}^k \mathcal{D}_{a^+,\nu,(\beta_i)}^{(\sigma_i),(\zeta_i)} f(x) \right) \\ &= {}_{s,\Theta}^k \mathcal{D}_{a^+,\mu+\nu,(\beta_i)}^{(\delta_i+\sigma_i),(\zeta_i)} f(x). \end{aligned}$$

**Proof.** By using Definition 13 and Theorem 5, we have

$$\begin{aligned} & {}_{s,\Theta}^k \mathcal{D}_{a^+,\mu,(\beta_i)}^{(\delta_i),(\zeta_i)} \left( {}_{s,\Theta}^k \mathcal{D}_{a^+,\nu,(\beta_i)}^{(\sigma_i),(\zeta_i)} f(x) \right) \\ &= \left( \frac{k}{\Theta^s(x)\Theta'(x)} \frac{d}{dx} \right)^n {}_{s,\Theta}^k \mathcal{J}_{a^+,nk-\mu,(\beta_i)}^{(-\delta_i),(\zeta_i)} \left( {}_{s,\Theta}^k \mathcal{D}_{a^+,\nu,(\beta_i)}^{(\sigma_i),(\zeta_i)} f(x) \right) \\ &= \left( \frac{k}{\Theta^s(x)\Theta'(x)} \frac{d}{dx} \right)^n {}_{s,\Theta}^k \mathcal{J}_{a^+,nk-\mu,(\beta_i)}^{(-\delta_i),(\zeta_i)} \left( {}_{s,\Theta}^k \mathcal{D}_{a^+,\nu,(\beta_i)}^{(\sigma_i),(\zeta_i)} f(x) \right) \\ &\quad \times \left( {}_{s,\Theta}^k \mathcal{J}_{a^+,\nu,(\beta_i)}^{(\sigma_i),(\zeta_i)} \right) \left( {}_{s,\Theta}^k \mathcal{J}_{a^+,-\nu,(\beta_i)}^{(-\sigma_i),(\zeta_i)} f(x) \right) \\ &= \left( \frac{k}{\Theta^s(x)\Theta'(x)} \frac{d}{dx} \right)^n {}_{s,\Theta}^k \mathcal{J}_{a^+,nk-\mu,(\beta_i)}^{(-\delta_i),(\zeta_i)} \\ &\quad \times \left( {}_{s,\Theta}^k \mathcal{J}_{a^+,-\nu,(\beta_i)}^{(-\sigma_i),(\zeta_i)} f(x) \right) \\ &= \left( \frac{k}{\Theta^s(x)\Theta'(x)} \frac{d}{dx} \right)^n \left( {}_{s,\Theta}^k \mathcal{J}_{a^+,nk-(\mu+\nu),(\beta_i)}^{(-(\delta_i+\sigma_i)),(\zeta_i)} f(x) \right) \\ &= {}_{s,\Theta}^k \mathcal{D}_{a^+,\mu+\nu,(\beta_i)}^{(\delta_i+\sigma_i),(\zeta_i)} f(x). \end{aligned}$$

This completes the proof.  $\square$

#### 4. Laplace Transform of the Fractional Operators

In this section, we find the Laplace transform of the fractional operators defined in Sections 2 and 3. It is notable that the classical Laplace transform fails to deal with the new fractional operators. For this purpose, we need to modify the Laplace transform presented by Fahad et al. [5] with a choice of  $w(x) = 1$ . First, we need the following integer-order integral and derivative operator to find the Laplace transform.

**Definition 16.** The integer-order integral of the function  $\Phi$  with respect to function  $\Theta$  is defined by

$${}_{\Theta}^s \mathcal{J}_{a^+}^n \Phi(\vartheta) = \frac{(s+1)^{1-n}}{\Gamma(n)} \int_a^\vartheta (\Theta^{s+1}(\vartheta) - \Theta^{s+1}(t))^{n-1} \Theta^s(t) \Theta'(t) \Phi(t) dt, \quad (14)$$

where  $s$  is a real number, except for  $-1$ ,  $n \in \mathbb{N}$ , and  $\Theta$  is a strictly increasing differentiable function. The corresponding derivative is defined by

$${}_{\Theta}^s \mathcal{D}_{a^+}^n \Phi(\vartheta) = \left( \frac{1}{\Theta^s(\vartheta)\Theta'(\vartheta)} \frac{d}{d\vartheta} \right)^n (\Phi(\vartheta)). \quad (15)$$

**Theorem 8.** Let  $s$  be a real number, except for  $-1$ , and let the parameters  $\mu, \beta_i, \delta_i, \zeta_i, \sigma_i$  be complex numbers with  $\Re(\beta_i) > 0$  for all  $i = 1, 2, \dots, n, \Re(\mu) > 0, \Re(v) > 0$ . Furthermore,  $f$  is in  $L[a, b]$ , and the function  $\Theta$  is strictly increasing; then, the following equation holds:

$${}_s^s \mathfrak{D}_{a^+}^n \left( {}_k^k \mathfrak{J}_{a^+, \mu, (\beta_i)}^{(\delta_i), (\zeta_i)} f(x) \right) = \frac{1}{k^n} {}_k^k \mathfrak{J}_{a^+, \mu - nk, (\beta_i)}^{(\delta_i), (\zeta_i)} f(x).$$

**Proof.** By using Definition 13 and the integer-order operator given by (15), we have

$$\begin{aligned} & {}_s^s \mathfrak{D}_{a^+}^n \left( {}_k^k \mathfrak{J}_{a^+, \mu, (\beta_i)}^{(\delta_i), (\zeta_i)} f(x) \right) \\ &= \frac{(s+1)^{1-\frac{\mu}{k}}}{k} \left( \frac{1}{\Theta^s(x) \Theta'(x)} \frac{d}{dx} \right)^n \int_a^x (\Theta^{s+1}(x) - \Theta^{s+1}(t))^{\frac{\mu}{k}-1} \\ &\quad \times \mathcal{E}_{k, \mu, (\beta_i)}^{(\delta_i)} \left( \zeta_1 (\Theta^{s+1}(x) - \Theta^{s+1}(t))^{\frac{\beta_1}{k}}, \zeta_2 (\Theta^{s+1}(x) - \Theta^{s+1}(t))^{\frac{\beta_2}{k}}, \dots, \right. \\ &\quad \left. \zeta_n (\Theta^{s+1}(x) - \Theta^{s+1}(t))^{\frac{\beta_n}{k}} \right) \Theta'(t) \Theta^s(t) f(t) dt \\ &= \frac{(s+1)^{1-\frac{\mu-nk}{k}}}{k^{n+1}} \int_a^x (\Theta^{s+1}(x) - \Theta^{s+1}(t))^{\frac{\mu-nk}{k}-1} \\ &\quad \times \mathcal{E}_{k, \mu-nk, (\beta_i)}^{(\delta_i)} \left( \zeta_1 (\Theta^{s+1}(x) - \Theta^{s+1}(t))^{\frac{\beta_1}{k}}, \zeta_2 (\Theta^{s+1}(x) - \Theta^{s+1}(t))^{\frac{\beta_2}{k}}, \dots, \right. \\ &\quad \left. \zeta_n (\Theta^{s+1}(x) - \Theta^{s+1}(t))^{\frac{\beta_n}{k}} \right) \Theta'(t) \Theta^s(t) f(t) dt \\ &= \frac{1}{k^n} {}_k^k \mathfrak{J}_{a^+, \mu-nk, (\beta_i)}^{(\delta_i), (\zeta_i)} f(x). \end{aligned}$$

Therefore, the result is proven.  $\square$

A modification of the Laplace transform [5] is given by the following definition.

**Definition 17.** Let  $\phi$  be defined on  $[a, \infty]$  and let  $\Theta$  be monotonically increasing on the interval  $[a, \infty)$ ,  $s \in \mathbb{R} \setminus \{-1\}$ ; then, the modified Laplace transform of  $\phi$  is defined by

$$L_{\Theta}(\phi)(u) = (s+1) \int_a^{\infty} e^{-u(\Theta^{s+1}(t) - \Theta^{s+1}(a))} \Theta^s(t) \Theta'(t) \phi(t) dt \quad (16)$$

for all values of  $u$  such that (16) is true.

**Definition 18.** The convolution of functions  $g$  and  $\phi$  is defined as

$$(\phi *_{\Theta^{s+1}} g)(x) = (s+1) \int_a^x \phi \left( \Theta^{-1} \left( \Theta^{s+1}(x) + \Theta^{s+1}(a) - \Theta^{s+1}(t) \right)^{\frac{1}{s+1}} \right) \Theta^s(t) \Theta'(t) g(t) dt,$$

where  $\Theta^{-1}$  is the inverse of  $\Theta$ .

**Theorem 9.** Let  $s$  be a real number, except for  $-1$ , and let the parameters  $\alpha, \beta_i, \delta_i, \zeta_i$  be complex numbers with  $\Re(\beta_i) > 0$  for all  $i = 1, 2, \dots, n, \Re(\mu) > 0$ . Furthermore, the function  $f$  is in  $L[a, b]$ ; then, we have the following result:

$$L_{\Theta} \left\{ {}_k^k \mathfrak{J}_{a^+, \mu, (\beta_i)}^{(\delta_i), (\zeta_i)} f(x) \right\} (u) = (s+1)^{-\frac{\mu}{k}} (ku)^{-\frac{\mu}{k}} \prod_{i=1}^n (1 - (k\zeta_i)(ku)^{-\frac{\beta_i}{k}})^{-\frac{\delta_i}{k}} L_{\Theta} \{ f(x) \} (u).$$

**Proof.** By using Definition 13, we can write

$$\begin{aligned}
& L_{\Theta} \left\{ {}^k\mathfrak{J}_{a^+, \mu, (\beta_i)}^{(\delta_i), (\zeta_i)} f(x) \right\} (u) \\
&= \frac{(s+1)^{-\frac{\mu}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i, k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\
&\quad \times L_{\Theta} \left\{ (\Theta^{s+1}(x) - \Theta^{s+1}(a))^{\frac{\mu + \sum_{i=1}^n (\beta_i \kappa_i)}{k} - 1} * f(x) \right\} (u) \\
&= \frac{(s+1)^{-\frac{\mu}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i, k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\
&\quad \times \frac{\Gamma(\frac{\mu + \sum_{i=1}^n (\beta_i \kappa_i)}{k})}{u^{\frac{\mu + \sum_{i=1}^n (\beta_i \kappa_i)}{k}}} L_{\Theta} \{f(x)\} (u) \\
&= \frac{(s+1)^{-\frac{\mu}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i, k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\
&\quad \times \frac{\Gamma_k(\frac{\mu + \sum_{i=1}^n (\beta_i \kappa_i)}{k})}{u^{\frac{\mu + \sum_{i=1}^n (\beta_i \kappa_i)}{k}}} L_{\Theta} \{f(x)\} (u) \\
&= \frac{(s+1)^{-\frac{\mu}{k}}}{k} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i, k} (\zeta_i)^{\kappa_i}}{\Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i)) (\kappa_1)! \dots (\kappa_n)!} \\
&\quad \times \frac{k^{\frac{1-\mu + \sum_{i=1}^n (\beta_i \kappa_i)}{k}} \Gamma_k(\mu + \sum_{i=1}^n (\beta_i \kappa_i))}{u^{\frac{\mu + \sum_{i=1}^n (\beta_i \kappa_i)}{k}}} L_{\Theta} \{f(x)\} (u) \\
&= \frac{(s+1)^{-\frac{\mu}{k}}}{(ku)^{\frac{\mu}{k}}} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i, k} (\zeta_i)}{(ku)^{\frac{\sum_{i=1}^n (\beta_i \kappa_i)}{k}} \prod_{i=1}^n (\kappa_i)!} L_{\Theta} \{f(x)\} (u) \\
&= \frac{(s+1)^{-\frac{\mu}{k}}}{(ku)^{\frac{\mu}{k}}} \sum_{\kappa_1, \dots, \kappa_n=0}^{\infty} \frac{\prod_{i=1}^n (\delta_i)_{\kappa_i, k} (\zeta_i)}{(ku)^{\frac{\sum_{i=1}^n (\beta_i \kappa_i)}{k}} \prod_{i=1}^n (\kappa_i)!} L_{\Theta} \{f(x)\} (u) \\
&= \frac{(s+1)^{-\frac{\mu}{k}}}{(ku)^{\frac{\mu}{k}}} \prod_{i=1}^n \frac{1}{\left(1 - \frac{k(\zeta_i)}{ku}\right)^{\frac{\beta_i}{k}}} L_{\Theta} \{f(x)\} (u) \\
&= (s+1)^{-\frac{\mu}{k}} (ku)^{-\frac{\mu}{k}} \prod_{i=1}^n \left(1 - \frac{k\zeta_i}{ku^{\frac{\beta_i}{k}}}\right)^{-\frac{\delta_i}{k}} L_{\Theta} \{f(x)\} (u) \\
&= (s+1)^{-\frac{\mu}{k}} (ku)^{-\frac{\mu}{k}} \prod_{i=1}^n \left(1 - (k\zeta_i)(ku)^{-\frac{\beta_i}{k}}\right)^{-\frac{\delta_i}{k}} L_{\Theta} \{f(x)\} (u).
\end{aligned}$$

Hence, the result is derived.  $\square$

In our next result, we discuss the generalized form of [37].

**Theorem 10.** Let the functions  $\phi \in AC_w[\alpha_1, \tau)$  and  ${}^s\mathfrak{D}_{a^+}^1 \phi$  be of the  $\Theta^{s+1}$  exponential order. Furthermore,  ${}^s\mathfrak{D}_{a^+}^1 \phi$  is a continuous function on  $[\alpha_1, \beta_1)$ ; then, the Laplace transform of  ${}^s\mathfrak{D}_{a^+}^1 \phi$  exists and is given by

$$L_{\Theta} \{ {}^s\mathfrak{D}_{a^+}^1 \phi \} (u) = u(s+1)^2 L_{\Theta} \{ \phi \} (u) - (s+1) \phi(a). \quad (17)$$

**Proof.** By applying the Laplace transform on  ${}^s_{\Theta}\mathfrak{D}_{a+}^1\phi$ , we can write

$$\begin{aligned} L_{\Theta}\{ {}^s_{\Theta}\mathfrak{D}_{a+}^1\phi\}(u) &= (s+1) \int_a^{\infty} e^{-u(\Theta^{s+1}(t)-\Theta^{s+1}(a))} \Theta^s(t) \Theta'(t) {}^s_{\Theta}\mathfrak{D}_{a+}^1\phi(t) dt \\ &= (s+1) \int_a^{\infty} e^{-u(\Theta^{s+1}(t)-\Theta^{s+1}(a))} \frac{d}{dt}(\phi(t)) dt. \end{aligned}$$

Integrating by parts, we get Equation (17).  $\square$

The generalization of the above theorem is defined in the subsequent corollary.

**Corollary 1.** Let the functions  $\phi \in AC^{n-1}[\alpha_1, \tau)$  and  ${}^s_{\Theta}\mathfrak{D}_{a+}^n\phi$  be of the  $\Theta^{s+1}$  exponential order. Furthermore,  ${}^s_{\Theta}\mathfrak{D}_{a+}^n\phi$  is a piecewise continuous function on  $[\alpha_1, \beta_1]$ ; then, the Laplace transform of  ${}^s_{\Theta}\mathfrak{D}_{a+}^n\phi$  exists and is given by

$$L_{\Theta}\{ {}^s_{\Theta}\mathfrak{D}_{a+}^n\phi\}(u) = u^n(s+1)^n L_{\Theta}\{\phi\}(u) - \sum_{p=0}^{n-1} (s+1)^{n-p} u^{n-p-1} \phi_p(\alpha_1),$$

where

$$\phi_p(\alpha_1) = \left( \frac{1}{\Theta^s(\tau)\Theta'(\tau)} \frac{d}{d\tau} \right)^p (\phi(\tau)) \Big|_{\tau=\alpha_1}$$

and  $p = 0, 1, 2, \dots, n-1$ .

**Theorem 11.** Let  $s$  be a real number, except for  $-1$ , and let the parameters  $\alpha, \beta_i, \delta_i, \zeta_i$  be complex numbers with  $\Re(\beta_i) > 0$  for all  $i = 1, 2, \dots, n$ ,  $\Re(\mu) > 0$ ; let the functions  $f \in AC[\alpha_1, \tau)$  and  ${}^s_{\Theta}\mathfrak{D}_{a+}^n f$ ,  $p = 0, 1, 2, \dots, n-1$  be of the  $\Theta^{s+1}$  exponential order. Furthermore,  ${}^s_{\Theta}\mathfrak{D}_{a+}^n f$  is a continuous function on  $[\alpha_1, \beta_1]$ ; then, we have

$$\begin{aligned} L_{\Theta}\left\{ {}^k_{s,\Theta}\mathfrak{D}_{a+,\mu,(\beta_i)}^{(\delta_i),(\zeta_i)} f(x) \right\}(u) &= (s+1)^{\frac{\mu}{k}} (ku)^{\frac{\mu}{k}} \prod_{i=1}^n ((1 - (k\zeta_i))(ku)^{-\frac{\beta_i}{k}})^{\frac{\delta_i}{k}} L_{\Theta}\{f(x)\}(u) \\ &\quad - \sum_{p=0}^{n-1} (s+1)^{n-p} u^{n+m-p-1} \left( {}^k_{s,\Theta}\mathfrak{D}_{a+,\mu-(n-m)p,(\beta_i)}^{(\delta_i),(\zeta_i)} f(a) \right)_k. \end{aligned}$$

**Proof.** By using Definition 13, we can write

$$\begin{aligned} &L_{\Theta}\left\{ {}^k_{s,\Theta}\mathfrak{D}_{a+,\mu,(\beta_i)}^{(\delta_i),(\zeta_i)} f(x) \right\}(u) \\ &= k^n L_{\Theta}\left\{ {}^s_{\Theta}\mathfrak{D}_{a+}^n \left\{ {}^k_{s,\Theta}\mathfrak{J}_{a+,nk-\mu,(\beta_i)}^{(-\delta_i),(\zeta_i)} f(a) \right\} \right\}(u) \\ &= (ku)^n (s+1)^n L_{\Theta}\left\{ {}^k_{s,\Theta}\mathfrak{J}_{a+,nk-\mu,(\beta_i)}^{(-\delta_i),(\zeta_i)} f(x) \right\}(u) \\ &\quad - \sum_{p=0}^{n-1} (s+1)^{n-p} u^{n-p-1} \left( {}^k_{s,\Theta}\mathfrak{J}_{a+,nk-\mu,(\beta_i)}^{(-\delta_i),(\zeta_i)} f(a) \right)_k \\ &= (ku)^n (s+1)^n (s+1)^{\frac{\mu-nk}{k}} (ku)^{\frac{\mu-nk}{k}} \prod_{i=1}^n \left( (1 - (k\zeta_i))(ku)^{-\frac{\beta_i}{k}} \right)^{\frac{\delta_i}{k}} L_{\Theta}\{f(x)\}(u) \end{aligned}$$

$$\begin{aligned}
& - \sum_{p=0}^{n-1} (s+1)^{n-p} u^{n-p-1} \left( {}^k_{s,\Theta} \mathfrak{J}_{a^+,nk-\mu,(\beta_i)}^{(-\delta_i),(\zeta_i)} f(a) \right)_p \\
& = (s+1)^{\frac{\mu}{k}} (ku)^{\frac{\mu}{k}} \prod_{i=1}^n \left( (1 - (k\zeta_i)) (ku)^{-\frac{\beta_i}{k}} \right)^{\frac{\delta_i}{k}} L_{\Theta} \{f(x)\}(u) \\
& - \sum_{p=0}^{n-1} (s+1)^{n-p} u^{n-p-1} \left( {}^k_{s,\Theta} \mathfrak{J}_{a^+,nk-\mu,(\beta_i)}^{(-\delta_i),(\zeta_i)} f(a) \right)_p.
\end{aligned}$$

Therefore, the proof is completed.  $\square$

## 5. Applications to a Real-Life Problem

The kinetic equation is the fundamental equation of mathematical physics and the natural sciences because it represents the continuity of the motion of a substance. The following is a generalization of the fractional kinetic differintegral equation involving new fractional operators.

$$\alpha \left( {}^k_{s,\Theta} \mathfrak{D}_{a^+,\mu,(\beta_i)}^{(\delta_i),(\zeta_i)} N \right)(x) - N_0 \Phi(x) = \beta \left( {}^k_{s,\Theta} \mathfrak{J}_{a^+,\mu,(\beta_i)}^{(\delta_i),(\zeta_i)} N \right)(x) \quad (18)$$

$${}^k_{s,F} \mathfrak{J}_{a^+,\mu,(\beta_i)}^{(\delta_i),(\zeta_i)} \Phi(a) = h, \quad h \geq 0 \quad (19)$$

where  $\Phi \in X_p^w(\alpha_1, \beta_1)$ ,  $\lambda_1 \in \mathbb{C}$ ,  $a, b \in \mathbb{R}$  ( $\alpha \neq 0$ ),  $\mu_1, \nu_1, \varrho_1, k > 0$  and  $\gamma_1, \sigma \geq 0$ .

**Theorem 12.** The solution of Equation (18) with the initial condition (19) is

$$\begin{aligned}
N(x) &= \frac{h_p}{k^{n-p-1}} \sum_{m=0}^{\infty} \left( \frac{\beta}{\alpha} \right)^m (\Theta^{s+1}(x) - \Theta^{s+1}(a))^{\frac{(\mu+\nu)m+\mu-nk+k(p+1)}{k}-1} \\
&\times \mathcal{E}_{k,(\mu+\nu)m+\mu-nk+k(p+1),(\beta_i)}^{((\delta_i+\sigma_i)m)} \left( \zeta_1 \left( \Theta^{s+1}(x) - \Theta^{s+1}(a) \right)^{\frac{\beta_1}{k}}, \zeta_2 \left( \Theta^{s+1}(x) - \Theta^{s+1}(a) \right)^{\frac{\beta_2}{k}}, \dots, \right. \\
&\left. \zeta_n \left( \Theta^{s+1}(x) - \Theta^{s+1}(a) \right)^{\frac{\beta_n}{k}} \right) + \frac{N_0}{\alpha} \sum_{m=0}^{\infty} \left( \frac{\beta}{\alpha} \right)^m {}^k_{s,\Theta} \mathfrak{J}_{a^+,\mu,(\beta_i)}^{((\delta_i+\sigma_i)m+\delta_i),(\zeta_i)} N(x). \quad (20)
\end{aligned}$$

**Proof.** Applying the generalized Laplace transform on both sides of Equation (18), we have

$$\begin{aligned}
& \alpha L_{\Theta} \left( {}^k_{s,\Theta} \mathfrak{D}_{a^+,\mu,(\beta_i)}^{(\delta_i),(\zeta_i)} N(\vartheta) \right)(u) - N_0 L_{\Theta} (\Phi(x))(u) \\
& = \beta L_{\Theta} \left( {}^k_{s,\Theta} \mathfrak{J}_{a^+,\mu,(\beta_i)}^{(\delta_i),(\zeta_i)} N(x) \right)(u) \\
& \alpha (s+1)^{\frac{\mu}{k}} (ku)^{\frac{\mu}{k}} \prod_{i=1}^n \left( (1 - (k\zeta_i)) (ku)^{-\frac{\beta_i}{k}} \right)^{\frac{\delta_i}{k}} L_{\Theta} \{N(x)\}(u) \\
& - \alpha \sum_{p=0}^{n-1} (s+1)^{n-p} u^{n-p-1} \left( {}^k_{s,F} \mathfrak{J}_{a^+,nk-\mu,(\beta_i)}^{(-\delta_i),(\zeta_i)} N(a) \right)_k - N_0 L_{\Theta} (\Phi(x))(u) \\
& = \beta (s+1)^{\frac{\mu}{k}} (ku)^{\frac{\mu}{k}} \prod_{i=1}^n \left( (1 - (k\zeta_i)) (ku)^{-\frac{\beta_i}{k}} \right)^{\frac{\delta_i}{k}} L_{\Theta} \{N(x)\}(u).
\end{aligned}$$

This can also be written as

$$\begin{aligned}
 & \frac{\alpha - \beta \left( (s+1)^{-\frac{\mu+\nu}{k}} (ku)^{-\frac{\mu+\nu}{k}} \prod_{i=1}^n (1 - (k\zeta_i)(ku)^{-\frac{\beta_i}{k}})^{-\frac{\delta_i+\sigma_i}{k}} \right)}{(s+1)^{-\frac{\mu}{k}} (ku)^{-\frac{\mu}{k}} \prod_{i=1}^n ((1 - (k\zeta_i))(ku)^{-\frac{\beta_i}{k}})^{-\frac{\delta_i}{k}}} L_{\Theta}\{N(x)\}(u) \\
 &= \alpha \sum_{p=0}^{n-1} (s+1)^{n-p} u^{n-p-1} h_p + N_0 L_{\Theta}(\Phi(x))(u) \\
 L_{\Theta}\{N(\vartheta)\}(u) &= \alpha \sum_{p=0}^{n-1} (s+1)^{n-p} u^{n-p-1} h_p \\
 &\times \frac{(s+1)^{-\frac{\mu}{k}} (ku)^{-\frac{\mu}{k}} \prod_{i=1}^n ((1 - (k\zeta_i))(ku)^{-\frac{\beta_i}{k}})^{-\frac{\delta_i}{k}}}{\alpha - \beta \left( (s+1)^{-\frac{\mu+\nu}{k}} (ku)^{-\frac{\mu+\nu}{k}} \prod_{i=1}^n (1 - (k\zeta_i)(ku)^{-\frac{\beta_i}{k}})^{-\frac{\delta_i+\sigma_i}{k}} \right)} \\
 &+ \frac{(s+1)^{-\frac{\mu}{k}} (ku)^{-\frac{\mu}{k}} \prod_{i=1}^n ((1 - (k\zeta_i))(ku)^{-\frac{\beta_i}{k}})^{-\frac{\delta_i}{k}}}{\alpha - \beta \left( (s+1)^{-\frac{\mu+\nu}{k}} (ku)^{-\frac{\mu+\nu}{k}} \prod_{i=1}^n (1 - (k\zeta_i)(ku)^{-\frac{\beta_i}{k}})^{-\frac{\delta_i+\sigma_i}{k}} \right)} N_0 L_{\Theta}(\Phi(x))(u).
 \end{aligned}$$

Taking  $|\frac{\beta}{\alpha} \left( (s+1)^{-\frac{\mu+\nu}{k}} (ku)^{-\frac{\mu+\nu}{k}} \prod_{i=1}^n (1 - (k\zeta_i)(ku)^{-\frac{\beta_i}{k}})^{-\frac{\delta_i+\sigma_i}{k}} \right)| < 1$ , we get

$$\begin{aligned}
 L_{\Theta}\{N(x)\}(u) &= \left( \sum_{p=0}^{n-1} (s+1)^{n-p} u^{n-p-1} h_p (s+1)^{-\frac{\mu}{k}} (ku)^{-\frac{\mu}{k}} \prod_{i=1}^n ((1 - (k\zeta_i))(ku)^{-\frac{\beta_i}{k}})^{-\frac{\delta_i}{k}} \right. \\
 &+ (s+1)^{-\frac{\mu}{k}} (ku)^{-\frac{\mu}{k}} \prod_{i=1}^n ((1 - (k\zeta_i))(ku)^{-\frac{\beta_i}{k}})^{-\frac{\delta_i}{k}} \frac{N_0}{\alpha} L_{\Theta}(\Phi(x))(u) \Big) \\
 &\times \left( \sum_{m=0}^{\infty} \left( \frac{\beta}{\alpha} \right)^m ((s+1)^{-\frac{(\mu+\nu)m}{k}} (ku)^{-\frac{(\mu+\nu)m}{k}} \prod_{i=1}^n (1 - (k\zeta_i)(ku)^{-\frac{\beta_i}{k}})^{-\frac{(\delta_i+\sigma_i)m}{k}}) \right) \\
 L_{\Theta}\{N(x)\}(u) &= \sum_{p=0}^{n-1} (s+1)^{n-p} u^{n-p-1} h_p \\
 &\left( \sum_{m=0}^{\infty} \left( \frac{\beta}{\alpha} \right)^m ((s+1)^{-\frac{(\mu+\nu)m-\mu}{k}} (ku)^{-\frac{(\mu+\nu)m-\mu}{k}} \prod_{i=1}^n (1 - (k\zeta_i)(ku)^{-\frac{\beta_i}{k}})^{-\frac{(\delta_i+\sigma_i)m-\delta_i}{k}}) \right) \\
 &+ \frac{N_0}{\alpha} \left( \sum_{m=0}^{\infty} \left( \frac{\beta}{\alpha} \right)^m ((s+1)^{-\frac{(\mu+\nu)m-\mu}{k}} (ku)^{-\frac{(\mu+\nu)m-\mu}{k}} \prod_{i=1}^n (1 - (k\zeta_i)(ku)^{-\frac{\beta_i}{k}})^{-\frac{(\delta_i+\sigma_i)m-\delta_i}{k}}) \right) \\
 &L_{\Theta}(\Phi(x))(u).
 \end{aligned}$$

By applying the inverse Laplace transform, we get the required result.  $\square$

**Example 1.** If we substitute  $s = 0$ ,  $\Theta(x) = x$ ,  $a = 0$ ,  $N_0 = 0$ ,  $\alpha = \beta = 1$ ,  $k = 1$ ,  $p = 1$ ,  $n = 2$ ,  $h_1 = 1$ ,  $\mu = \nu = 1$ , and  $\delta_i = \sigma_i = 0$  for all  $i$ , then the solution (20) of Equation (18) with the condition (19) becomes

$$N(x) = \sum_{m=0}^{\infty} \frac{x^{2m}}{\Gamma(2m+1)} \quad (21)$$

A graph of the function (21) is given below (Figure 1).

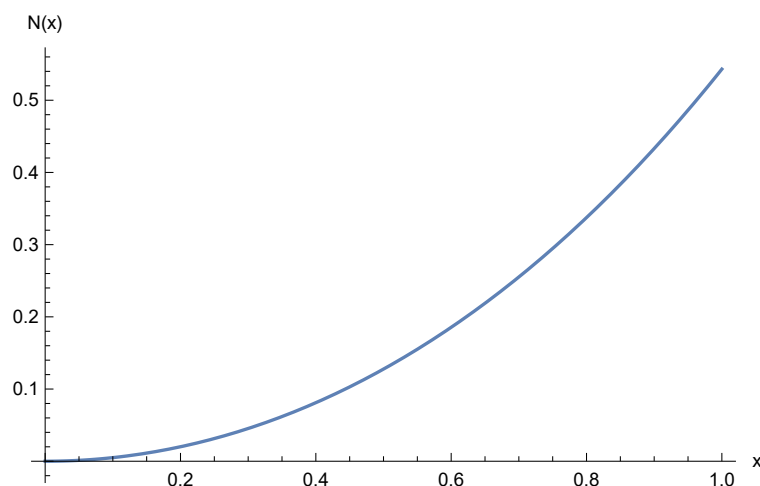


**Figure 1.** For the function  $N(x)$ , we get Figure 1, where  $0 < x < 1$ .

## 6. Conclusions and Discussion

The importance of fractional calculus has a significant role in all sciences due to its widespread applications. Fractional calculus represents nature more accurately than integer-order calculus does. Numerous mathematicians have studied the Mittag–Leffler function and its applications. In the present work, we introduced the multivariate Mittag–Leffler function and used it to explore a fractional integral operator and its inverse derivative operator. The cases presented just after the definitions proved the generalization of the introduced operators. Both operators are bounded in the  $X^p[a, b]$  space. The fundamental properties of the fractional operators were established. Moreover, we evaluated the modified Laplace transform of both the derivative and integral operators. These results cannot be obtained with the classical Laplace transform. By using the new operators, a fractional kinetic differintegral equation was developed, and its solution was obtained via the modified Laplace transform. A growth model was explored as a real-life application, and its graph was sketched. The authors of this study strongly encourage readers to explore more multivariate forms of special functions and new forms of fractional operators.

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