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Conic Duality for Multi-Objective Robust Optimization Problem

Khoirunnisa Rohadatul Aisy Muslihin *, Endang Rusyaman and Diah Chaerani

Department of Mathematics, Universitas Padjadjaran, Sumedang 45363, Indonesia
* Correspondence: khoirunnisa17002@mail.unpad.ac.id

Abstract: Duality theory is important in finding solutions to optimization problems. For example, in linear programming problems, the primal and dual problem pairs are closely related, i.e., if the optimal solution of one problem is known, then the optimal solution for the other problem can be obtained easily. In order for an optimization problem to be solved through the dual, the first step is to formulate its dual problem and analyze its characteristics. In this paper, we construct the dual model of an uncertain linear multi-objective optimization problem as well as its weak and strong duality criteria via conic duality. The multi-objective form of the problem is solved using the utility function method. In addition, the uncertainty is handled using robust optimization with ellipsoidal and polyhedral uncertainty sets. The robust counterpart formulation for the two uncertainty sets belongs to the conic optimization problem class; therefore, the dual problem can be built through conic duality. The results of the analysis show that the dual model obtained meets the weak duality, while the criteria for strong duality are identified based on the strict feasibility, boundedness, and solvability of the primal and dual problems.

Keywords: conic duality; robust optimization; multi-objective; weak duality; strong duality

MSC: 90C17; 90C25; 90C29; 90C46

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1. Introduction

In mathematical optimization, data uncertainty may be due to various factors, such as measurement/prediction errors or unknown future demands [1]. Robust optimization (RO) is a methodology that can be used to deal with problems affected by data uncertainty, where there is no probability distribution that satisfies the parameter uncertainty [2]. RO was first introduced by [3], which was later developed by [4]. This method assumes that the data uncertainty is in the uncertainty set, including the box, ellipsoidal, and polyhedral set. This uncertainty assumption can then be handled by formulating the problem into a robust counterpart (RC) [5].

The RC formulation for these uncertainty sets is included in the conic optimization problem class (linear and conic quadratic programming). Conic optimization (CO) is a class of convex optimization which is related to the minimization of linear functions at the intersection of the affine subspace and convex cone [6]. It is important to study CO because many practical non-linear problems can be treated as CO, and a wide class of CO problems can be solved using the interior point method [7].

The dual formulation of the CO problem is an important step for determining the optimal solution. Duality can be used to detect infeasibility and to check the accuracy of the primal solution [8]. Several studies that discuss optimality and duality are [9–11]. In the convex optimization problem, duality plays a role in determining the solution of primal problem to determine the lower bound of the solution [12,13]. The development of the dual problem in CO is an extension of the duality in linear programming (LP), where in conic duality, strict feasibility is required in order to establish strong duality [14]. In fact,

the linear optimization problem is a special case of the simplest CO problem, where the convex cone is a non-negative orthant $\mathbb{R}^m_+$.

The term "dual problem" usually refers to the Lagrangian dual problem. However, several studies also use other types of duality in constructing dual problems, such as the Wolfe duality discussed by [15] in [16] and the Fenchel dual problem discussed by [17]. In addition, Shapiro [18] discusses duality for the linear conic problem through conjugation duality. Another approach is Mond–Weir duality, which is a modification of the Wolfe-type duality that weakens the convexity requirement [19]. Furthermore, Komodakis and Pesquet in [20] discuss the application of the primal–dual approach in solving large scale optimization problems such as alternating direction method of multipliers (ADMM).

Research on duality for the RO problem has been carried out, including that of Beck and Ben-tal [21] which shows that the primal worst is equal to the dual best. Moreover, ref. [22] extends the study for the multi-objective case. The cone-convexity is used in this article to determine the primal worst and dual best in robust multi-objective optimization with the RC represented as a set-valued function. Other studies that discuss duality for multi-objective RO problems can be seen in [23–26]. In RO, conic duality also plays a role in constructing RC from nonlinear inequalities and turning them into computationally tractable constraint sets [27]. Research that uses dual principles in solving RO problems includes [28–30].

The main purpose of this paper is to construct the dual model for an uncertain multi-objective linear optimization problem in which multiple objectives must be solved simultaneously. Whereas a single-objective optimization problem may have a unique optimal solution, a multi-objective problem can have an uncountable set of solutions [31]. Some such multi-objective problems do not have a single solution that simultaneously optimizes each goal. The purpose of solving this problem is to find a representative Pareto optimal solution, i.e., a solution in which there is no function that can be increased in value without decreasing some other objective value [32].

In this paper, the multi-objective problem is handled using the weighted sum utility function method so that the optimization problem to be solved remains linear with the single objective function [33]. In addition to making problems easier to solve, this method ensures the achievement of the Pareto optimality by selecting non-negative weights [34]. To overcome the uncertainty that exists in the model, we use the RC formulation by assuming the uncertainty is in the coefficients of the objective function and the constraints and is contained in the ellipsoidal and polyhedral uncertainty sets. We use the recipe for building conic dual by [7,35] to construct the dual problem of the RC formulation. Along with this, we also address the weak and strong duality criteria from the primal and dual forms.

This paper is constructed as follows. In Section 2, we recall some basic theories, such as the convex cone, CO, RO, and utility function method. Section 3 provides the method used in the paper. We build the dual problem of the multi-objective RO problem through conic duality in Section 4. The weak and strong duality analysis is given in Section 5. Section 6 concludes the finding.

2. Supporting Theory

In this section, we provide the theories used, i.e., cone and convex cone, conic optimization and conic duality, robust optimization, and the utility function method. In this paper, we consider vectors to be column vectors, and they are written in bold and italic. In addition, the inequality $\boldsymbol{a} \geq \boldsymbol{b}$ for vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ is defined as

$$a_i \geq b_i, \text{ for } i = 1, \ldots, n.$$

Other descriptions are given throughout the paper.

2.1. Convex Cone

In this subsection, the theory regarding convex cone is given as an introduction to convex optimization and conic optimization. This subsection is sourced from [12,36].

Definition 1. *A set $\mathcal{K}$ is called a cone if for every $\boldsymbol{x} \in \mathcal{K}$ and $\theta \geq 0$ implies $\theta \boldsymbol{x} \in \mathcal{K}$. $\mathcal{K}$ is called convex cone if it is a cone and convex, i.e., for $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{K}$ and $\theta_1, \theta_2 \geq 0$ implies*

$$\theta_1 \boldsymbol{x}_1 + \theta_2 \boldsymbol{x}_2 \in \mathcal{K}. \tag{1}$$

Definition 2. *Let $\mathcal{K}$ be a cone. The dual cone of $\mathcal{K}$ is given by*

$$\mathcal{K}^* = \{\boldsymbol{y} \ : \ \boldsymbol{x}^T \boldsymbol{y} \geq 0, \forall \boldsymbol{x} \in \mathcal{K}\}. \tag{2}$$

Convex cone $\mathcal{K}$ is called pointed if it contains no lines, i.e., satisfies

$$\boldsymbol{a} \in \mathcal{K}, -\boldsymbol{a} \in \mathcal{K} \Rightarrow \boldsymbol{a} = \boldsymbol{0}. \tag{3}$$

$\mathcal{K}$ is closed if it satisfies

$$\boldsymbol{a}_i \in \mathcal{K} \ (i = 1, 2, \ldots), \ \boldsymbol{a} = \lim_{i \to \infty} \boldsymbol{a}_i \ \Rightarrow \boldsymbol{a} \in \mathcal{K}. \tag{4}$$

Moreover, $\mathcal{K}$ is solid if $\mathcal{K}$ has a nonempty interior. If cone $\mathcal{K}$ is pointed, closed and solid, then $\mathcal{K}$ is called a proper cone. For example, the linear cone is

$$\mathbb{R}^m_+ = \{\boldsymbol{x} \in \mathbb{R}^m \ : \ \boldsymbol{x} \geq \boldsymbol{0}\},$$

and the Lorentz cone is

$$\mathbf{L}^m = \left\{ \boldsymbol{x} \in \mathbb{R}^m \ : \ \left(\sum_{i=1}^{m-1} x_i^2 \right)^{\frac{1}{2}} \leq x_m \right\}.$$

Cone $\mathcal{K}$ is called self-dual if $\mathcal{K}^* = \mathcal{K}$. Furthermore, the dual cone of the direct product

$$\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_m,$$

is direct product of the dual cones $\mathcal{K}_i^*$, i.e.,

$$\mathcal{K}^* = \mathcal{K}_1^* \times \cdots \times \mathcal{K}_m^*. \tag{5}$$

2.2. Conic Optimization and Conic Duality

The conic optimization (CO) problem is part of the convex optimization in the form of minimizing a linear function with convex constraint functions contained in a convex cone [37]. The general form of the CO problem is given by

$$\min\{\boldsymbol{c}^T \boldsymbol{x} \ : \ A\boldsymbol{x} - \boldsymbol{b} \in \mathcal{K}\}, \tag{6}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the decision variable vector, $A \in \mathbb{R}^{m \times n}$, $\boldsymbol{c} \in \mathbb{R}^n$, $\mathcal{K} \subseteq \mathbb{R}^m$ is a proper cone, and $x \mapsto A\boldsymbol{x} - \boldsymbol{b}$ is an affine mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$.

The dual form of CO problem (6) is given by

$$\max\{\boldsymbol{b}^T \boldsymbol{y} \ : \ A^T \boldsymbol{y} = \boldsymbol{c}, \boldsymbol{y} \in \mathcal{K}^*\}, \tag{7}$$

where $\mathcal{K}^*$ is the dual cone of $\mathcal{K}$ [7]. Weak and strong duality theorems in conic duality are given in the following theorems [35].

Theorem 1. *Given the primal problem (6) and the dual (7), then $\boldsymbol{b}^T \boldsymbol{y} \leq \boldsymbol{c}^T \boldsymbol{x}$.*

Theorem 2. *Let $\boldsymbol{x}$ be a feasible solution of (6) and y be a feasible solution of (7), then*

1. *The duality is symmetric: the dual problem is conic, and the problem dual of the dual problem is equivalent to the primal problem.*

2. *Suppose that at least one of the two problems (6) and (7) is bounded and strictly feasible. Then a primal–dual feasible pair (x, y) is comprised of optimal solutions to the respective problems:*
   (a) *If and only if $\boldsymbol{b}^T\boldsymbol{y} = \boldsymbol{c}^T\boldsymbol{x}$ (zero duality gap);*
   (b) *If and only if $\boldsymbol{y}^T(A\boldsymbol{x} - \boldsymbol{b}) = 0$ (complementary slackness).*

In practice, a typical conic optimization problem can consist of several constraints with different convex cones as follows.

$$\min\{\boldsymbol{c}^T\boldsymbol{x} \ : \ A_i\boldsymbol{x} - \boldsymbol{b}_i \in \mathcal{K}_i, i = 1, \ldots, m\}, \tag{8}$$

where $\mathcal{K}_i$ for $i = 1, \ldots, m$ are different convex cones. Therefore, the dual problem of (8) can be obtained by combining all constraints into one conical constraint $A\boldsymbol{x} - \boldsymbol{b} \in \mathcal{K}$ assuming

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \vdots \\ \boldsymbol{b}_m \end{bmatrix}, \text{ and } \mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \ldots \mathcal{K}_m,$$

so that it can be transformed into a (6) conic problem. By writing the dual variable as

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ \vdots \\ \boldsymbol{y}_m \end{bmatrix}, \ \boldsymbol{y}_i \in \mathcal{K}_i^*, i = 1, 2, \ldots, m,$$

we obtain $A^T\boldsymbol{y} = A_1^T\boldsymbol{y}_1 + A_2^T\boldsymbol{y}_2 + \cdots + A_m^T\boldsymbol{y}_m$. Moreover, due to the dual property of the direct multiplication of convex cone, we obtain the dual cone of $\mathcal{K}$ as

$$\mathcal{K}^* = \mathcal{K}_1^* \times \mathcal{K}_2^* \times \ldots \mathcal{K}_m^*.$$

Hence, the dual problem of (8) is given by

$$\max\left\{\sum_{i=1}^{m} \boldsymbol{b}_i^T\boldsymbol{y}_i^T \ : \ \sum_{i=1}^{m} A_i^T\boldsymbol{y}_i = \boldsymbol{c}, \boldsymbol{y}_i \in \mathcal{K}_i^*, i = 1, 2, \ldots, m\right\}. \tag{9}$$

2.3. Robust Optimization

Robust optimization (RO) is a method for solving optimization problems with uncertain data and is only known in an uncertainty set [5]. The general model of the uncertain linear optimization problem is given as follows [38]:

$$\min\{\boldsymbol{c}^T\boldsymbol{x} \ : \ (\boldsymbol{c}, A, \boldsymbol{b}) \in \mathcal{U}\}, \tag{10}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the decision variable vector, $\boldsymbol{c} \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, and $\mathcal{U}$ is uncertainty set.

In RO, an uncertain linear optimization problem model can always be formed into a problem that only contains the uncertainty in the constraint functions. In the RO approach, uncertainty is eliminated by formulating the problem into a deterministic problem called the robust counterpart (RC). According to Ben-Tal and Nemirovski [5] we can analyze the computational tractability of the optimization model by representing RC into the linear programming (LP), conic quadratic programming (CQP), or semidefinite programming

(SDP) problem. In this paper, we use the box and ellipsoidal uncertainty sets. The RC reformulation for these uncertainty sets is given in Table 1.

Table 1. RC reformulation for ellipsoidal and polyhedral uncertainty sets.

Uncertainty Set	RC Formulation	Tractability	Description
Ellipsoidal	$\boldsymbol{a}^T\boldsymbol{x} + \left\|P^T\boldsymbol{x}\right\|_2 \leq \boldsymbol{b}$	CQP	$P \in \mathbb{R}^{n\times L}$
Polyhedral	$\boldsymbol{a}^T\boldsymbol{x} + \boldsymbol{q}^T\boldsymbol{y} \leq \boldsymbol{b}$ $D^T\boldsymbol{y} = -P\boldsymbol{x}$ $\boldsymbol{y} \geq \boldsymbol{0}$	LP	$D \in \mathbb{R}^{m\times L}$ $P \in \mathbb{R}^{n\times L}$ $\boldsymbol{q}, \boldsymbol{y} \in \mathbb{R}^m$

2.4. Utility Function Method

Multi-objective optimization is an optimization problem that has more than one objective function. In general, the multi-objective optimization problem is given by [32]:

$$\begin{aligned} &\min(f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \ldots, f_k(\boldsymbol{x})) \\ &\text{s.t. } g_j(\boldsymbol{x}) \leq 0,\ j = 1, \ldots, m. \end{aligned} \tag{11}$$

where f_i, for $i = 1, \ldots, l$, are the objective functions, g_j, for $j = 1, \ldots, m$, are the constraint functions, and $\boldsymbol{x}$ denotes the decision variable.

Multi-objective optimization problems can be solved using utility functions. In the field of economics, utility, which is represented by the utility function, is used to measure the level of satisfaction or benefits felt by an individual or group obtained when consuming goods or services [39]. In the case of multi-objective optimization, the utility function is a combination of the individual utility functions of each objective function and is a mathematical expression to model the decision maker's desires [33]. This method requires preference information before the problem can be solved. One of the simplest forms of utility functions is the weighted sum, which has the form

$$U = \sum_{i=1}^{l} \alpha_i f_i(\boldsymbol{x}),\ \alpha_i \geq 0, i = 1, \ldots, l. \tag{12}$$

where α_i is the scalar weighting, and $\sum_{i=1}^{l} \alpha_i = 1$. In the utility function method, the solution $\boldsymbol{x}^*$ obtained by maximizing the utility function U with constraints, i.e.,

$$\begin{aligned} &\max U \\ &\text{s.t. } g_j(\boldsymbol{x}) \leq 0; j = 1, \ldots, m. \end{aligned} \tag{13}$$

The value of $\boldsymbol{x}^*$ of (13) can also be obtained by changing the problem to a minimization problem; based on the transformation of the objective function, we obtain the following.

$$\begin{aligned} &\min\ (-U) \\ &\text{s.t. } g_j(\boldsymbol{x}) \leq 0; j = 1, \ldots, m. \end{aligned} \tag{14}$$

3. Method

The initial model used in this study is a linear multi-objective optimization model with finite constraints and non-negative decision variables. This form of multi-objective optimization is then handled by the utility function method so that it becomes an optimization problem with a single objective function. This method allows the problem to remain linear, and the optimal solution $\boldsymbol{x}^*$ can be obtained by considering all objective functions together.

The uncertainty is assumed to exist in the coefficients of the objective and constraint functions and is contained in the ellipsoidal and polyhedral sets of uncertainties. Furthermore, the RC formulation is formed for each form of the set of uncertainty. The dual

problem form is formulated from the RC formulation using conic duality by redefining the constraints into the conic form. Moreover, we also analyze weak and strong duality based on the primal and dual problems that have been obtained.

4. Results

4.1. Initial Model and Utility Function Method

In this paper, we use the following linear multi-objective optimization problem as the initial model:

$$\begin{aligned} &\min \left(\boldsymbol{c}_1^T \boldsymbol{x}, \boldsymbol{c}_2^T \boldsymbol{x}, \ldots, \boldsymbol{c}_k^T \boldsymbol{x} \right) \\ &\text{s.t. } A\boldsymbol{x} \leq \boldsymbol{b} \\ &\quad \boldsymbol{x} \geq \boldsymbol{0}, \end{aligned} \tag{15}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{c}_i \in \mathbb{R}^n, i = 1, \ldots, k$, $A \in \mathbb{R}^{m \times n}$, and $\boldsymbol{b} \in \mathbb{R}^m$. Furthermore, the utility function method is used to solve the multi-objective form of the (15). Using the weighted sum (12), we obtain the utility function

$$U = \sum_{i=1}^{k} \alpha_i \boldsymbol{c}_i^T \boldsymbol{x}, \tag{16}$$

where $\alpha_i \geq 0$ for $i = 1, \ldots, k$ and $\sum_{i=1}^{k} \alpha_i = 1$. The optimal solution $\boldsymbol{x}^*$ is found by maximizing the utility function (16) with constraint, i.e.,

$$\begin{aligned} &\max U \\ &\text{s.t. } A\boldsymbol{x} \leq \boldsymbol{b} \\ &\quad \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned} \tag{17}$$

The optimal solution can also be obtained by changing (17) to a minimization problem:

$$\begin{aligned} &\min (-U) \\ &\text{s.t. } A\boldsymbol{x} \leq \boldsymbol{b} \\ &\quad \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned} \tag{18}$$

Note that the objective function in (18) can be changed to

$$-U = -\sum_{i=1}^{k} \alpha_i \boldsymbol{c}_i^T \boldsymbol{x} = -\left[\sum_{i=1}^{k} \alpha_i \boldsymbol{c}_i \right]^T \boldsymbol{x}.$$

Therefore, suppose that $\boldsymbol{w} = \sum_{i=1}^{k} \alpha_i \boldsymbol{c}_i \in \mathbb{R}^n$, and model (18) can also be written as

$$\begin{aligned} &\min \boldsymbol{w}^T \boldsymbol{x} \\ &\text{s.t. } A\boldsymbol{x} \leq \boldsymbol{b} \\ &\quad \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned} \tag{19}$$

4.2. Robust Counterpart Formulation for Multi-Objective Linear Optimization Problem

In this study, we assume that the uncertainty lies in the coefficient of the objective function ($\boldsymbol{w}$) and the constraint functions (A). In this case, we formulate the RC for (19). The first step is to form a nominal model as follows.

We modify the model so that the objective function can be replaced by an additional variable $t \in \mathbb{R}$ with $t \geq \boldsymbol{w}^T \boldsymbol{x}$ to eliminate the uncertainty in the objective function,

$$\begin{aligned} &\min t \\ &\text{s.t. } \boldsymbol{w}^T\boldsymbol{x} - t \leq 0 \\ &\quad A\boldsymbol{x} \leq \boldsymbol{b} \\ &\quad \boldsymbol{x} \geq \mathbf{0},\ t \in \mathbb{R} \\ &\quad (\boldsymbol{w}, A) \in \mathcal{U}. \end{aligned} \tag{20}$$

In the (20) problem, t is unsigned. According to the standard form of linear programming, it must be converted into a non-negative variable, that is, by assuming t as $t = \delta_1 - \delta_2$ where $\delta_1, \delta_1 \in \mathbb{R}$ and $\delta_1, \delta_2 \geq 0$. Therefore, (20) becomes

$$\begin{aligned} &\min \delta_1 - \delta_2 \\ &\text{s.t. } \boldsymbol{w}^T\boldsymbol{x} - \delta_1 + \delta_2 \leq 0 \\ &\quad A\boldsymbol{x} \leq \boldsymbol{b} \\ &\quad \boldsymbol{x} \geq \mathbf{0} \\ &\quad \delta_1, \delta_2 \geq 0 \\ &\quad (\boldsymbol{w}, A) \in \mathcal{U}. \end{aligned} \tag{21}$$

The (21) problem is equivalent to the following nominal problem.

$$\begin{aligned} &\min \begin{bmatrix} \mathbf{0} \\ 1 \\ -1 \end{bmatrix}^T \begin{bmatrix} \boldsymbol{x} \\ \delta_1 \\ \delta_2 \end{bmatrix} \\ &\text{s.t. } \begin{bmatrix} \boldsymbol{w}^T & -1 & 1 \\ A & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \delta_1 \\ \delta_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ \boldsymbol{b} \end{bmatrix} \\ &\quad \begin{bmatrix} \boldsymbol{x} \\ \delta_1 \\ \delta_2 \end{bmatrix} \geq \mathbf{0} \\ &\quad (\boldsymbol{w}, A) \in \mathcal{U}. \end{aligned} \tag{22}$$

The uncertainty in (22) only contained in the constraint coefficient. Because of the constraint-wise robustness, the RC can be formulated for each constraint. First, we construct the RC formulation for the first constraint, namely

$$\boldsymbol{w}^T\boldsymbol{x} - \delta_1 + \delta_2 \leq 0. \tag{23}$$

Note that the assumption of uncertainty in $\boldsymbol{w}$ can be substituted for uncertainty in $\boldsymbol{c}_i$ for $i = 1, \ldots, k$. Suppose $\boldsymbol{c}_i \in \mathcal{U}_0^{(i)}$ for $i = 1, \ldots, k$ can be expressed as an affine function of the primitive uncertain parameter $\boldsymbol{\zeta}_i \in Z_0^{(i)} \subseteq \mathbb{R}^{M_i}$, i.e.,

$$\boldsymbol{c}_i = \bar{\boldsymbol{c}}_i + C_i\boldsymbol{\zeta}_i, \forall \boldsymbol{\zeta}_i \in Z_0^{(i)}, \tag{24}$$

where $C_i \in \mathbb{R}^{n \times M_i}$ and $Z_0^{(i)}$ be a primitive uncertainty set. Hence, we obtain

$$\begin{aligned} \boldsymbol{w} &= -\sum_{i=1}^{k} \alpha_i \boldsymbol{c}_i \\ &= -\sum_{i=1}^{k} \alpha_i (\bar{\boldsymbol{c}}_i + C_i \boldsymbol{\zeta}_i) \\ &= -\sum_{i=1}^{k} (\alpha_i \bar{\boldsymbol{c}}_i + \alpha_i C_i \boldsymbol{\zeta}_i) \\ &= -\sum_{i=1}^{k} \alpha_i \bar{\boldsymbol{c}}_i - \sum_{i=1}^{k} \alpha_i C_i \boldsymbol{\zeta}_i. \end{aligned} \tag{25}$$

Note that the second term on the right side of (25) can be written as

$$\begin{aligned} -\sum_{i=1}^{k} \alpha_i C_i \boldsymbol{\zeta}_i &= -\alpha_1 C_1 \boldsymbol{\zeta}_1 - \alpha_2 C_2 \boldsymbol{\zeta}_2 - \cdots - \alpha_k C_k \boldsymbol{\zeta}_k \\ &= \begin{bmatrix} -\alpha_1 C_1 & -\alpha_2 C_2 & \cdots & -\alpha_k C_k \end{bmatrix} \begin{bmatrix} \boldsymbol{\zeta}_1 \\ \boldsymbol{\zeta}_2 \\ \vdots \\ \boldsymbol{\zeta}_k \end{bmatrix}. \end{aligned}$$

Let

$$\bar{\boldsymbol{w}} = -\sum_{i=1}^{k} \alpha_i \bar{\boldsymbol{c}}_i,\ P_0 = \begin{bmatrix} -\alpha_1 C_1 & -\alpha_2 C_2 & \cdots & -\alpha_k C_k \end{bmatrix},\ \text{and } \boldsymbol{\zeta}_0 = \begin{bmatrix} \boldsymbol{\zeta}_1 \\ \boldsymbol{\zeta}_2 \\ \vdots \\ \boldsymbol{\zeta}_k \end{bmatrix}.$$

We have

$$\boldsymbol{w} = \bar{\boldsymbol{w}} + P_0 \boldsymbol{\zeta}_0 \in \mathcal{U}_0, \tag{26}$$

which is an affine function of $\boldsymbol{\zeta}_0 \in Z_0$ where $Z_0 = \left(Z_0^{(1)} \times Z_0^{(2)} \times \cdots \times Z_0^{(k)} \right)$ and $\mathcal{U}_0 = \{ \boldsymbol{w} \mid \boldsymbol{w} = \bar{\boldsymbol{w}} + P_0 \boldsymbol{\zeta}_0, \zeta_0 \in Z_0 \}$. By substituting (26) in constraint (23) we obtain

$$(\bar{\boldsymbol{w}} + P_0 \boldsymbol{\zeta}_0)^T \boldsymbol{x} - \delta_1 + \delta_2 \leq 0. \tag{27}$$

Furthermore, the A constraint matrix can be assumed as follows,

$$A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_m \end{bmatrix}^T,$$

where $\boldsymbol{a}_j \in \mathbb{R}^n$ and $b_j \in \mathbb{R}$ for $j = 1, \ldots, m$. The constraint $A\boldsymbol{x} \leq \boldsymbol{b}$ can be modified to

$$\boldsymbol{a}_j^T \boldsymbol{x} \leq b_j, j = 1, \ldots, m. \tag{28}$$

The uncertainty in A can be formulated as an uncertainty in $\boldsymbol{a}_j$ for $j = 1, \ldots, m$. Suppose that $\boldsymbol{a}_j \in \mathcal{U}_j$ is defined as an affine function over the primitive indefinite parameter $\boldsymbol{\zeta}_j$

$$\boldsymbol{a}_j = \bar{\boldsymbol{a}}_j + P_j \boldsymbol{\zeta}_j, \forall \boldsymbol{\zeta}_j \in Z_j, \tag{29}$$

where $\bar{\boldsymbol{a}}_j$ is nominal value vector, $P_j \in \mathbb{R}^{n \times L_j}$, and $Z_j \subseteq \mathbb{R}^{L_j}$. Substituting $\boldsymbol{a}_j$ in (28) with (29), we obtain

$$\left(\bar{\boldsymbol{a}}_j + P_j \boldsymbol{\zeta}_j \right)^T \boldsymbol{x} \leq b_j, \forall \boldsymbol{\zeta}_j \in Z_j, j = 1, \ldots, m. \tag{30}$$

The RC formulation for problem (19) with ellipsoidal and polyhedral uncertainty set is given by Table 2.

Table 2. RC formulation for problem (19).

Uncertainty Set	RC Formulation	Description
Ellipsoidal	$\bar{\boldsymbol{w}}^T\boldsymbol{x} + \left\|P_0^T\boldsymbol{x}\right\|_2 - \delta_1 + \delta_2 \leq 0$ $\bar{\boldsymbol{a}}_j^T\boldsymbol{x} + \left\|P_j^T\boldsymbol{x}\right\|_2 \leq b_j$	$P_0 \in \mathbb{R}^{n\times L_0}$ $P_j \in \mathbb{R}^{n\times L_j}$
Polyhedral	$\bar{\boldsymbol{w}}^T\boldsymbol{x} + \boldsymbol{q}_0^T\boldsymbol{y}_0 - \delta_1 + \delta_2 \leq 0$ $D_0^T\boldsymbol{y}_0 = P_0^T\boldsymbol{x}$ $\boldsymbol{y} \geq \boldsymbol{0}$ $\bar{\boldsymbol{a}}_j^T\boldsymbol{x} + \boldsymbol{q}_j^T\boldsymbol{y}_j \leq b_j$ $D_j^T\boldsymbol{y}_j = P_j^T\boldsymbol{x}$ $\boldsymbol{y}_j \geq \boldsymbol{0}$	$P_0 \in \mathbb{R}^{n\times L_0}$ $D_0 \in \mathbb{R}^{M_0\times L_0}$ $\boldsymbol{y}_0, \boldsymbol{q}_0 \in \mathbb{R}^{M_0}$ $P_j \in \mathbb{R}^{n\times L_j}$ $D_j \in \mathbb{R}^{M_j\times L_j}$ $\boldsymbol{y}_j, \boldsymbol{q}_j \in \mathbb{R}^{M_j}$

4.3. Dual Problems

In this section, the dual problem form of the RC formulation is given for the two cases of the set of uncertainties that have been obtained previously.

4.3.1. Dual Problem for Problem with Ellipsoidal Uncertainty Sets

We formulate the dual problem of (22) using the RC of constraints with the ellipsoidal set of uncertainties in Table 2, namely

$$\bar{\boldsymbol{w}}^T\boldsymbol{x} + \left\|P_0^T\boldsymbol{x}\right\|_2 - \delta_1 + \delta_2 \leq 0 \tag{31}$$

$$\bar{\boldsymbol{a}}_j^T\boldsymbol{x} + \left\|P_j^T\boldsymbol{x}\right\|_2 \leq b_j \tag{32}$$

(31) and (32) constraints can be converted to the form of a conic constraint

$$\begin{bmatrix} P_0^T & 0 & 0 \\ -\bar{\boldsymbol{w}}^T & 1 & -1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \delta_1 \\ \delta_2 \end{bmatrix} \in \mathbf{L}^{L_0+1}, \tag{33}$$

$$\begin{bmatrix} P_j^T & 0 & 0 \\ -\bar{\boldsymbol{a}}_j^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \delta_1 \\ \delta_2 \end{bmatrix} - \begin{bmatrix} \boldsymbol{0} \\ -b_j \end{bmatrix} \in \mathbf{L}^{L_j+1}. \tag{34}$$

Problem (22) can be changed to

$$\begin{aligned} \min \ & \begin{bmatrix} \boldsymbol{0} \\ 1 \\ -1 \end{bmatrix}^T \begin{bmatrix} \boldsymbol{x} \\ \delta_1 \\ \delta_2 \end{bmatrix} \\ \text{s.t.} \ & \begin{bmatrix} P_0^T & 0 & 0 \\ -\bar{\boldsymbol{w}}^T & 1 & -1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \delta_1 \\ \delta_2 \end{bmatrix} \in \mathbf{L}^{L_0+1} \\ & \begin{bmatrix} P_j^T & 0 & 0 \\ -\bar{\boldsymbol{a}}_j^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \delta_1 \\ \delta_2 \end{bmatrix} - \begin{bmatrix} \boldsymbol{0} \\ -b_j \end{bmatrix} \in \mathbf{L}^{L_j+1} \\ & \begin{bmatrix} \boldsymbol{x} \\ \delta_1 \\ \delta_2 \end{bmatrix} \geq \boldsymbol{0}. \end{aligned} \tag{35}$$

Let

$$z = \begin{bmatrix} \boldsymbol{x} \\ \delta_1 \\ \delta_2 \end{bmatrix}, \ g = \begin{bmatrix} \mathbf{0} \\ 1 \\ -1 \end{bmatrix}, \ H_0 = \begin{bmatrix} P_0^T & 0 & 0 \\ -\bar{\boldsymbol{w}}^T & 1 & -1 \end{bmatrix},$$

$$H_j = \begin{bmatrix} P_j^T & 0 & 0 \\ -\bar{\boldsymbol{a}}_j^T & 0 & 0 \end{bmatrix}, \ \boldsymbol{v}_j = \begin{bmatrix} \mathbf{0} \\ -b_j \end{bmatrix}, j = 1, \ldots, m,$$

and I be the identity matrix. In addition, define the convex cones

$$\mathcal{K}_0 = \mathbf{L}^{L_0+1}, \ \mathcal{K}_j = \mathbf{L}^{L_j+1}, j = 1, \ldots, m, \ \mathbb{R}^N = \mathbb{R}^{n+2}.$$

Model (35) can be simplified to

$$\begin{aligned} &\min \boldsymbol{g}^T \boldsymbol{z} \\ &\text{s.t. } H_0 \boldsymbol{z} \in \mathcal{K}_0 \\ &\quad H_j \boldsymbol{z} - \boldsymbol{v}_j \in \mathcal{K}_j, \ j = 1, \ldots, m \\ &\quad I \boldsymbol{z} \in \mathbb{R}_+^N. \end{aligned} \tag{36}$$

Based on the conic duality formulation, the dual model of (36) is given as follows:

$$\begin{aligned} &\max \sum_{j=1}^{m} \boldsymbol{v}_j^T \boldsymbol{\lambda}_j \\ &\text{s.t. } H_0^T \boldsymbol{\lambda}_0 + \sum_{j=1}^{m} H_j^T \boldsymbol{\lambda}_j + I \boldsymbol{\lambda}_{m+1} = \boldsymbol{g} \\ &\quad \boldsymbol{\lambda}_0 \in \mathcal{K}_0^* \\ &\quad \boldsymbol{\lambda}_j \in \mathcal{K}_j^*, j = 1, \ldots, m \\ &\quad \boldsymbol{\lambda}_{m+1} \in \left(\mathbb{R}_+^N\right)^*, \end{aligned} \tag{37}$$

with dual variable $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_m, \boldsymbol{\lambda}_{m+1})$.

Returning to the initial form and the fact that the Lorentz cone and linear cone are self-dual,

$$\begin{aligned} &\mathcal{K}_0^* = \left(\mathbf{L}^{L_0+1}\right)^* = \mathbf{L}^{L_0+1} \\ &\mathcal{K}_j^* = \left(\mathbf{L}^{L_j+1}\right)^* = \mathbf{L}^{L_j+1}, j = 1, \ldots, m, \\ &\left(\mathbb{R}_+^N\right)^* = \mathbb{R}_+^N = \mathbb{R}_+^{n+2}. \end{aligned}$$

Therefore, (37) can be changed to

$$\begin{aligned} &\max \sum_{j=1}^{m} \begin{bmatrix} \mathbf{0} \\ -b_j \end{bmatrix}^T \boldsymbol{\lambda}_j \\ &\text{s.t. } \begin{bmatrix} P_0^T & 0 & 0 \\ -\bar{\boldsymbol{w}}^T & 1 & -1 \end{bmatrix}^T \boldsymbol{\lambda}_0 + \sum_{j=1}^{m} \begin{bmatrix} P_j^T & 0 & 0 \\ -\bar{\boldsymbol{a}}_j^T & 0 & 0 \end{bmatrix}^T \boldsymbol{\lambda}_j + I \boldsymbol{\lambda}_{m+1} = \begin{bmatrix} \mathbf{0} \\ 1 \\ -1 \end{bmatrix} \\ &\quad \boldsymbol{\lambda}_0 \in \mathbf{L}^{L_0+1} \\ &\quad \boldsymbol{\lambda}_j \in \mathbf{L}^{L_j+1}, j = 1, \ldots, m \\ &\quad \boldsymbol{\lambda}_{m+1} \in \mathbb{R}_+^{n+2}. \end{aligned} \tag{38}$$

Furthermore, define

$$\boldsymbol{\lambda}_0 = \begin{bmatrix} \boldsymbol{\mu}_0 \\ \tau_0 \end{bmatrix} \in \mathbf{L}^{L_0+1} \text{ where } \boldsymbol{\mu}_0 \in \mathbb{R}^{L_0},\ \tau_0 \in \mathbb{R}, \text{ and } \|\boldsymbol{\mu}_0\|_2 \leq \tau_0,$$

$$\boldsymbol{\lambda}_j = \begin{bmatrix} \boldsymbol{\mu}_j \\ \tau_j \end{bmatrix} \in \mathbf{L}^{L_j+1} \text{ where } \boldsymbol{\mu}_j \in \mathbb{R}^{L_j},\ \tau_j \in \mathbb{R}, \text{ and } \|\boldsymbol{\mu}_j\|_2 \leq \tau_j, \text{ untuk } j = 1, \ldots, m,$$

$$\boldsymbol{\lambda}_{m+1} = \begin{bmatrix} \boldsymbol{s} \\ \sigma_1 \\ \sigma_2 \end{bmatrix} \in \mathbb{R}^{n+2}_{+} \text{ where } \boldsymbol{s} \geq \mathbf{0},\ \sigma_1 \geq 0, \text{ and } \sigma_2 \geq 0.$$

Problem (38) can be modified to

$$\begin{aligned} \max\ & \sum_{j=1}^{m} \begin{bmatrix} \mathbf{0}^T & -b_j \end{bmatrix} \begin{bmatrix} \boldsymbol{\mu}_j \\ \tau_j \end{bmatrix} \\ \text{s.t. } & \begin{bmatrix} P_0 & -\bar{\boldsymbol{w}} \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \boldsymbol{\mu}_0 \\ \tau_0 \end{bmatrix} + \sum_{j=1}^{m} \begin{bmatrix} P_j & -\bar{\boldsymbol{a}}_j \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\mu}_j \\ \tau_j \end{bmatrix} + \begin{bmatrix} \boldsymbol{s} \\ \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \\ -1 \end{bmatrix} \\ & \|\boldsymbol{\mu}_0\|_2 \leq \tau_0 \\ & \|\boldsymbol{\mu}_j\|_2 \leq \tau_j, j = 1, \ldots, m \\ & \boldsymbol{s} \geq \mathbf{0},\ \sigma_1 \geq 0,\ \sigma_2 \geq 0. \end{aligned} \tag{39}$$

By describing each form of the constraint and the objective function, (39) becomes

$$\begin{aligned} \max\ & \left(-\sum_{j=1}^{m} b_j \tau_j \right) \\ \text{s.t. } & P_0 \boldsymbol{\mu}_0 - \bar{\boldsymbol{w}} \tau_0 + \sum_{j=1}^{m} \left(P_j \boldsymbol{\mu}_j - \bar{\boldsymbol{a}}_j \tau_j \right) + \boldsymbol{s} = \mathbf{0} \\ & \tau_0 + \sigma_1 = 1 \\ & -\tau_0 + \sigma_2 = -1 \\ & \|\boldsymbol{\mu}_0\|_2 \leq \tau_0 \\ & \|\boldsymbol{\mu}_j\|_2 \leq \tau_j, j = 1, \ldots, m \\ & \boldsymbol{s} \geq \mathbf{0},\ \sigma_1 \geq 0,\ \sigma_2 \geq 0. \end{aligned} \tag{40}$$

4.3.2. Dual Problem for Problem with Polyhedral Uncertainty Sets

The dual problem formulation is built from the (22) problem with RC constraints, i.e.,

$$\begin{aligned} \min\ & \delta_1 - \delta_2 \\ \text{s.t. } & \bar{\boldsymbol{w}}^T \boldsymbol{x} + \boldsymbol{q}_0^T \boldsymbol{y}_0 - \delta_1 + \delta_2 \leq 0, \\ & D_0^T \boldsymbol{y}_0 = P_0^T \boldsymbol{x}, \\ & \bar{\boldsymbol{a}}_j^T \boldsymbol{x} + \boldsymbol{q}_j^T \boldsymbol{y}_j \leq b_j, \\ & D_j^T \boldsymbol{y}_j = P_j^T \boldsymbol{x}, \\ & \boldsymbol{x} \geq \mathbf{0},\ \delta_1 \geq 0,\ \delta_2 \geq 0, \\ & \boldsymbol{y}_0 \geq 0,\ \boldsymbol{y}_j \geq 0 \\ & j = 1, \ldots, m. \end{aligned} \tag{41}$$

The first step is to convert the above problem into a CO problem. Define the decision variable

$$\boldsymbol{z} = \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y}_0 & \cdots & \boldsymbol{y}_m & \delta_1 & \delta_2 \end{bmatrix}^T \in \mathbb{R}^N, \text{ where } N = n + \sum_{j=0}^{m} L_j + 2.$$

The objective function of (41) can be changed to

$$\delta_1 - \delta_2 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ 1 \\ -1 \end{bmatrix}^T \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_0 \\ \vdots \\ \boldsymbol{y}_m \\ \delta_1 \\ \delta_2 \end{bmatrix} = \boldsymbol{g}^T \boldsymbol{z}, \text{ where } \boldsymbol{g} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ 1 \\ -1 \end{bmatrix}. \tag{42}$$

Then the first constraint on (41) becomes

$$-\bar{\boldsymbol{w}}^T \boldsymbol{x} - \boldsymbol{q}_0^T \boldsymbol{y}_0 + \delta_1 - \delta_2 \geq 0 \tag{43}$$

$$\begin{bmatrix} -\bar{\boldsymbol{w}}^T & -\boldsymbol{q}_0 & \mathbf{0}^T & \cdots & \mathbf{0}^T & 1 & -1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_0 \\ \boldsymbol{y}_1 \\ \vdots \\ \boldsymbol{y}_m \\ \delta_1 \\ \delta_2 \end{bmatrix} \geq 0 \tag{44}$$

$$\boldsymbol{r}_0^T \boldsymbol{z} \in \mathbb{R}_+. \tag{45}$$

where $\boldsymbol{r}_0 = \begin{bmatrix} -\bar{\boldsymbol{w}}^T & -\boldsymbol{q}_0 & \mathbf{0}^T & \cdots & \mathbf{0}^T & 1 & -1 \end{bmatrix}^T$. Furthermore, the second constraint from (41) becomes

$$P_0^T \boldsymbol{x} - D_0^T \boldsymbol{y}_0 = 0 \tag{46}$$

$$\begin{bmatrix} P_0^T & -D_0^T & O_{L_1,M_1} & \cdots & O_{L_m,M_m} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_0 \\ \boldsymbol{y}_1 \\ \vdots \\ \boldsymbol{y}_m \\ \delta_1 \\ \delta_2 \end{bmatrix} = 0 \tag{47}$$

$$Q_0 \boldsymbol{z} = 0 \tag{48}$$

where $Q_0 = \begin{bmatrix} P_0^T & -D_0^T & O_{L_1,M_1} & \cdots & O_{L_m,M_m} & \mathbf{0} & \mathbf{0} \end{bmatrix}$ and O_{L_j,M_j} is $L_j \times M_j$ zero matrix. Equation (48) can be converted into the form of a conic constraint

$$\begin{bmatrix} Q_j \\ \mathbf{0}^T \end{bmatrix} \boldsymbol{z} \in \mathbf{L}^{L_0+1} \tag{49}$$

Combining constraints (45) and (49), we obtain

$$\begin{bmatrix} Q_0 \\ \mathbf{0}^T \\ \boldsymbol{r}_0^T \end{bmatrix} \boldsymbol{z} \in \mathbf{L}^{L_0+1} \times \mathbb{R}_+. \tag{50}$$

Furthermore, the third and fourth constraints of (41) can be changed to

$$-\bar{\boldsymbol{a}}_j^T \boldsymbol{x} - \boldsymbol{q}_j^T \boldsymbol{y}_j + b_j \geq 0 \tag{51}$$

$$P_j^T \boldsymbol{x} - D_j^T \boldsymbol{y}_j = 0, \tag{52}$$

for $j = 1, \ldots, m$. By defining the $\boldsymbol{r}_j$ given by

$$\begin{aligned}
\boldsymbol{r}_1 &= \begin{bmatrix} -\bar{\boldsymbol{a}}_1^T & \mathbf{0}^T & -\boldsymbol{q}_1 & \mathbf{0}^T & \cdots & \mathbf{0}^T & 0 & 0 \end{bmatrix}^T \\
\boldsymbol{r}_2 &= \begin{bmatrix} -\bar{\boldsymbol{a}}_2^T & \mathbf{0}^T & \mathbf{0}^T & -\boldsymbol{q}_2 & \cdots & \mathbf{0}^T & 0 & 0 \end{bmatrix}^T \\
&\vdots \\
\boldsymbol{r}_m &= \begin{bmatrix} -\bar{\boldsymbol{a}}_m^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \cdots & -\boldsymbol{q}_m & 0 & 0 \end{bmatrix}^T,
\end{aligned}$$

and Q_j given by

$$\begin{aligned}
Q_1 &= \begin{bmatrix} P_1^T & O_{L_0,M_0} & -D_1^T & O_{L_2,M_2} & \cdots & O_{L_m,M_m} & \mathbf{0} & \mathbf{0} \end{bmatrix} \\
Q_2 &= \begin{bmatrix} P_2^T & O_{L_0,M_0} & O_{L_1,M_1} & -D_2^T & \cdots & O_{L_m,M_m} & \mathbf{0} & \mathbf{0} \end{bmatrix} \\
&\vdots \\
Q_m &= \begin{bmatrix} P_2^T & O_{L_0,M_0} & O_{L_1,M_1} & O_{L_2,M_2} & \cdots & -D_m^T & \mathbf{0} & \mathbf{0} \end{bmatrix},
\end{aligned}$$

In the same way, (51) and (52) can be converted into conic constraint forms and combined into

$$\begin{bmatrix} Q_j \\ \mathbf{0}^T \\ \boldsymbol{r}_j^T \end{bmatrix} \boldsymbol{z} - \begin{bmatrix} \mathbf{0} \\ -b_j \end{bmatrix} \in \mathbf{L}^{L_j+1} \times \mathbb{R}_+. \tag{53}$$

Let

$$H_0 = \begin{bmatrix} Q_0 \\ \mathbf{0}^T \\ \boldsymbol{r}_0^T \end{bmatrix}, \; H_j = \begin{bmatrix} Q_j \\ \mathbf{0}^T \\ \boldsymbol{r}_j^T \end{bmatrix}, \; \boldsymbol{v}_j = \begin{bmatrix} \mathbf{0} \\ -b_j \end{bmatrix}, j = 1, \cdots$$

and cone

$$\mathcal{K}_0 = \mathbf{L}^{L_0+1} \times \mathbb{R}_+, \; \mathcal{K}_j = \mathbf{L}^{L_j+1} \times \mathbb{R}_+, j = 1, \cdots,$$

the constraints (50) and (53) become

$$H_0 \boldsymbol{z} \in \mathcal{K}_0, \tag{54}$$

$$H_j \boldsymbol{z} - \boldsymbol{v}_j \in \mathcal{K}_j, j = 1, \ldots, m, \tag{55}$$

and the model can be simplified to

$$\begin{aligned}
&\min \boldsymbol{g}^T \boldsymbol{z} \\
&\text{s.t. } H_0 \boldsymbol{z} \in \mathcal{K}_0 \\
&\quad H_j \boldsymbol{z} - \boldsymbol{v}_j \in \mathcal{K}_j, \; j = 1, \ldots, m \\
&\quad I \boldsymbol{z} \in \mathbb{R}_+^N.
\end{aligned} \tag{56}$$

Based on the conic duality formulation, the dual model of (56) is as follows:

$$\begin{aligned} \max \ & \sum_{j=1}^{m} \boldsymbol{v}_j^T \boldsymbol{\lambda}_j \\ \text{s.t. } & H_0^T \boldsymbol{\lambda}_0 + \sum_{j=1}^{m} H_j^T \boldsymbol{\lambda}_j + I \boldsymbol{\lambda}_{m+1} = \boldsymbol{g} \\ & \boldsymbol{\lambda}_0 \in \mathcal{K}_0^* \\ & \boldsymbol{\lambda}_j \in \mathcal{K}_j^*, j = 1, \ldots, m \\ & \boldsymbol{\lambda}_{m+1} \in \left(\mathbb{R}_+^N\right)^*, \end{aligned} \tag{57}$$

with dual variable $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_m, \boldsymbol{\lambda}_{m+1})$.

Furthermore, considering that the Lorentz cone and linear cone are self-dual and based on the dual property of the direct product of the convex cone, the following is obtained:

$$\begin{aligned} \mathcal{K}_0^* &= \left(\mathbf{L}^{L_0+1} \times \mathbb{R}_+\right)^* = \mathbf{L}^{L_0+1} \times \mathbb{R}_+, \\ \mathcal{K}_j^* &= \left(\mathbf{L}^{L_j+1} \times \mathbb{R}_+\right)^* = \mathbf{L}^{L_j+1} \times \mathbb{R}_+, \ j = 1, \ldots, m, \\ \left(\mathbb{R}_+^N\right)^* &= \mathbb{R}_+^N. \end{aligned}$$

Let

$$\boldsymbol{\lambda}_0 = \begin{bmatrix} \boldsymbol{\mu}_0 \\ \tau_0 \\ \theta_0 \end{bmatrix} \in \mathbf{L}^{L_0+1} \times \mathbb{R}_+$$

where $\boldsymbol{\mu}_0 \in \mathbb{R}^{L_0}$, $\tau_0 \in \mathbb{R}$, $\|\boldsymbol{\mu}_0\|_2 \leq \tau_0$, and $\theta_0 \geq 0$,

$$\boldsymbol{\lambda}_j = \begin{bmatrix} \boldsymbol{\mu}_j \\ \tau_j \\ \theta_j \end{bmatrix} \in \mathbf{L}^{L_j+1} \times \mathbb{R}_+,$$

where $\boldsymbol{\mu}_j \in \mathbb{R}^{L_j}$, $\tau_j \in \mathbb{R}$, $\left\|\boldsymbol{\mu}_j\right\|_2 \leq \tau_j$, $\theta_j \geq 0$ for $j = 1, \ldots, m$, and

$$\boldsymbol{\lambda}_{m+1} = \begin{bmatrix} \boldsymbol{s} \\ \boldsymbol{d}_0 \\ \vdots \\ \boldsymbol{d}_m \\ \sigma_1 \\ \sigma_2 \end{bmatrix} \in \mathbb{R}_+^N,$$

where $\boldsymbol{s} \in \mathbb{R}_+^n$, $\boldsymbol{d}_0 \in \mathbb{R}_+^{L_0}$, $\boldsymbol{d}_j \in \mathbb{R}_+^{L_j}$ for $j = 1, \ldots, m$, $\sigma_1 \geq 0$, and $\sigma_2 \geq 0$. The (57) problem becomes

$$\begin{aligned}
&\max \sum_{j=1}^{m} \begin{bmatrix} \mathbf{0} \\ -b_j \end{bmatrix}^T \begin{bmatrix} \boldsymbol{\mu}_0 \\ \tau_0 \\ \theta_0 \end{bmatrix} \\
&\text{s.t.} \begin{bmatrix} Q_0 \\ \mathbf{0}^T \\ \boldsymbol{r}_0^T \end{bmatrix}^T \begin{bmatrix} \boldsymbol{\mu}_0 \\ \tau_0 \\ \theta_0 \end{bmatrix} + \sum_{j=1}^{m} \begin{bmatrix} Q_j \\ \mathbf{0}^T \\ \boldsymbol{r}_j^T \end{bmatrix}^T \begin{bmatrix} \boldsymbol{\mu}_0 \\ \tau_0 \\ \theta_0 \end{bmatrix} + \begin{bmatrix} \boldsymbol{s} \\ \boldsymbol{d}_0 \\ \vdots \\ \boldsymbol{d}_m \\ \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ 1 \\ -1 \end{bmatrix} \\
&\quad \|\boldsymbol{\mu}_0\|_2 \le \tau_0,\ \|\boldsymbol{\mu}_j\|_2 \le \tau_j,\ j = 1, \ldots, m, \\
&\quad \theta_0 \ge 0,\ \theta_j \ge 0,\ j = 1, \ldots, m, \\
&\quad \boldsymbol{s} \ge \mathbf{0},\ \boldsymbol{d}_0 \ge \mathbf{0},\ \boldsymbol{d}_j \ge \mathbf{0},\ j = 1, \ldots, m, \\
&\quad \sigma_1 \ge 0,\ \sigma_2 \ge 0.
\end{aligned} \tag{58}$$

By describing the respective forms of the objective function and the constraint, problem (58) becomes

$$\begin{aligned}
&\max \sum_{j=1}^{m} \left(-b_j \theta_j\right) \\
&\text{s.t. } Q_0^T \boldsymbol{\mu}_0 + \sum_{j=1}^{m} \left(Q_j^T \boldsymbol{\mu}_j + r_j \theta_j\right) + \begin{bmatrix} \boldsymbol{s} \\ \boldsymbol{d}_0 \\ \vdots \\ \boldsymbol{d}_m \\ \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ 1 \\ -1 \end{bmatrix} \\
&\quad \|\boldsymbol{\mu}_0\|_2 \le \tau_0,\ \|\boldsymbol{\mu}_j\|_2 \le \tau_j,\ j = 1, \ldots, m, \\
&\quad \theta_0 \ge 0,\ \theta_j \ge 0,\ j = 1, \ldots, m, \\
&\quad \boldsymbol{s} \ge \mathbf{0},\ \boldsymbol{d}_0 \ge \mathbf{0},\ \boldsymbol{d}_j \ge \mathbf{0},\ j = 1, \ldots, m, \\
&\quad \sigma_1 \ge 0,\ \sigma_2 \ge 0.
\end{aligned} \tag{59}$$

Note that

$$Q_0^T \boldsymbol{\mu}_0 = \begin{bmatrix} P_0 \boldsymbol{\mu}_0 \\ -D_0 \boldsymbol{\mu}_0 \\ O_{M_1, L_1} \\ \vdots \\ O_{M_m, L_m} \\ \mathbf{0}^T \\ \mathbf{0}^T \end{bmatrix},\ \boldsymbol{r}_0 \theta_0 = \begin{bmatrix} -\bar{\boldsymbol{w}} \theta_0 \\ -\boldsymbol{q}_0 \theta_0 \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \theta_0 \\ -\theta_0 \end{bmatrix}.$$

Moreover, we also obtain

$$Q_1^T\boldsymbol{\mu}_1 = \begin{bmatrix} P_1\boldsymbol{\mu}_1 \\ O_{M_0,L_0} \\ -D_1\boldsymbol{\mu}_1 \\ \vdots \\ O_{M_m,L_m} \\ \mathbf{0}^T \\ \mathbf{0}^T \end{bmatrix}, \ldots, Q_m^T\boldsymbol{\mu}_m = \begin{bmatrix} P_m\boldsymbol{\mu}_m \\ O_{M_0,L_0} \\ O_{M_1,L_1} \\ \vdots \\ -D_m\boldsymbol{\mu}_m \\ \mathbf{0}^T \\ \mathbf{0}^T \end{bmatrix}$$

$$\boldsymbol{r}_1\theta_1 = \begin{bmatrix} -\bar{\boldsymbol{a}}_1\theta_1 \\ \mathbf{0} \\ -\boldsymbol{q}_1\theta_1 \\ \vdots \\ \mathbf{0} \\ 0 \\ 0 \end{bmatrix}, \ldots, \boldsymbol{r}_m\theta_m = \begin{bmatrix} -\bar{\boldsymbol{a}}_m\theta_m \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ -\boldsymbol{q}_m\theta_m \\ 0 \\ 0 \end{bmatrix}.$$

Therefore the first constraint of (59) can be changed to

$$\begin{bmatrix} P_0\boldsymbol{\mu}_0 + \sum_{j=1}^m P_j\boldsymbol{\mu}_j \\ -D_0\boldsymbol{\mu}_0 \\ -D_1\boldsymbol{\mu}_1 \\ \vdots \\ -D_m\boldsymbol{\mu}_m \\ \mathbf{0}^T \\ \mathbf{0}^T \end{bmatrix} + \begin{bmatrix} -\bar{\boldsymbol{w}}\theta_0 - \sum_{j=1}^m \bar{\boldsymbol{a}}_j\theta_j \\ -\boldsymbol{q}_0\theta_0 \\ -\boldsymbol{q}_1\theta_1 \\ \vdots \\ -\boldsymbol{q}_m\theta_m \\ \theta_0 \\ -\theta_0 \end{bmatrix} + \begin{bmatrix} \boldsymbol{s} \\ \boldsymbol{d}_0 \\ \boldsymbol{d}_1 \\ \vdots \\ \boldsymbol{d}_m \\ \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ 1 \\ -1 \end{bmatrix}.$$

$\|\boldsymbol{\mu}_0\|_2 \leq \tau_0$ and $\|\boldsymbol{\mu}_j\|_2 \leq \tau_j$, $j = 1, \ldots, m$ can be ignored because the objective function does not contain $\boldsymbol{\mu}_0$, $\boldsymbol{\mu}_j$, τ_0, and τ_j, hence (59) can be written as

$$\begin{aligned} \max \ & \left(-\sum_{j=1}^m b_j\theta_j \right) \\ \text{s.t. } & P_0\boldsymbol{\mu}_0 + \sum_{j=1}^m P_j\boldsymbol{\mu}_j - \bar{\boldsymbol{w}}\theta_0 - \sum_{j=1}^m \bar{\boldsymbol{a}}_j\theta_j + \boldsymbol{s} = \mathbf{0} \\ & -D_0\boldsymbol{\mu}_0 - \boldsymbol{q}_j\theta_0 + \boldsymbol{d}_0 = \mathbf{0} \\ & -D_j\boldsymbol{\mu}_j - \boldsymbol{q}_j\theta_j + \boldsymbol{d}_j = \mathbf{0}, \ j = 1, \ldots, m \\ & \theta_0 + \sigma_1 = 1 \\ & -\theta_0 + \sigma_2 = -1 \\ & \theta_0 \geq 0, \ \theta_j \geq 0, \ j = 1, \ldots, m, \\ & \boldsymbol{s} \geq \mathbf{0}, \ \boldsymbol{d}_0 \geq \mathbf{0}, \ \boldsymbol{d}_j \geq \mathbf{0}, \ j = 1, \ldots, m, \\ & \sigma_1 \geq 0, \ \sigma_2 \geq 0. \end{aligned} \tag{60}$$

5. Discussion

The primal problem forms for each of the problems in Section 4.3, namely (36) and (56), can be simplified to

$$\begin{aligned} &\min \boldsymbol{g}^T \boldsymbol{z} \\ &\text{s.t. } H_0 \boldsymbol{z} \in \mathcal{K}_0 \\ &\quad H_j \boldsymbol{z} - \boldsymbol{v}_j \in \mathcal{K}_j,\ j = 1, \ldots, m \\ &\quad \boldsymbol{z} \in \mathbb{R}^N_+. \end{aligned} \tag{61}$$

In addition, the duals, namely (37) and (57), are given in the form

$$\begin{aligned} &\max \sum_{j=1}^{m} \boldsymbol{v}_j^T \boldsymbol{\lambda}_j \\ &\text{s.t. } H_0^T \boldsymbol{\lambda}_0 + \sum_{j=1}^{m} H_j^T \boldsymbol{\lambda}_j + \boldsymbol{\lambda}_{m+1} = \boldsymbol{g} \\ &\quad \boldsymbol{\lambda}_0 \in \mathcal{K}_0^* \\ &\quad \boldsymbol{\lambda}_j \in \mathcal{K}_j^*, j = 1, \ldots, m \\ &\quad \boldsymbol{\lambda}_{m+1} \in \left(\mathbb{R}^N_+\right)^*, \end{aligned} \tag{62}$$

where $\boldsymbol{g}, \boldsymbol{z}, H_0, , \boldsymbol{v}_j, H_j$ $(j = 1, \ldots, m)$ and the convex cones $\mathcal{K}_0, \mathcal{K}_j, \mathbb{R}^N_+$ are defined for each problem. Therefore, duality analysis is performed on the primal–dual pairs, (61) and (62).

5.1. Duality Symmetry

Based on (61) and (62), we obtain the following.

1. The dual variable of (62), $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_{m+1})$, corresponds to the number of rows of each constraint of (61).
2. There is a natural one-to-one correspondence between the constraints of (61) and (62). Moreover, the underlying convex cone of the primal–dual pair are dual to each other.
3. Dual problem of (62) is equivalent to (61).

To see the duality symmetry in point 3 above, we construct the dual of (62) as follows. First, the objective function of (62) is changed to the minimization form as follows.

$$\begin{aligned} &\min \ -\sum_{j=1}^{m} \boldsymbol{v}_j^T \boldsymbol{\lambda}_j \\ &\text{s.t. } H_0^T \boldsymbol{\lambda}_0 + \sum_{j=1}^{m} H_j^T \boldsymbol{\lambda}_j + \boldsymbol{\lambda}_{m+1} = \boldsymbol{g} \\ &\quad \boldsymbol{\lambda}_0 \in \mathcal{K}_0^* \\ &\quad \boldsymbol{\lambda}_j \in \mathcal{K}_j^*, j = 1, \ldots, m \\ &\quad \boldsymbol{\lambda}_{m+1} \in \left(\mathbb{R}^N_+\right)^*. \end{aligned} \tag{63}$$

Furthermore, let $\boldsymbol{z}'$ correspond to the equation constraint of (63) and let the dimension of $\boldsymbol{z}'$ be equal to the dimension of $\boldsymbol{g}$. Let $\boldsymbol{\eta}_j, j = 0, \ldots, m+1$ be responsible for other corresponding constraints, respectively. Hence, the dual problem of (63) is given by

$$\begin{aligned} &\max \boldsymbol{g}^T \boldsymbol{z}' \\ &\text{s.t. } H_0 \boldsymbol{z}' + \boldsymbol{\eta}_0 = \boldsymbol{0} \\ &\quad H_j \boldsymbol{z}' + \boldsymbol{\eta}_j = -\boldsymbol{v}_j,\ j = 1, \ldots, m \\ &\quad \boldsymbol{z}' + \boldsymbol{\eta}_{m+1} = \boldsymbol{0} \\ &\quad \boldsymbol{\eta}_0 \in \mathcal{K}_0 \\ &\quad \boldsymbol{\eta}_j \in \mathcal{K}_j, j = 1, \ldots, m \\ &\quad \boldsymbol{\eta}_{m+1} \in \mathbb{R}^N_+. \end{aligned} \tag{64}$$

Note that by changing the objective function to minimization and setting $z = -z'$, we can see that (64) is equivalent to (61).

5.2. Weak Duality

In this section, we examine the weak duality for the primal–dual problem pairs (61) and (62). Fulfillment of the weak duality condition is seen by examining

$$\sum_{j=1}^{m} v_j^T \lambda_j \leq g^T z. \tag{65}$$

Let z be a feasible solution of (61) and $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_m, \lambda_{m+1})$ be a feasible solution (62). Because $\lambda_0 \in \mathcal{K}_0^*$ and z satisfies $H_0 z \in \mathcal{K}_0$, then by the dual cone definition, we obtain

$$\lambda_0^T H_0 z \geq 0. \tag{66}$$

Variables $\lambda_j \in \mathcal{K}_j$ for $j = 1, \ldots, m$ satisfy

$$\begin{aligned} \lambda_j^T \left(H_j z - v_j\right) \geq & 0 \\ \lambda_j^T H_j z - \lambda_j^T v_j \geq & 0 \\ \lambda_j^T H_j z \geq & \lambda_j^T v_j. \end{aligned} \tag{67}$$

Because (67) holds for every $j = 1, \ldots, m$, then

$$\sum_{j=1}^{m} \lambda_j^T H_j z \geq \sum_{j=1}^{m} \lambda_j v_j. \tag{68}$$

Furthermore, for λ_{m+1} satisfies

$$\lambda_{m+1}^T z = \lambda_{m+1}^T z \geq 0. \tag{69}$$

Moreover, λ satisfies the constraint of (62). Consequently,

$$\begin{aligned} g^T z &= \left(H_0^T \lambda_0 + \sum_{j=1}^{m} H_j^T \lambda_j + \lambda_{m+1} \right)^T z \\ &= \left(H_0^T \lambda_0 \right)^T z + \sum_{j=1}^{m} \left(H_j^T \lambda_j \right)^T z + \lambda_{m+1}^T z \\ &= \lambda_0^T H_0 z + \sum_{j=1}^{m} \lambda_j^T H_j z + \lambda_{m+1}^T z. \end{aligned}$$

By (66), (68), and (69), we obtain

$$g^T z \geq 0 + \sum_{j=1}^{m} \lambda_j^T v_j + 0 = \sum_{j=1}^{m} v_j^T \lambda_j. \tag{70}$$

The inequality (70) is equivalent to (65), so we can conclude that the primal–dual pair problems, (61) and (62), satisfy the weak duality.

5.3. Strong Duality

In this section, we provide strong duality criteria for the primal–dual problem pairs (61) and (62). Based on the strong assumptions for the conic problem stated by [7,35], we obtain the following.

1. No vector $z \neq \mathbf{0}$ that satisfies $H_0 z = \mathbf{0}$ and $H_j z = \mathbf{0}$ for $j = 1, \ldots, m$.

2. The duality is symmetric, i.e., the dual of (62) is equivalent to (61). This statement is explained in Section 5.1.
3. Primal problem (61) is strictly feasible if there is a feasible solution $\boldsymbol{z}$ that satisfies

$$H_0\boldsymbol{z} \in \text{int } \mathcal{K}_0,\ H_j\boldsymbol{z} - \boldsymbol{v}_j \in \text{int } \mathcal{K}_j\ (j = 1, \ldots, m), \text{ and } \boldsymbol{z} \in \text{int } \mathbb{R}^N_+. \tag{71}$$

4. Dual problem (62) is strictly feasible if there is a feasible solution $\boldsymbol{\lambda}$ that satisfies

$$\boldsymbol{\lambda}_0 \in \text{int } \mathcal{K}_0^*,\ \boldsymbol{\lambda}_j \in \text{int } \mathcal{K}_j^*, \text{ and } \boldsymbol{\lambda}_{m+1} \in \text{int } \left(\mathbb{R}^N_+\right)^*. \tag{72}$$

5. Weak duality: The optimal value of (62) is less than or equal to (61). This is explained in Section 5.2.
6. Strong duality: If either (61) or (62) is strictly feasible and finite, then the other is solvable, and the optimal values are equal. If (61) and (62) are strictly feasible, then both are solvable with the same optimal value.

Returning to the original problem, (35), the strictly feasible condition (71) for the problem with the ellipsoidal uncertainty set can be broken down as follows.

$$\begin{aligned} H_0\boldsymbol{z} &\in \text{int } \mathcal{K}_0 \\ \begin{bmatrix} P_0^T & 0 & 0 \\ -\bar{\boldsymbol{w}}^T & 1 & -1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \delta_1 \\ \delta_2 \end{bmatrix} &\in \text{int } \mathbf{L}^{L_0+1} \\ \begin{bmatrix} P_0^T\boldsymbol{x} \\ -\bar{\boldsymbol{w}}^T\boldsymbol{x} + \delta_1 - \delta_2 \end{bmatrix} &\in \text{int } \mathbf{L}^{L_0+1}, \end{aligned} \tag{73}$$

$$\begin{aligned} H_j\boldsymbol{z} - \boldsymbol{v}_j &\in \text{int } \mathcal{K}_j \\ \begin{bmatrix} P_j^T & 0 & 0 \\ -\bar{\boldsymbol{a}}_j^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \delta_1 \\ \delta_2 \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ -b_j \end{bmatrix} &\in \text{int } \mathbf{L}^{L_j+1} \\ \begin{bmatrix} P_j^T\boldsymbol{x} \\ -\bar{\boldsymbol{a}}_j^T\boldsymbol{x} + b_j \end{bmatrix} &\in \text{int } \mathbf{L}^{L_j+1}, \end{aligned} \tag{74}$$

and

$$\begin{aligned} \boldsymbol{z} &\in \text{int } \mathbb{R}^N_+ \\ \begin{bmatrix} \boldsymbol{x} \\ \delta_1 \\ \delta_2 \end{bmatrix} &\in \text{int } \mathbb{R}^{m+2}_+. \end{aligned} \tag{75}$$

Note that the interiors of a linear cone and a Lorentz cone are given by

$$\begin{aligned} \text{int } \mathbb{R}^m_+ &= \{\boldsymbol{x} \in \mathbb{R}^m : x_i > 0, i = 1, \ldots, m\}, \text{ and} \\ \text{int } \mathbf{L}^m &= \left\{\boldsymbol{x} \in \mathbb{R}^m : \sqrt{x_1^2 + \ldots + x_{m-1}^2} < x_m\right\}. \end{aligned}$$

Therefore, (73)–(75) become

$$\begin{aligned} &\left\|P_0^T\boldsymbol{x}\right\|_2 < -\bar{\boldsymbol{w}}^T\boldsymbol{x} + \delta_1 - \delta_2, \\ &\left\|P_j^T\boldsymbol{x}\right\|_2 < -\bar{\boldsymbol{a}}_j^T\boldsymbol{x} + b_j,\ j = 1, \ldots, m, \\ &\boldsymbol{x} > \mathbf{0},\ \delta_1 > 0,\ \delta_2 > 0. \end{aligned}$$

Furthermore, the strictly feasible condition (72) for the dual problem (37) is satisfied if there is a feasible solution $\boldsymbol{\lambda}$ that satisfies

$$\begin{aligned}
&\boldsymbol{\lambda}_0 \in \mathbf{L}^{L_0+1} \\
&\boldsymbol{\lambda}_j \in \operatorname{int} \mathbf{L}^{L_j+1},\ j = 1, \ldots, m, \\
&\boldsymbol{\lambda}_{m+1} \in \operatorname{int} \mathbb{R}_+^{n+2},
\end{aligned}$$

or there is a feasible solution that satisfies

$$\begin{aligned}
&\|\mu_0\|_2 < 1, \\
&\|\mu_j\|_2 < \tau_j,\ j = 1, \ldots, m, \\
&\boldsymbol{s} > \mathbf{0}.
\end{aligned}$$

The strictly feasible condition (71) for the problem with polyhedral uncertainty set (56) is equivalent to the existence of a feasible solution $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y_0}, \ldots, \boldsymbol{y}_m, \delta_1, \delta_2)$ that satisfies the following conditions:

$$\begin{aligned}
&\bar{\boldsymbol{w}}^T \boldsymbol{x} + \boldsymbol{q}_0^T \boldsymbol{y}_0 - \delta_1 + \delta_2 > 0, \\
&\bar{\boldsymbol{a}}_j^T \boldsymbol{x} + \boldsymbol{q}_j^T \boldsymbol{y}_0 - b_j > 0, \\
&\boldsymbol{x} > \mathbf{0}, \boldsymbol{y}_0 > \mathbf{0}, \boldsymbol{y}_j > \mathbf{0}\ (j = 1, \ldots, m), \\
&\delta_1 > 0, \delta_2 > 0.
\end{aligned}$$

The strictly feasible condition (72) for the dual problem is equivalent to the existence of a feasible solution that satisfies

$$\begin{aligned}
&\theta_0 > 0,\ \theta_j > 0,\ j = 1, \ldots, m, \\
&\boldsymbol{s} > \mathbf{0},\ \boldsymbol{d}_0 > \mathbf{0},\ \boldsymbol{d}_j > \mathbf{0},\ j = 1, \ldots, m, \\
&\sigma_1 > 0,\ \sigma_2 > 0.
\end{aligned}$$

Furthermore, the optimality condition can be stated as follows. Let $\boldsymbol{z}$ be a feasible solution of (61) and $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_m, \boldsymbol{\lambda}_{m+1})$ be a feasible solution of (62). The duality gap of the pair $(\boldsymbol{z}, \boldsymbol{\lambda})$ given by

$$\Delta(\boldsymbol{z}, \boldsymbol{\lambda}) = \boldsymbol{g}^T \boldsymbol{z} - \sum_{j=1}^{m} \boldsymbol{v}_j^T \boldsymbol{\lambda}_j,$$

is non-negative and is equal to

$$\boldsymbol{\lambda}_0^T (H_0 \boldsymbol{z}) + \sum_{j=1}^{m} \boldsymbol{\lambda}_j^T (H_j \boldsymbol{z} - \boldsymbol{v}_j) + \lambda_{m+1}^T \boldsymbol{z}.$$

The duality gap is zero if and only if the complementary slackness holds, i.e.,

$$\begin{aligned}
&\boldsymbol{\lambda}_0^T (H_0 \boldsymbol{z}) = 0, \\
&\boldsymbol{\lambda}_j^T (H_j \boldsymbol{z} - \boldsymbol{v}_j) = 0,\ j = 1, \ldots, m, \\
&\lambda_{m+1}^T \boldsymbol{z} = 0.
\end{aligned}$$

This means that the duality gap is zero if and only if $\boldsymbol{z}$ and $\boldsymbol{\lambda}$ are the optimal solutions of (61) and (62), respectively, and the optimal values are equal.

6. Conclusions

In this paper, the dual of a multi-objective linear RO problem is formed through conic duality. This is based on the fact that the RC obtained using ellipsoidal and polyhedral uncertainty sets forms a CO problem, namely the CQP and LP. We also use the utility function method in solving multi-objective forms so that the problem solved is a single-objective optimization. Based on the analysis results, the primal and dual problems obtained meet the weak duality theorem. Moreover, the strong duality criteria are identified based on the strictly feasible, boundedness, and solvability properties of the primal and dual problem.

The selection of weights in the weighted sum utility function method does not always guarantee that the final or optimal solution is acceptable because this weight is the preference of each user. Therefore, the weight selection can be performed using the guidelines and methods discussed by [34]. In addition, further research can also use other types of utility function or other multi-objective problem solving methods, such as the lexicographic method or goal programming.

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Abbreviations

The following abbreviations are used in this manuscript:

CO	Conic Optimization
CQP	Conic Quadratic Programming
LP	Linear Programming
RO	Robust Optimization
RC	Robust Counterpart

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