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Integral Representations of Ratios of the Gauss Hypergeometric Functions with Parameters Shifted by Integers

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Abstract: Given real parameters a, b, c and integer shifts n_1, n_2, m, we consider the ratio $R(z) = {}_2F_1(a+n_1, b+n_2; c+m; z)/{}_2F_1(a, b; c; z)$ of the Gauss hypergeometric functions. We find a formula for $\operatorname{Im} R(x \pm i0)$ with $x > 1$ in terms of real hypergeometric polynomial P, beta density and the absolute value of the Gauss hypergeometric function. This allows us to construct explicit integral representations for R when the asymptotic behaviour at unity is mild and the denominator does not vanish. The results are illustrated with a large number of examples.

Keywords: gauss hypergeometric function; gauss continued fraction; integral representation

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1. Introduction

The Gauss hypergeometric functions ([1], [2] (Chapter II), [3] (Chapter 15))

$$
{}_2F_1(a, b; c; z) = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \tag{1}
$$

and ${}_2F_1(a+n_1, b+n_2; c+m; z)$ are called contiguous in a wide sense if $n_1, n_2, m \in \mathbb{Z}$; see [4]. Any three functions of this type satisfy a linear relation with coefficients rational in a, b, c, z. If $n_1, n_2, m \in \{-1, 0, 1\}$, then the coefficients of this relation are linear in z, and the functions are called contiguous in a narrow sense. Such a contiguous relation was used by Euler to derive a continued fraction (much later termed T-fraction) for the ratio ${}_2F_1(a, b+1; c+1; z)/{}_2F_1(a, b; c; z)$. Gauss described all three-term relations among the functions contiguous in the narrow sense and found another continued fraction for the above ratio, which has the following form [1] (p. 134) (see also [5] (Formula (89.9)) or [6] (p. 123)):

$$
G(z) = \frac{F(a, b+1; c+1; z)}{F(a, b; c; z)} = \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 z}{1 - \cfrac{\alpha_2 z}{1 - \cdots}}}, \tag{2}
$$

where $\alpha_0 = 1$, and for $n \geq 0$,

$$
\alpha_{2n+1} = \frac{(a+n)(c-b+n)}{(c+2n)(c+2n+1)}, \quad \alpha_{2n+2} = \frac{(b+n+1)(c-a+n+1)}{(c+2n+1)(c+2n+2)}. \tag{3}
$$

Clearly, we have $\lim_{n\to\infty} \alpha_n = 1/4$, while $\sup_n |\alpha_n| =: \gamma/4 \geq 1/4$. So, if $\alpha_n > 0$, $n = 1, 2, \ldots$, then it follows from [7] that there exists a unique positive measure $d\mu(s)$

on $[0, \gamma]$ whose support is dense in $[0, 1]$ and has at most finitely many points in $(1, \gamma]$, such that

$$G(z) = \int_{[0,\gamma]} \frac{d\mu(s)}{1 - sz}. \tag{4}$$

(The fact that $d\mu(s)$ has at most finitely many atoms in this interval directly follows from the fact that ${}_2F_1(a, b; c; z)$ has finitely many zeros in $[0, 1)$. The latter is given by Theorem 4, a corollary of [8].) In general, on sending γ to infinity in (4) so that the integration is over $[0, +\infty)$, and letting $d\mu(s)$ run over all positive measures $d\mu(s)$ such that $\int_0^\infty (1-s)^{-1} d\mu(s) < \infty$, we obtain the collection of functions called the Stieltjes class $\mathcal{S}$. For functions asymptotically behaving as $\sum_{k=0}^{\infty} s_k z^k$ at the origin, the class $\mathcal{S}$ is characterized by a continued fraction $\alpha_0 / \big(1 - \alpha_1 z / (1 - \cdots)\big)$ with $\alpha_j \geq 0$ for all j, see [7] or, for example, [9] (p. 6). Such functions arise often in different areas, ranging from analysis and operator theory to combinatorics and probability.

The tighter collection of functions obtained by taking $\gamma = 1$ in (4) and letting $d\mu(s)$ run over all positive measures, making the integral convergent, is known as the Markov class $\mathcal{M}$. The same class can be described as the collection of generating functions of the Hausdorff moment sequences; see [5] (Chapter XIV). Certainly, if $\gamma < \infty$, we can re-scale the integration variable to make γ equal 1.

Theorem 69.2 from [5] asserts that one may take $\gamma = 1$ in (4) if $\alpha_n = (1 - g_{n-1})g_n$ for all $n \geq 1$ with some numbers $g_n \in [0, 1]$ (the cases where g_n for some n is 0, or 1 corresponds to rational $G(z)$). It is immediate to see that the condition $\alpha_n > 0$ is satisfied for the Gauss continued fraction for all n when $-1 < b < c$ and $0 < a < c + 1$. The more restrictive condition $g_n \in [0, 1]$ holds true if $0 \leq a \leq c + 1$, $0 \leq b \leq c$; see [10] (Proof of Theorem 1.1) for details. Surprisingly enough, the representing measure $d\mu$ in (4) for the Gauss continued fraction was only computed in 1982 by Vitold Belevitch [11]. Around the same time, Jet Wimp [12] constructed explicit formulae for the odd convergence of the continued fraction (2) in terms of hypergeometric polynomials.

The main protagonist of this paper is the following generalization of the Gauss ratio (2)

$$R_{n_1,n_2,m}(z) = \frac{{}_2F_1(a + n_1, b + n_2; c + m; z)}{{}_2F_1(a, b; c; z)}, \tag{5}$$

where $n_1, n_2, m \in \mathbb{Z}$ are arbitrary. This ratio was studied in our recent preprint [13]. The ideas presented in this preprint were developed further in [14]. The present work constitutes a corrected, streamlined and elaborated version of a part of [13]. The main objectives are to furnish a complete derivation of the integral representation of $R_{n_1,n_2,m}(z)$, including all detailed proofs omitted in [14], and to illustrate its structure with numerous examples. As a by-product, each example contains sufficient conditions for $R_{n_1,n_2,m} \in \mathcal{M}$ in terms of the parameters a, b, c.

The ratios of the Gauss hypergeometric functions are a recurring theme in the literature. An important particular case of this ratio is the logarithmic derivative of the Gauss hypergeometric function. Its Stieltjes transform representation can be used to study the infinite divisibility of certain ratios of beta-distributed beta variables in a way similar to the investigation of the ratios of the gamma random variables in [15]. Furthermore, integral representations of the ratios of the Gauss hypergeometric functions are useful when determining whether they belong to certain important functional classes. For example, the authors of [16] applied such a representation to verify that $R_{0,1,0}(z)$ can be written as (4), and hence a certain pair of hypergeometric weights forms the so-called Nikishin system—an important property in the realm of multiple orthogonal polynomials.

Concerning further applications, observe that the membership of $R_{n_1,n_2,m}$ in the Markov class $\mathcal{M}$, conditions for which we give in each of the examples of Section 3, has a number of important implications. These include the normality of all Padé approximants and uniform convergence to $R_{n_1,n_2,m}(z)$ of the para-diagonal Padé approximants on all compact subsets of $\mathbb{C} \backslash [1, \infty)$; two sided bounds on the real line in terms of Padé approximants;

the univalence of $R_{n_1,n_2,m}(z)$ and $zR_{n_1,n_2,m}(z)$ in $\mathrm{Re}(z) < 1$ and its various consequences; and the starlikeness of $zR_{n_1,n_2,m}(z)$ in the disk $|z| < r^*$ with $r^* = \sqrt{13\sqrt{13} - 46} \approx 0.934$. Details regarding these claims and further references can be found in [17,18].

There are many intriguing open questions related to our work. For example, the case when the shifts are no longer an integer is also of interest for applications, but requires additional tools. For the Jacobi polynomials, certain relevant results are presented in [19]. For the non-polynomial case, there are only very fragmentary results of this type, such as [20] (Lemma 4.5).

Another compelling problem is to extend the results of this paper to the ratios of the generalized hypergeometric functions ${}_pF_q$ which, for certain integer shifts, have explicitly known branched continued fractions generalizing the Gauss continued fraction (2); see [9] (Sections 13–14). Similar problems may be posed, mutatis mutandis, for the basic hypergeometric functions, cf. [9] (Section 15). The basic analogue of the Gauss continued fraction is considered in detail in [21,22].

This paper is organized as follows. Section 2.1 deals with the asymptotic behavior of $R_{n_1,n_2,m}(z)$ near the point $z = 1$ and at infinity. In Section 2.2, we derive a formula for the values of $\mathrm{Im}(R_{n_1,n_2,m}(x \pm i0))$ for $x > 1$ using a recent duality identity for the Gauss hypergeometric function. Section 2.3 is at the heart of our work: it contains the integral representation for $R_{n_1,n_2,m}(z)$. The basic ingredients are Theorem 4, which is a corollary of Runckel's theorem from [8] and Lemma 4 connecting $\mathrm{Im}(R_{n_1,n_2,m}(x \pm i0))$ with a Cauchy-type integral. The largest section of this paper—Section 3—illustrates our study with 15 different examples. In the last section, we show how our results may help to calculate "generalized beta integrals", as well as obtaining integral representations of such functions as $z/\,\mathrm{Log}(1+z)$.

2. Main Results

2.1. Asymptotic Behavior

In this section, we will record the behavior of $R_{n_1,n_2,m}(z)$ in the neighborhood of the singular points $z = 1$ and $z = \infty$. It will be convenient to use the following notation: if a is a real number, then

$$(a)_- := \min(a, 0) \quad \text{and} \quad (a)_+ := \max(a, 0).$$

Denote also $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, \dots\}$. We will use the standard symbols $\phi_1(z) = o(\phi(z))$ and $\phi_2(z) = O(\phi(z))$ as $z \to A$ to denote the functions satisfying the relations

$$\lim_{z \to A} \frac{\phi_1(z)}{\phi(z)} = 0 \quad \text{and} \quad |\phi_2(z)| \le C|\phi(z)| \text{ for } z \text{ near } A,$$

respectively (C is a positive constant independent of z). The goal of this section is the following theorem, which is a slightly corrected version of [14] (Lemma 1) presented there without proof.

Theorem 1. *Let $a, b, c \in \mathbb{R}$ and $c, c+m \notin -\mathbb{N}_0$. Then there exist four constants $\varepsilon_1, \varepsilon_\infty \in \{-1, 0, 1\}$ and $L_1, L_\infty \neq 0$ independent of z such that*

$$R_{n_1,n_2,m}(z) = L_1\,(1-z)^{\eta(a+n_1,b+n_2,c+m)-\eta(a,b,c)}\,[\log(1-z)]^{\varepsilon_1}\,(1+o(1)) \qquad \text{as } z \to 1; \tag{6}$$

$$R_{n_1,n_2,m}(-z) = L_\infty\,z^{\zeta(a+n_1,b+n_2,c+m)-\zeta(a,b,c)}\,[\log(z)]^{\varepsilon_\infty}\,(1+o(1)) \qquad \text{as } z \to \infty, \tag{7}$$

where

$$\eta(a,b,c) = \begin{cases} (c-a-b)_+, & \text{if } \ -a, b-c \in \mathbb{N}_0 \ \text{or} \ -b, a-c \in \mathbb{N}_0; \\ 0, & \text{if } -a \in \mathbb{N}_0 \text{ and/or } -b \in \mathbb{N}_0 \text{ while } a-c, b-c \notin \mathbb{N}_0; \\ c-a-b, & \text{if } -a, -b \notin \mathbb{N}_0, \text{ while } a-c \in \mathbb{N}_0 \text{ and/or } b-c \in \mathbb{N}_0; \\ (c-a-b)_-, & \text{otherwise} \end{cases} \tag{8}$$

and

$$\zeta(a,b,c) = \begin{cases} -a, & \text{if } A(b,a,c) = 0 \\ -b, & \text{if } A(a,b,c) = 0 \\ -\min(a,b), & \text{otherwise} \end{cases} \quad \text{where } A(x_1,x_2,x_3) = \frac{\Gamma(x_3)\Gamma(x_2-x_1)}{\Gamma(x_2)\Gamma(x_3-x_1)}. \tag{9}$$

Remark 1. *The function $A(x_1, x_2, x_3)$ is defined by continuity if some of the arguments of the gamma functions become non-positive integers. Details can be found in this section below Formula* (12).

The above theorem is a corollary of three lemmas giving a more precise description of the behavior of $R_{n_1,n_2,m}(z)$ in the neighborhood of the singular points $z = 1$ and $z = \infty$. We will furnish a detailed proofs of these lemmas below. Before formulating the first lemma, note that the condition

$$\{a, a+n_1, b, b+n_2, c-a, c+m-a-n_1, c-b, c+m-b-n_2\} \cap -\mathbb{N}_0 = \varnothing \tag{10}$$

is equivalent to the claim that neither ${}_2F_1(a+n_1, b+n_2; c+m; z)$ nor ${}_2F_1(a,b;c;z)$ reduce to a polynomial or polynomial multiple of a power of $(1-z)$—the cases we will refer to as degenerate. Note that Formulae (6) and (7) also hold for such degenerate cases.

Our first lemma deals with the singular point $z = 1$.

Lemma 1. *Suppose that $-c, -c-m \notin \mathbb{N}_0$ and condition* (10) *hold true for some $a, b, c \in \mathbb{R}$ and $n_1, n_2, m \in \mathbb{Z}$. Denote $\rho = c-a-b$, $q = m-n_1-n_2$ and write $\delta_{x,y}$ for the Kronecker delta. Then*

$$R_{n_1,n_2,m}(z) = M\frac{(1-z)^{(\rho+q)_-}[1-\delta_{\rho+q,0}+\delta_{\rho+q,0}\log(1-z)]}{(1-z)^{(\rho)_-}[1-\delta_{\rho,0}+\delta_{\rho,0}\log(1-z)]}(1+o(1)) \tag{11}$$

as $z \to 1$ with some constant $M \neq 0$ independent of z. If (10) *is violated, Formula* (11) *should be modified as follows:*

(a) *If $-a \in \mathbb{N}_0$ ($-b \in \mathbb{N}_0$), then the denominator should be replaced by 1, except when $-a \geq \rho \in \mathbb{N}$ ($-b \geq \rho \in \mathbb{N}$) in which case it should be replaced by $(1-z)^{\rho}$.*

(b) *If $-a-n_1 \in \mathbb{N}_0$ ($-b-n_2 \in \mathbb{N}_0$), then the numerator should be replaced by 1, except when $-a-n_1 \geq \rho+q \in \mathbb{N}$ ($-b-n_2 \geq \rho+q \in \mathbb{N}$), in which case it should be replaced by $(1-z)^{\rho+q}$.*

(c) *If $-a, -b \notin \mathbb{N}_0$ but $a-c \in \mathbb{N}_0$ and/or $b-c \in \mathbb{N}_0$, then the denominator should be replaced by $(1-z)^{\rho}$.*

(d) *If $-a-n_1, -b-n_2 \notin \mathbb{N}_0$, but $a+n_1-c-m \in \mathbb{N}_0$ and/or $b+n_2-c-m \in \mathbb{N}_0$, then the numerator should be replaced by $(1-z)^{\rho+q}$.*

Proof. Suppose first that (10) is satisfied. Then, if $\rho = c-a-b \notin \mathbb{Z}$, according to [3] (15.8.4), we have

$${}_2F_1(a,b;c;1-z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1(a,b;1+\rho;z) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}z^{\rho}{}_2F_1(c-a,c-b;1-\rho;z).$$

If $\rho = c-a-b = s \in \mathbb{N}_0$, then according to [2] (2.10(12–13)) or [3] (15.8.10), we have

$$\begin{aligned} {}_2F_1(a,b;c;1-z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\sum_{n=0}^{s-1}\frac{(a)_n(b)_n}{(1-\rho)_n n!}z^n + \frac{(-1)^s\Gamma(c)}{\Gamma(a)\Gamma(b)s!}z^{\rho}\sum_{n=0}^{\infty}\frac{(c-a)_n(c-b)_n}{(1+\rho)_n n!}H_n z^n \\ - \frac{(-1)^s\Gamma(c)}{\Gamma(a)\Gamma(b)s!}z^{\rho}\log(z){}_2F_1(c-a,c-b;1+\rho;z), \end{aligned}$$

where the sum over the empty index set equals zero, and

$$H_n = \psi(n+1) + \psi(n+s+1) - \psi(a+n+s) - \psi(b+n+s), \quad \psi(z) = \Gamma'(z)/\Gamma(z).$$

If $\rho = c - a - b = -s$ for some $s \in \mathbb{N}_0$, according to [2] (2.10(14-15)), we have

$$
\begin{aligned}
{}_2F_1(a,b;c;1-z) &= \frac{\Gamma(c)\Gamma(a+b-c)z^{\rho}}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{s-1}\frac{(c-a)_n(c-b)_n}{(1+\rho)_n n!}z^n \\
&+ \frac{(-1)^s\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)s!}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(1-\rho)_n n!}\hat{H}_n z^n - \frac{(-1)^s\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)s!}\log(z){}_2F_1(a,b;1-\rho;z),
\end{aligned}
$$

and

$$\hat{H}_n = \psi(n+1) + \psi(n+s+1) - \psi(a+n) - \psi(b+n).$$

These formulae imply that

$$
{}_2F_1(a,b;c;1-z) = \begin{cases} A(1+\alpha_1 z+\alpha_2 z^2+\cdots) + Bz^{\rho}(1+\beta_1 z+\beta_2 z^2+\cdots), & \rho \notin \mathbb{Z}; \\ \hat{A}(1+\hat{\alpha}_1 z+\hat{\alpha}_2 z^2+\cdots) + \hat{B}z^{\rho}\log(z)(1+\hat{\beta}_1 z+\hat{\beta}_2 z^2+\cdots), & \rho \in \mathbb{N}_0; \\ \tilde{A}z^{\rho}(1+\tilde{\alpha}_1 z+\tilde{\alpha}_2 z^2+\cdots) + \tilde{B}\log(z)(1+\tilde{\beta}_1 z+\tilde{\beta}_2 z^2+\cdots), & -\rho \in \mathbb{N}, \end{cases}
$$

where the constants $A, \hat{A}, \tilde{A}, B, \hat{B}, \tilde{B}$ do not vanish due to condition (10). In a similar fashion,

$$
\begin{aligned}
&{}_2F_1(a+n_1,b+n_2;c+m;1-z) \\
&= \begin{cases} C(1+\delta_1 z+\delta_2 z^2+\cdots) + Dz^{\rho+q}(1+\gamma_1 z+\gamma_2 z^2+\cdots), & \rho+q \notin \mathbb{Z}; \\ \hat{C}(1+\hat{\delta}_1 z+\hat{\delta}_2 z^2+\cdots) + \hat{D}z^{\rho+q}\log(z)(1+\hat{\gamma}_1 z+\hat{\gamma}_2 z^2+\cdots), & \rho+q \in \mathbb{N}_0; \\ \tilde{C}z^{\rho+q}(1+\tilde{\delta}_1 z+\tilde{\delta}_2 z^2+\cdots) + \tilde{D}\log(z)(1+\tilde{\gamma}_1 z+\tilde{\gamma}_2 z^2+\cdots), & -\rho-q \in \mathbb{N}, \end{cases}
\end{aligned}
$$

where the constants $C, \hat{C}, \tilde{C}, D, \hat{D}, \tilde{D}$ do not vanish due to condition (10). Substituting these formulae into definition (5) of the function $R_{n_1,n_2,m}(z)$ and analyzing the principal asymptotic term in each of the five possible cases (1) $\rho \notin \mathbb{Z}$; (2) $\rho \in \mathbb{N}_0$ and $\rho+q \in \mathbb{N}_0$; (3) $\rho \in \mathbb{N}_0$ and $-\rho-q \in \mathbb{N}$; (4) $-\rho \in \mathbb{N}$ and $\rho+q \in \mathbb{N}_0$; (5) $-\rho \in \mathbb{N}$ and $-\rho-q \in \mathbb{N}$, we arrive at Formula (11).

If condition (10) is violated, then claims (a)–(d) of the lemma follow from the following two facts: (1) If $-a \in \mathbb{N}_0$ and /or $-b \in \mathbb{N}_0$, then ${}_2F_1(a,b;c;z)$ reduces to a polynomial; (2) If $-a, -b \notin \mathbb{N}_0$, but $a-c \in \mathbb{N}_0$ and /or $b-c \in \mathbb{N}_0$, then Euler's transformation

$${}_2F_1(a,b;c;z) = (1-z)^{\rho}{}_2F_1(c-a,c-b;c;z)$$

implies that ${}_2F_1(a,b;c;z) = (1-z)^{\rho} \times$ polynomial. In view of a similar statement for ${}_2F_1(a+n_1,b+n_2;c+m;z)$, we arrive at the conclusions contained in claims (a)–(d) of the lemma on the basis of case-by-case analysis. □

We now turn our attention to the neighborhood of the point $z = \infty$. According to [23] (2.3.12) or [3] (15.8.2), as long as $a - b \notin \mathbb{Z}$, we have

$$ {}_2F_1(a,b;c;-z) = A(a,b,c)z^{-a}\left(1+\sum\nolimits_{j=1}^{\infty}\hat{\alpha}_j z^{-j}\right) + A(b,a,c)z^{-b}\left(1+\sum\nolimits_{j=1}^{\infty}\tilde{\alpha}_j z^{-j}\right) \quad (12)$$

for some finite numbers $\hat{\alpha}_j$, $\tilde{\alpha}_j$, where, in accord with (9),

$$A(x_1,x_2,x_3) = \frac{\Gamma(x_3)\Gamma(x_2-x_1)}{\Gamma(x_2)\Gamma(x_3-x_1)}.$$

Note that the situation $A(a,b,c) = A(b,a,c) = 0$ is not possible as long as we assume that $-c \notin \mathbb{N}_0$. With that, one of the numbers $A(a,b,c)$ or $A(b,a,c)$ vanishes when $\{-a,-b,a-c,b-c\} \cap \mathbb{N}_0 \neq \emptyset$. This is precisely the degenerate case: ${}_2F_1(a,b;c;z)$ reduces to a polynomial, possibly times a power of $(1-z)$. A brief analysis shows that (12) remains valid in this degenerate case, despite the possibility that $a - b \in \mathbb{Z}$. In such a situation, one of the numbers $A(a,b,c)$ or $A(b,c,a)$ vanishes, while the other is well defined under the convention

$$\frac{\Gamma(-k)}{\Gamma(-n)} = (-1)^{n-k}\frac{n!}{k!},$$

which results from computing the limit of $\Gamma(-k+\varepsilon)/\Gamma(-n+\varepsilon)$ as $\varepsilon \to 0$. We will assume this extended definition of $A(x_1,x_2,x_3)$ in what follows. Define further for brevity

$$\begin{aligned} &A_1 = A(a+n_1, b+n_2, c+m), \quad A_2 = A(b+n_2, a+n_1, c+m), \\ &A_3 = A(a,b,c), \quad A_4 = A(b,a,c). \end{aligned} \tag{13}$$

Note that the condition $A_1A_2A_3A_4 \neq 0$ is equivalent to (10). The following quantities will play an important role for the sequel. Put $\alpha = \zeta(a+n_1, b+n_2, c+m)$ and $\gamma = \zeta(a,b,c)$ with ζ from (9). In detail,

$$\alpha = \begin{cases} -\min(a+n_1, b+n_2) \text{ if } A_1A_2 \neq 0 \\ -a-n_1 \text{ if } A_2 = 0 \\ -b-n_2 \text{ if } A_1 = 0 \end{cases}, \quad \gamma = \begin{cases} -\min(a,b) \text{ if } A_3A_4 \neq 0 \\ -a \text{ if } A_4 = 0 \\ -b \text{ if } A_3 = 0 \end{cases}. \tag{14}$$

Note that α is well defined since $A_1^2 + A_2^2 \neq 0$ as long as $-c-m \notin \mathbb{N}_0$, including the case when one of A_1, A_2 is infinite; similarly, γ is well defined since $A_3^2 + A_4^2 \neq 0$ as long as $-c \notin \mathbb{N}_0$, including the case when one of A_3, A_4 is infinite. Put further

$$A_\alpha = \begin{cases} A_1, \text{ if } \alpha = -a-n_1 \\ A_2, \text{ if } \alpha = -b-n_2 \end{cases}, \quad A_\gamma = \begin{cases} A_3, \text{ if } \gamma = -a \\ A_4, \text{ if } \gamma = -b \end{cases}. \tag{15}$$

The above definition implies that both A_α and A_γ do not vanish as long as $-c, -c-m \notin \mathbb{N}_0$. We will break the result in two sub-cases. The following lemma treats the case when no logarithmic terms appear in the asymptotics.

Lemma 2. *Suppose that the numbers A_1, A_2, A_3, A_4 defined in (13) are all finite. Then the principal asymptotics of $R_{n_1,n_2,m}(-z)$ have the form*

$$R_{n_1,n_2,m}(-z) \sim \frac{A_\alpha}{A_\gamma} z^{\alpha-\gamma}(1+o(1)) \ \text{ as } \ z \to \infty, \tag{16}$$

where α,γ are defined in (14) and A_α, A_γ are defined in (15). The term $o(1)$ is a (generally infinite) linear combination of negative powers of z.

Proof. In view of (12) and definitions (14) and (15), we have

$$ {}_2F_1(a,b;c;-z) = A_\gamma z^\gamma \Big(z^{-\delta} f(z) + g(z)\Big), \tag{17}$$

where $\delta = |a-b|$ in the non-degenerate case, or $\delta = 1$ in the degenerate case, while $f(z) = \sum_{j=0}^{\infty} \alpha_j' z^{-j}$ and $g(z) = 1 + \sum_{j=1}^{\infty} \bar{\alpha}_j z^{-j}$ for some numbers α_j', $\bar{\alpha}_j$. Now, for $y \to 0$

$$\frac{1}{yf(z)+g(z)} = \frac{1/g(z)}{1-(-yf(z)/g(z))} = \sum_{k=0}^{\infty} (-y)^k \frac{f^k(z)}{g^{k+1}(z)}, \tag{18}$$

so the left-hand side is just a sum of the geometric series on the right-hand side. On plugging $y = z^{-\delta}$ into (18) and writing expansions of the ratios $f^k(z)/g^{k+1}(z)$ in powers of z^{-1} through $f(z)$ and $g(z)$ using the standard recursion formulae (see also [24] (p. 141, notation on p. 6)), we arrive at

$$[{}_2F_1(a,b;c;-z)]^{-1} = (A_\gamma)^{-1} z^{-\gamma} \left(1 + \sum_{k=1}^{\infty} \frac{\alpha_k}{z^{\hat{\sigma}_k}}\right) \tag{19}$$

for some positive numbers $\hat{\sigma}_k$. (In our case $f(z)$, $g(z)$ and, hence, $f^k(z)/g^{k+1}(z)$ actually converge to functions analytic near infinity; this makes the proof even simpler.) Analogous to (17),

$$ {}_2F_1(a+n_1,b+n_1;c+m;-z) = A_\alpha z^\alpha\left(1+\beta_1' z^{-\varepsilon}+\beta_2' z^{-\varepsilon-1}+\cdots+\bar{\beta}_1 z^{-1}+\bar{\beta}_2 z^{-2}+\cdots\right) \tag{20} $$

for some numbers β_j', $\bar{\beta}_j$ and $\varepsilon = |a+n_1-b-n_2|$ in the non-degenerate case or $\varepsilon = 1$ in the degenerate case. Multiplying (19) by (20), we arrive at (16). □

The condition $a-b \notin \mathbb{Z}$ in Lemma 2 ensures that no logarithms appear in the asymptotics. If, on the contrary, $a-b \in \mathbb{Z}$ such that also $a+n_1-b-n_2 \in \mathbb{Z}$, the asymptotic expansions of the hypergeometric functions in both the numerator and denominator of $R_{n_1,n_2,m}(-z)$ will contain logarithmic terms if (10) holds true (i.e., $A_1A_2A_3A_4 \neq 0$). We will treat this situation in the lemma below. If (10) is violated, however, then either the numerator (if $A_1A_2 = 0$) or denominator (if $A_3A_4 = 0$) or both will reduce to a polynomial possibly times a power of $(1-z)$ in which case the logarithmic terms are missing, and (12) holds. Note also that in the non-degenerate case when $a-b \in \mathbb{Z}\setminus\{0\}$, we have $0 \neq |A_3| < \infty$, $|A_4| = \infty$ if $a<b$ and $0 \neq |A_4| < \infty$, $|A_3| = \infty$ if $a > b$. This implies that $\gamma = -\min(a,b)$ in (14), and A_γ in (15) is well defined. Similar claims hold for α and A_α when $a+n_1-b-n_2 \in \mathbb{Z}\setminus\{0\}$.

Lemma 3. *Suppose that $n_2-n_1 \neq a-b \in \mathbb{Z}\setminus\{0\}$, $A_1A_2A_3A_4 \neq 0$ ($\Leftrightarrow$ condition (10) holds) such that $\alpha = -\min(a+n_1,b+n_2)$ and $\gamma = -\min(a,b)$. Let A_α, A_γ be defined in (15). Then the asymptotic expansion of $R_{n_1,n_2,m}(-z)$ as $z\to\infty$ has the form*

$$ R_{n_1,n_2,m}(-z) \sim \frac{A_\alpha}{A_\gamma} z^{\alpha-\gamma}\left(1+\sum_{k=1}^{\min(\delta,\varepsilon)-1}\frac{a_k}{z^k}+\sum_{k=\min(\delta,\varepsilon)}^{\infty}\frac{a_k}{z^k}\left[1+b_{1,k}\log(z)+\cdots+b_{k,k}\log^k(z)\right]\right), \tag{21} $$

where the sum over the empty index set is zero, a_k and $b_{j,k}$ are real numbers (possibly vanishing), $\varepsilon = |a+n_1-b-n_2|$ and $\delta = |a-b|$ are positive integers.

Proof. Indeed if $|a-b| \geq 1$, we apply [3] (15.8.8), which can be written in the form:

$$ {}_2F_1(a,b;c;-z) = A_\gamma z^\gamma\left(1+\sum_{j=1}^{\infty}\frac{f_j}{z^j}+\log(z)\sum_{k=\delta}^{\infty}\frac{e_k}{z^k}\right), $$

where as before $\gamma = -\min(a,b)$, $\delta = |a-b| \in \mathbb{N}$, and A_γ is defined in (15). Hence, letting $y = z^{-1}\log(z)$, $f(z) = \sum_{k=\delta}^{\infty} e_k/z^{k-1}$ and $g(z) = 1+\sum_{j=1}^{\infty} f_j/z^j$ in (18) yields

$$ [{}_2F_1(a,b;c;-z)]^{-1} = (A_\gamma)^{-1}z^{-\gamma}\left(1+\sum_{j=1}^{\infty}\frac{\hat{f}_j}{z^j}\left[1+\hat{e}_{j,1}\frac{\log(z)}{z^{\delta-1}}+\hat{e}_{j,2}\frac{\log^2(z)}{z^{2(\delta-1)}}+\cdots+\hat{e}_{j,j}\frac{\log^j(z)}{z^{j(\delta-1)}}\right]\right). \tag{22} $$

In a similar fashion,

$$ {}_2F_1(a+n_1,b+n_2;c+m;-z) = A_\alpha z^\alpha\left(1+\sum_{j=1}^{\infty}\frac{g_j}{z^j}+\log(z)\sum_{k=\epsilon}^{\infty}\frac{q_k}{z^k}\right), \tag{23} $$

where as before $\alpha = -\min(a+n_1,b+n_2)$, $\varepsilon = |a+n_1-b-n_2| \in \mathbb{N}$ and A_α is defined in (15). The multiplication of (22) and (23) yields (21). □

Note that in the above lemma, $\alpha-\gamma \in \mathbb{Z}$. The remaining cases not covered by Lemmas 2 and 3 are the following. If $a=b$, but $-a, a-c \notin \mathbb{N}_0$, according to [3] (15.8.8), we have

$$ {}_2F_1(a,a;c;-z) = \frac{\log(z)\Gamma(c)}{\Gamma(a)\Gamma(c-a)z^a}\left(1+\frac{f_0}{\log(z)}+\sum_{k=1}^{\infty}\frac{e_k}{z^k}\left[1+\frac{f_k}{\log(z)}\right]\right), $$

so that (18) with $y = 1/\log(z)$, $f(z) = f_0 + \sum_{k=1}^{\infty} e_k f_k / z^k$ and $g(z) = 1 + \sum_{k=1}^{\infty} e_k / z^k$ implies

$$[{}_2F_1(a,a;c;-z)]^{-1} = \frac{\Gamma(a)\Gamma(c-a)z^a}{\Gamma(c)\log(z)}\left(1 + \sum_{k=1}^{\infty}\frac{f_0^k}{[\log(z)]^k} + \sum_{j=1}^{\infty}\frac{\hat{f}_j}{z^j}\left[1 + \frac{\hat{e}_{j,1}}{\log(z)} + \cdots + \frac{\hat{e}_{j,j}}{[\log(z)]^j}\right]\right). \tag{24}$$

In a similar fashion, if $a + n_1 = b + n_2$, but $-a - n_1, a + n_1 - c - m \notin \mathbb{N}_0$, we will have

$$ {}_2F_1(a+n_1, a+n_1; c+m; -z) = \frac{z^{-a-n_1}\log(z)\Gamma(c+m)}{\Gamma(a+n_1)\Gamma(c-a+m-n_1)}\left(1 + \frac{g_0}{\log(z)} + \sum_{k=1}^{\infty}\frac{q_k}{z^k}\left[1 + \frac{g_k}{\log(z)}\right]\right). \tag{25}$$

Hence, when both $a = b$ and $a + n_1 = b + n_2$, but there are no non-negative integers among the numbers $-a, a - c, -a - n_1, a + n_1 - c - m$, the asymptotic expansion of $R_{n_1,n_2,m}(-z)$ is obtained by the multiplication of (24) and (25). If $a = b$ but $a + n_1 \neq b + n_2$, we have to multiply (24) by (23) or, if $a + n_1 = b + n_2$ but $a \neq b$, then multiply (25) by (22). Finally, if the denominator is degenerate while the numerator is not, we multiply (19) by (23) when $a + n_1 \neq b + n_2$ or by (25) when $a + n_1 = b + n_2$. Similarly, if the numerator is degenerate while the denominator is not, we multiply (20) by (22) when $a \neq b$ or by (24) when $a = b$.

2.2. Boundary Values

For any integer r, define the Pochhammer symbol by $(z)_r = \Gamma(z+r)/\Gamma(z)$. Given three integers $n_1, n_2, m \in \mathbb{Z}$, define the following related quantities:

$$\begin{gathered} \underline{n} = \min(n_1,n_2),\ \overline{n} = \max(n_1,n_2),\ p = (m - n_1 - n_2)_+,\ l = (n_1 + n_2 - m)_+ \\ r = l + (m)_+ - \underline{n} - 1 = \begin{cases} \max(m - \underline{n}, \overline{n}) - 1, & m \geq 0 \\ \max(-\underline{n}, \overline{n} - m) - 1, & m \leq 0. \end{cases} \end{gathered} \tag{26}$$

Note that $p - l = m - n_1 - n_2$ and r may only be negative when $n_1 = n_2 = m = 0$, in which case $r = -1$. In the following theorem, which forms the main result of this subsection, we give an explicit formula for the imaginary part of $R_{n_1,n_2,m}(z)$ on the banks of the branch cut $[1,\infty)$. Note that for $x > 1$, the function ${}_2F_1(a,b;c;x \pm i0)$ may vanish at finitely many points in the degenerate case $\{-a,-b,c-a,c-b\} \cap \mathbb{N}_0 \neq \varnothing$, but does not vanish otherwise, see, respectively, Theorem 4 and [8] (Lemma 2, p. 54).

Theorem 2. *Suppose that $n_1, n_2, m \in \mathbb{Z}$, $a, b, c \in \mathbb{R}$ and $c, c + m \notin -\mathbb{N}_0$. The following identity holds on the banks of the branch cut $x > 1$:*

$$\mathrm{Im}[R_{n_1,n_2,m}(x \pm i0)] = \pm\pi B_{n_1,n_2,m}(a,b,c)\frac{x^{l-\underline{n}-c}(x-1)^{c-a-b-l}P_r(1/x)}{|{}_2F_1(a,b;c;x)|^2}, \tag{27a}$$

where

$$B_{n_1,n_2,m}(a,b,c) = -\frac{\Gamma(c)\Gamma(c+m)}{\Gamma(a)\Gamma(b)\Gamma(c-a+m-n_1)\Gamma(c-b+m-n_2)} \tag{27b}$$

and $P_r(t)$ is a polynomial of degree r $(P_{-1} \equiv 0)$ given by

$$P_r(t) = (-1)^{\overline{n}}\sum_{k=0}^{r}(-t)^k \sum_{j=(k-p)_+-\overline{n}}^{k-\overline{n}}(-1)^j\binom{p}{k-\overline{n}-j}K_j, \tag{28a}$$

where, with the convention $1/(-i)! = 0$ for $i \in \mathbb{N}$,

$$K_j = \frac{(1-a)_j(c-a)_{m+j}}{(b-a)_{n_2+j+1}(j+n_1)!}\,{}_4F_3\left(\begin{matrix}-j-n_1, a, 1+a-c, a-b-n_2-j\\ a-j, 1+a-c-m-j, 1+a-b\end{matrix}\middle|\,1\right)$$
$$+\frac{(1-b)_j(c-b)_{m+j}}{(a-b)_{n_1+j+1}(j+n_2)!}\,{}_4F_3\left(\begin{matrix}-j-n_2, b, 1+b-c, b-a-n_1-j\\ b-j, 1+b-c-m-j, 1+b-a\end{matrix}\middle|\,1\right). \quad (28b)$$

Remark 2. *Note that the coefficients of P_r depend on the parameters, so the whole expression (27a) may remain nonzero even when $B_{n_1,n_2,m}(a,b,c)$ vanishes. Furthermore, the polynomial $P_r(t) = P_r(t;n_1,n_2,m)$ depends on all three indices n_1, n_2, m and not only on the degree r. For example, rather, the straightforward calculation yields*

$$P_0(t;0;1;1) = -\frac{1}{b},\quad P_0(t;1,1,1) = -\frac{1}{ab},$$
$$P_1(t;0,2,2) = -\frac{ct+b-a+1}{b(b+1)},\quad P_1(t;0,0,2) = ct+a+b-2c-1.$$

The key fact that we will need for the proof of the above theorem is a more precise version of a particular case of [25] (Theorem 1), which (after some change of notation), reads:

Theorem 3. *Assume that $n_1, n_2, m \in \mathbb{Z}$. Then*

$$\frac{(\gamma-\alpha)_{-n_2}(\gamma-\beta)_{m-n_2}t^{n_1}}{(\gamma-1)_{n_1-n_2+1}}\,{}_2F_1\left(\begin{matrix}1-\gamma+\alpha, 1-\gamma+\beta\\ 2-\gamma\end{matrix}\middle|\,t\right){}_2F_1\left(\begin{matrix}\gamma-\alpha-n_2, \gamma-\beta+m-n_2\\ \gamma+n_1-n_2\end{matrix}\middle|\,t\right)+$$
$$\frac{(1-\alpha)_{-n_1}(1-\beta)_{m-n_1}t^{n_2}}{(1-\gamma)_{n_2-n_1+1}}\,{}_2F_1\left(\begin{matrix}\alpha, \beta\\ \gamma\end{matrix}\middle|\,t\right){}_2F_1\left(\begin{matrix}1-\alpha-n_1, 1-\beta+m-n_1\\ 2-\gamma+n_2-n_1\end{matrix}\middle|\,t\right) = \frac{t^{\underline{n}}P_r(t)}{(1-t)^p}, \quad (29)$$

where $P_r(t)$ is the polynomial (28) of degree r ($P_{-1} \equiv 0$) with parameters $a = \alpha$, $b = 1+\alpha-\gamma$, $c+\alpha-\beta$. This polynomial can also be computed by multiplying the left hand side of (29) by $t^{-\underline{n}}(1-t)^p$ and calculating the first $r+1$ Taylor coefficients on the left-hand side.

Remark 3. *The particular ${}_2F_1$ case of our general identity [25] (Theorem 1) given in (29) was essentially discovered by Ebisu in [26]. Namely, it can be derived by combining Theorem 3.7 with Proposition 3.4 from [26].*

Remark 4. *Our identity from [25] (Theorem 1) does not contain explicit expression (28) for the polynomial P_r. This expression is found in [27] (Lemma 6.1). It can also be computed by taking the limit $q \to 1$ in [28] (Theorem 2). For specific values of n_1, n_2, m, the second method of computing $P_r(t)$ indicated in the above theorem is more practical.*

Proof. The boundary values of the generalized hypergeometric function on the cut $[1,\infty)$ was found in [18] (Theorem 3). For the case of the Gauss function ${}_2F_1$, this theorem takes the form ($x > 1$):

$${}_2F_1(a,b;c;x\pm i0) = -\frac{\pi\Gamma(c)}{\Gamma(a)\Gamma(b)}G^{2,1}_{3,3}\left(\frac{1}{x}\,\middle|\,\begin{matrix}1, 3/2, c\\ a, b, 3/2\end{matrix}\right) \pm \pi i\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}G^{2,0}_{2,2}\left(\frac{1}{x}\,\middle|\,\begin{matrix}1, c\\ a, b\end{matrix}\right),$$

where $G^{m,n}_{p,q}$ denotes Meijer's G function defined by the Mellin–Barnes integral

$$G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right) := \frac{1}{2\pi i}\int_{\mathcal{L}}\frac{\Gamma(1-a_1-s)\cdots\Gamma(1-a_n-s)\Gamma(b_1+s)\cdots\Gamma(b_m+s)}{\Gamma(a_{n+1}+s)\cdots\Gamma(a_p+s)\Gamma(1-b_{m+1}-s)\cdots\Gamma(1-b_q-s)}z^{-s}ds, \quad (30)$$

where the contour $\mathcal{L}$ is a simple loop that starts and ends at infinity and separates the poles of $s \to \Gamma(b_j+s)$, $j = 1,\dots,m$, leaving them on the left from those of $s \to \Gamma(1-a_j-s)$,

$j = 1, \ldots, n$, leaving them on the right. Details regarding the choice of the contour $\mathcal{L}$ and the convergence of the above integral can be found, for instance, in [3] (Section 16.17), [27] (Formula (1.2)). As

$$\operatorname{Im}\left(\frac{\alpha + i\beta}{\gamma + i\delta}\right) = \frac{\beta\gamma - \alpha\delta}{|\gamma + i\delta|^2},$$

by writing $\phi_\pm(x) = \operatorname{Im}[R_{n_1,n_2,m}(x \pm i0)]$, we will get

$$\phi_\pm(x) = \operatorname{Im}\left[\frac{{}_2F_1(a+n_1, b+n_2; c+m; x \pm i0)}{{}_2F_1(a,b;c;x \pm i0)}\right] = \pm \frac{\pi^2 \Gamma(c)\Gamma(c+m)}{|{}_2F_1(a,b;c;x)|^2 \Gamma(a)\Gamma(b)\Gamma(a+n_1)\Gamma(b+n_2)}$$
$$\left\{ G^{2,1}_{3,3}\left(\frac{1}{x}\,\middle|\, \begin{matrix} 1, 3/2, c+m \\ a+n_1, b+n_2, 3/2 \end{matrix}\right) G^{2,0}_{2,2}\left(\frac{1}{x}\,\middle|\, \begin{matrix} 1, c \\ a, b \end{matrix}\right) - G^{2,1}_{3,3}\left(\frac{1}{x}\,\middle|\, \begin{matrix} 1, 3/2, c \\ a, b, 3/2 \end{matrix}\right) G^{2,0}_{2,2}\left(\frac{1}{x}\,\middle|\, \begin{matrix} 1, c+m \\ a+n_1, b+n_2 \end{matrix}\right) \right\}.$$

Meijer's G function here can be expanded as follows [18] (Proof of Theorem 3):

$$-G^{2,1}_{3,3}\left(t\,\middle|\, \begin{matrix} 1, 3/2, c \\ a, b, 3/2 \end{matrix}\right) = \frac{\Gamma(b-a)\Gamma(a)t^a}{\pi\Gamma(c-a)} {}_2F_1\left(\begin{matrix} a, 1-c+a \\ 1-b+a \end{matrix}\,\middle|\, t\right)\cos(\pi a)$$
$$+ \frac{\Gamma(a-b)\Gamma(b)t^b}{\pi\Gamma(c-b)} {}_2F_1\left(\begin{matrix} b, 1-c+b \\ 1-a+b \end{matrix}\,\middle|\, t\right)\cos(\pi b)$$

and

$$G^{2,0}_{2,2}\left(t\,\middle|\, \begin{matrix} 1, c \\ a, b \end{matrix}\right) = \frac{\Gamma(b-a)\Gamma(a)t^a}{\pi\Gamma(c-a)} {}_2F_1\left(\begin{matrix} a, 1-c+a \\ 1-b+a \end{matrix}\,\middle|\, t\right)\sin(\pi a)$$
$$+ \frac{\Gamma(a-b)\Gamma(b)t^b}{\pi\Gamma(c-b)} {}_2F_1\left(\begin{matrix} b, 1-c+b \\ 1-a+b \end{matrix}\,\middle|\, t\right)\sin(\pi b).$$

Substituting these expansions into the above formula for $\phi_\pm(x)$ and collecting terms, the expression in the braces becomes

$$\frac{\Gamma(b-a)\Gamma(a+n_1-b-n_2)\Gamma(b+n_2)\Gamma(a)x^{-a-b-n_2}}{\pi^2\Gamma(c-a)\Gamma(c+m-b-n_2)} {}_2F_1\left(\begin{matrix} a, 1-c+a \\ 1-b+a \end{matrix}\,\middle|\, \frac{1}{x}\right)$$
$$\times {}_2F_1\left(\begin{matrix} b+n_2, 1-c-m+b+n_2 \\ 1-a-n_1+b+n_2 \end{matrix}\,\middle|\, \frac{1}{x}\right)\sin(\pi(b+n_2-a))$$
$$+ \frac{\Gamma(a-b)\Gamma(b+n_2-a-n_1)\Gamma(a+n_1)\Gamma(b)x^{-b-a-n_1}}{\pi^2\Gamma(c-b)\Gamma(c+m-a-n_1)} {}_2F_1\left(\begin{matrix} b, 1-c+b \\ 1-a+b \end{matrix}\,\middle|\, \frac{1}{x}\right)$$
$$\times {}_2F_1\left(\begin{matrix} a+n_1, 1-c-m+a+n_1 \\ 1-b-n_2+a+n_1 \end{matrix}\,\middle|\, \frac{1}{x}\right)\sin(\pi(a+n_1-b)).$$

Then, writing $t = 1/x$, applying Euler's transformation and the reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, we obtain

$$\frac{|{}_2F_1(a,b;c;1/t)|^2\Gamma(a)\Gamma(b)\Gamma(a+n_1)\Gamma(b+n_2)}{\Gamma(c)\Gamma(c+m)}\phi_+(1/t)=$$
$$=\frac{\Gamma(a-b)\Gamma(b+n_2-a-n_1)\Gamma(a+n_1)\Gamma(b)t^{a+b+n_1}}{\Gamma(c-b)\Gamma(c+m-a-n_1)}$$
$$\times\,{}_2F_1\left(\begin{matrix}b,1-c+b\\1-a+b\end{matrix}\middle|\,t\right){}_2F_1\left(\begin{matrix}a+n_1,1-c-m+a+n_1\\1-b-n_2+a+n_1\end{matrix}\middle|\,t\right)\sin(\pi(a+n_1-b))$$
$$+\frac{\Gamma(b-a)\Gamma(a+n_1-b-n_2)\Gamma(b+n_2)\Gamma(a)t^{a+b+n_2}}{\Gamma(c-a)\Gamma(c+m-b-n_2)}$$
$$\times\,{}_2F_1\left(\begin{matrix}a,1-c+a\\1-b+a\end{matrix}\middle|\,t\right){}_2F_1\left(\begin{matrix}b+n_2,1-c-m+b+n_2\\1-a-n_1+b+n_2\end{matrix}\middle|\,t\right)\sin(\pi(b+n_2-a))$$
$$=\frac{\pi\Gamma(a-b)\Gamma(b-a+n_2-n_1)\Gamma(a+n_1)\Gamma(b)t^{a+b+n_1}(1-t)^{c-a-b+m-n_1-n_2}}{\Gamma(c-b)\Gamma(c-a+m-n_1)\Gamma(a-b+n_1)\Gamma(1+b-a-n_1)}$$
$$\times\,{}_2F_1\left(\begin{matrix}b,1-c+b\\1-a+b\end{matrix}\middle|\,t\right){}_2F_1\left(\begin{matrix}1-b-n_2,c-b+m-n_2\\1+a-b+n_1-n_2\end{matrix}\middle|\,t\right)$$
$$+\frac{\pi\Gamma(b-a)\Gamma(a-b+n_1-n_2)\Gamma(b+n_2)\Gamma(a)t^{a+b+n_2}(1-t)^{c-a-b+m-n_1-n_2}}{\Gamma(c-a)\Gamma(c+m-b-n_2)\Gamma(b-a+n_2)\Gamma(1+a-b-n_2)}$$
$$\times\,{}_2F_1\left(\begin{matrix}a,1-c+a\\1-b+a\end{matrix}\middle|\,t\right){}_2F_1\left(\begin{matrix}1-a-n_1,c-a+m-n_1\\1+b-a+n_2-n_1\end{matrix}\middle|\,t\right).$$

Further, writing $a=\alpha$, $b=1-\gamma+\alpha$, $c=1-\beta+\alpha$ after tedious but elementary transformations with the use of the relations

$$(1-z)_{-k}=\frac{(-1)^k}{(z)_k}\quad\text{and}\quad(z-r)_k=\frac{(z)_{k-r}}{(z)_{-r}}=(-1)^k\frac{(1-z)_r}{(1-z)_{r-k}},$$

the above expression reduces to

$$\frac{|{}_2F_1(a,b;c;1/t)|^2\Gamma(a)\Gamma(b)\Gamma(a+n_1)\Gamma(b+n_2)}{\Gamma(c)\Gamma(c+m)}\phi_+(1/t)$$
$$=-\frac{\pi t^{2\alpha-\gamma+1}(1-t)^{\gamma-\alpha-\beta+m-n_1-n_2}\Gamma(\alpha+n_1)\Gamma(1+\alpha-\gamma+n_2)}{\Gamma(1-\beta+m-n_1)\Gamma(\gamma-\beta+m-n_2)}\times$$
$$\Bigg\{\frac{(\gamma-\alpha)_{-n_2}(\gamma-\beta)_{m-n_2}t^{n_1}}{(\gamma-1)_{n_1-n_2+1}}{}_2F_1\left(\begin{matrix}1-\gamma+\alpha,1-\gamma+\beta\\2-\gamma\end{matrix}\middle|\,t\right){}_2F_1\left(\begin{matrix}\gamma-\alpha-n_2,\gamma-\beta+m-n_2\\\gamma+n_1-n_2\end{matrix}\middle|\,t\right)$$
$$+\frac{(1-\alpha)_{-n_1}(1-\beta)_{m-n_1}t^{n_2}}{(1-\gamma)_{n_2-n_1+1}}{}_2F_1\left(\begin{matrix}\alpha,\beta\\\gamma\end{matrix}\middle|\,t\right){}_2F_1\left(\begin{matrix}1-\alpha-n_1,1-\beta+m-n_1\\2-\gamma+n_2-n_1\end{matrix}\middle|\,t\right)\Bigg\}$$
$$=-\frac{\pi t^{2\alpha-\gamma+1+\underline{n}}(1-t)^{\gamma-\alpha-\beta-l}\Gamma(\alpha+n_1)\Gamma(1+\alpha-\gamma+n_2)}{\Gamma(1-\beta+m-n_1)\Gamma(\gamma-\beta+m-n_2)}P_r(t),$$

where the ultimate equality is an application of Theorem 3 with the notation introduced in (26).

Now substituting back $\alpha=a$, $\beta=1-c+a$, $\gamma=1-b+a$, we obtain

$$|{}_2F_1(a,b;c;1/t)|^2\phi_+(1/t)=-\frac{\pi\Gamma(c)\Gamma(c+m)t^{a+b}(1-t)^{c-a-b}}{\Gamma(a)\Gamma(b)\Gamma(c-a+m-n_1)\Gamma(c-b+m-n_2)}\frac{t^{\underline{n}}P_r(t)}{(1-t)^l}.$$

It remains to plug here $x=1/t$ to arrive at (27a). □

2.3. Integral Representation

The goal of this subsection is to construct an explicit integral representation for $R_{n_1,n_2,m}(z)$—the central result of this paper. It will be based on a polynomial correction of the standard Schwarz formula expressing the analytic function in the upper half-plane via the boundary values of its real part. The Schwarz formula is a particular case of the Stieltjes–Perron inversion formula (the measure in the Stieltjes–Perron inversion formula is often assumed to be positive (see [7] (no. 39), [6] (p. 188) or [5] (p. 250)), although this requirement can be relaxed) applied for recovering the representing measure in (4). However, the integral representation of the form (4) may already be too restrictive for the Gauss ratio $G(z) = R_{0,1,1}(z)$, let alone $R_{n_1,n_2,m}(z)$. The two main reasons are that (1) the right-hand side of (4) is analytic in $\mathbb{C} \setminus \mathbb{R}$, while $R_{0,1,1}(z)$ may have complex poles for certain values of a, b, c, and (2) the representing measure may grow too fast for the integral in (4) to be convergent. We deal with the first problem in Theorem 4 below containing conditions ensuring that there are no poles in $\mathbb{C} \setminus [1, \infty)$ as well as on the banks of the branch cut. Under these conditions, the corresponding signed measure (or charge) is supported on $[1, +\infty)$ and has an analytic density. Thus, to obtain an integral representation we only need to deal with the asymptotic behavior of $R_{n_1,n_2,m}(z)$ near the points $z = 1$ and $z = \infty$ to handle the second problem. This was solved by our rational correction presented in [14] (Lemma 4). In this paper, we will only use a particular case of [14] (Lemma 4) containing a polynomial correction at infinity. Recall that an analytic function is called real if $f(\overline{z}) = \overline{f(z)}$ in the appropriate domain.

Lemma 4. *Let $f(z)$ be a real analytic function defined in the cut plane $\mathbb{C} \setminus [1, +\infty)$ and suppose that $u(x) := \frac{1}{\pi} \operatorname{Im} f(x + i0)$ is continuous on $(1, +\infty)$. Suppose that there exists $n \in \mathbb{N}_0$ such that*

$$\lim_{|z-1|\to 0} \left| f(z)(1-z) \right| = \lim_{|z|\to\infty} \left| f(z) z^{-n} \right| = 0 \tag{31}$$

and $u(x)x^{-n-1}$ is absolutely integrable over $(1, +\infty)$. Then

$$f(z) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0) z^k}{k!} + z^n \int_1^{+\infty} \frac{u(x)\, dx}{(x-z)x^n}. \tag{32}$$

The above lemma assumes that the function f is analytic in $\mathbb{C} \setminus [1, +\infty)$. Hence, in order to apply it to $R_{n_1,n_2,m}(z)$, we need to make sure that the denominator ${}_2F_1(a, b; c; z)$ does not vanish in this domain. Such conditions will follow from an important theorem due to Runckel [8]. We will denote by $\lfloor \xi \rfloor$ the maximal integer number $\le \xi$ for any $\xi \in \mathbb{R}$. Note that if ξ is non-integer, then $\lfloor -\xi \rfloor = -\lfloor \xi \rfloor - 1$.

Theorem 4. *The function ${}_2F_1(a, b; c; z)$ does not vanish for $z \in \mathbb{C} \setminus [1, +\infty)$, including on the banks of the branch cut if and only if $(a, b, c) \in V \subset \mathbf{R}^3$, where V is the set of points (a, b, c) with $c \neq 0$ satisfying any of the following conditions:*

(I) $-1 < \min(a, b) \le c \le \max(a, b) \le 0$;

(II) $-1 < \min(a, b) \le 0 \le \max(a, b) \le c$;

(III) $-1 < c \le \min(a, b) \le 0 \le \max(a, b) < c + 1$;

(IV) $0 \le \min(a, b) \le c$ *and* $\max(a, b) < c + 1$;

(V) *$a, b, c, c-a, c-b$ are non-integer negative numbers, such that $\lfloor \xi_1 \rfloor + 1 = \lfloor \xi_4 \rfloor$ and $\lfloor \xi_2 \rfloor = \lfloor \xi_3 \rfloor$, where $\xi_1, \ldots, \xi_4$ are the numbers $a, b, c-a, c-b$ taken in non-decreasing order:*

$$\min(a, b, c-a, c-b) = \xi_1 \le \xi_2 \le \xi_3 \le \xi_4 = \max(a, b, c-a, c-b);$$

(VI) $0 \in \{a, b, c-a, c-b\}$.

In this form, this theorem was formulated and proved by us in [14] (Corollary 2).

Remark 5. *Under condition (V), one necessarily has $c - \xi_4 = \xi_1 < \xi_2$ and $c - \xi_2 = \xi_3 < \xi_4$. Indeed, $\xi_1 + \xi_4 = c = \xi_2 + \xi_3$ in view of $a + (c-a) = c = b + (c-b)$. So, if we have one of* *the equalities $\xi_1 = \xi_2$ and $\xi_3 = \xi_4$, we automatically have the other, assuming that the last two equalities together will contradict to $\lfloor \xi_1 \rfloor + 1 = \lfloor \xi_4 \rfloor$ on account of $\lfloor \xi_2 \rfloor = \lfloor \xi_3 \rfloor$.*

In fact, (V) is generated by the following two basic cases,

$$
\begin{aligned}
&-k-1 < a < \min(b, c-b) \leq \max(b, c-b) < -k < c-a < -k+1, \quad k \in \mathbb{N}, \quad \text{and} \\
&-k-1 < a < -k < \min(b, c-b) \leq \max(b, c-b) < c-a < -k+1, \quad k \in \mathbb{N},
\end{aligned}
$$

further extended through the symmetry $a \leftrightarrow b$ and Euler's transformation exchanging $(a, b) \leftrightarrow (c-a, c-b)$.

Another important fact established by Runckel is the following corollary of [8] (Lemma 2):

Lemma 5. *If $a, b, c-a, c-b \notin -\mathbb{N}_0$ and $x > 1$, then ${}_2F_1(a, b; c; x \pm i0) \neq 0$.*

Lemmas 2 and 3 and the subsequent remarks show that the asymptotic expansion of $R_{n_1,n_2,m}(z)$ at infinity is a combination of terms of the form $Az^{\mu}[\log(z)]^k$, where A and μ are real numbers, while k is an integer. Condition (34) in the theorem below requires each exponent μ satisfying $\mu \geq N$, $N \in \mathbb{N}_0$ to be an integer and the corresponding k to be zero (no logarithms at powers $\mu \geq N$). The following theorem is the main result of this section.

Theorem 5. *Suppose that $(a, b, c) \in V$, with V defined in Theorem 4, and $\eta(\cdot)$ given in (8) satisfies*

$$\eta(a+n_1, b+n_2, c+m) - \eta(a, b, c) > -1. \tag{33}$$

Assume further that there exists $N \in \mathbb{N}_0$ such that the asymptotics of $R_{n_1,n_2,m}(z)$ at infinity have the form

$$R_{n_1,n_2,m}(z) = Q_{a,b,c}(z) + o(z^N) \text{ as } z \to \infty, \tag{34}$$

where $Q_{a,b,c}(z)$ is a (possibly vanishing) polynomial with real coefficients and the lowest degree non-vanishing term $\sim z^N$. Then the following representation holds true:

$$
\begin{aligned}
R_{n_1,n_2,m}(z) = Q_{a,b,c}(z) &+ \sum_{k=0}^{N-1} \frac{R^{(k)}_{n_1,n_2,m}(0)}{k!} z^k \\
&+ z^N B_{n_1,n_2,m}(a, b, c) \int_1^{\infty} \frac{x^{l-\underline{n}-c-N}(x-1)^{c-a-b-l} P_r(1/x)}{|{}_2F_1(a, b; c; x)|^2 (x-z)} dx,
\end{aligned} \tag{35}
$$

where r, l and $B_{n_1,n_2,m}(a, b, c)$ retain their meanings from Theorem 2 and P_r is defined in (28). If (34) holds with $N = 0$, we obtain

$$R_{n_1,n_2,m}(z) = Q_{a,b,c}(z) + B_{n_1,n_2,m}(a, b, c) \int_1^{\infty} \frac{x^{l-\underline{n}-c}(x-1)^{c-a-b-l} P_r(1/x)}{|{}_2F_1(a, b; c; x)|^2 (x-z)} dx \tag{36}$$

In particular, (34) holds for $N = 0$, $Q_{a,b,c}(z)$ being a constant if $n_1, n_2 \geq 0$ and (10) is satisfied.

Remark 6. *Note that the choice of N and $Q_{a,b,c}(z)$ in (34) is not unique. In particular, it follows from Lemmas 2 and 3 that we can always take $Q_{a,b,c}(z) = 0$ by choosing a large enough N.*

Remark 7. *The first two terms of the Taylor expansion of $R_{n_1,n_2,m}(z)$ are given by*

$$R_{n_1,n_2,m}(z) = 1 + \frac{(an_2 + bn_1 + n_1 n_2)c - abm}{c(c+m)} z + O(z^2).$$

Remark 8. *Substitution $x = 1/t$ brings Formula (36) to the form (we write $B = B_{n_1,n_2,m}(a, b, c)$ for brevity):*

$$R_{n_1,n_2,m}(z) = Q_{a,b,c}(z) + \sum_{k=0}^{N-1} \frac{R^{(k)}_{n_1,n_2,m}(0)}{k!} z^k + z^N B \int_0^1 \frac{t^{a+b+\underline{n}+N-1}(1-t)^{c-a-b-l} P_r(t)}{|{}_2F_1(a,b;c;1/t)|^2 (1-zt)} dt. \tag{37}$$

This form turns out to be more convenient in most applications. Moreover, taking $z = 0$ or $z = 1$, we obtain the following curious integral evaluations:

$$\int_0^1 \frac{t^{a+b+\underline{n}+N-1}(1-t)^{c-a-b-l} P_r(t)}{|{}_2F_1(a,b;c;1/t)|^2} dt = \frac{R^{(N)}_{n_1,n_2,m}(0) - Q_N N!}{N! B}, \tag{38}$$

where Q_N denotes the coefficient at z^N in $Q_{a,b,c}(z)$, and

$$\int_0^1 \frac{t^{a+b+\underline{n}+N-1}(1-t)^{c-a-b-l-1} P_r(t)}{|{}_2F_1(a,b;c;1/t)|^2} dt = \frac{R_{n_1,n_2,m}(1) - Q_{a,b,c}(1)}{B} - \frac{1}{B} \sum_{k=0}^{N-1} \frac{R^{(k)}_{n_1,n_2,m}(0)}{k!}, \tag{39}$$

where, in view of the Gauss summation formula,

$$R_{n_1,n_2,m}(1) = \frac{(c)_m (c-a-b)_{m-n_1-n_2}}{(c-a)_{m-n_1} (c-b)_{m-n_2}}.$$

Multiplying the integrand in (39) by $(1-t)$, splitting the result in two summands, and using both formulae (38) and (39), we also obtain

$$\int_0^1 \frac{t^{a+b+\underline{n}+N}(1-t)^{c-a-b-l-1} P_r(t)}{|{}_2F_1(a,b;c;1/t)|^2} dt = \frac{R_{n_1,n_2,m}(1) - Q_{a,b,c}(1) + Q_N}{B} - \frac{1}{B} \sum_{k=0}^{N} \frac{R^{(k)}_{n_1,n_2,m}(0)}{k!}. \tag{40}$$

Remark 9. *The absolute value of ${}_2F_1$ on the branch cut in the integrands in (35) and (36) can be computed as follows ($x > 1$):*

$$\begin{aligned} |{}_2F_1(a,b;c;x)|^2 = \frac{\pi^2 \Gamma(c)^2}{\Gamma(a)^2 \Gamma(b)^2} \Bigg\{ & \frac{(x-1)^{2(c-a-b)}}{[\Gamma(c-a-b)]^2} \left[{}_2F_1\left(\begin{matrix} c-a, c-b \\ c-a-b \end{matrix} \middle| 1-x \right) \right]^2 \\ & + \left[\frac{\Gamma(b-a)\Gamma(a) x^{-a}}{\Gamma(c-a)\Gamma(1/2-a)\Gamma(1/2+a)} {}_2F_1\left(\begin{matrix} a, 1-c+a \\ 1-b+a \end{matrix} \middle| 1/x \right) \right. \\ & \left. + \frac{\Gamma(a-b)\Gamma(b) x^{-b}}{\Gamma(c-b)\Gamma(1/2-b)\Gamma(1/2+b)} {}_2F_1\left(\begin{matrix} b, 1-c+b \\ 1-a+b \end{matrix} \middle| 1/x \right) \right]^2 \Bigg\}. \end{aligned}$$

Proof of Theorem 5. Define $f(z) = R_{n_1,n_2,m}(z) - Q_{a,b,c}(z)$. As the lowest degree term in $Q_{a,b,c}(z)$ is $\sim z^N$, in view of the condition $(a,b,c) \in V$, Theorem 4 implies that the function

$$\hat{f}_N(z) = R_{n_1,n_2,m}(z) - Q_{a,b,c}(z) - \sum_{k=0}^{N-1} \frac{f^{(k)}(0)}{k!} z^k = R_{n_1,n_2,m}(z) - Q_{a,b,c}(z) - \sum_{k=0}^{N-1} \frac{R^{(k)}_{n_1,n_2,m}(0)}{k!} z^k$$

is holomorphic in $z \in \mathbb{C} \setminus [1,\infty)$ and has no singularities on the banks of the branch cut other than $z = 1$ and $z = \infty$. We aim at the application of Lemma 4 to the function $\hat{f}_N(z)$. Denote $u(x) = \mathrm{Im}(\hat{f}_N(x+i0))$. As $Q_{a,b,c}(z)$ has real coefficients, we conclude that $u(x) = \mathrm{Im}[R_{n_1,n_2,m}(x+i0)]$.

If condition (33) is satisfied, then Formula (6) from Theorem 1 guarantees that the first limit in (31) in Lemma 4 is indeed equal to zero for $R_{n_1,n_2,m}(z)$ and hence also for $\hat{f}_N(z)$. Moreover, for any $-1 < \theta < \eta(a+n_1, b+n_2, c+m) - \eta(a,b,c)$, we will have

$$|u(x) x^{-N-1}| \le |R_{n_1,n_2,m}(x+i0)| \le M|1-x|^{\theta} \tag{41}$$

for some $M > 0$ in certain neighborhood of $x = 1$. Hence, $u(x)x^{-N-1}$ is integrable in the neighborhood of $x = 1$.

Further, condition (34) leads to the second equality in (31) with $n = N$ for the function $\hat{f}_N(z)$. Indeed, the condition (34) gives precisely (31) for the function $f(z)$ by the definition of o symbol and the extra terms in $\hat{f}_N(z)$ go to zero as $z \to \infty$ after division by z^N. Then Lemmas 2 and 3 imply that the asymptotics at infinity must have one of the forms

$$z^{-N}\hat{f}_N(z) = \frac{C}{\log(z)}\Big(1 + O\Big([\log(z)]^{-1}\Big)\Big) \text{ as } z \to \infty$$

or

$$z^{-N}\hat{f}_N(z) = \frac{C}{z^\tau}(1 + o(1)) \text{ as } z \to \infty$$

for some $\tau > 0$. In view of

$$\left|\operatorname{Im}\frac{1}{\log(x+i0)}\right| \le \frac{\pi}{\log^2|x| + \pi^2}$$

which leads to

$$\frac{u(x)}{x^{N+1}} = O\left(\frac{1}{x\log^2(x)}\right) \quad \text{or} \quad \frac{u(x)}{x^{N+1}} = O\left(\frac{1}{x^{1+\tau}}\right) \text{ as } x \to \infty.$$

This implies the absolute integrability of $x^{-N-1}u(x)$ on $(1, +\infty)$. Hence, we are in the position to apply Lemma 4 leading to formula (35) by an application of Theorem 2. The ultimate claim of the Theorem follows directly from Lemmas 2 and 3. □

3. Examples

In this section, we will apply Theorem 5 to 15 specific triples (n_1, n_2, m) to obtain integral representations of the ratio $R_{n_1,n_2,m}(z)$ defined in (5). These representations are only valid if $R_{n_1,n_2,m}(z)$ is well behaved near $z = 1$ and its denominator ${}_2F_1(a,b;c;z) \neq 0$ in the cut plane $\mathbb{C} \setminus [1, +\infty)$ and on the banks of the branch cut. Conditions for the latter are given in Theorem 4, while the former in ensured by the inequality (33). To relax these restrictions, one needs a kind of regularization near the point $z = 1$ as well as near all zeros of the denominator. Such regularizations were explored by us in [14]. We will further mention conditions for $R_{n_1,n_2,m}(z)$ to belong the Markov $\mathcal{M}$ and the Stieltjes $\mathcal{S}$ classes, whose definitions can be found below formula (4).

Example 1. *For the Gauss ratio $R_{0,1,1}(z)$ according to (26), we obtain $p = l = r = 0$. Theorem 3 and definition (27b) yield*

$$B_{0,1,1}P_0(t) \equiv \frac{\Gamma(c)\Gamma(c+1)}{\Gamma(a)\Gamma(b+1)\Gamma(c-a+1)\Gamma(c-b)}.$$

Next, using (16) and (21), or directly, it is easy to verify that

$$Q_{a,b,c} = \lim_{z\to\infty} R_{0,1,1}(z) = \begin{cases} 0, & b \le a \\ [c(b-a)]/[b(c-a)], & b > a. \end{cases}$$

Then, Theorem 5 with $N = 0$ yields

$$R_{0,1,1}(z) = Q_{a,b,c} + \frac{\Gamma(c)\Gamma(c+1)}{\Gamma(a)\Gamma(b+1)\Gamma(c-b)\Gamma(c-a+1)}\int_0^1 \frac{t^{a+b-1}(1-t)^{c-a-b}dt}{(1-zt)|{}_2F_1(a,b;c;t^{-1})|^2}.$$

In order for this representation to hold, we need to assume that $(a,b,c) \in V$, that is to say (a,b,c) satisfies at least one of the conditions (I)–(VI) from Theorem 4. For $(a,b,c) \in V$ the condition (33) from Theorem 5 holds automatically since the parameter $q = m - n_1 - n_2$ in Lemma 1 vanishes such that $R_{0,1,1}(z)$ is integrable in the neighborhood of $z = 1$. We remark that the integrand is symmetric with respect to the interchange of a and b, and the asymmetry of $R_{0,1,1}(z)$ is only reflected in the constants $Q_{a,b,c}$ and $B_{0,1,1}$.

The above integral representation was first found by V. Belevitch in [11] (Formula (72)) under the restrictions $0 \le a, b \le c$, $c \ge 1$ (there is a small mistake in Belevitch's paper—a superfluous 2 in the denominator of the constant $Q_{a,b,c}$). Independently, using the Gauss continued fraction (2) and Wall's theorem, Küstner [10] (Theorem 1.5) proved that $R_{0,1,1}(z)$ is a Markov function (generating function of a Hausdorff moment sequence) if $0 < a \le c+1$, $0 < b \le c$. As we mentioned in introduction, the coefficients of the Gauss continued fraction (2) for $R_{0,1,1}(z)$ are all positive if (a) $-1 < a < 0$ and either $-1 < b < c < 0$ or $0 < c < b < c+1$ or (b) $0 < a < c+1$, $c > 0$ and $-1 < b < c$. If these conditions hold, while conditions of Runckel's Theorem 4 are violated, i.e., $(a,b,c) \notin V$, then representation (4) is true while the above integral representation is not. Hence, in this situation, $R_{0,1,1}(z)$ has pole(s) in the interval $(0,1)$, which are reflected by the atoms of the representing measure in (4) at some real points $s_k > 1$. This is the case, for instance, if $0 < c < a < c+1$ and $-1 < b < 0$. In this situation $R_{0,1,1}(z)$ still belongs to the Stieltjes class $\mathcal{S}$.

Example 2. *For the ratio $R_{0,1,0}(z)$ according to (26), we obtain $l = 1$, $p = r = 0$. Theorem 3 and definition (27b) yield*

$$B_{0,1,0}P_0(t) \equiv \frac{[\Gamma(c)]^2}{\Gamma(a)\Gamma(b+1)\Gamma(c-a)\Gamma(c-b)}.$$

Next, using (16) and (21), or directly, we can verify that

$$Q_{a,b} = \lim_{z\to\infty} R_{0,1,0}(z) = \begin{cases} 0, & b \le a \\ (b-a)/b, & b > a. \end{cases}$$

Then Theorem 5 with $N = 0$ yields

$$R_{0,1,0}(z) = Q_{a,b} + \frac{[\Gamma(c)]^2}{\Gamma(a)\Gamma(b+1)\Gamma(c-a)\Gamma(c-b)} \int_0^1 \frac{t^{a+b-1}(1-t)^{c-a-b-1}dt}{(1-zt)|{}_2F_1(a,b;c;1/t)|^2}.$$

Note that similarly to Example 1, the integrand is symmetric with respect to the interchange of a and b, and the asymmetry of the left-hand side is only reflected in the constants. In order for this representation to hold, we need to assume that $(a,b,c) \in V$ in Theorem 4. Under this restriction and except for the degenerate cases $ab = 0$ and $(c-a)(c-b) = 0$, the condition (33) reads

$$(c-a-b-1)_- - (c-a-b)_- > -1,$$

which is easily seen to be equivalent to $c > a+b$. The above set of conditions holds, for example, if $-1 < a < 0$ and $0 < b < c$ or $a > 0$ and $-1 < b < c-a$. Note that the degenerate cases $ab = 0$ and $(c-a)(c-b) = 0$ yield the standard Euler's integral [23] (Theorem 2.2.4) in the above representation (although the integral may disappear when multiplied by zero). This remark is also true for all subsequent examples, so we will omit it in the sequel.

Using continued fractions, Küstner [10] (Theorem 1.5) proved that $R_{0,1,0}(z) \in \mathcal{M}$ (the Markov class) if $-1 \le b \le c$ and $0 < a \le c$. Askitis [29] (Lemma 6.2.2) found another proof for the this claim (without a use of continued fractions). We also remark that the continued fraction for $R_{0,1,0}$ was also found by Gauss; see [1] (Equation (26)) or [10] (Equation (2.7)), in the form

$$\cfrac{1}{1-\cfrac{\alpha_1 z}{1-\cfrac{\alpha_2 z}{1-\ddots}}},$$

where $\alpha_1 = a/c$, *and for* $k \geq 1$

$$\alpha_{2k} = \frac{(b+k)(c-a+k-1)}{(c+2k-2)(c+2k-1)}, \quad \alpha_{2k+1} = \frac{(a+k)(c-b+k-1)}{(c+2k-1)(c+2k)}.$$

From these formulae, it is also not difficult to formulate sufficient conditions for $\alpha_n \geq 0$ *ensuring that* $R_{0,1,0} \in \mathcal{S}$ *(the Stieltjes class).*

Example 3. *For the ratio* $R_{1,1,1}(z)$ *according to (26), we obtain* $l = 1$, $p = r = 0$. *Theorem 3 and definition (27b) yield*

$$B_{1,1,1}P_0(t) = \frac{\Gamma(c)\Gamma(c+1)}{\Gamma(a+1)\Gamma(b+1)\Gamma(c-a)\Gamma(c-b)}$$

Next, it is easy to verify using (16) and (21) or directly that

$$Q_{a,b,c} = \lim_{z\to\infty} R_{1,1,1}(z) = 0.$$

Then, according to the case $N = 0$ *of Theorem 5, we obtain*

$$R_{1,1,1}(z) = \frac{\Gamma(c)\Gamma(c+1)}{\Gamma(a+1)\Gamma(b+1)\Gamma(c-a)\Gamma(c-b)} \int_0^1 \frac{t^{a+b}(1-t)^{c-a-b-1}dt}{(1-zt)|{}_2F_1(a,b;c;1/t)|^2}.$$

In order for this representation to hold, we need to assume that $(a,b,c) \in V$ *in Theorem 4. Under this restriction and except for the degenerate cases* $ab = 0$ *and* $(c-a)(c-b) = 0$, *the condition (33) reads*

$$(c-a-b-1)_- - (c-a-b)_- > -1,$$

which is easily seen to be equivalent to $c > a+b$. *All these conditions are satisfied, for example, if (a)* $-1 < a < 0$ *and* $0 < b < c$ *or (b)* $0 < a < c$ *and* $-1 < b < c-a$. *The above integral representation obviously implies that* $R_{1,1,1} \in \mathcal{M}$ *if the constant in front of the integral is positive (or* $-R_{1,1,1} \in \mathcal{M}$ *otherwise).*

Example 4. *For the ratio* $R_{1,1,2}(z)$ *according to (26), we obtain* $l = p = r = 0$. *Theorem 3 and definition (27b) yield*

$$B_{1,1,2}P_0(t) = B_{1,1,2}P_0 = \frac{\Gamma(c+1)\Gamma(c+2)}{\Gamma(a+1)\Gamma(b+1)\Gamma(c-a+1)\Gamma(c-b+1)}.$$

Next, it is easy to verify using (16) and (21) or directly that

$$Q_{a,b,c} = \lim_{z\to\infty} R_{1,1,2}(z) = 0.$$

Then, according to the case $N = 0$ *of Theorem 5, we obtain*

$$R_{1,1,2}(z) = \frac{\Gamma(c+1)\Gamma(c+2)}{\Gamma(a+1)\Gamma(b+1)\Gamma(c-a+1)\Gamma(c-b+1)} \int_0^1 \frac{t^{a+b}(1-t)^{c-a-b}dt}{(1-zt)|{}_2F_1(a,b;c;1/t)|^2}.$$

In order for this representation to hold, we need to assume that $(a,b,c) \in V$ *in Theorem 4. Under this restriction and except for the degenerate cases* $ab = 0$ *and* $(c-a)(c-b) = 0$, *the condition (33) reads*

$$(c-a-b)_- - (c-a-b)_- > -1$$

and is trivially satisfied. If the above integral representation holds true, then $R_{1,1,2} \in \mathcal{M}$ *once the constant in front of the integral is positive, which is the case for parameters satisfying any of the conditions (I)–(V) of Theorem 4.*

Example 5. *For the ratio $R_{0,2,2}(z)$ according to (26), we obtain $l = p = 0$, $r = 1$. Theorem 3 and definition (27b) yield*

$$B_{0,2,2}P_1(t) = \frac{\Gamma(c)\Gamma(c+2)(ct+b-a+1)}{\Gamma(a)\Gamma(b+2)\Gamma(c-a+2)\Gamma(c-b)}.$$

Next, it is easy to verify using (16) and (21) or directly that

$$Q_{a,b,c} = \lim_{z\to\infty} R_{0,2,2}(z) = \begin{cases} 0, & b \le a \\ c(c+1)(b-a)(b-a+1)/[b(b+1)(c-a)(c-a+1)], & b > a. \end{cases}$$

Then, according to the case $N = 0$ of Theorem 5, we obtain

$$R_{0,2,2}(z) = Q_{a,b,c} + \frac{\Gamma(c)\Gamma(c+2)}{\Gamma(a)\Gamma(b+2)\Gamma(c-a+2)\Gamma(c-b)} \int_0^1 \frac{t^{a+b-1}(ct+b-a+1)(1-t)^{c-a-b}dt}{(1-zt)|{}_2F_1(a,b;c;1/t)|^2}.$$

In order for this representation to hold, we need to assume that $(a,b,c) \in V$ in Theorem 4. Under this restriction and except for the degenerate cases $ab = 0$ and $(c-a)(c-b) = 0$, the condition (33) reads

$$(c-a-b)_- - (c-a-b)_- > -1$$

and is trivially satisfied. Here, we need to require that the zero $t^* = (a-b-1)/c$ of the polynomial $ct+b-a+1$ lies outside the interval $(0,1)$ in order that $R_{0,2,2} \in \mathcal{M}$ or $-R_{0,2,2} \in \mathcal{M}$ (depending on the signs of the measure and the constant).

Example 6. *For the ratio $R_{0,2,0}(z)$ according to (26), we obtain $p = 0$, $l = 2$, $r = 1$. Theorem 3 and definition (27b) yield*

$$B_{0,2,0}P_1(t) = \frac{[\Gamma(c)]^2(t(c-2b-2)+b+1-a)}{\Gamma(a)\Gamma(b+2)\Gamma(c-a)\Gamma(c-b)}.$$

Next, it is easy to verify using (16) and (21) or directly that

$$Q_{a,b} = \lim_{z\to\infty} R_{0,2,0}(z) = \begin{cases} 0, & b \le a \\ (b-a)(b-a+1)/[b(b+1)], & b > a. \end{cases}$$

Then, according to the case $N = 0$ of Theorem 5, we obtain

$$R_{0,2,0}(z) = Q_{a,b} + \frac{[\Gamma(c)]^2}{\Gamma(a)\Gamma(b+2)\Gamma(c-a)\Gamma(c-b)} \int_0^1 \frac{t^{a+b-1}(b-a+1+t(c-2b-2))(1-t)^{c-a-b-2}dt}{(1-zt)|{}_2F_1(a,b;c;1/t)|^2}.$$

In order for this representation to hold, we need to assume that $(a,b,c) \in V$ in Theorem 4. Under this restriction and except for the degenerate cases $ab = 0$ and $(c-a)(c-b) = 0$, the condition (33) reads

$$(c-a-b-2)_- - (c-a-b)_- > -1$$

which is easily seen to be equivalent to $c > a+b+1$. Similar to the previous example, a necessary condition for $R_{0,2,0} \in \mathcal{M}$ or $-R_{0,2,0} \in \mathcal{M}$ is that the zero $t^* = (a-b-1)/(c-2b-2)$ of the polynomial $b-a+1+t(c-2b-2)$ lies outside the interval $(0,1)$.

Example 7. *For the ratio $R_{1,1,0}(z)$ according to (26), we obtain $p = 0$, $l = 2$, $r = 0$. Theorem 3 and definition (27b) yield*

$$B_{1,1,0}P_0(t) = -\frac{[\Gamma(c)]^2(c-a-b-1)}{\Gamma(a+1)\Gamma(b+1)\Gamma(c-a)\Gamma(c-b)}.$$

Next, it is easy to verify using (16) and (21) or directly that

$$Q_{a,b,c} = \lim_{z\to\infty} R_{1,1,0}(z) = 0.$$

Then, according to the case $N = 0$ of Theorem 5, we obtain

$$R_{1,1,0}(z) = -\frac{[\Gamma(c)]^2(c-a-b-1)}{\Gamma(a+1)\Gamma(b+1)\Gamma(c-a)\Gamma(c-b)} \int_0^1 \frac{t^{a+b}(1-t)^{c-a-b-2}dt}{(1-zt)|{}_2F_1(a,b;c;1/t)|^2}.$$

In order for this representation to hold, we need to assume that $(a,b,c) \in V$ in Theorem 4. Under this restriction and except for the degenerate cases $ab = 0$ and $(c-a)(c-b) = 0$, the condition (33) reads

$$(c-a-b-2)_- - (c-a-b)_- > -1$$

which is easily seen to be equivalent to $c > a+b+1$. Now, if the above integral representation for $R_{1,1,0}$ holds true, then either $-R_{1,1,0}$ or $R_{1,1,0}$ belong to the class $\mathcal{M}$ (depending on the sign of the constant in front of the integral).

Example 8. *For the ratio $R_{0,0,1}(z)$ according to (26), we obtain $p = 1$, $l = r = 0$. Theorem 3 and definition (27b) yield*

$$B_{0,0,1}P_0(t) = -\frac{\Gamma(c)\Gamma(c+1)}{\Gamma(a)\Gamma(b)\Gamma(c-a+1)\Gamma(c-b+1)}.$$

Next, it is easy to verify using (16) and (21) or directly that

$$Q_{a,b,c} = \lim_{z\to\infty} R_{0,0,1}(z) = \begin{cases} c/(c-b), & b \le a \\ c/(c-a), & b > a, \end{cases}$$

unless $c = \min(a,b)$. Then, the case $N = 0$ of Theorem 5 leads to the representation

$$R_{0,0,1}(z) = Q_{a,b,c} - \frac{\Gamma(c)\Gamma(c+1)}{\Gamma(a)\Gamma(b)\Gamma(c-a+1)\Gamma(c-b+1)} \int_0^1 \frac{t^{a+b-1}(1-t)^{c-a-b}dt}{(1-zt)|{}_2F_1(a,b;c;1/t)|^2}.$$

In order for this representation to hold, we need to assume that $(a,b,c) \in V$ in Theorem 4. Under this restriction and except for the degenerate cases $ab = 0$ and $(c-a)(c-b) = 0$, the condition (33) reads

$$(c-a-b+1)_- - (c-a-b)_- > -1$$

which is easily seen to be satisfied for all real a, b, c. Here, $R_{0,0,1}(z) - Q_{a,b,c}$ or $Q_{a,b,c} - R_{0,0,1}(z)$ is a Markov function under conditions (I)–(II) or, respectively, (III)–(V) of Theorem 4.

Example 9. *For the ratio $R_{0,0,-1}(z)$ according to (26) we obtain $l = 1$, $p = r = 0$. Theorem 3 and definition (27b) then yield*

$$B_{0,0,-1}P_0(t) = \frac{\Gamma(c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}.$$

Next, it is easy to verify using (16) and (21) or directly that

$$Q_{a,b,c} = \lim_{z\to\infty} R_{0,0,-1}(z) = \begin{cases} (c-b-1)/(c-1), & b \le a \\ (c-a-1)/(c-1), & b > a. \end{cases}$$

Then, the case $N = 0$ of Theorem 5 leads to the representation

$$R_{0,0,-1}(z) = Q_{a,b,c} + \frac{\Gamma(c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} \int_0^1 \frac{t^{a+b-1}(1-t)^{c-a-b-1}dt}{(1-zt)|{}_2F_1(a,b;c;1/t)|^2}.$$

In order for this representation to hold, we need to assume that $(a,b,c) \in V$ in Theorem 4. Under this restriction and except for the degenerate cases $ab = 0$ and $(c-a)(c-b) = 0$, the condition (33) reads

$$(c-a-b-1)_- - (c-a-b)_- > -1$$

which is easily seen to be equivalent to $c > a+b$. All these conditions are satisfied, for example, if (a) $-1 < a < 0$ and $0 < b < c-a$ or (b) $0 < a < c$ and $-1 < b < c-a$. Here, the representing measure is again positive for all values of parameters so that $R_{0,0,-1} \in \mathcal{M}$ provided the above integral representation holds and the constants are positive.

Example 10. *For the ratio $R_{0,0,2}(z)$ according to (26), we obtain $p=2$, $l=0$, $r=1$. The application of Theorem 3 and definition (27b) yields*

$$B_{0,0,2}P_1(t) = \frac{\Gamma(c)\Gamma(c+2)[ct+a+b-2c-1]}{\Gamma(a)\Gamma(b)\Gamma(c-a+2)\Gamma(c-b+2)}.$$

Next, it is easy to verify using (16) and (21) or directly that

$$Q_{a,b,c} = \lim_{z\to\infty} R_{0,0,2}(z) = \begin{cases} c(c+1)/[(c-b)(c-b+1)], & b \le a \\ c(c+1)/[(c-a)(c-a+1)], & b > a. \end{cases}$$

Then, the case $N=0$ of Theorem 5 leads to the representation

$$R_{0,0,2}(z) = Q_{a,b,c} + \frac{\Gamma(c)\Gamma(c+2)}{\Gamma(a)\Gamma(b)\Gamma(c-a+2)\Gamma(c-b+2)} \int_0^1 \frac{t^{a+b-1}(ct+a+b-2c-1)(1-t)^{c-a-b}dt}{(1-zt)|{}_2F_1(a,b;c;1/t)|^2}.$$

In order for this representation to hold, we need to assume that $(a,b,c) \in V$ in Theorem 4. Under this restriction and except for the degenerate cases $ab = 0$ and $(c-a)(c-b) = 0$, the condition (33) reads

$$(c-a-b+2)_- - (c-a-b)_- > -1$$

which is true for all real a,b,c. The necessary condition for $R_{0,0,2} \in \mathcal{M}$ or $-R_{0,0,2} \in \mathcal{M}$ is that the zero $t^* = (2c-a-b+1)/c$ of the polynomial $ct+a+b-2c-1$ lies outside the interval $(0,1)$. Under this condition, $R_{0,0,2} \in \mathcal{M}$ for the values of parameters, making the constants positive.

Example 11. *For the ratio $R_{0,1,2}(z)$ according to (26), we obtain $p=1$, $l=0$, $r=1$. Theorem 3 and definition (27b) yield*

$$B_{0,1,2}P_1(t) = -\frac{\Gamma(c)\Gamma(c+2)(ct+b-c)}{\Gamma(a)\Gamma(b+1)\Gamma(c-a+2)\Gamma(c-b+1)}.$$

Next, it is easy to verify using (16) and (21) or directly that

$$Q_{a,b,c} = \lim_{z\to\infty} R_{0,1,2}(z) = \begin{cases} 0, & b \le a \\ c(c+1)(b-a)/[b(c-a)(c-a+1)], & b > a. \end{cases}$$

Then, the case $N=0$ of Theorem 5 leads to the representation

$$R_{0,1,2}(z) = Q_{a,b,c} - \frac{\Gamma(c)\Gamma(c+2)}{\Gamma(a)\Gamma(b+1)\Gamma(c-a+2)\Gamma(c-b+1)} \int_0^1 \frac{t^{a+b-1}(ct+b-c)(1-t)^{c-a-b}dt}{(1-zt)|{}_2F_1(a,b;c;1/t)|^2}.$$

In order for this representation to hold, we need to assume that $(a,b,c) \in V$ in Theorem 4. Under this restriction and except for the degenerate cases $ab = 0$ and $(c-a)(c-b) = 0$, the condition (33) reads

$$(c-a-b+1)_- - (c-a-b)_- > -1$$

which is true for all real a, b, c. Similar to the previous example, the additional condition that $t^* = (c-b)/c \notin (0,1)$ yields $Q_{a,b,c} - R_{0,1,2} \in \mathcal{M}$ or $R_{0,1,2} - Q_{a,b,c} \in \mathcal{M}$ depending on whether the constant near the integral is positive or negative.

Example 12. *For the ratio $R_{0,-1,0}(z)$ according to (26) we obtain $l = 1$, $p = r = 0$. Theorem 3 and definition (27b) yield*

$$B_{0,-1,0}P_0(t) = -\frac{[\Gamma(c)]^2}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b+1)}.$$

Using Lemmas 2 and 3 or by direct, albeit tedious calculation, we obtain the following asymptotic approximations:

(1) *If $b+1 < a$, then $R_{0,-1,0}(z) = Az + B + o(1)$ as $z \to \infty$;*

(2) *If $b < a \le b+1$, then $R_{0,-1,0}(z) = Az + o(z)$ as $z \to \infty$;*

(3) *If $b-1 \le a \le b$, then $R_{0,-1,0}(z) = o(z)$ as $z \to \infty$;*

(4) *If $a < b-1$, then $R_{0,-1,0}(z) = C + o(1)$ as $z \to \infty$,*

where

$$A = \frac{b-a}{c-b}, \quad B = \frac{b(b+1) - 2ab + c(a-1)}{(c-b)(a-b-1)}, \quad C = \frac{b-1}{b-a-1}.$$

Hence, if $|a-b| > 1$, we have $R_{0,-1,0}(z) = \beta z + \alpha + o(1)$ as $z \to \infty$, with $(\beta,\alpha) = (A,B)$ if $a > b+1$ and $(\beta,\alpha) = (0,C)$ if $a < b-1$. Then for $|a-b| > 1$ we can choose $N = 0$ in Theorem 5 leading to the representation

$$R_{0,-1,0}(z) = \alpha + \beta z - \frac{[\Gamma(c)]^2}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b+1)} \int_0^1 \frac{t^{a+b-2}(1-t)^{c-a-b}}{(1-zt)|{}_2F_1(a,b;c;1/t)|^2} dt. \quad (42)$$

In addition to the condition $|a-b| > 1$, we need to assume that $(a,b,c) \in V$ in Theorem 4. Under these restrictions and except for the degenerate cases $ab = 0$ and $(c-a)(c-b) = 0$, the condition (33) reads

$$(c-a-b+1)_- - (c-a-b)_- > -1$$

which is true for all real a, b, c. If the above representation holds, we see that $\alpha + \beta z - R_{0,-1,0}(z) \in \mathcal{M}$ if the constant in front of the integral is positive.

For arbitrary a, b, we obtain $R_{0,-1,0}(z) = \beta z + o(z)$ as $z \to \infty$, with $\beta = A$ if $b < a$ and $\beta = 0$ if $a \le b$. Hence, we can remove the restriction $|a-b| > 1$ by taking $N = 1$ in Theorem 5, which leads to

$$R_{0,-1,0}(z) = 1 + \beta z - \frac{z[\Gamma(c)]^2}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b+1)} \int_0^1 \frac{t^{a+b-1}(1-t)^{c-a-b}}{(1-zt)|{}_2F_1(a,b;c;1/t)|^2} dt, \quad (43)$$

or, by taking $N = 2$, we obtain

$$R_{0,-1,0}(z) = 1 - \frac{ac}{c^2} z - \frac{z^2[\Gamma(c)]^2}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b+1)} \int_0^1 \frac{t^{a+b}(1-t)^{c-a-b}}{(1-zt)|{}_2F_1(a,b;c;1/t)|^2} dt. \quad (44)$$

Example 13. *For the ratio $R_{-1,-1,0}(z)$ according to (26), we obtain $p = 2$, $l = r = 0$. Theorem 3 and definition (27b) yields*

$$B_{-1,-1,0}P_0(t) = -\frac{[\Gamma(c)]^2(c-a-b+1)}{\Gamma(a)\Gamma(b)\Gamma(c-a+1)\Gamma(c-b+1)}.$$

Using Lemmas 2 and 3 or by direct, albeit tedious, calculation, we obtain the asymptotic approximations

(1) *If $a > b+1$, then $R_{-1,-1,0}(z) = B(a,b)z + A(a,b) + o(1)$ as $z \to \infty$;*

(2) *If $b \le a \le b+1$, then $R_{-1,-1,0}(z) = B(a,b)z + o(z)$ as $z \to \infty$;*

(3) *If* $b-1 \le a \le b$, *then* $R_{-1,-1,0}(z) = B(b,a)z + o(z)$ *as* $z \to \infty$;

(4) *If* $a < b-1$, *then* $R_{1,-1,0}(z) = B(b,a)z + A(b,a) + o(1)$ *as* $z \to \infty$,

where

$$B(a,b) = \frac{a-1}{b-c}, \quad A(a,b) = \frac{(a-1)(2b-c)}{(c-b)(1+b-a)}.$$

Hence, if $|a-b| > 1$, *then* $R_{1,-1,0}(z) = \beta z + \alpha + o(1)$ *as* $z \to \infty$, *where* $(\beta,\alpha) = (B(a,b), A(a,b))$ *if* $a > b+1$ *and* $(\beta,\alpha) = (B(b,a), A(b,a))$ *if* $a < b-1$. *Hence, for* $|a-b| > 1$, *the* $N = 1$ *case of Theorem 5 leads to the representation*

$$R_{-1,-1,0}(z) = \alpha + \beta z - \frac{[\Gamma(c)]^2(c-a-b+1)}{\Gamma(a)\Gamma(b)\Gamma(c-a+1)\Gamma(c-b+1)} \int_0^1 \frac{t^{a+b-2}(1-t)^{c-a-b}}{|{}_2F_1(a,b;c;1/t)|^2(1-zt)} dt. \tag{45}$$

In addition to the condition $|a-b| > 1$, *we need to assume that* $(a,b,c) \in V$ *in Theorem 4. Under these restrictions and except for the degenerate cases* $ab = 0$ *and* $(c-a)(c-b) = 0$ *the condition (33) reads*

$$(c-a-b+2)_- - (c-a-b)_- > -1,$$

which is true for all real a,b,c. *The above representation implies that* $\alpha + \beta z - R_{-1,-1,0}(z) \in \mathcal{M}$ *if the constant in front of the integral is positive.*

As $R_{-1,-1,0}(z) = \beta z + o(z)$ *as* $z \to \infty$, *where* $\beta = B(a,b)$ *if* $a \ge b$ *and* $\beta = B(b,a)$ *if* $a \ge b$, *we can lift the restriction* $|a-b| > 1$ *by taking* $N = 1$ *in Theorem 5, which leads to*

$$R_{-1,-1,0}(z) = 1 + \beta z - \frac{z[\Gamma(c)]^2(c-a-b+1)}{\Gamma(a)\Gamma(b)\Gamma(c-a+1)\Gamma(c-b+1)} \int_0^1 \frac{t^{a+b-1}(1-t)^{c-a-b}}{|{}_2F_1(a,b;c;1/t)|^2(1-zt)} dt, \tag{46}$$

or, by taking $N = 2$, *we obtain*

$$R_{-1,-1,0}(z) = 1 + \frac{(a+b-1)c}{c^2} z - \frac{z^2[\Gamma(c)]^2(c-a-b+1)}{\Gamma(a)\Gamma(b)\Gamma(c-a+1)\Gamma(c-b+1)} \int_0^1 \frac{t^{a+b}(1-t)^{c-a-b}}{|{}_2F_1(a,b;c;1/t)|^2(1-zt)} dt. \tag{47}$$

Example 14. *For the ratio* $R_{-1,1,0}(z)$ *according to (26), we obtain* $p = l = r = 0$. *Theorem 3 and definition (27b) yield:*

$$B_{-1,1,0}P_0 = \frac{[\Gamma(c)]^2(a-b-1)}{\Gamma(a)\Gamma(b+1)\Gamma(c-a+1)\Gamma(c-b)}.$$

The asymptotic behavior of $R_{-1,1,0}(z)$ *as* $z \to \infty$ *is rather complicated and depends on the relation between* a *and* b. *The application of Lemmas 2 and 3 yields the following:*

(1) *If* $b+1 < a$, *then* $R_{-1,1,0}(z) = o(1)$ *as* $z \to \infty$;

(2) *If* $b \le a \le b+1$, *then* $R_{-1,1,0}(z) = o(z)$ *as* $z \to \infty$;

(3) *If* $b-1 \le a < b$, *then* $R_{-1,1,0}(z) = Bz + o(z)$ *as* $z \to \infty$;

(4) *If* $a < b-1$, *then* $R_{-1,1,0}(z) = Bz + C + o(1)$ *as* $z \to \infty$,

where

$$B = \frac{(b-a)(b-a+1)}{b(a-c)}, \quad C = \frac{(b-a)(b-a+1)(c(a+b-1)-2ab)}{b(c-a)(a-b-1)(a-b+1)}.$$

Hence, if $|a-b| > 1$ *we have* $R_{-1,1,0}(z) = \beta z + \alpha + o(1)$ *as* $z \to \infty$, *where* $(\beta,\alpha) = (0,0)$ *when* $a > b+1$ *and* $(\beta,\alpha) = (B,C)$ *when* $a < b-1$. *Then, for* $|a-b| > 1$ *the* $N = 0$ *case of Theorem 5 leads to the representation*

$$R_{-1,1,0}(z) = \alpha + \beta z + \frac{[\Gamma(c)]^2(a-b-1)}{\Gamma(a)\Gamma(b+1)\Gamma(c-a+1)\Gamma(c-b)} \int_0^1 \frac{t^{a+b-2}(1-t)^{c-a-b}}{|{}_2F_1(a,b;c;1/t)|^2(1-zt)} dt. \tag{48}$$

In addition to the condition $|a-b|>1$, we need to assume that $(a,b,c)\in V$ in Theorem 4. Under these restrictions and except for the degenerate cases $ab=0$ and $(c-a)(c-b)=0$ the condition (33) reads

$$(c-a-b)_- - (c-a-b)_- > -1,$$

which is true for all real a,b,c. Here, $R_{-1,1,0}(z)-\alpha-\beta z\in\mathcal{M}$ provided that the above representation holds and the constant in front of the integral is positive.

For arbitrary values of a,b, we have $R_{-1,1,0}(z)=\beta z+o(z)$ as $z\to\infty$, where $\beta=0$ when $a\geq b$ and $\beta=B$ when $a<b$. Hence, we can use representation (35) with $N=1$ yielding

$$R_{-1,1,0}(z)=1+\beta z+\frac{z[\Gamma(c)]^2(a-b-1)}{\Gamma(a)\Gamma(b+1)\Gamma(c-a+1)\Gamma(c-b)}\int_0^1\frac{t^{a+b-1}(1-t)^{c-a-b}}{|{}_2F_1(a,b;c;1/t)|^2(1-zt)}dt \tag{49}$$

or with $N=2$ yielding

$$R_{-1,1,0}(z)=1+\frac{(a-b-1)c}{c^2}z+\frac{z^2[\Gamma(c)]^2(a-b-1)}{\Gamma(a)\Gamma(b+1)\Gamma(c-a+1)\Gamma(c-b)}\int_0^1\frac{t^{a+b}(1-t)^{c-a-b}}{|{}_2F_1(a,b;c;1/t)|^2(1-zt)}dt. \tag{50}$$

Example 15. *For the ratio $R_{-2,-2,0}(z)$ according to (26), we obtain $p=4$, $l=0$, $r=1$. Theorem 3 and definition (27b) yield*

$$B_{-2,-2,0}P_1(t)=-\frac{[\Gamma(c)]^2(c-a-b+2)(\rho_0+\rho_1 t)}{\Gamma(a)\Gamma(b)\Gamma(c-a+2)\Gamma(c-b+2)},$$

where $\rho_0=a^2+b^2-(c+2)(a+b)+3c+1$, $\rho_1=c(c-a-b+1)+2(ab-a-b+1)$. Using Lemmas 2 and 3 or by direct, albeit tedious, calculation, we obtain the following asymptotic approximations:

(1) *If $a>b+2$, then $R_{-2,-2,0}(z)=\gamma_{a,b,c}z^2+\beta_{a,b,c}z+\alpha_{a,b,c}+o(1)$ as $z\to\infty$;*

(2) *If $b+1<a\leq b+2$, then $R_{-2,-2,0}(z)=\gamma_{a,b,c}z^2+\beta_{a,b,c}z+o(z)$ as $z\to\infty$;*

(3) *If $b\leq a\leq b+1$, then $R_{-2,-2,0}(z)=\gamma_{a,b,c}z^2+o(z^2)$ as $z\to\infty$;*

(4) *If $b-1\leq a\leq b$, then $R_{-2,-2,0}(z)=\gamma_{b,a,c}z^2+o(z^2)$ as $z\to\infty$;*

(5) *If $b-2\leq a<b-1$, then $R_{-2,-2,0}(z)=\gamma_{b,a,c}z^2+\beta_{b,a,c}z+o(z)$ as $z\to\infty$;*

(6) *If $a<b-2$, then $R_{-2,-2,0}(z)=\gamma_{b,a,c}z^2+\beta_{b,a,c}z+\alpha_{b,a,c}+o(1)$ as $z\to\infty$,*

where

$$\gamma_{a,b,c}=\frac{(a-2)(a-1)}{(c-b)(c-b+1)},\quad \beta_{a,b,c}=\frac{2(a-2)(a-1)(c+1-2b)}{(c-b)(c-b+1)(b-a+1)},$$

$$\alpha_{a,b,c}=\gamma_{a,b,c}\frac{c(c+1)(a-1)+2b^2(a+4c-3b)-2ab(c+2)-b(3c^2-c-6)}{(a-b-2)(a-b-1)^2}.$$

Hence, for $|a-b|>2$, we have $R_{-2,-2,0}(z)=\gamma z^2+\beta z+\alpha+o(1)$ as $z\to\infty$, where $(\gamma,\beta,\alpha)=(\gamma_{a,b,c},\beta_{a,b,c},\alpha_{a,b,c})$ when $a>b+2$ and $(\gamma,\beta,\alpha)=(\gamma_{b,a,c},\beta_{b,a,c},\alpha_{b,a,c})$ when $a<b-2$. Then, for $|a-b|>2$, the case $N=0$ of Theorem 5 leads to the representation

$$R_{-2,-2,0}(z)=\gamma z^2+\beta z+\alpha-\frac{[\Gamma(c)]^2(c-a-b+2)}{\Gamma(a)\Gamma(b)\Gamma(c-a+2)\Gamma(c-b+2)}\int_0^1\frac{(\rho_0+\rho_1 t)t^{a+b-3}(1-t)^{c-a-b}}{|{}_2F_1(a,b;c;1/t)|^2(1-zt)}dt. \tag{51}$$

In addition to condition $|a-b|>2$, we need to assume that $(a,b,c)\in V$ in Theorem 4. Under these restrictions and except for the degenerate cases $ab=0$ and $(c-a)(c-b)=0$, the condition (33) reads

$$(c-a-b+4)_- - (c-a-b)_- > -1,$$

which is true for all real a,b,c. The above integral representation implies that either $\gamma z^2+\beta z+\alpha-R_{-2,-2,0}(z)\in\mathcal{M}$ or $R_{-2,-2,0}(z)-\gamma z^2-\beta z-\alpha\in\mathcal{M}$ if the zero $t^*=-\rho_0/\rho_1$ of the polynomial $\rho_0+\rho_1 t$ lies outside of the interval $(0,1)$.

If $1 < |a-b| \le 2$, we see that the asymptotics takes the form $R_{-2,-2,0}(z) = \gamma z^2 + \beta z + o(z)$ as $z \to \infty$, where $(\gamma, \beta) = (\gamma_{a,b,c}, \beta_{a,b,c})$ when $a > b+1$ and $(\gamma, \beta) = (\gamma_{b,a,c}, \beta_{b,a,c})$ when $a < b-1$. Hence, for $1 < |a-b|$ according to (35) with $N = 1$, we obtain

$$R_{-2,-2,0}(z) = \gamma z^2 + \beta z + 1 - \frac{z[\Gamma(c)]^2(c-a-b+2)}{\Gamma(a)\Gamma(b)\Gamma(c-a+2)\Gamma(c-b+2)} \int_0^1 \frac{(\rho_0+\rho_1 t)t^{a+b-2}(1-t)^{c-a-b}}{|{}_2F_1(a,b;c;1/t)|^2(1-zt)}dt. \tag{52}$$

Similarly, for $|a-b| \le 1$, the asymptotics takes the form $R_{-2,-2,0}(z) = \gamma z^2 + o(z^2)$, where $\gamma = \gamma_{a,b,c}$ when $a \ge b$ and $\gamma = \gamma_{b,a,c}$ when $a \le b$. Hence, without additional restrictions according to (35) with $N = 2$, we obtain

$$\begin{aligned} R_{-2,-2,0}(z) &= 1 + \frac{2(2-a-b)}{c^2} z + \gamma z^2 \\ &\quad - \frac{z^2[\Gamma(c)]^2(c-a-b+2)}{\Gamma(a)\Gamma(b)\Gamma(c-a+2)\Gamma(c-b+2)} \int_0^1 \frac{(\rho_0+\rho_1 t)t^{a+b-1}(1-t)^{c-a-b}}{|{}_2F_1(a,b;c;1/t)|^2(1-zt)}dt \end{aligned} \tag{53}$$

under the condition $(a,b,c) \in V$ from Theorem 4, but without any other restrictions.

4. Concluding Remarks

It turns out that our results may help in finding integral representations of elementary and special functions. For instance, Formulas (43) and (44) with $a = b = 1$ and $c = 2$ yield the following curious identity:

$$\frac{z}{\mathrm{Log}(1+z)} = 1 + z\int_1^\infty \frac{dx}{(\log^2(x-1)+\pi^2)(x+z)} = 1 + \frac{z}{2} - z^2 \int_1^\infty \frac{dx}{(\log^2(x-1)+\pi^2)(x+z)x}.$$

The first equality here after division by z corrects the representation [30] (Formula (34)). This identity may be easily generalized by applying (37) with arbitrary $N \in \mathbb{N}$, $a = b = 1$ and $c = 2$ to the results of Example 12:

$$\frac{z}{\mathrm{Log}(1+z)} = \sum_{k=0}^{N-1} \frac{\mathcal{C}_k z^k}{k!} - (-z)^N \int_1^\infty \frac{dx}{(\log^2(x-1)+\pi^2)(x+z)x^{N-1}},$$

where $N = 1, 2, 3, \ldots$ and $\mathcal{C}_k$ is the kth Cauchy number [24] (p. 294).

Moreover, Theorem 5, in view of Remark 8, gives a way for calculating the "generalized beta integrals" of the form

$$I_{a,b}(j,k) := \int_0^1 \frac{t^{a+b+j}(1-t)^{c-a-b-k}}{|{}_2F_1(a,b;c;1/t)|^2}dt.$$

In particular, Examples 1–4, 7–9 and 12 lead immediately to explicit evaluations in terms of gamma functions of the integral $I_{a,b}(j,k)$ for the following pairs (j,k): $(-2,-1)$, $(-2,0)$, $(-1,-2)$, $(-1,-1)$, $(-1,0)$, $(0,-3)$, $(0,-2)$, $(0,-1)$, $(0,0)$, $(1,-3)$, $(1,-2)$, $(1,-1)$. This list can be extended by invoking Examples 6 and 15 with the following pairs: $(-3,-1)$, $(-3,0)$, $(-1,-3)$. For instance, for $j \in \{-1,0\}$ and $k \in \{0,1\}$ we get:

$$I_{a,b}(j,k) = \frac{\Gamma(a+1+j)\Gamma(b+1)\Gamma(c-a+1-k)\Gamma(c-b+n)}{\Gamma(c+n)\Gamma(c+2+j-k)}, \text{ where } n = \min(1+j, 1-k) \in \{0,1\},$$

provided that $a \ge b$; the case $a \le b$ follows by exchanging $a \leftrightarrow b$. Further examples are

$$I_{a,b}(0,2) = \frac{\Gamma(a+1)\Gamma(b+1)\Gamma(c-a)\Gamma(c-b)}{[\Gamma(c)]^2(a+b-c+1)}$$

and, if $a > b+1$,

$$I_{a,b}(-2,0) = \frac{\Gamma(a)\Gamma(b+1)\Gamma(c-a+1)\Gamma(c-b)}{(a-b-1)\Gamma^2(c)}.$$

Note that the value of j in the above 15 pairs (j,k) may be increased by any positive integer (and hence made as large as desired) by choosing larger values of $N \in \mathbb{N}_0$ in (37). A natural limitation of the above integral evaluations is that the hypergeometric function in the denominator has to be non-vanishing in $\mathbb{C} \setminus [1,\infty)$ and on the branch cut, which can be verified via Theorem 4. For a general pair of integers (j,k), we can use formulae (26) to choose the corresponding shifts n_1, n_2, m and use Remark 8 to calculate the corresponding integral. The details of this algorithm will be elaborated in a separate publication.

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