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# An Integrable Evolution System and Its Analytical Solutions with the Help of Mixed Spectral AKNS Matrix Problem

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**Abstract:** In this work, a novel integrable evolution system in the sense of Lax's scheme associated with a mixed spectral Ablowitz-Kaup-Newell-Segur (AKNS) matrix problem is first derived. Then, the time dependences of scattering data corresponding to the mixed spectral AKNS matrix problem are given in the inverse scattering analysis. Based on the given time dependences of scattering data, the reconstruction of potentials is carried out, and finally analytical solutions with four arbitrary functions of the derived integrable evolution system are formulated. This study shows that some other systems of integrable evolution equations under the resolvable framework of the inverse scattering method with mixed spectral parameters can be constructed by embedding different spectral parameters and time-varying coefficient functions to the known AKNS matrix spectral problem.

**Keywords:** integrable evolution system; mixed spectral AKNS matrix problem; spectral parameter; scattering data; analytical solution; inverse scattering method

**MSC:** 35Q58; 35P30; 35G30; 35A20

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## 1. Introduction

In nonlinear mathematical physics, the derivation, solution and integrability of equations are important topics [1–14]. Generally, an evolution equation is called integrable in the sense of Lax if it can be written as the compatibility condition between the related linear spectral problem and the adjoint time evolution equation [2]. For example [5], the well-known Korteweg-de Vries (KdV) equation $u_t + 6uu_x + u_{xxx} = 0$ has the Lax integrability owing to the compatibility condition [8]:

$$[L, N - \partial_t] \equiv L(N - \partial_t) - (N - \partial_t)L = 0, \tag{1}$$

of a pair of given linear problems:

$$L\phi = \lambda\phi, \ L = \partial_x^2 + u, \tag{2}$$

$$L\phi = \lambda\phi, \ N = -4\partial_x^3 - 6u\partial_x - 3u_x - \partial_t, \tag{3}$$

where the eigenfunction $\phi$ and the potential function $u$ are dependent on the space variable $x$ and the time variable $t$, and the spectral parameter $\lambda$ is a constant.

Since the isospectral AKNS matrix problem [2]:

$$\phi_x = M\phi, \ M = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix}, \ \phi = \begin{pmatrix} \phi_1(x,t) \\ \phi_2(x,t) \end{pmatrix}, \ \lambda = \mathrm{i}k, \ \frac{\mathrm{d}k}{\mathrm{d}t} = 0 \tag{4}$$

and its adjoint time evolution equation:

$$\phi_t = N\phi, \ N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \tag{5}$$

were proposed in 1974, a large number of important integrable equations [1–9] have been derived from the compatibility condition of Equations (4) and (5):

$$M_t - N_x + [M, N] = 0, \ [M, N] \equiv MN - NM, \tag{6}$$

such as the KdV equation, the modified KdV (mKdV) equation, the nonlinear Schrödinger (NLS) equation, and the sine-Gordon equation. In Equations (4) and (5), $q$ and $r$ are two smooth potential functions of $x$ and $t$; $A$, $B$ and $C$ are three undetermined functions of $x$, $t$, $q$, $r$ and $\lambda$; and i is the imaginary unit. The findings of a large number of integrable equations are due to the pioneering work of Lax's scheme [1], including Equations (1) and (6) and their generalizations [5–9]. The generalizations of Equations (4) and (5) can be summarized as follows: (i) extending the isospectrum $\lambda$, which is independent of $t$, to the nonisospectral case depending on $t$; (ii) embedding some coefficient functions into the evolution equation satisfied by the nonisospectrum $\lambda$ and/or the function $A$ for the derivation of time-varying nonisospectral equations or isospectral equations with time-varying coefficient functions; (iii) coupling isospectral equations and nonisospectral equations to mixed spectral equations; (iv) modifying local equations to nonlocal equations; (v) extension of the equations with integer-order derivatives to fractional-order equations.

From the view of physics, the variable-coefficient equations and nonisospectral equations can be used to describe solitary waves in nonuniform media, and they have their own advantages [8] in being more suitable for approaching the essence of nonlinear phenomena than the constant-coefficient equations or isospectral equations. This work aims at generalizing Equations (4) and (5) to other different forms by proposing that the spectral parameter $\lambda = \mathrm{i}k$ and the undetermined function $A$ satisfy the following time evolution equation:

$$\mathrm{i}\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{1}{2}[\delta(t) + 2\mathrm{i}k\beta(t)], \tag{7}$$

and assumption:

$$A = \partial^{-1}(r, q)\begin{pmatrix} -B \\ C \end{pmatrix} - \frac{1}{2}[\delta(t) + 2\mathrm{i}k\beta(t)]x - \frac{1}{2}\alpha(t)(2\mathrm{i}k)^3 - \frac{1}{2}\gamma(t), \tag{8}$$

respectively. Here $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\delta(t)$ are time-varying integrable functions, and $B$ and $C$ are supposed as:

$$\begin{pmatrix} -B \\ C \end{pmatrix} = \alpha(t)L^2\begin{pmatrix} -q \\ r \end{pmatrix} + \beta(t)\begin{pmatrix} -xq \\ xr \end{pmatrix} + 2\mathrm{i}k\alpha(t)L\begin{pmatrix} -q \\ r \end{pmatrix} + \alpha(t)(2\mathrm{i}k)^2\begin{pmatrix} -q \\ r \end{pmatrix}, \tag{9}$$

with

$$L = \sigma\partial + 2\begin{pmatrix} q \\ -r \end{pmatrix}\partial^{-1}(r, q), \ \partial = \frac{\partial}{\partial x}, \ \partial^{-1} = \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{+\infty}\right)\mathrm{d}x, \ \sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{10}$$

As a results, a novel system of integrable evolution equations:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} \alpha(t)q_{xxx} - 6\alpha(t)qrq_x + \beta(t)q + \beta(t)xq_x - \delta(t)xq - \gamma(t)q \\ \alpha(t)r_{xxx} - 6\alpha(t)qrr_x + \beta(t)r + \beta(t)xr_x + \delta(t)xr + \gamma(t)r \end{pmatrix}, \tag{11}$$

is derived in Section 2 for the first time. Equation (11) is a mixed spectral system and this, due to Equation (7), contains two kinds of spectra. One is isospectrum under the case of $\delta(t) + 2\mathrm{i}k\beta(t) = 0$, and the other becomes nonisospectrum when $\delta(t) + 2\mathrm{i}k\beta(t) \neq 0$. Thus, we call such a parameter i$k$ in Equation (7) a mixed spectrum. Meanwhile, Equation (11) is called a mixed spectral system. Several special cases of Equation (11) and their corresponding simplified forms of Equations (7) and (8) can be found in Section 3. In Section 4, the inverse scattering method [2,3,9] combined with the mixed spectral parameter i$k$ satisfying Equation (7) is established to solve Equation (11), and implicit solutions are obtained.

Considering the reflectionless potential, the explicit unified formulae are reduced from the obtained implicit analytical solutions in Section 4. As a conclusion, we summarize the results of this article in Section 5.

## 2. Derivation of Equation (11) by Lax's Scheme

Substituting the matrices $M$ and $N$ in Equations (4) and (5) into Equation (6), we have:

$$-\mathrm{i}\frac{\mathrm{d}k}{\mathrm{d}t} - A_x + qC - rB = 0, \tag{12}$$

$$q_t - B_x - 2\mathrm{i}kB - 2qA = 0, \tag{13}$$

$$r_t - C_x + 2\mathrm{i}kC + 2rA = 0. \tag{14}$$

Then, the substitution of Equations (7) and (8) into Equations (12)–(14) shows that Equation (12) holds automatically, and Equations (13) and (14) are converted as follows:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L\begin{pmatrix} -B \\ C \end{pmatrix} - 2\mathrm{i}k\begin{pmatrix} -B \\ C \end{pmatrix} + [\alpha(t)(2\mathrm{i}k)^3 + \gamma(t)]\begin{pmatrix} -q \\ r \end{pmatrix} + [\delta(t) + 2\mathrm{i}k\beta(t)]\begin{pmatrix} -xq \\ xr \end{pmatrix}. \tag{15}$$

Further, we suppose that:

$$\begin{pmatrix} -B \\ C \end{pmatrix} = \sum_{i=1}^{4}\begin{pmatrix} -b_i \\ c_i \end{pmatrix}(2\mathrm{i}k)^{4-i}, \tag{16}$$

where $b_i$ and $c_i$ are all undetermined functions of $x$ and $t$. Substituting Equation (16) into Equation (15) and comparing the coefficients of the same powers of $2\mathrm{i}k$ yields:

$$(2\mathrm{i}k)^0\colon \begin{pmatrix} q \\ r \end{pmatrix}_t = L\begin{pmatrix} -b_4 \\ c_4 \end{pmatrix} + \delta(t)\begin{pmatrix} -xq \\ xr \end{pmatrix} + \gamma(t)\begin{pmatrix} -q \\ r \end{pmatrix}, \tag{17}$$

$$(2\mathrm{i}k)^1\colon \begin{pmatrix} -b_4 \\ c_4 \end{pmatrix} = L\begin{pmatrix} -b_3 \\ c_3 \end{pmatrix} + \beta(t)\begin{pmatrix} -xq \\ xr \end{pmatrix}, \tag{18}$$

$$(2\mathrm{i}k)^2\colon \begin{pmatrix} -b_3 \\ c_3 \end{pmatrix} = L\begin{pmatrix} -b_2 \\ c_2 \end{pmatrix}, \tag{19}$$

$$(2\mathrm{i}k)^3\colon \begin{pmatrix} -b_2 \\ c_2 \end{pmatrix} = L\begin{pmatrix} -b_1 \\ c_1 \end{pmatrix} + \alpha\begin{pmatrix} -q \\ r \end{pmatrix}, \tag{20}$$

$$(2\mathrm{i}k)^4\colon \begin{pmatrix} -b_1 \\ c_1 \end{pmatrix} = 0. \tag{21}$$

Using Equations (18)–(21) we have:

$$\begin{pmatrix} -b_2 \\ c_2 \end{pmatrix} = L\begin{pmatrix} -b_1 \\ c_1 \end{pmatrix} + \alpha(t)\begin{pmatrix} -q \\ r \end{pmatrix} = \alpha(t)\begin{pmatrix} -q \\ r \end{pmatrix}, \tag{22}$$

$$\begin{pmatrix} -b_3 \\ c_3 \end{pmatrix} = L\begin{pmatrix} -b_2 \\ c_2 \end{pmatrix} = \alpha(t)L\begin{pmatrix} -q \\ r \end{pmatrix}, \tag{23}$$

$$\begin{pmatrix} -b_4 \\ c_4 \end{pmatrix} = L\begin{pmatrix} -b_3 \\ c_3 \end{pmatrix} + \beta(t)\begin{pmatrix} -xq \\ xr \end{pmatrix} = \alpha(t)L^2\begin{pmatrix} -q \\ r \end{pmatrix} + \beta(t)\begin{pmatrix} -xq \\ xr \end{pmatrix}, \tag{24}$$

and then Equation (17) gives:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \alpha(t)L^3\begin{pmatrix} -q \\ r \end{pmatrix} + \beta(t)L\begin{pmatrix} -xq \\ xr \end{pmatrix} + \delta(t)\begin{pmatrix} -xq \\ xr \end{pmatrix} + \gamma(t)\begin{pmatrix} -q \\ r \end{pmatrix}. \tag{25}$$

Employing Equation (10), we easily find:

$$L\begin{pmatrix} -xq \\ xr \end{pmatrix} = \begin{pmatrix} q + xq_x \\ r + xr_x \end{pmatrix} + 2\begin{pmatrix} q \\ -r \end{pmatrix}\partial^{-1}(-rxq + qxr) = \begin{pmatrix} q + xq_x \\ r + xr_x \end{pmatrix}, \tag{26}$$

$$L^3\begin{pmatrix} -q \\ r \end{pmatrix} = L\begin{pmatrix} -q_{xx} + 2q^2r \\ r_{xx} - 2qr^2 \end{pmatrix} = \begin{pmatrix} q_{xxx} - 6qrq_x \\ r_{xxx} - 6qrr_x \end{pmatrix}, \tag{27}$$

and finally arrive at Equation (11) by means of Equations (25)–(27).

It should be noted that Equation (11) or Equation (25) cannot be included in the known mixed spectral AKNS hierarchy [7]:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L^{2n+1}\begin{pmatrix} -xq \\ xr \end{pmatrix} + L^{2n}\begin{pmatrix} -q \\ r \end{pmatrix}, \ (n = 0, 1, 2, \cdots). \tag{28}$$

In fact, Equation (25) contains one sum of two nonisospectral terms:

$$L\begin{pmatrix} -xq \\ xr \end{pmatrix} = \begin{pmatrix} q + xq_x \\ r + xr_x \end{pmatrix}, \begin{pmatrix} -xq \\ xr \end{pmatrix}, \tag{29}$$

which cannot occur simultaneously in Equation (28). Similarly, Equation (28) cannot contain the other sum of two isospectral terms:

$$L^3\begin{pmatrix} -q \\ r \end{pmatrix} = \begin{pmatrix} q_{xxx} - 6qrq_x \\ r_{xxx} - 6qrr_x \end{pmatrix}, \begin{pmatrix} -q \\ r \end{pmatrix}. \tag{30}$$

In addition, all the four time-varying coefficient functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\delta(t)$ are absent in Equation (28).

## 3. Special Cases of Equation (11)

Proper selections of $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\delta(t)$ can give some special cases of Equation (11), including the known equations.

**Special case 1.** Constant-coefficient mixed spectral AKNS equations under the case of $\alpha(t) = \beta(t) = \gamma(t) = \delta(t) = 1$:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} q_{xxx} - 6qrq_x + q + xq_x - xq - q \\ r_{xxx} - 6qrr_x + r + xr_x - xr - r \end{pmatrix}, \tag{31}$$

associated with:

$$\mathrm{i}\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{1}{2} + \mathrm{i}k, \tag{32}$$

$$A = \partial^{-1}(r, q)\begin{pmatrix} -B \\ C \end{pmatrix} - \frac{1}{2}(1 + 2\mathrm{i}k)x - \frac{1}{2}(2\mathrm{i}k)^3 - \frac{1}{2}, \tag{33}$$

$$\begin{pmatrix} -B \\ C \end{pmatrix} = L^2\begin{pmatrix} -q \\ r \end{pmatrix} + \begin{pmatrix} -xq \\ xr \end{pmatrix} + 2\mathrm{i}kL\begin{pmatrix} -q \\ r \end{pmatrix} + (2ik)^2\begin{pmatrix} -q \\ r \end{pmatrix}. \tag{34}$$

**Special case 2.** Constant-coefficient isospectral AKNS equations [5] under the case of $\alpha(t) = 1$ and $\beta(t) = \gamma(t) = \delta(t) = 0$:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} q_{xxx} - 6qrq_x \\ r_{xxx} - 6qrr_x \end{pmatrix}, \tag{35}$$

associated with:

$$\mathrm{i}\frac{\mathrm{d}k}{\mathrm{d}t} = 0, \tag{36}$$

$$A = \partial^{-1}(r,q)\begin{pmatrix}-B\\C\end{pmatrix} - \frac{1}{2}(2\mathrm{i}k)^3,\ \begin{pmatrix}-B\\C\end{pmatrix} = L^2\begin{pmatrix}-q\\r\end{pmatrix} + 2\mathrm{i}kL\begin{pmatrix}-q\\r\end{pmatrix} + (2ik)^2\begin{pmatrix}-q\\r\end{pmatrix}. \quad (37)$$

**Special case 3.** Constant-coefficient nonisospectral AKNS equations [5] under the case of $\alpha(t) = \gamma(t) = \delta(t) = 0$ and $\beta(t) = 1$:

$$\begin{pmatrix}q\\r\end{pmatrix}_t = \begin{pmatrix}q + xq_x\\r + xr_x\end{pmatrix}, \quad (38)$$

associated with:

$$\mathrm{i}\frac{\mathrm{d}k}{\mathrm{d}t} = \mathrm{i}k, \quad (39)$$

$$A = \partial^{-1}(r,q)\begin{pmatrix}-B\\C\end{pmatrix} - \mathrm{i}kx,\ \begin{pmatrix}-B\\C\end{pmatrix} = \begin{pmatrix}-xq\\xr\end{pmatrix}. \quad (40)$$

**Special case 4.** Variable-coefficient mixed spectral KdV equation under the case of $q = 1$ and $r = -u$:

$$u_t = \alpha(t)u_{xxx} + 6\alpha(t)uu_x + \beta(t)u + \beta(t)xu_x - \delta(t)xu - \gamma(t)u, \quad (41)$$

associated with Equation (7) and:

$$A = \partial^{-1}(-u,1)\begin{pmatrix}-B\\C\end{pmatrix} - \frac{1}{2}[\delta(t) + 2\mathrm{i}k\beta(t)]x - \frac{1}{2}\alpha(t)(2\mathrm{i}k)^3 - \frac{1}{2}\gamma(t), \quad (42)$$

$$\begin{pmatrix}-B\\C\end{pmatrix} = \alpha(t)L^2\begin{pmatrix}-1\\-u\end{pmatrix} + \beta(t)\begin{pmatrix}-x\\-xu\end{pmatrix} + \alpha(t)2\mathrm{i}kL\begin{pmatrix}-1\\-u\end{pmatrix} + \alpha(t)(2\mathrm{i}k)^2\begin{pmatrix}-1\\-u\end{pmatrix}. \quad (43)$$

**Special case 5.** Constant-coefficient isospectral KdV equation [5] under the case of $q = 1$ and $r = -u$, $\alpha(t) = 1$ and $\beta(t) = \gamma(t) = \delta(t) = 0$:

$$u_t = u_{xxx} + 6uu_x, \quad (44)$$

associated with Equation (36) and:

$$A = \partial^{-1}(-u,1)\begin{pmatrix}-B\\C\end{pmatrix} - \frac{1}{2}(2\mathrm{i}k)^3, \quad (45)$$

$$\begin{pmatrix}-B\\C\end{pmatrix} = L^2\begin{pmatrix}-1\\-u\end{pmatrix} + 2\mathrm{i}kL\begin{pmatrix}-1\\-u\end{pmatrix} + (2\mathrm{i}k)^2\begin{pmatrix}-1\\-u\end{pmatrix}. \quad (46)$$

**Special case 6.** Constant-coefficient isospectral mKdV equation [5] under the case of $q = v$ and $r = \mp v$, $\alpha(t) = 1$ and $\beta(t) = \gamma(t) = \delta(t) = 0$:

$$v_t = v_{xxx} - 6v^2v_x, \quad (47)$$

associated with Equation (36) and:

$$A = \partial^{-1}(\mp v, v)\begin{pmatrix}-B\\C\end{pmatrix} - \frac{1}{2}(2\mathrm{i}k)^3, \quad (48)$$

$$\begin{pmatrix}-B\\C\end{pmatrix} = L^2\begin{pmatrix}-v\\\mp v\end{pmatrix} + 2\mathrm{i}kL\begin{pmatrix}-v\\\mp v\end{pmatrix} + (2\mathrm{i}k)^2\begin{pmatrix}-v\\\mp v\end{pmatrix}. \quad (49)$$

**Special case 7.** Constant-coefficient isospectral sine-Gordon equation [5] under the case of $q = u_x/2$ and $r = -u_x/2$, $\alpha(t) = 1$ and $\beta(t) = \gamma(t) = \delta(t) = 0$:

$$u_{xt} = \sin u, \tag{50}$$

associated with Equation (36) and:

$$A = \partial^{-1}\left(-\frac{1}{2}u_x, \frac{1}{2}u_x\right)\begin{pmatrix}-B\\ C\end{pmatrix} - \frac{1}{2}(2\mathrm{i}k)^3, \tag{51}$$

$$\begin{pmatrix}-B\\ C\end{pmatrix} = L^2\begin{pmatrix}-\frac{1}{2}u_x\\ -\frac{1}{2}u_x\end{pmatrix} + 2\mathrm{i}kL\begin{pmatrix}-\frac{1}{2}u_x\\ -\frac{1}{2}u_x\end{pmatrix} + (2\mathrm{i}k)^2\begin{pmatrix}-\frac{1}{2}u_x\\ -\frac{1}{2}u_x\end{pmatrix}. \tag{52}$$

**Special case 8.** Variable-coefficient nonisospectral mKdV equation [5] under the case of $q = v$ and $r = \mp v$, $\beta(t) = 1$ and $\alpha(t) = \gamma(t) = \delta(t) = 0$:

$$v_t = v + xv_x, \tag{53}$$

associated with Equations (39) and (40).

**Special case 9.** Variable-coefficient nonisospectral mKdV equation [5] under the case of $q = 1$ and $r = -u$, $\beta(t) = \gamma(t) = 1$ and $\alpha(t) = \delta(t) = 0$:

$$u_t = xu_x, \tag{54}$$

associated with Equations (39) and:

$$A = \partial^{-1}(-u, 1)\begin{pmatrix}-B\\ C\end{pmatrix} - \mathrm{i}kx - \frac{1}{2},\ \begin{pmatrix}-B\\ C\end{pmatrix} = \begin{pmatrix}-x\\ -xu\end{pmatrix}. \tag{55}$$

## 4. Implicit Solutions of Equation (11)

In what follows, we assume that the potentials $q$, $r$ and their derivatives of each order with respect to $x$ and $t$ are smooth functions, and when $|x| \to \infty$, they all tend to 0.

**Theorem 1.** *Let us assume that $q(x,t)$ and $r(x,t)$ evolve according to Equation (11). Then, the time-dependences of scattering data:*

$$\left\{\mathrm{Im}k = 0,\ R(t,k) = \frac{b(t,k)}{a(t,k)},\ \kappa_j(t),\ c_j(t),\ j = 1,2,\cdots,n\right\}, \tag{56}$$

$$\left\{\mathrm{Im}k = 0,\ \overline{R}(t,k) = \frac{\overline{b}(t,k)}{\overline{a}(t,k)},\ \overline{\kappa}_j(t),\ \overline{c}_j(t),\ j = 1,2,\cdots,\overline{n}\right\}, \tag{57}$$

*which correspond to the mixed spectral AKNS matrix problem:*

$$\phi_x = M\phi,\ M = \begin{pmatrix}-\lambda & q\\ r & \lambda\end{pmatrix},\ \phi = \begin{pmatrix}\phi_1(x,t)\\ \phi_2(x,t)\end{pmatrix},\ \mathrm{i}\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{1}{2}[\delta(t) + 2\mathrm{i}k\beta(t)]. \tag{58}$$

*are as follows:*

$$\kappa_j(t) = \mathrm{e}^{\int_0^t \beta(\tau)\mathrm{d}\tau}\left[\kappa_j(0) - \frac{\mathrm{i}}{2}\int_0^t \mathrm{e}^{\int_0^\tau \beta(w)\mathrm{d}w}\delta(\tau)\mathrm{d}\tau\right], \tag{59}$$

$$c_j(t) = c_j(0)\mathrm{e}^{\frac{1}{2}\int_0^t [\alpha(\tau)(2\mathrm{i}k_j(\tau))^3 + \beta(\tau) + \gamma(\tau)]\mathrm{d}\tau}, \tag{60}$$

$$a(t,k) = a(0,k),\ b(t,k) = b(0,k)\mathrm{e}^{\int_0^t [\alpha(\tau)(2\mathrm{i}\kappa_j(\tau))^3 + \gamma(\tau)]d\tau}, \tag{61}$$

$$\overline{\kappa}_j(t) = \mathrm{e}^{\int_0^t \beta(\tau)\mathrm{d}\tau}\left[\overline{\kappa}_j(0) - \frac{\mathrm{i}}{2}\int_0^t \mathrm{e}^{\int_0^\tau \beta(w)\mathrm{d}w}\delta(\tau)\mathrm{d}\tau\right], \tag{62}$$

$$\overline{c}_j(t) = \overline{c}_j(0)\mathrm{e}^{-\frac{1}{2}\int_0^t [\alpha(2\mathrm{i}\overline{k}_j(\tau))^3+\beta(\tau)+\gamma(\tau)]\mathrm{d}\tau}, \tag{63}$$

$$\overline{a}(t,k) = \overline{a}(0,k),\ \overline{b}(t,k) = \overline{b}(0,k)\mathrm{e}^{-\int_0^t [\alpha(\tau)(2\mathrm{i}\overline{\kappa}_j(\tau))^3+\gamma(\tau)]\mathrm{d}\tau}, \tag{64}$$

where $c_j(0)$, $\overline{c}_j(0)$, $R(0,k) = b(0,k)/a(0,k)$ and $\overline{R}(0,k) = \overline{b}(0,k)/\overline{a}(0,k)$ are the normalization factors and reflection coefficients when $q(x,0)$ and $r(x,0)$ are potentials of the mixed spectral AKNS matrix problem (59).

**Proof of Theorem 1.** Since $\phi(x,k)$ satisfies Equation (58), $P(x,k) = \phi_t(x,k) - N\phi(x,k)$ also satisfies Equation (58) and then can be expressed by a pair of linearly independent basic solutions $\phi(x,k)$ and $\widetilde{\phi}(x,k)$ [5] of Equation (58):

$$P(x,k) = \phi_t(x,k) - N\phi(x,k) = \zeta(t,k)\phi(x,k) + \tau(t,k)\widetilde{\phi}(x,k), \tag{65}$$

where $\zeta(t,k)$ and $\tau(t,k)$ are two undetermined functions.

Firstly, we start from the discrete spectrum $k = \kappa_j$ ($\mathrm{Im}\kappa_j > 0$). Because when $x \to +\infty$, $\phi(x,k)$ decreases exponentially while $\widetilde{\phi}(x,k)$ increases exponentially, $\tau(t,k) = 0$. In this case, Equation (65) becomes:

$$\phi_t(x,\kappa_j(t)) - N\phi(x,\kappa_j(t)) = \zeta(t,\kappa_j(t))\phi(x,\kappa_j(t)). \tag{66}$$

Using $(\phi_2(x,\kappa_j(t)), \phi_1(x,\kappa_j(t)))$ to multiply the left-hand side of Equation (66), we have:

$$[\phi_1(x,\kappa_j(t))\phi_2(x,\kappa_j(t))]_t - [C\phi_1^2(x,\kappa_j(t)) + B\phi_2^2(x,\kappa_j(t))] = 2\zeta(t,\kappa_j(t))\phi_1(x,\kappa_j(t))\phi_2(x,\kappa_j(t)). \tag{67}$$

Integrating Equation (67) with respect to $x$ from $-\infty$ to $+\infty$, and considering the assumption [5]:

$$2\int_{-\infty}^{\infty} c_j^2(t)\phi_1(x,\kappa_j(t))\phi_2(x,\kappa_j(t))\mathrm{d}x = 1 \tag{68}$$

between the normalization function $\phi(x,\kappa_j(t))$ and the normalization factor $c_j(t)$, we can find:

$$\zeta(t,\kappa_j(t)) = -c_j^2(t)\int_{-\infty}^{\infty}[C\phi_1^2(x,\kappa_j(t)) + B\phi_2^2(x,\kappa_j(t))]\mathrm{d}x, \tag{69}$$

which has the inner product form:

$$\zeta(t,\kappa_j(t)) = -c_j^2(t)((\phi_2^2(x,\kappa_j(t)), \phi_1^2(x,\kappa_j(t)))^T, (B,C)^T), \tag{70}$$

And then it has:

$$\zeta(t,\kappa_j(t)) = -c_j^2(t)((\phi_2^2(x,\kappa_j(t)), \phi_1^2(x,\kappa_j(t)))^T, (B,C)^T) = \frac{1}{2}\beta(t), \tag{71}$$

Here, the following results have been used:

$$\int_{-\infty}^{\infty}[q(x)\phi_2^2(x,\kappa_j(t)) + r(x)\phi_1^2(x,\kappa_j(t))]\mathrm{d}x = \int_{-\infty}^{\infty}[\phi_1(x,\kappa_j(t))\phi_2(x,\kappa_j(t))]_x\,\mathrm{d}x = 0, \tag{72}$$

$$\begin{pmatrix} B \\ C \end{pmatrix} = \beta(t)\begin{pmatrix} xq \\ xr \end{pmatrix} + \alpha(t)\sum_{l=2}^{4} L^{l-2}\begin{pmatrix} q \\ r \end{pmatrix}(2\mathrm{i}k)^{4-l}, \tag{73}$$

$$\left((\phi_2^2(x,\kappa_j(t)), \phi_1^2(x,\kappa_j(t)))^T, \begin{pmatrix} xq \\ xr \end{pmatrix}\right) = \int_{-\infty}^{\infty} x(\phi_1(x,\kappa_j(t))\phi_2(x,\kappa_j(t)))_x\mathrm{d}x = -\frac{1}{2c_j^2(t)}. \tag{74}$$

Thus, Equation (66) reads:

$$\phi_t(x,\kappa_j(t)) - N\phi(x,\kappa_j(t)) = \frac{1}{2}\beta(t)\phi(x,\kappa_j(t)). \tag{75}$$

Note the asymptotic properties when $x \to +\infty$:

$$N \to \begin{pmatrix} n_{11} & 0 \\ 0 & -n_{11} \end{pmatrix},\ n_{11} = \frac{1}{2}[\delta(t) + \beta(t)2\mathrm{i}\kappa_j(t)]x + \frac{1}{2}\alpha(t)(2\mathrm{i}\kappa_j(t))^3 + \frac{1}{2}\gamma(t), \tag{76}$$

$$\phi(x,\kappa_j(t)) \to c_j(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathrm{e}^{\mathrm{i}\kappa_j(t)x}, \tag{77}$$

$$\phi_t \to \frac{\mathrm{d}c_j(t)}{\mathrm{d}t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathrm{e}^{\mathrm{i}\kappa_j(t)x} + \mathrm{i}\frac{\mathrm{d}\kappa_j(t)}{\mathrm{d}t}c_j(t)x\begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathrm{e}^{\mathrm{i}\kappa_j(t)x}, \tag{78}$$

from Equation (75) we reach:

$$\frac{\mathrm{d}\kappa_j(t)}{\mathrm{d}t} = -\frac{\mathrm{i}}{2}[\delta(t) + 2\mathrm{i}\kappa_j(t)\beta(t)], \tag{79}$$

$$\frac{\mathrm{d}c_j(t)}{\mathrm{d}t} = \left\{\left[\frac{1}{2}\alpha(t)(2\mathrm{i}\kappa_j(t))^3 + \frac{1}{2}\gamma(t)\right] + \frac{1}{2}\beta(t)\right\}c_j(t). \tag{80}$$

Directly solving Equations (79) and (80) yields Equations (59) and (60). By a similar way, we also obtain:

$$\frac{\mathrm{d}\overline{\kappa}_j(t)}{\mathrm{d}t} = -\frac{\mathrm{i}}{2}[\delta(t) + 2\mathrm{i}\overline{\kappa}_j(t)\beta(t)], \tag{81}$$

$$\frac{\mathrm{d}\overline{c}_j(t)}{\mathrm{d}t} = -\left\{\left[\frac{1}{2}\alpha(t)(2\mathrm{i}\overline{\kappa}_j(t))^3 + \frac{1}{2}\gamma(t)\right] + \frac{1}{2}\beta(t)\right\}\overline{c}_j(t), \tag{82}$$

and hence reach Equations (62) and (63).

Secondly, we deal with the real continuous spectrum $k$. Taking a solution $\varphi(x,k)$ of Equation (58), then we can see that $Q(x,k) = \varphi_t(x,k) - N\varphi(x,k)$ solves Equation (58). Therefore, there are two linearly independent fundamental solutions $\varphi(x,k)$ and $\widetilde{\varphi}(x,k)$ of Equation (58), so that:

$$\varphi_t(x,k) - N\varphi(x,k) = \varpi(t,k)\varphi(x,k) + \theta(t,k)\overline{\varphi}(x,k), \tag{83}$$

where $\varpi(t,k)$ and $\theta(t,k)$ are two functions to be determined. Setting $x \to -\infty$ and using the asymptotic properties:

$$\varphi_t(x,k) \to -\mathrm{i}\frac{\mathrm{d}k}{\mathrm{d}t}x\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{e}^{-\mathrm{i}kx},\ \varphi(x,k) \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{e}^{-\mathrm{i}kx},\ \overline{\varphi}(x,k) \to \begin{pmatrix} 0 \\ -1 \end{pmatrix}\mathrm{e}^{\mathrm{i}kx}, \tag{84}$$

we have:

$$\mathrm{i}\frac{\mathrm{d}k(t)}{\mathrm{d}t} = \frac{1}{2}[\delta(t) + 2\mathrm{i}\beta(t)k(t)],\ \theta(t,k) = 0, \varpi(t,k) = \frac{1}{2}\alpha(t)(2\mathrm{i}k(t))^3 + \frac{1}{2}\gamma(t). \tag{85}$$

Substituting the Jost relationship:

$$\varphi(x,k) = a(t,k)\overline{\phi}(x,k) + b(t,k)\phi(x,k) \tag{86}$$

into Equation (83) and using asymptotic properties:

$$\phi(x,k) \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathrm{e}^{\mathrm{i}kx},\ \overline{\phi}(x,k) \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{e}^{-\mathrm{i}kx},\ (x \to +\infty), \tag{87}$$

we easily derive from Equation (83):

$$\frac{\mathrm{d}a(t,k)}{\mathrm{d}t}=0,\ \frac{\mathrm{d}b(t,k)}{\mathrm{d}t}=[\alpha(t)(2\mathrm{i}k(t))^3+\gamma(t)]b(t,k). \tag{88}$$

Similarly, we can also have:

$$\frac{\mathrm{d}\overline{a}(t,k)}{\mathrm{d}t}=0,\ \frac{\mathrm{d}\overline{b}(t,k)}{\mathrm{d}t}=-[\alpha(t)(2\mathrm{i}k(t))^3+\gamma(t)]b(t,k). \tag{89}$$

Solving Equations (88) and (89) arrives at Equations (61) and (64). The proof is completed. □

**Theorem 2.** *Based on the time-dependences of scattering data in Equations (56) and (57) corresponding to the mixed spectral AKNS matrix problem (58), the implicit solutions of Equation (11) can be expressed by:*

$$q=-2K_1(t,x,x), \tag{90}$$

$$r=\frac{K_{2x}(t,x,x)}{K_1(t,x,x)}, \tag{91}$$

*where* $K(t,x,y)=(K_1(t,x,y),K_2(t,x,y))^T$ *satisfies the Gel'fand-Levitan-Marchenko (GLM) integral equation:*

$$K(t,x,y)-\begin{pmatrix}1\\0\end{pmatrix}\overline{F}(t,x+y)+\begin{pmatrix}0\\1\end{pmatrix}\int_x^\infty F(t,z+x)\overline{F}(t,z+y)\mathrm{d}z+\int_x^\infty K(t,x,s)\int_x^\infty F(t,z+s)\overline{F}(t,z+y)\mathrm{d}z\mathrm{d}s=0 \tag{92}$$

*with:*

$$F(t,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}R(t,k)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k+\sum_{j=1}^{n}c_j^2(t)\mathrm{e}^{\mathrm{i}\kappa_j(t)x}, \tag{93}$$

$$\overline{F}(t,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{R}(t,k)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k-\sum_{j=1}^{\overline{n}}\overline{c}_j^2(t)\mathrm{e}^{-\mathrm{i}\overline{\kappa}_j(t)x}, \tag{94}$$

*where* $R(t,k)=b(t,k)/a(t,k)$, $\overline{R}(t,k)=\overline{b}(t,k)/\overline{a}(t,k)$, $\kappa_j(t)$, $\overline{\kappa}_j(t)$, $c_j(t)$ *and* $\overline{c}_j(t)$ *are determined by Equations (59)–(64).*

**Proof of Theorem 2.** Since the proof is similar to that in [5], we omit it here. However, it is worth noting that the scattering data in Equations (93) and (94) are different. The proof is finished. □

## 5. Reflectiveless Potential Solutions of Equation (11)

**Theorem 3.** *In the case of the reflection potentials* $R(t,k)=\overline{R}(t,k)=0$, *explicit solutions of Equation (11) can be formulated as follows:*

$$q=2\mathrm{tr}(W^{-1}(x,t)\overline{\Lambda}(x,t)\overline{\Lambda}^T(x,t)), \tag{95}$$

$$r=-\frac{\frac{\partial}{\partial x}\mathrm{tr}(W^{-1}(x,t)P(x,t)\frac{\partial}{\partial x}P^T(x,t))}{\mathrm{tr}(W^{-1}(x,t)\overline{\Lambda}(x,t)\overline{\Lambda}^T(x,t))}, \tag{96}$$

*with:*

$$W(x,t)=I+P(x,t)P^T(x,t),\ P(x,t)=\left(\frac{c_j(t)\overline{c}_j(t)}{\kappa_j(t)-\overline{\kappa}_j(t)}\mathrm{e}^{\mathrm{i}(\kappa_j(t)-\overline{\kappa}_j(t))x}\right)_{\overline{n}\times n}, \tag{97}$$

$$\overline{\Lambda}=(\overline{c}_1(t)\mathrm{e}^{-\mathrm{i}\overline{\kappa}_1(t)x},\overline{c}_2(t)\mathrm{e}^{-\mathrm{i}\overline{\kappa}_2(t)x},\cdots,\overline{c}_n(t)\mathrm{e}^{-\mathrm{i}\overline{\kappa}_n(t)x})^T, \tag{98}$$

*where $tr(\cdot)$ represents the trace of matrix, $I$ is the $\overline{n} \times \overline{n}$ identity matrix, while $\kappa_j(t)$, $\overline{\kappa}_j(t)$, $c_j(t)$ and $\overline{c}_j(t)$ are determined by Equations (59), (60), (62) and (63).*

**Proof of Theorem 3.** We use $K(t,x,y) = (K_1(t,x,y), K_2(t,x,y))^T$ to rewrite Equation (92) as:

$$K_1(t,x,y) - \overline{F}_d(t,x+y) + \int_x^\infty K_1(t,x,s)\int_x^\infty F_d(t,z+s)\overline{F}_d(t,z+y)\mathrm{d}z\mathrm{d}s = 0, \tag{99}$$

$$K_2(t,x,y) + \int_x^\infty F_d(t,z+x)\overline{F}_d(t,z+y)\mathrm{d}z + \int_x^\infty K_2(t,x,s)\int_x^\infty F_d(t,z+s)\overline{F}_d(t,z+y)\mathrm{d}z\mathrm{d}s = 0. \tag{100}$$

Considering $R(t,k) = \overline{R}(t,k) = 0$, we simplify Equations (93) and (94) as:

$$\int_x^\infty F_d(t,s+z)\overline{F}_d(t,z+y)\mathrm{d}z = -\sum_{j=1}^{n}\sum_{m=1}^{\overline{n}} \frac{\mathrm{i}c_j^2(t)\overline{c}_m^2(t)}{k_j - \overline{k}_m}\mathrm{e}^{\mathrm{i}k_j(x+s)-\mathrm{i}\overline{k}_m(x+y)}. \tag{101}$$

We suppose that:

$$K_1(x,y,t) = \sum_{p=1}^{\overline{n}} \overline{c}_p(t)g_p(x,t)\mathrm{e}^{-\mathrm{i}\overline{k}_p y}, \tag{102}$$

$$K_2(x,y,t) = \sum_{p=1}^{\overline{n}} \overline{c}_p(t)h_p(x,t)\mathrm{e}^{-\mathrm{i}\overline{k}_p y}, \tag{103}$$

and substitute them into Equations (99) and (100), then the following equations are derived for $m = 1, 2, \cdots, \overline{n}$:

$$g_m(x,t) + \overline{c}_m(t)\mathrm{e}^{-\mathrm{i}\overline{k}_m x} + \sum_{j=1}^{n}\sum_{p=1}^{\overline{n}} \frac{c_j^2(t)\overline{c}_m(t)\overline{c}_p(t)}{(k_j - \overline{k}_m)(k_j - \overline{k}_p)}\mathrm{e}^{\mathrm{i}(2k_j - \overline{k}_m - \overline{k}_p)x}g_p(x,t) = 0, \tag{104}$$

$$h_m(x,t) - \sum_{j=1}^{n} \frac{1}{(k_j - \overline{k}_m)}c_j^2(t)\overline{c}_m(t)\mathrm{e}^{\mathrm{i}(2k_j - \overline{k}_m)x} + \sum_{j=1}^{n}\sum_{p=1}^{\overline{n}} \frac{c_j^2(t)\overline{c}_m(t)\overline{c}_p(t)}{(k_j - \overline{k}_m)(k_j - \overline{k}_p)}\mathrm{e}^{\mathrm{i}(2k_j - \overline{k}_m - \overline{k}_p)x}h_p(x,t) = 0. \tag{105}$$

Using the notations:

$$g(x,t) = (g_1(x,t), g_2(x,t), \cdots, g_{\overline{n}}(x,t))^T, \tag{106}$$

$$h(x,t) = (h_1(x,t), h_2(x,t), \cdots, h_{\overline{n}}(x,t))^T, \tag{107}$$

$$\Lambda = (c_1(t)\mathrm{e}^{-\mathrm{i}\kappa_1(t)x}, c_2(t)\mathrm{e}^{-\mathrm{i}\kappa_2(t)x}, \cdots, c_n(t)\mathrm{e}^{-\mathrm{i}\kappa_n(t)x})^T, \tag{108}$$

we can rewrite Equations (104) and (105) as:

$$W(x,t)g(x,t) = -\overline{\Lambda}(x,t), \tag{109}$$

$$W(x,t)h(x,t) = \mathrm{i}P(x,t)\Lambda(x,t). \tag{110}$$

When $W^{-1}(x,t)$ exists, Equations (109) and (110) give:

$$g(x,t) = -W^{-1}(x,t)\overline{\Lambda}(x,t), \tag{111}$$

$$h(x,t) = \mathrm{i}W^{-1}(x,t)P(x,t)\Lambda(x,t). \tag{112}$$

Substituting Equations (111) and (112) into Equations (102) and (103), we have:

$$K_1(x,y,t) = -\mathrm{tr}(W^{-1}(x,t)\overline{\Lambda}(x,t)\overline{\Lambda}^T(y,t)), \tag{113}$$

$$K_2(x,y,t) = \mathrm{i}\mathrm{tr}(W^{-1}(x,t)E(x,t)\Lambda(x,t)\overline{\Lambda}^T(y,t)). \tag{114}$$

We finally obtain Equations (95) and (96) by the substitution of Equations (113) and (114) into Equations (90) and (91). The proof is finished. □

As two special cases of Equations (95) and (96), we first consider $n = \overline{n} = 1$, then Equations (95) and (96) become:

$$q = \frac{2\bar{c}_1^2(t)\mathrm{e}^{-2\mathrm{i}\bar{\kappa}_1(t)x}}{1+\frac{c_1^2(t)\bar{c}_1^2(t)}{(\kappa_1(t)-\bar{\kappa}_1(t))^2}\mathrm{e}^{2\mathrm{i}(\kappa_1(t)-\bar{\kappa}_1(t))x}}, \tag{115}$$

$$r = \frac{2c_1^2(t)\mathrm{e}^{2\mathrm{i}\kappa_1 x}}{1+\frac{c_1^2(t)\bar{c}_1^2(t)}{(\kappa_1(t)-\bar{\kappa}_1(t))^2}\mathrm{e}^{2\mathrm{i}(\kappa_1(t)-\bar{\kappa}_1(t))x}}, \tag{116}$$

where:

$$\kappa_1(t) = \mathrm{e}^{\int_0^t \beta(\tau)\mathrm{d}\tau}\left[\kappa_1(0) - \frac{\mathrm{i}}{2}\int_0^t \mathrm{e}^{\int_0^\tau \beta(w)\mathrm{d}w}\delta(\tau)\mathrm{d}\tau\right], \tag{117}$$

$$c_1^2(t) = c_1^2(0)\mathrm{e}^{\int_0^t [\alpha(t)(2\mathrm{i}k_j(\tau))^3+\beta(\tau)+\gamma(\tau)]\mathrm{d}t}, \tag{118}$$

$$\overline{\kappa}_1(t) = \mathrm{e}^{\int_0^t \beta(\tau)\mathrm{d}\tau}\left[\overline{\kappa}_1(0) - \frac{\mathrm{i}}{2}\int_0^t \mathrm{e}^{\int_0^\tau \beta(w)\mathrm{d}w}\delta(\tau)\mathrm{d}\tau\right], \tag{119}$$

$$\bar{c}_1^2(t) = \bar{c}_1^2(0)\mathrm{e}^{-\int_0^t [\alpha(t)(2\mathrm{i}k_j(\tau))^3+\beta(\tau)+\gamma(\tau)]\mathrm{d}t}. \tag{120}$$

Selecting $\kappa_1(0) = 0.5$, $\alpha(t) = t-1$, $\beta(t) = t$, $\gamma(t) = t^2+1$ and $\delta(t) = i$, from Equation (109) we have:

$$\kappa_1(t) = \mathrm{e}^{\frac{t^2}{2}}\left[0.5 + \frac{1}{2}\sqrt{\frac{\pi}{2}}\mathrm{Erf}\left(\frac{t}{\sqrt{2}}\right)\right], \tag{121}$$

where $\mathrm{Erf}(\cdot)$ is the error function. We depict in Figure 1 the dynamical evolution of the spectrum $\kappa_1$. It can be seen from Figure 1 that the dynamical evolution of $\kappa_1$ presents nonlinear characteristics.

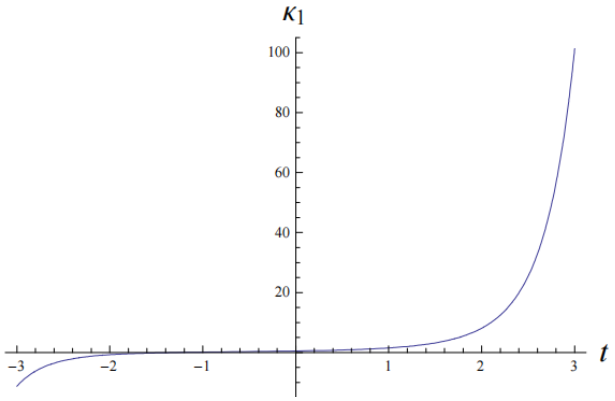



**Figure 1.** Nonlinear dynamical evolution of the spectrum $\kappa_1$ in Equation (121).

In Figures 2 and 3, the space–time dynamical evolutions of solutions (115) and (116) are shown by setting $\kappa_1(0) = 0.5$, $\overline{\kappa}_1(0) = 0.3$, $c_1(0) = 1$, $\bar{c}_1(0) = -2\times 10^{-15}$, $\alpha(t) = t-1$, $\beta(t) = t$, $\gamma(t) = t^2+1$ and $\delta(t) = i$. We can see from Figure 2 that the space–time dynamical evolution of solution (115) has the characteristics of a bell-shaped soliton. However, Figure 3 shows that the space–time dynamical evolution of solution (116) does not have the characteristics of a soliton, but its amplitude increases infinitely with the increase in time.

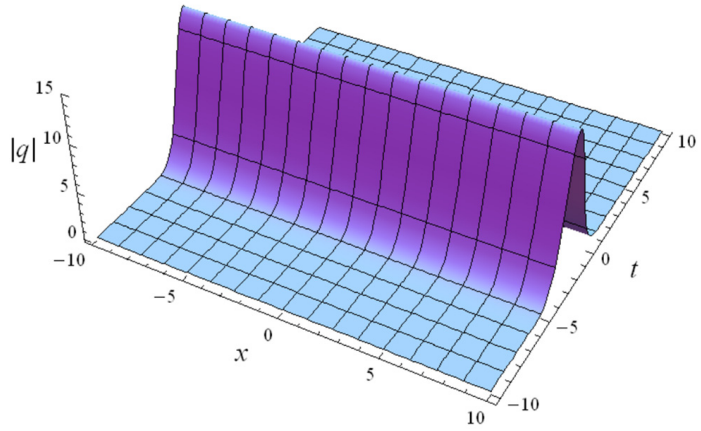



**Figure 2.** Space–time dynamical evolution of solution (115) with $\kappa_1(0) = 0.5$, $\overline{\kappa}_1(0) = 0.3$, $c_1(0) = 1$, $\bar{c}_1(0) = -2 \times 10^{-15}$, $\alpha(t) = t - 1$, $\beta(t) = t$, $\gamma(t) = t^2 + 1$ and $\delta(t) = i$.

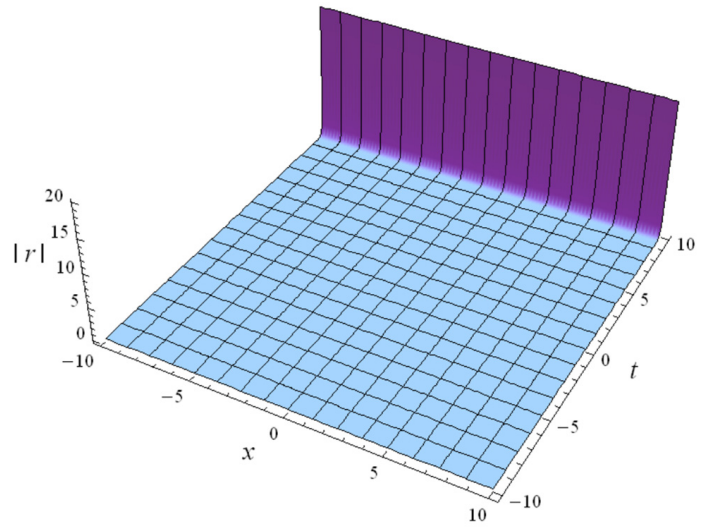



**Figure 3.** Space–time dynamical evolution of solution (116) with $\kappa_1(0) = 0.5$, $\overline{\kappa}_1(0) = 0.3$, $c_1(0) = 1$, $\bar{c}_1(0) = -2 \times 10^{-15}$, $\alpha(t) = t - 1$, $\beta(t) = t$, $\gamma(t) = t^2 + 1$ and $\delta(t) = i$.

For $n = \overline{n} = 2$, we select $\kappa_1(0) = 0.5$, $\kappa_2(0) = -1$, $\alpha = t - 1$, $\beta = 1$, $\gamma = t + 1$ and $\delta(t) = i$, then two cases of Equation (59) for $j = 1$ and $j = 2$ give:

$$\kappa_1(t) = \mathrm{e}^t \left[ 0.5 + \frac{1}{2}(1 - \cosh t + \sinh t) \right], \tag{122}$$

$$\kappa_2(t) = \mathrm{e}^t \left[ -1 + \frac{1}{2}(1 - \cosh t + \sinh t) \right]. \tag{123}$$

In Figures 4 and 5, we depict the dynamical evolution of the spectrum $\kappa_1$ in Equation (122) and $\kappa_2$ in Equation (123), respectively. It can be seen from Figures 4 and 5 that the dynamical evolution of $\kappa_1$ and $\kappa_2$ presents nonlinear characteristics.

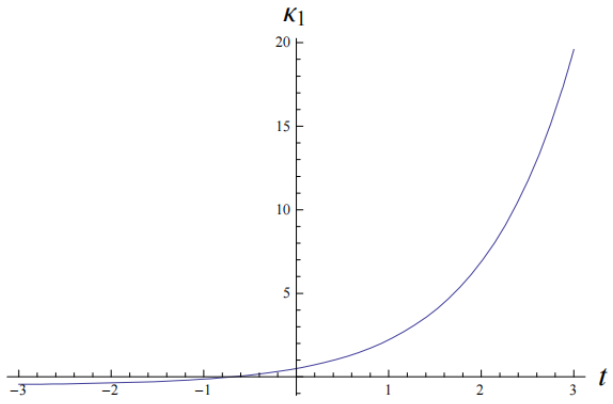



**Figure 4.** Nonlinear dynamical evolution of the spectrum $\kappa_1$ in Equation (122).

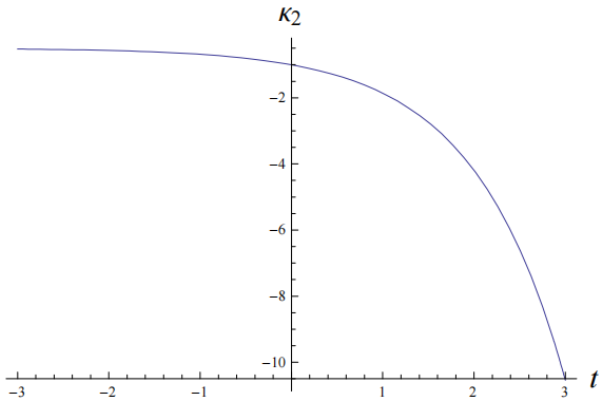



**Figure 5.** Nonlinear dynamical evolution of the spectrum $\kappa_2$ in Equation (123).

It can be seen from Figures 6 and 7 that the space–time dynamical evolution of solution determined by Equation (95) shows a multipoint feature. However, Figures 8 and 9 show that in addition to the multipoint feature of the space–time dynamical evolution of the solution determined by Equation (96), its amplitude also shows a feature of increase with time.

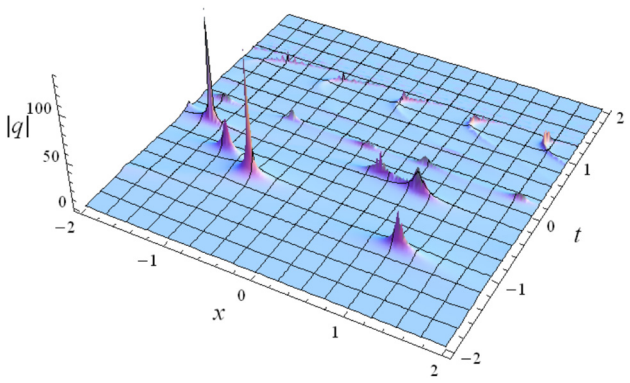



**Figure 6.** Space–time dynamical evolution of the solution determined by Equation (95) with $\kappa_1(0) = 0.5$, $\kappa_2(0) = -1$, $\alpha = t - 1$, $\beta = 1$, $\gamma = t + 1$ and $\delta(t) = i$.

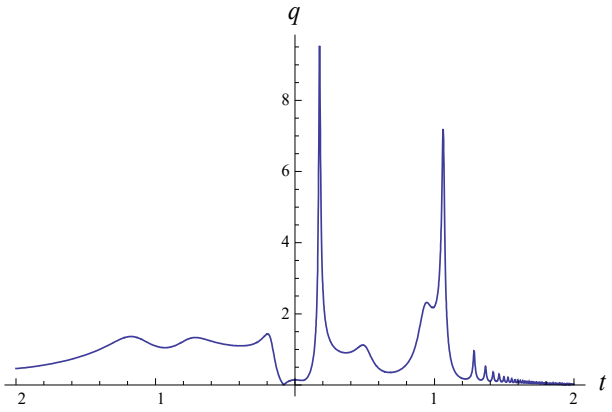



**Figure 7.** Profile at the position $x = 0$ of space–time of dynamical evolution of solution determined by Equation (95) with $\kappa_1(0) = 0.5$, $\kappa_2(0) = -1$, $\alpha = t - 1$, $\beta = 1$, $\gamma = t + 1$ and $\delta(t) = i$.

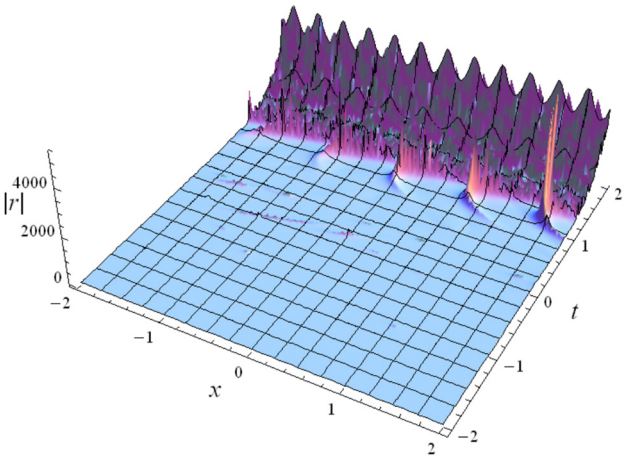



**Figure 8.** Space–time dynamical evolution of solution determined by solution (96) with $\kappa_1(0) = 0.5$, $\kappa_2(0) = -1$, $\alpha = t - 1$, $\beta = 1$, $\gamma = t + 1$ and $\delta(t) = i$.

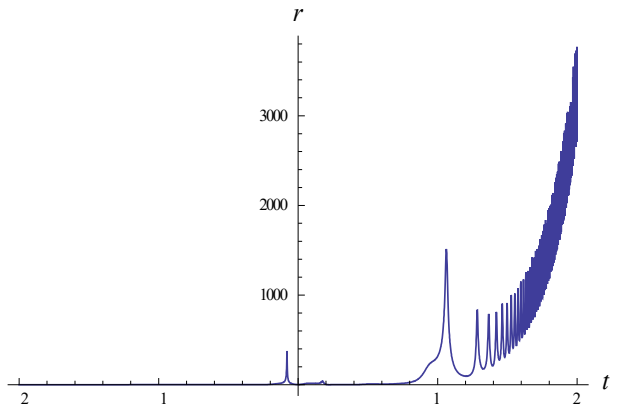



**Figure 9.** Profile at the position $x = 0$ of space–time of dynamical evolution of solution determined by Equation (96) with $\kappa_1(0) = 0.5$, $\kappa_2(0) = -1$, $\alpha = t - 1$, $\beta = 1$, $\gamma = t + 1$ and $\delta(t) = i$.

## 6. Conclusions

In short, we have derived the mixed spectral integrable Equation (11), time-dependences of scattering data (59)–(64), implicit solutions (90) and (91), and explicit reflectionless potential solutions (95) and (96). As far as we know, these obtained results are novel. Especially, the spectra with error function and hyperbolic functions in Equations (113)–(115) are new, by which the solutions (95) and (96) with $n = \overline{n} = 1$ and $n = \overline{n} = 2$ depicted in

Figures 2, 3 and 6–9 are obtained. Compared with the mixed spectral AKNS hierarchy [7] mentioned earlier in Equation (28) and the other results in [15–18], the work of this paper has some differences. Specifically, Equation (11) or its operator form (25) are different from the following equations [15–18]:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L^3 \begin{pmatrix} -q \\ r \end{pmatrix} + \sum_{n=0}^{5} L^n \begin{pmatrix} -xq \\ xr \end{pmatrix}, \tag{124}$$

associated with [15]:

$$\mathrm{i}\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{1}{2}\sum_{n=0}^{5} (2\mathrm{i}k)^5, \ A = \partial^{-1}(r,q)\begin{pmatrix} -B \\ C \end{pmatrix} - \frac{1}{2}(2\mathrm{i}k)^3 - \frac{1}{2}\left[\sum_{n=0}^{5} (2\mathrm{i}k)^n\right] x; \tag{125}$$

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L \begin{pmatrix} -q \\ r \end{pmatrix} + \sum_{n=0}^{3} L^n \begin{pmatrix} -xq \\ xr \end{pmatrix}, \tag{126}$$

associated with [16]:

$$\mathrm{i}\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{1}{2}\sum_{n=0}^{3} (2\mathrm{i}k)^n, \ A = \partial^{-1}(r,q)\begin{pmatrix} -B \\ C \end{pmatrix} - \frac{1}{2}(2\mathrm{i}k)^3 - \frac{1}{2}\left[\sum_{n=0}^{3} (2\mathrm{i}k)^n\right] x; \tag{127}$$

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = t \begin{pmatrix} -q \\ r \end{pmatrix} + \sum_{n=0}^{2} L^n \begin{pmatrix} -xq \\ xr \end{pmatrix}, \tag{128}$$

associated with [17]:

$$\mathrm{i}\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{1}{2}\sum_{n=0}^{2} (2\mathrm{i}k)^n, \ A = \partial^{-1}(r,q)\begin{pmatrix} -B \\ C \end{pmatrix} - \frac{1}{2}t - \frac{1}{2}\left[\sum_{n=0}^{2} (2\mathrm{i}k)^n\right] x; \tag{129}$$

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \alpha(t) L \begin{pmatrix} -q \\ r \end{pmatrix} + \sum_{n=0}^{2} \beta_n(t) L^n \begin{pmatrix} -xq \\ xr \end{pmatrix}, \tag{130}$$

associated with [18]:

$$\mathrm{i}\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{1}{2}\sum_{n=0}^{2} \beta_n(t)(2\mathrm{i}k)^n, \ A = \partial^{-1}(r,q)\begin{pmatrix} -B \\ C \end{pmatrix} - \mathrm{i}k - \frac{1}{2}\left[\sum_{n=0}^{2} \beta_n(t)(2\mathrm{i}k)^n\right] x. \tag{131}$$

The construction of meaningful integrable evolution equations based on the AKNS matrix problem (4) with some other different spectra and their exact solutions are worth studying. This paper gives the feasibility of constructing mixed spectral integrable evolution equations which are solvable in the framework of the inverse scattering method with time-varying spectrum. Therefore, we also conclude that Equation (11) constructed in this paper is also integrable in the sense of inverse scattering.

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