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Nonlocal Impulsive Fractional Integral Boundary Value Problem for (ρ_k, ϕ_k)-Hilfer Fractional Integro-Differential Equations

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Abstract: In this paper, we establish the existence and stability results for the (ρ_k, ϕ_k)-Hilfer fractional integro-differential equations under instantaneous impulse with non-local multi-point fractional integral boundary conditions. We achieve the formulation of the solution to the (ρ_k, ϕ_k)-Hilfer fractional differential equation with constant coefficients in term of the Mittag–Leffler kernel. The uniqueness result is proved by applying Banach's fixed point theory with the Mittag–Leffler properties, and the existence result is derived by using a fixed point theorem due to O'Regan. Furthermore, Ulam–Hyers stability and Ulam–Hyers–Rassias stability results are demonstrated via the non-linear functional analysis method. In addition, numerical examples are designed to demonstrate the application of the main results.

Keywords: (ρ, ϕ)-Hilfer fractional derivative; impulsive conditions; integral multi-point boundary conditions; fixed point theorems; Ulam–Hyers stability

MSC: 26A33; 33E12; 34A37; 34B10; 34D20

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1. Introduction

Fractional calculus ($\mathbb{FC}$) is discussed as the fractional integral operator ($\mathbb{FIO}$) and fractional derivative operator ($\mathbb{FDO}$), which have a long and illustrious history. $\mathbb{FC}$ is popularly used to analyze phenomena in the branch of mathematical analysis, which is noticed to be of outstanding assistance in modifying complex real-world problems in many fields, such as physical sciences [1], financial economics [2], dynamics of particles, fields and media [3], bio-engineering [4], Zika [5], HIV [6], COVID-19 [7], ecology [8], continuum mechanics [9], Navier–Stokes problem [10], social media addiction [11], and references cited therein. For more theoretical details on this topic, see: [12–16]. A variety of types of $\mathbb{FDO}$s are regularly settled in the sense of $\mathbb{FIO}$s. Various types of $\mathbb{FDO}$s with different kernel terms, such as Riemann–Liouville ($\mathbb{RL}$), Caputo, Hadamard, Katugampola, Hilfer, and others, are shown in the literature survey on $\mathbb{FC}$.

Recently, in 2018, the concept of $\mathbb{FDO}$ with respect to another function was developed by Sousa and Oliveira [17], which is known as the ϕ-Hilfer $\mathbb{FDO}$. Some existence and stability results of the solutions for fractional differential equations ($\mathbb{FDE}$s) were created in the context of ϕ-Hilfer $\mathbb{FDO}$ [18–22] and the references therein. After that, in 2021, Kucche and Mali [23] introduced and demonstrated some properties of the (ρ, ϕ)-Hilfer $\mathbb{FDO}$. They applied Banach's type to analyzed the uniqueness result for the non-linear $\mathbb{FDE}$s under (ρ,ϕ)-Hilfer $\mathbb{FDO}$:

$$\begin{cases} {}^{H}_{\rho}\mathfrak{D}^{\alpha,\beta;\phi}_{a^+}u(t) = f(t,u(t)), \quad t \in (a,b], \quad \alpha \in (0,\rho),\ \beta \in [0,1], \\ {}_{\rho}\mathcal{I}^{\rho-\gamma_r;\phi}_{a^+}u(a) = u_a \in \mathbb{R}, \quad \gamma_r = \alpha + \beta(\rho-\alpha), \quad \rho > 0, \end{cases} \tag{1}$$

where ${}^{H}_{\rho}\mathfrak{D}^{\alpha,\beta;\phi}_{a^+}$ is the (ρ,ϕ)-Hilfer $\mathbb{FDO}$ of order α and type β with $\alpha \in (0,1)$, and $f \in \mathcal{C}([a,b]\times\mathbb{R},\mathbb{R})$, $0 \leq a < b < \infty$. It is worth noting the (ρ,ϕ)-Hilfer $\mathbb{FDO}$, which can be generalized as various known $\mathbb{FDO}$s (see more details in Remark 2).

The physical and social sciences are explained by applying impulsive differential equations with integer order and fractional order. They are also applicable to dynamical systems, such as evolutionary processes, which show instantaneous state changes at some points. The qualitative theory of impulsive $\mathbb{FDE}$s, such as existence theory and stability results, has been widely employed in engineering and applied sciences throughout the last several decades (see [24–26]). Many researchers will attempt to operate in the area of impulsive $\mathbb{FDE}$s with impulses and have presented essential and interesting results through the years that have contributed greatly to the mathematical analysis of $\mathbb{FDE}$s with impulses effect. In 2009, Benchohra and Slimani [27] studied a variety of conditions for the existence of the solutions for the impulsive Caputo-type $\mathbb{FDE}$ with initial condition by using Banach's, Leray–Schauder's, and Schaefer's fixed point theorems. Later, the impulsive $\mathbb{FDE}$s in [27] have been extended and studied for their existence results in Banach spaces by Benchohra and Seba [28]. In 2012, Wang et al. [29] investigated the piecewise continuous solutions to the problem in [27,28]. The existence, uniqueness, and Ulam's stability results of solutions for the impulsive boundary value problems ($\mathbb{BVP}$s) are obtained by using a fixed point theorem via generalized Gronwall inequalities. In 2014, Wang and Lin [30] investigated the existence of solutions to impulsive Caputo $\mathbb{FDE}$s under anti-periodic boundary conditions via constant coefficients. The formula of solutions to the problem in [30] was constructed in the sense of Mittag–Leffler kernels. At the same time, the Lipschitz and non-linear growth conditions were used to establish the existence results of solutions to the problem in [30]. In 2017, Zuo et al. [31] established the existence and uniqueness results for impulsive anti-periodic $\mathbb{BVP}$s through fractional integro-differential equation ($\mathbb{FIDE}$) with constant coefficient based on Banach's and Krasnoselskii's types. In 2020, Kucche et al. [32] developed the existence results of solutions for the non-linear ϕ-Hilfer impulsive $\mathbb{FDE}$ with initial condition:

$$\begin{cases} {}^{H}\mathfrak{D}^{\alpha,\beta;\phi}_{a^+}u(t) = f(t,u(t)),\ t \in \mathcal{J}_k \subseteq (a,T],\ t \neq t_k,\ T > a, \\ \Delta\mathcal{I}^{1-\gamma;\phi}_{a^+}u(t_k) = \xi_k \in \mathbb{R}, \quad k = 1,2,\ldots,m, \\ \mathcal{I}^{1-\gamma;\phi}_{a^+}u(a) = \delta \in \mathbb{R}, \quad \gamma = \alpha+\beta-\alpha\beta, \end{cases} \tag{2}$$

where ${}^{H}\mathfrak{D}^{\alpha,\beta;\phi}_{a^+}$ is the ϕ-Hilfer $\mathbb{FDO}$ of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$, $\mathcal{I}^{1-\gamma;\phi}_{a^+}$ is the ϕ-$\mathbb{RL}$-$\mathbb{FIO}$ of order $1-\gamma > 0$. In addition, they extended the problem (2) to the non-local ϕ-Hilfer $\mathbb{FDE}$. In 2022, based on Banach's and Schauder's types, Salim et al. [33] proved the existence and uniqueness of solutions for the non-linear implicit ρ-generalized ϕ-Hilfer $\mathbb{FDE}$-$\mathbb{BVP}$s via retardation and anticipation:

$$\begin{cases} {}^{H}_{\rho}\mathfrak{D}^{\alpha,\beta;\phi}_{a^+}u(t) = f(t,u_t(t),{}^{H}_{\rho}\mathfrak{D}^{\alpha,\beta;\phi}_{a^+}u(t)),\ t \in (a,b], \\ a_1{}_{\rho}\mathcal{I}^{\rho(1-\xi);\phi}_{a^+}u(a^+) + a_2{}_{\rho}\mathcal{I}^{\rho(1-\gamma);\phi}_{a^+}u(b) = a_3, \quad \gamma = \dfrac{1}{\rho}(\beta(\rho-\alpha)+\alpha), \\ u(t) = \omega(t), \quad t \in [a-\lambda,a],\ \lambda > 0, \quad a_1,a_2,a_3 \in \mathbb{R}, \\ u(t) = \overline{\omega}(t), \quad t \in [b,b+\overline{\lambda}],\ \overline{\lambda} > 0, \quad a_1+a_2 \neq 0, \end{cases} \tag{3}$$

where ${}^{H}_{\rho}\mathfrak{D}^{\alpha,\beta;\phi}_{a^+}$ and ${}_{\rho}\mathcal{I}^{\rho(1-\gamma);\phi}_{a^+}$ are the ρ-generalized ϕ-Hilfer $\mathbb{FDO}$ of order $\alpha \in (0,\rho)$, $\rho > 0$ and type $\beta \in [0,1]$, and the ρ-generalized ϕ-Hilfer $\mathbb{FIO}$ of order $\rho(1-\gamma) \in (0,\rho)$, respectively, and $u_t(\tau) = u(t+\tau)$ for $\tau \in [-\lambda, \overline{\lambda}]$. Note that several works have been published using concentrated and important tools in mathematical analysis. We suggest modern works on impulsive $\mathbb{FDE}$s on existence, uniqueness, and Ulam's stability and the reference given therein [34–46].

To motivate the enrichment of novel literature for interested researchers, in this paper, we establish qualitative results of the solutions for the following non-linear impulsive (ρ_k,ϕ_k)-Hilfer $\mathbb{FIDE}$s with non-local multi-point fractional integral boundary conditions ($\mathbb{NMP}$-$\mathbb{FIBC}$s) as:

$$\begin{cases} {}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\phi_k}_{t_k^+}u(t) = \lambda_k u(t) + f(t, u(t), {}_{\rho_k}\mathcal{I}^{\delta_k;\phi_k}_{t_k}u(t), {}_{\rho_k}\mathcal{I}^{\theta_k;\phi_k}_{t_k}u(t)), \ t \in \mathcal{J}_k, \ t \neq t_k, \\ \Delta_{\rho_k}\mathcal{I}^{\rho_k(1-\gamma_k);\phi_k}_{t_k^+}u(t_k) = \varphi_k(u(t_k)), \quad k = 1,2,\ldots,m, \\ \displaystyle\sum_{i=0}^{m}\kappa_i u(\eta_i) = \sum_{j=0}^{n}\omega_{j\rho_j}\mathcal{I}^{\mu_j;\phi_j}_{t_j^+}u(\xi_j) + A, \quad \eta_i \in (t_i, t_{i+1}], \xi_j \in (t_j, t_{j+1}]. \end{cases} \tag{4}$$

where ${}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\phi_k}_{t_k^+}$ denotes the (ρ_k,ϕ_k)-Hilfer $\mathbb{FDO}$ of order α_k and type β_k on $\mathcal{J}_k$, $0 < \alpha_k < 1$, $0 \le \beta_k \le 1$, $\rho_k > 0$, $\mathcal{J}_k = (t_k, t_{k+1}] \subseteq (0,T]$, $k = 0,1,\ldots,m$, $\mathcal{J}_0 = [a,t_1]$, $\mathcal{J} = [a,b]$, $0 \le a = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, ${}_{\rho_k}\mathcal{I}^{q;\phi_k}_{t_k}$ is the (ρ_k,ϕ_k)-$\mathbb{RL}$ $\mathbb{FIO}$ of order $q \in \{\delta_k, \theta_k, \mu_k\} > 0$, $k = 0,1,\ldots,m$, $\varphi_k \in \mathcal{C}(\mathbb{R},\mathbb{R})$, $\Delta_{\rho_k}\mathcal{I}^{\rho_k(1-\gamma_k);\phi_k}_{t_k^+}u(t_k) = {}_{\rho_k}\mathcal{I}^{\rho_k(1-\gamma_k);\phi_k}_{t_k^+}u(t_k^+) - {}_{\rho_{k-1}}\mathcal{I}^{\rho_{k-1}(1-\gamma_{k-1});\phi_{k-1}}_{t_{k-1}^+}u(t_k^-)$ where ${}_{\rho_k}\mathcal{I}^{\rho_k(1-\gamma_k);\phi_k}_{t_k^+}u(t_k^+) = \lim_{h\to 0^+}{}_{\rho_k}\mathcal{I}^{\rho_k(1-\gamma_k);\phi_k}_{t_k^+}u(t_k+h)$, $k = 1,2,\ldots,m$, $f \in \mathcal{C}(\mathcal{J},\mathbb{R}^3,\mathbb{R})$, κ_i, $\omega_j \in \mathbb{R}$, $\eta_i \in (t_i,t_{i+1}]$, $\xi_j \in (t_j,t_{j+1}]$, $i = 0,1,\ldots,m$, $j = 0,1,\ldots,n$, $\lambda < 0$ and $A \in \mathbb{R}$. For the sake of use, the problem (4) can be called the impulsive (ρ_k,ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s.

The remaining sections of this work are structured as follows: in Section 2, some concepts of the (ρ,ϕ)-Hilfer fractional operators related to our discussion are defined along with some essential lemmas are proved. Additionally, the solution of the linear variant of the (ρ,ϕ)-Hilfer fractional Cauchy problem (11) is derived in the form of the generalized Mittag–Leffler kernel. After that, an equivalent integral equation to the impulsive (ρ_k,ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (4). The essential lemma is very important to transform the proposed problem (4) into a fixed-point problem. In Section 3, presenting the first main results of the problem (4), the uniqueness result is proved by Banach's type and the existence result is studied by a fixed point theorem due to O'Regan. In addition, a variety of Ulam's stability results for problem (4) are investigated in Section 4. Finally, Section 5 shows illustrative examples to verify the main results.

2. Preliminaries

This section introduces fundamental concepts and constructs several properties of the (ρ,ϕ)-Hilfer fractional calculus relevant to our results.

2.1. The (ρ,ϕ)-Hilfer Fractional Calculus and Its Properties

Definition 1 ([47]). *Let $f \in L^1([a,b],b)$ and an increasing function $\phi(t) : [a,b] \to \mathbb{R}$ via $\phi'(t) \neq 0$ for $t \in [a,b]$. The (ρ,ϕ)-$\mathbb{RL}$-$\mathbb{FIO}$ of a function f of order $\alpha > 0$ is defined by*

$${}_{\rho}\mathcal{I}^{\alpha;\phi}_{a^+}f(t) = \frac{1}{\rho\Gamma_\rho(\alpha)}\int_a^t (\phi(t)-\phi(s))^{\frac{\alpha}{\rho}-1}\phi'(s)f(s)ds, \quad \rho > 0,$$

where $\Gamma_\rho(\cdot)$ is the ρ-Gamma function which is introduced by Diaz and Pariguan [48],

$$\Gamma_\rho(z) = \int_0^\infty t^{z-1}e^{-\frac{t^\rho}{\rho}}dt, \quad z \in \mathbb{C}, \ Re(z) > 0, \ \rho > 0. \tag{5}$$

Some other useful properties of (5) *are well known as follows:*

$$\Gamma_\rho(z+\rho) = z\Gamma_\rho(z), \quad \Gamma_\rho(\rho) = 1, \quad \Gamma_\rho(z) = (\rho)^{\frac{z}{\rho}-1}\Gamma\left(\frac{z}{\rho}\right), \quad \Gamma(z) = \lim_{\rho\to 1}\Gamma_\rho(z). \tag{6}$$

Definition 2 ([23])**.** *Let $f \in \mathcal{C}^n([a,b],\mathbb{R})$, $\phi \in \mathcal{C}^n([a,b],\mathbb{R})$, $\phi'(t) \neq 0$, for $t \in [a,b]$, $\alpha, \rho \in \mathbb{R}^+$, and $\beta \in [0,1]$. The (ρ,ϕ)-Hilfer $\mathbb{FDO}$ of a function f of order α and type β is given by*

$${}^{H}_{\rho}\mathfrak{D}^{\alpha,\beta;\phi}_{a^+} f(t) = {}_{\rho}\mathcal{I}^{\beta(\rho n-\alpha);\phi}_{a^+}\, \delta^n_{\phi\rho}\mathcal{I}^{(1-\beta)(\rho n-\alpha);\phi}_{a^+} f(t), \tag{7}$$

where $\delta^n_\phi = \left(\frac{\rho}{\phi'(t)}\frac{d}{dt}\right)^n$ and $n = \left\lceil \frac{\alpha}{\rho} \right\rceil$.

Remark 1. *The (ρ,ϕ)-Hilfer $\mathbb{FDO}$ can be rewritten in the sense of the (ρ,ϕ)-$\mathbb{RL}$-$\mathbb{FDO}$ as follows:*

$${}^{H}_{\rho}\mathfrak{D}^{\alpha,\beta;\phi}_{a^+} f(t) = {}_{\rho}\mathcal{I}^{\gamma-\alpha;\phi}_{a^+}\left(\frac{\rho}{\phi'(t)}\frac{d}{dt}\right)^n {}_{\rho}\mathcal{I}^{n\rho-\gamma;\phi}_{a^+} f(t) = {}_{\rho}\mathcal{I}^{\gamma-\alpha;\phi}_{a^+}\left({}^{\mathbb{RL}}_{\rho}\mathfrak{D}^{\gamma;\phi}_{a^+} f\right)(t),$$

where

$${}^{\mathbb{RL}}_{\rho}\mathfrak{D}^{\gamma;\phi}_{a^+} f(t) = \left(\frac{\rho}{\phi'(t)}\frac{d}{dt}\right)^n {}_{\rho}\mathcal{I}^{n\rho-\gamma;\phi}_{a^+} f(t),$$

and $n\rho - \gamma = (1-\beta)(n\rho - \alpha)$ with $\frac{\gamma}{\rho} \in (n-1, n]$.

Remark 2. *It is noticed that:*

($\mathbb{A}_1$) *If we take $\beta = 0$ in* (7)*, then we have the (ρ,ϕ)-$\mathbb{RL}$-$\mathbb{FDO}$ defined in [23], while if $\phi(t) = t$ with $\beta = 0$, then we obtain the ρ-$\mathbb{RL}$-$\mathbb{FDO}$ defined in [49].*

($\mathbb{A}_2$) *If we take $\beta = 1$ in* (7)*, then we have the (ρ,ϕ)-Caputo $\mathbb{FDO}$ defined in [23], while if we take $\phi(t) = t$ with $\beta = 0$, then we obtain the ρ-Caputo $\mathbb{FDO}$ defined in [23].*

($\mathbb{A}_3$) *If we take $\phi(t) = t^\sigma$ in* (7)*, then we have the ρ-Hilfer–Katugampola $\mathbb{FDO}$, that is:*

- *(i) If we take $\phi(t) = t^\sigma$ with $\beta = 0$ in* (7)*, then we have the ρ-Katugampola $\mathbb{FDO}$ defined in [50].*
- *(ii) If we take $\phi(t) = t^\sigma$ with $\beta = 1$ in* (7)*, then we have the ρ-Caputo–Katugampola $\mathbb{FDO}$ defined in [50].*

($\mathbb{A}_4$) *If we take $\phi(t) = \log t$ in* (7)*, then we have the ρ-Hilfer–Hadamard $\mathbb{FDO}$, that is:*

- *(i) If we take $\phi(t) = \log t$ with $\beta = 0$ in* (7)*, then we have the ρ-Hadamard $\mathbb{FDO}$ defined in [23].*
- *(ii) If we take $\phi(t) = \log t$ with $\beta = 1$ in* (7)*, then we have the ρ-Caputo–Hadamard $\mathbb{FDO}$ defined in [23].*

Some important basic properties, which are used throughout this paper, are as follows:

Lemma 1 ([23])**.** *Let $\alpha, \rho > 0$ and $\beta \in \mathbb{R}$, such that $\frac{\beta}{\rho} > -1$. Then, we have*

(i) ${}_{\rho}\mathcal{I}^{\alpha;\phi}_{a^+}\left[(\phi(t)-\phi(a))^{\frac{\beta}{\rho}}\right] = \frac{\Gamma_\rho(\beta+\rho)}{\Gamma_\rho(\beta+\rho+\alpha)}(\phi(t)-\phi(a))^{\frac{\beta+\alpha}{\rho}}.$

(ii) ${}^{RL}_{\rho}\mathfrak{D}^{\alpha;\phi}_{a^+}\left[(\phi(t)-\phi(a))^{\frac{\beta}{\rho}}\right] = \frac{\Gamma_\rho(\beta+\rho)}{\Gamma_\rho(\beta+\rho-\alpha)}(\phi(t)-\phi(a))^{\frac{\beta-\alpha}{\rho}}.$

(iii) ${}_{\rho}\mathcal{I}^{\alpha;\phi}_{a^+}\,{}_{\rho}\mathcal{I}^{\beta;\phi}_{a^+} f(t) = {}_{\rho}\mathcal{I}^{\alpha+\beta;\phi}_{a^+} f(t) = {}_{\rho}\mathcal{I}^{\beta;\phi}_{a^+}\,{}_{\rho}\mathcal{I}^{\alpha;\phi}_{a^+} f(t).$

Lemma 2 ([33])**.** *If $f \in \mathcal{C}^n([a,b],\mathbb{R}$, $\frac{\alpha}{\rho} \in (n-1,n)$, $\beta \in [0,1]$, where $n \in \mathbb{N}$ and $\rho > 0$, then*

$$\left({}_{\rho}\mathcal{I}^{\alpha;\phi}_{a^+}\,{}^{H}_{\rho}\mathfrak{D}^{\alpha,\beta;\phi}_{a^+} f\right)(t) = f(t) - \sum_{i=1}^{n}\frac{(\phi(t)-\phi(a))^{\gamma-i}}{\rho^{i-n}\Gamma_\rho(\rho(\gamma-i+1))}\left[\delta^{n-i}_\phi\left({}_{\rho}\mathcal{I}^{\rho(n-\gamma);\phi}_{a^+} f(a)\right)\right],$$

where $\gamma = \frac{1}{\rho}(\beta(\rho n - \alpha) + \alpha)$ and $n = \left\lceil \frac{\alpha}{\rho} \right\rceil$.

Next, we provide the Mittag–Leffler functions $\mathbb{E}_\alpha(\cdot)$ and $\mathbb{E}_{\alpha,\beta}(\cdot)$ that will be employed throughout in this paper.

Lemma 3 ([51,52])**.** *Take $\alpha \in (0,1)$, $c > 0$. Hence, $\mathbb{E}_\alpha(\cdot)$ and $\mathbb{E}_{\alpha,c}(\cdot)$ are non-negative functions, and for each $u < 0$, $\mathbb{E}_\alpha(u) \leq 1$, $\mathbb{E}_{\alpha,c}(u) \leq 1/\Gamma(c)$, with*

$$\mathbb{E}_\alpha(u) = \sum_{n=0}^{\infty} \frac{u^n}{\Gamma(\alpha n + 1)} \qquad \text{and} \qquad \mathbb{E}_{\alpha,c}(u) = \sum_{n=0}^{\infty} \frac{u^n}{\Gamma(\alpha n + c)}, \quad u \in \mathbb{R}.$$

For the sake of easy for calculation in this paper, we define the symbols:

$$\Phi_\phi^{c-1}(t,a) = (\phi(t) - \phi(a))^{c-1}. \tag{8}$$

Lemma 4. *Assume that $\alpha > 0$, $c > 0$, $\rho > 0$, $q > 0$, and $\lambda \in \mathbb{R}$. Then, we obtain*

$${}_\rho\mathcal{I}_{a^+}^{\alpha,\phi}\left[\Phi_\phi^{q-1}(t,a)\mathbb{E}_{c,q}\left(\lambda\left(\rho^{-1}\Phi_\phi(t,a)\right)^c\right)\right] = \rho^{-\frac{\alpha}{\rho}}\Phi_\phi^{q+\frac{\alpha}{\rho}-1}(t,a)\,\mathbb{E}_{c,q+\frac{\alpha}{\rho}}\left(\lambda\left(\rho^{-1}\Phi_\phi(t,a)\right)^c\right), \tag{9}$$

where $\Phi_\phi^{(\cdot)}(t,a)$ and $\mathbb{E}_{u,v}(\cdot)$ are given as in (8) and Lemma 3, respectively.

Proof. By applying Definition 1 and Lemma 3 we have

$$\begin{aligned}
&{}_\rho\mathcal{I}_{a^+}^{\alpha,\phi}\left[\Phi_\phi^{q-1}(t,a)\mathbb{E}_{c,q}\left(\lambda\left(\rho^{-1}\Phi_\phi(t,a)\right)^c\right)\right] \\
=\;& \frac{1}{\rho\Gamma_\rho(\alpha)}\int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(s,t)\left[\Phi_\phi^{q-1}(s,a)\mathbb{E}_{c,q}\left(\lambda\left(\rho^{-1}\Phi_\phi(s,a)\right)^c\right)\right]\phi'(s)ds \\
=\;& \frac{1}{\rho\Gamma_\rho(\alpha)}\int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)\Phi_\phi^{q-1}(s,a)\left[\sum_{n=0}^{\infty}\frac{\left(\lambda\left(\rho^{-1}\Phi_\phi(s,a)\right)^c\right)^n}{\Gamma(cn+q)}\right]\phi'(s)ds \\
=\;& \frac{1}{\rho\Gamma_\rho(\alpha)}\sum_{n=0}^{\infty}\frac{(\lambda\rho^{-c})^n}{\Gamma(cn+q)}\int_a^t(\phi(t)-\phi(s))^{\frac{\alpha}{\rho}-1}(\phi(s)-\phi(a))^{cn+q-1}\phi'(s)ds \\
=\;& \sum_{n=0}^{\infty}\frac{(\lambda\rho^{-c})^n}{\Gamma(cn+q)}\,{}_\rho\mathcal{I}_{a^+}^{\alpha,\phi}\left[(\phi(t)-\phi(a))^{\frac{\rho(cn+q)}{\rho}-1}\right].
\end{aligned}$$

By using (ii) of Lemma 1, we have

$$\begin{aligned}
&{}_\rho\mathcal{I}_{a^+}^{\alpha,\phi}\left[\Phi_\phi^{q-1}(t,a)\mathbb{E}_{c,q}\left(\lambda\left(\rho^{-1}\Phi_\phi(t,a)\right)^c\right)\right] \\
=\;& \sum_{n=0}^{\infty}\frac{(\lambda\rho^{-c})^n}{\Gamma(cn+q)}\left[\frac{\Gamma_\rho(\rho(cn+q))}{\Gamma_\rho(\rho(cn+q)+\alpha)}(\phi(t)-\phi(a))^{\frac{\rho(cn+q)+\alpha}{\rho}-1}\right] \\
=\;& \sum_{n=0}^{\infty}\frac{(\lambda\rho^{-c})^n}{\Gamma(cn+q)}\left[\frac{\rho^{cn+q-1}\Gamma(cn+q)}{\rho^{cn+q+\frac{\alpha}{\rho}-1}\Gamma(cn+q+\frac{\alpha}{\rho})}(\phi(t)-\phi(a))^{cn+q+\frac{\alpha}{\rho}-1}\right],
\end{aligned}$$

which provides the desired (9). □

Lemma 5. *Suppose that $\alpha > 0$, $c > 0$, $\rho > 0$, $\lambda \in \mathbb{R}$, and $h \in \mathcal{C}([a,b])$. Then, we have*

$$\begin{aligned}
&{}_\rho\mathcal{I}_{a^+}^{\alpha,\phi}\left[\int_a^t \Phi_\phi^{c-1}(t,s)\mathbb{E}_{c,c}\left(\lambda\left(\rho^{-1}\Phi_\phi(t,s)\right)^c\right)h(s)\phi'(s)ds\right] \\
&\qquad = \rho^{-\frac{\alpha}{\rho}}\int_a^t \Phi_\phi^{c+\frac{\alpha}{\rho}-1}(s,r)\mathbb{E}_{c,c+\frac{\alpha}{\rho}}\left(\lambda\left(\rho^{-1}\Phi_\phi(s,r)\right)^c\right)h(r)\phi'(r)dr,
\end{aligned} \tag{10}$$

where $\Phi_\phi^{(\cdot)}(t,a)$ and $\mathbb{E}_{u,v}(\cdot)$ are given as in (8) and Lemma 3, respectively.

Proof. By applying Definition 1 and Lemma 3 we have

$$\begin{aligned}
& {}_{\rho}\mathcal{I}_{a^+}^{\alpha,\phi}\left[\int_a^t \Phi_\phi^{c-1}(t,s)\mathbb{E}_{c,c}\left(\lambda\left(\rho^{-1}\Phi_\phi(t,s)\right)^c\right)h(s)\phi'(s)ds\right] \\
= & \frac{1}{\rho\Gamma_\rho(\alpha)}\int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)\left[\int_a^s \Phi_\phi^{c-1}(s,r)\mathbb{E}_{c,c}\left(\lambda\left(\rho^{-1}\Phi_\phi(s,r)\right)^c\right)h(r)\phi'(r)dr\right]\phi'(s)ds \\
= & \int_a^t \frac{1}{\rho\Gamma_\rho(\alpha)}\int_r^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)\left[\Phi_\phi^{c-1}(s,r)\mathbb{E}_{c,c}\left(\lambda\left(\rho^{-1}\Phi_\phi(s,r)\right)^c\right)\right]\phi'(s)ds h(r)\phi'(r)dr \\
= & \int_a^t {}_{\rho}\mathcal{I}_{r^+}^{\alpha,\phi}\left[\Phi_\phi^{c-1}(s,r)\mathbb{E}_{c,c}\left(\lambda\left(\rho^{-1}\Phi_\phi(s,r)\right)^c\right)\right]h(r)\phi'(r)dr.
\end{aligned}$$

By using Lemma 4, the equality (10) is obtained. □

2.2. The Linear (ρ,ϕ)-Hilfer Fractional Cauchy Problem

Consider the linear variant of the (ρ,ϕ)-Hilfer fractional Cauchy problem with constant coefficient as follows:

$$\begin{cases} {}_{\rho}^{H}\mathfrak{D}_{a^+}^{\alpha,\beta;\phi}u(t) = \lambda u(t) + h(t), \quad \alpha\in(n-1,n), \quad \beta\in[0,1], \quad t\in(a,b], \\ \delta_\phi^{n-i}\,{}_{\rho}\mathcal{I}_{a^+}^{\rho(n-\gamma);\phi}u(a) = c_i, \quad i=1,2,\ldots,n, \quad \alpha\le\gamma=\dfrac{1}{\rho}(\beta(\rho n-\alpha)+\alpha), \end{cases} \tag{11}$$

where ${}_{\rho}^{H}\mathfrak{D}_{a^+}^{\alpha,\beta;\phi}$ denotes the (ρ,ϕ)-Hilfer $\mathbb{FDO}$ of order α and type β, ${}_{\rho}\mathcal{I}_{a^+}^{\rho(n-\gamma),\phi}$ denotes the (ρ,ϕ)-$\mathbb{RL}$-$\mathbb{FIO}$ of order $\rho(n-\gamma)>0$, $c_j\in\mathbb{R}$, $j=1,2,\ldots,n$, and $\lambda<0$. By applying the Picard's successive approximation technique, we derive to construct an explicit solution to the problem (11) in form of the Mittag–Leffler kernel.

Lemma 6. *Let $h\in\mathcal{C}([a,b],\mathbb{R})$, $\lambda\in\mathbb{R}$, $\alpha\in(n-1,n)$, $\beta\in[0,1]$, and $\rho>0$. Then, the explicit solution of the problem (11) is provided by*

$$\begin{aligned}
u(t) = & \frac{1}{\rho^{\frac{\alpha}{\rho}}}\int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)\mathbb{E}_{\frac{\alpha}{\rho},\frac{\alpha}{\rho}}\left(\lambda\left(\rho^{-1}\Phi_\phi(t,s)\right)^{\frac{\alpha}{\rho}}\right)h(s)\phi'(s)ds \\
& +\sum_{i=1}^n \frac{c_i}{\rho^{\gamma-n}}\Phi_\phi^{\gamma-i}(t,a)\mathbb{E}_{\frac{\alpha}{\rho},\gamma-i+1}\left(\lambda\left(\rho^{-1}\Phi_\phi(t,a)\right)^{\frac{\alpha}{\rho}}\right).
\end{aligned} \tag{12}$$

Proof. Assume u is a solution of the problem (11). By applying Lemma 2, The corresponding an integral equation of the problem (11) can be represented as

$$\begin{aligned}
u(t) = & \sum_{i=1}^n \frac{\Phi_\phi^{\gamma-i}(t,a)c_i}{\rho^{i-n}\Gamma_\rho(\rho(\gamma-i+1))} + \frac{\lambda}{\rho\Gamma_\rho(\alpha)}\int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)u(s)\phi'(s)ds \\
& +\frac{1}{\rho\Gamma_\rho(\alpha)}\int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)h(s)\phi'(s)ds.
\end{aligned}$$

The method of successive approximation is applied to develop an explicit form for the solution in our results. Define

$$\begin{aligned}
u_0(t) = & \sum_{i=1}^n \frac{\Phi_\phi^{\gamma-i}(t,a)c_i}{\rho^{i-n}\Gamma_\rho(\rho(\gamma-i+1))}, \\
u_k(t) = & u_0(t) + \frac{\lambda}{\rho\Gamma_\rho(\alpha)}\int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)u_{k-1}(s)\phi'(s)ds \\
& +\frac{1}{\rho\Gamma_\rho(\alpha)}\int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)h(s)\phi'(s)ds, \quad k=1,2,3,\ldots.
\end{aligned}$$

For $k = 1$, by using Definition 1, we obtain

$$\begin{aligned}
u_1(t) &= u_0(t) + \frac{\lambda}{\rho\Gamma_\rho(\alpha)} \int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s) u_0(s)\phi'(s)ds + \frac{1}{\rho\Gamma_\rho(\alpha)} \int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)h(s)\phi'(s)ds \\
&= \frac{1}{\rho\Gamma_\rho(\alpha)} \int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)h(s)\phi'(s)ds + \sum_{i=1}^n \frac{\Phi_\phi^{\gamma-i}(t,a)c_i}{\rho^{i-n}\Gamma_\rho(\rho(\gamma-i+1))} \\
&\quad + \frac{\lambda}{\rho\Gamma_\rho(\alpha)} \int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s) \left(\sum_{i=1}^n \frac{\Phi_\phi^{\gamma-i}(s,a)c_i}{\rho^{i-n}\Gamma_\rho(\rho(\gamma-i+1))} \right) \phi'(s)ds \\
&= \frac{1}{\rho\Gamma_\rho(\alpha)} \int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)h(s)\phi'(s)ds + \sum_{i=1}^n \frac{\Phi_\phi^{\gamma-i}(t,a)c_i}{\rho^{i-n}\Gamma_\rho(\rho(\gamma-i+1))} \\
&\quad + \sum_{i=1}^n \frac{\lambda c_i}{\rho^{i-n}\Gamma_\rho(\rho(\gamma-i+1))} \left[\frac{1}{\rho\Gamma_\rho(\alpha)} \int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s) \left(\Phi_\phi^{\frac{\rho(\gamma-i+1)}{\rho}-1}(s,a) \right) \phi'(s)ds \right] \\
&= \frac{1}{\rho\Gamma_\rho(\alpha)} \int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)h(s)\phi'(s)ds + \sum_{i=1}^n \frac{\Phi_\phi^{\gamma-i}(t,a)c_i}{\rho^{i-n}\Gamma_\rho(\rho(\gamma-i+1))} \\
&\quad + \sum_{i=1}^n \frac{\lambda c_i}{\rho^{i-n}\Gamma_\rho(\rho(\gamma-i+1))} {}_\rho\mathcal{I}_{a^+}^{\alpha;\phi} \left[\Phi_\phi^{\frac{\rho(\gamma-i+1)}{\rho}-1}(t,a) \right].
\end{aligned}$$

From (i) of Lemma 1, we have

$$\begin{aligned}
u_1(t) &= \frac{1}{\rho\Gamma_\rho(\alpha)} \int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)h(s)\phi'(s)ds \\
&\quad + \sum_{i=1}^n \frac{c_i\Phi_\phi^{\gamma-i}(t,a)}{\rho^{i-n}\Gamma_\rho(\rho(\gamma-i+1))} + \sum_{i=1}^n \frac{\lambda c_i \Phi_\phi^{\frac{\rho(\gamma-i+1)+\alpha}{\rho}-1}(t,a)}{\rho^{i-n}\Gamma_\rho(\rho(\gamma-i+1)+\alpha)} \\
&= \frac{1}{\rho\Gamma_\rho(\alpha)} \int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)h(s)\phi'(s)ds \\
&\quad + \sum_{i=1}^n \frac{c_i\Phi_\phi^{\gamma-i}(t,a)}{\rho^{i-n}} \left(\frac{1}{\Gamma_\rho(\rho(\gamma-i+1))} + \frac{\lambda \Phi_\phi^{\frac{\alpha}{\rho}}(t,a)}{\Gamma_\rho(\rho(\gamma-i+1)+\alpha)} \right) \\
&= \frac{1}{\rho\Gamma_\rho(\alpha)} \int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)h(s)\phi'(s)ds \\
&\quad + \sum_{i=1}^n \frac{c_i\Phi_\phi^{\gamma-i}(t,a)}{\rho^{i-n}} \left(\sum_{j=1}^2 \frac{\lambda^{j-1}\Phi_\phi^{(j-1)\frac{\alpha}{\rho}}(t,a)}{\Gamma_\rho(\rho(\gamma-i+1)+(j-1)\alpha)} \right).
\end{aligned}$$

By the same process, for $k = 2$, one has

$$\begin{aligned}
u_2(t) &= u_0(t) + \frac{\lambda}{\rho\Gamma_\rho(\alpha)} \int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s) u_1(s)\phi'(s)ds + \frac{1}{\rho\Gamma_\rho(\alpha)} \int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)h(s)\phi'(s)ds \\
&= \frac{1}{\rho\Gamma_\rho(\alpha)} \int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)h(s)\phi'(s)ds + \sum_{i=1}^n \frac{c_i\Phi_\phi^{\gamma-i}(t,a)}{\rho^{i-n}\Gamma_\rho(\rho(\gamma-i+1))} \\
&\quad + \frac{\lambda}{\rho\Gamma_\rho(\alpha)} \int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s) \Bigg\{ \frac{1}{\rho\Gamma_\rho(\alpha)} \int_a^s \Phi_\phi^{\frac{\alpha}{\rho}-1}(s,r)h(r)\phi'(r)dr \\
&\quad + \sum_{i=1}^n \frac{c_i\Phi_\phi^{\gamma-i}(s,a)}{\rho^{i-n}} \left(\sum_{j=1}^2 \frac{\lambda^{j-1}\Phi_\phi^{(j-1)\frac{\alpha}{\rho}}(s,a)}{\Gamma_\rho(\rho(\gamma-i+1)+(j-1)\alpha)} \right) \Bigg\} \phi'(s)ds
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\rho\Gamma_\rho(\alpha)}\int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)h(s)\phi'(s)ds + \sum_{i=1}^n \frac{c_i\Phi_\phi^{\gamma-i}(t,a)}{\rho^{i-n}\Gamma_\rho(\rho(\gamma-i+1))} \\
&\quad + \frac{\lambda}{\rho\Gamma_\rho(2\alpha)}\int_a^t \Phi_\phi^{\frac{2\alpha}{\rho}-1}(t,s)h(s)\phi'(s)ds \\
&\quad + \sum_{i=1}^n \frac{c_i}{\rho^{i-n}}\left(\sum_{j=1}^2 \frac{\lambda^j}{\Gamma_\rho(\rho(\gamma-i+1)+(j-1)\alpha)}\right) \\
&\quad \times \frac{1}{\rho\Gamma_\rho(\alpha)}\int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)\Phi_\phi^{\frac{\rho(\gamma-i+1)+(j-1)\alpha}{\rho}-1}(s,a)\phi'(s)ds \\
&= \frac{1}{\rho\Gamma_\rho(\alpha)}\int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)h(s)\phi'(s)ds + \frac{\lambda}{\rho\Gamma_\rho(2\alpha)}\int_a^t \Phi_\phi^{\frac{2\alpha}{\rho}-1}(t,s)h(s)\phi'(s)ds \\
&\quad + \sum_{i=1}^n \frac{c_i\Phi_\phi^{\gamma-i}(t,a)}{\rho^{i-n}\Gamma_\rho(\rho(\gamma-i+1))} + \sum_{i=1}^n \frac{c_i}{\rho^{i-n}}\left\{\sum_{j=1}^2 \frac{\lambda^j {}_\rho\mathcal{I}_{a^+}^{\alpha;\phi}\left[\Phi_\phi^{\frac{\rho(\gamma-i+1)+(j-1)\alpha}{\rho}-1}(t,a)\right]}{\Gamma_\rho(\rho(\gamma-i+1)+(j-1)\alpha)}\right\} \\
&= \frac{1}{\rho}\int_a^t \left(\frac{\Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)}{\Gamma_\rho(\alpha)} + \frac{\lambda\Phi_\phi^{\frac{2\alpha}{\rho}-1}(t,s)}{\Gamma_\rho(2\alpha)}\right)h(s)\phi'(s)ds \\
&\quad + \sum_{i=1}^n \frac{c_i\Phi_\phi^{\gamma-i}(t,a)}{\rho^{i-n}\Gamma_\rho(\rho(\gamma-i+1))} + \sum_{i=1}^n \frac{c_i}{\rho^{i-n}}\left(\sum_{j=1}^2 \frac{\lambda^j\Phi_\phi^{\frac{\rho(\gamma-i+1)+j\alpha}{\rho}-1}(t,a)}{\Gamma_\rho(\rho(\gamma-i+1)+j\alpha)}\right) \\
&= \frac{1}{\rho}\int_a^t \left(\frac{\Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)}{\Gamma_\rho(\alpha)} + \frac{\lambda\Phi_\phi^{\frac{2\alpha}{\rho}-1}(t,s)}{\Gamma_\rho(2\alpha)}\right)h(s)\phi'(s)ds \\
&\quad + \sum_{i=1}^n \frac{c_i\Phi_\phi^{\gamma-i}(t,a)}{\rho^{i-n}}\left(\frac{1}{\Gamma_\rho(\rho(\gamma-i+1))} + \sum_{j=1}^2 \frac{\lambda^j\Phi_\phi^{\frac{j\alpha}{\rho}}(t,a)}{\Gamma_\rho(\rho(\gamma-i+1)+j\alpha)}\right) \\
&= \frac{1}{\rho}\int_a^t \left(\sum_{j=1}^2 \frac{\lambda^{j-1}\Phi_\phi^{\frac{j\alpha}{\rho}-1}(t,s)}{\Gamma_\rho(j\alpha)}\right)h(s)\phi'(s)ds \\
&\quad + \sum_{i=1}^n \frac{c_i\Phi_\phi^{\gamma-i}(t,a)}{\rho^{i-n}}\left(\sum_{j=1}^3 \frac{\lambda^{j-1}\Phi_\phi^{(j-1)\frac{\alpha}{\rho}}(t,a)}{\Gamma_\rho(\rho(\gamma-i+1)+(j-1)\alpha)}\right).
\end{aligned}
$$

For $k = 1, 2, \ldots$, we obtain that

$$
\begin{aligned}
u_k(t) &= \frac{1}{\rho}\int_a^t \left(\sum_{j=1}^k \frac{\lambda^{j-1}\Phi_\phi^{\frac{j\alpha}{\rho}-1}(t,s)}{\Gamma_\rho(j\alpha)}\right)h(s)\phi'(s)ds \\
&\quad + \sum_{i=1}^n \frac{c_i\Phi_\phi^{\gamma-i}(t,a)}{\rho^{i-n}}\left(\sum_{j=1}^{k+1} \frac{\lambda^{j-1}\Phi_\phi^{(j-1)\frac{\alpha}{\rho}}(t,a)}{\Gamma_\rho(\rho(\gamma-i+1)+(j-1)\alpha)}\right).
\end{aligned}
$$

Taking $k \to \infty$ with changing the summation index in the last expression, $j \to j+1$, we obtain

$$
\begin{aligned}
u(t) &= \frac{1}{\rho}\int_a^t \left(\sum_{j=1}^{\infty} \frac{\lambda^{j-1}\Phi_\phi^{\frac{j\alpha}{\rho}-1}(t,s)}{\Gamma_\rho(j\alpha)}\right) h(s)\phi'(s)ds \\
&\quad + \sum_{i=1}^{n} \frac{c_i \Phi_\phi^{\gamma-i}(t,a)}{\rho^{i-n}} \left(\sum_{j=1}^{\infty} \frac{\lambda^{j-1}\Phi_\phi^{(j-1)\frac{\alpha}{\rho}}(t,a)}{\Gamma_\rho(\rho(\gamma-i+1)+(j-1)\alpha)}\right) \\
&= \frac{1}{\rho}\int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)\left(\sum_{j=0}^{\infty} \frac{\lambda^{j}\Phi_\phi^{\frac{j\alpha}{\rho}}(t,s)}{\Gamma_\rho((j+1)\alpha)}\right) h(s)\phi'(s)ds \\
&\quad + \sum_{i=1}^{n} \frac{c_i \Phi_\phi^{\gamma-i}(t,a)}{\rho^{i-n}} \left(\sum_{j=0}^{\infty} \frac{\lambda^{j}\Phi_\phi^{j\frac{\alpha}{\rho}}(t,a)}{\Gamma_\rho(\rho(\gamma-i+1)+j\alpha)}\right).
\end{aligned}
$$

By using the property (6), we obtain

$$
\begin{aligned}
u(t) &= \frac{1}{\rho^{\frac{\alpha}{\rho}}}\int_a^t \Phi_\phi^{\frac{\alpha}{\rho}-1}(t,s)\left(\sum_{j=0}^{\infty} \frac{\lambda^{j}\Phi_\phi^{\frac{j\alpha}{\rho}}(t,s)}{\rho^{\frac{j\alpha}{\rho}}\Gamma\left(\frac{\alpha}{\rho}j+\frac{\alpha}{\rho}\right)}\right) h(s)\phi'(s)ds \\
&\quad + \sum_{i=1}^{n} \frac{c_i \Phi_\phi^{\gamma-i}(t,a)}{\rho^{\gamma-n}} \left(\sum_{j=0}^{\infty} \frac{\lambda^{j}\Phi_\phi^{j\frac{\alpha}{\rho}}(t,a)}{\rho^{\frac{j\alpha}{\rho}}\Gamma\left(\frac{\alpha}{\rho}j+\gamma-i+1\right)}\right).
\end{aligned}
$$

Applying Lemma 3, we find that the explicit solution (12). □

2.3. An Auxiliary Lemma

Let us denote the weighted space

$$
\mathcal{C}_{\phi_k}^{1-\gamma_k}(\mathcal{J},\mathbb{R}) = \left\{u : (a,b] \to \mathbb{R} | u(a^+) \text{ exists and } \Phi_\phi^{1-\gamma}(t,a)u(t) \in \mathcal{C}(\mathcal{J},\mathbb{R})\right\},\ \gamma \in (0,1],
$$

where $\mathcal{C}_{\phi_k}^{1-\gamma_k} = \mathcal{C}_{\phi_k}^{1-\gamma_k}(\mathcal{J},\mathbb{R})$. Next, we provide the weighted space of piecewise continuous functions as follows:

$$
\begin{aligned}
\mathcal{PC}_{\phi_k}^{1-\gamma_k}(\mathcal{J},\mathbb{R}) = &\Big\{u : (a,b] \to \mathbb{R} | u \in \mathcal{C}_{\phi_k}^{1-\gamma_k}, k = 0,1,2,\ldots,m, \\
&{}_{\rho_k}\mathcal{I}_{t_k^+}^{\rho_k(1-\gamma_k);\phi_k} u(t_k^+), {}_{\rho_{k-1}}\mathcal{I}_{t_{k-1}^+}^{\rho_{k-1}(1-\gamma_{k-1});\phi_{k-1}} u(t_k^-) \text{ exist and} \\
&{}_{\rho_{k-1}}\mathcal{I}_{t_{k-1}^+}^{\rho_{k-1}(1-\gamma_{k-1});\phi_{k-1}} u(t_k^-) = {}_{\rho_{k-1}}\mathcal{I}_{t_{k-1}^+}^{\rho_{k-1}(1-\gamma_{k-1});\phi_{k-1}} u(t_k),\ k = 1,\ldots,m\Big\}.
\end{aligned}
$$

Observe that $\mathcal{PC}_{\phi_k}^{1-\gamma_k} = \mathcal{PC}_{\phi_k}^{1-\gamma_k}(\mathcal{J},\mathbb{R})$ is a Banach space equipped with

$$
\|u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}} = \sup_{t\in\mathcal{J}} \left|\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)u(t)\right|.
$$

For the easy to prove, we set the symbol that will be used throughout this paper.

$$
\mathcal{P}_k(j) = \prod_{l=j}^{k-1} \mathbb{E}_{\frac{\alpha_l}{\rho_l}}\left(\lambda_l\left(\rho_l^{-1}\Phi_{\phi_l}(t_{l+1},t_l)\right)^{\frac{\alpha_l}{\rho_l}}\right), \quad |\mathcal{P}_k(j)| \le 1, \tag{13}
$$

$$\begin{aligned}
\Xi \;=\; & \sum_{i=0}^{m} \frac{\kappa_i \mathcal{P}_i(0) \Phi_{\phi_i}^{\gamma_i - 1}(\eta_i, t_i)}{\rho_i^{\gamma_i - 1}} \, \mathbb{E}_{\frac{\alpha_i}{\rho_i}, \gamma_i}\big(\lambda_i \big(\rho_i^{-1} \Phi_{\phi_i}(\eta_i, t_i)\big)^{\frac{\alpha_i}{\rho_i}}\big) \\
& - \sum_{j=0}^{n} \frac{\omega_j \mathcal{P}_j(0) \Phi_{\phi_j}^{\gamma_j + \frac{\mu_j}{\rho_j} - 1}(\xi_j, t_j)}{\rho_j^{\frac{\mu_j}{\rho_j} + \gamma_j - 1}} \, \mathbb{E}_{\frac{\alpha_j}{\rho_j}, \gamma_j + \frac{\mu_j}{\rho_j}}\big(\lambda_j \big(\rho_j^{-1} \Phi_{\phi_j}(\xi_j, t_j)\big)^{\frac{\alpha_j}{\rho_j}}\big).
\end{aligned} \tag{14}$$

Lemma 7. *Assume that* $\alpha_k \in (0,1)$*,* $\beta_k \in [0,1]$*,* $\rho_k > 0$*,* $\gamma_k = (\beta_k(\rho_k - \alpha_k) + \alpha_k)/\rho_k$*,* $\lambda_k < 0$*,* $\phi_k \in \mathcal{C}(\mathcal{J}, \mathbb{R})$ *with* $\phi_k' > 0$*,* $k = 0, 1, 2 \ldots, m$*,* κ_i*,* $\omega_r \in \mathbb{R}$*,* $i = 0, 1, 2 \ldots, m$*,* $j = 0, 1, 2 \ldots, n$*,* $h \in \mathcal{C}_{\phi_k}^{1-\gamma_k}$*, and* $\Xi \neq 0$*. Then, the following impulsive* (ρ_k, ϕ_k)*-Hilfer* $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$*s:*

$$\begin{cases}
{}^{H}_{\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k, \beta_k; \phi_k} u(t) = \lambda_k u(t) + h(t), \quad t \in \mathcal{J}_k \subseteq \mathcal{J}, t \neq t_k, \\
\Delta_{\rho_k} \mathcal{I}_{t_k^+}^{\rho_k(1-\gamma_k); \phi_k} u(t_k) = \varphi_k(u(t_k)), \quad k = 1, 2, \ldots, m, \\
\displaystyle\sum_{i=0}^{m} \kappa_i u(\eta_i) = \sum_{j=0}^{n} \omega_{j \rho_j} \mathcal{I}_{t_j^+}^{\mu_j; \phi_j} u(\xi_j) + A, \quad \eta_i \in (t_i, t_{i+1}], \xi_j \in (t_j, t_{j+1}],
\end{cases} \tag{15}$$

is corresponding to the following integral equation, $u \in \mathcal{PC}_{\phi_k}^{1-\gamma_k}$*,*

$$\begin{aligned}
u(t) \;=\; & \frac{1}{\rho_k^{\frac{\alpha_k}{\rho_k}}} \int_{t_k}^{t} \Phi_{\phi_k}^{\frac{\alpha_k}{\rho_k} - 1}(t, s) \mathbb{E}_{\frac{\alpha_k}{\rho_k}, \frac{\alpha_k}{\rho_k}}\big(\lambda_k \big(\rho_k^{-1} \Phi_{\phi_k}(t, s)\big)^{\frac{\alpha_k}{\rho_k}}\big) h(s) \phi_k'(s) ds \\
& + \Bigg[\sum_{r=0}^{k-1} \frac{\mathcal{P}_k(r+1)}{\rho_r^{\frac{\alpha_r}{\rho_r} + 1 - \gamma_r}} \int_{t_r}^{t_{r+1}} \Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r} - \gamma_r}(t_{r+1}, s) \mathbb{E}_{\frac{\alpha_r}{\rho_r}, \frac{\alpha_r}{\rho_r} + 1 - \gamma_r}\big(\lambda_r \big(\rho_r^{-1} \Phi_{\phi_r}(t_{r+1}, s)\big)^{\frac{\alpha_r}{\rho_r}}\big) h(s) \phi_r'(s) ds \\
& + \sum_{r=1}^{k} \varphi_r(u(t_r)) \mathcal{P}_k(r) \Bigg] \frac{\Phi_{\phi_k}^{\gamma_k - 1}(t, t_k)}{\rho_k^{\gamma_k - 1}} \mathbb{E}_{\frac{\alpha_k}{\rho_k}, \gamma_k}\big(\lambda_k \big(\rho_k^{-1} \Phi_{\phi_k}(t, t_k)\big)^{\frac{\alpha_k}{\rho_k}}\big) \\
& + \frac{1}{\Xi} \Bigg(\sum_{j=0}^{n} \frac{\omega_j}{\rho_j^{\frac{\alpha_j + \mu_j}{\rho_j}}} \int_{t_j}^{\xi_j} \Phi_{\phi_j}^{\frac{\alpha_j + \mu_j}{\rho_j} - 1}(\xi_j, s) \mathbb{E}_{\frac{\alpha_j}{\rho_j}, \frac{\alpha_j + \mu_j}{\rho_j}}\big(\lambda_j \big(\rho_j^{-1} \Phi_{\phi_j}(\xi_j, s)\big)^{\frac{\alpha_j}{\rho_j}}\big) h(s) \phi_j'(s) ds \\
& + \sum_{j=0}^{n} \Bigg\{ \Bigg[\sum_{r=0}^{j-1} \frac{\mathcal{P}_j(r+1)}{\rho_r^{\frac{\alpha_r}{\rho_r} + 1 - \gamma_r}} \int_{t_r}^{t_{r+1}} \Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r} - \gamma_r}(t_{r+1}, s) \mathbb{E}_{\frac{\alpha_r}{\rho_r}, \frac{\alpha_r}{\rho_r} + 1 - \gamma_r}\big(\lambda_r \big(\rho_r^{-1} \Phi_{\phi_r}(t_{r+1}, s)\big)^{\frac{\alpha_r}{\rho_r}}\big) h(s) \phi_r'(s) ds \\
& + \sum_{r=1}^{j} \varphi_r(u(t_r)) \mathcal{P}_j(r) \Bigg] \frac{\omega_j \Phi_{\phi_j}^{\gamma_j + \frac{\mu_j}{\rho_j} - 1}(\xi_j, t_j)}{\rho_j^{\frac{\mu_j}{\rho_j} + \gamma_j - 1}} \mathbb{E}_{\frac{\alpha_j}{\rho_j}, \gamma_j + \frac{\mu_j}{\rho_j}}\big(\lambda_j \big(\rho_j^{-1} \Phi_{\phi_j}(\xi_j, t_j)\big)^{\frac{\alpha_j}{\rho_j}}\big) \Bigg\} \\
& + A - \sum_{i=0}^{m} \frac{\kappa_i}{\rho_i^{\frac{\alpha_i}{\rho_i}}} \int_{t_i}^{\eta_i} \Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i} - 1}(\eta_i, s) \mathbb{E}_{\frac{\alpha_i}{\rho_i}, \frac{\alpha_i}{\rho_i}}\big(\lambda_i \big(\rho_i^{-1} \Phi_{\phi_i}(\eta_i, s)\big)^{\frac{\alpha_i}{\rho_i}}\big) h(s) \phi_r'(s) ds \\
& - \sum_{i=0}^{m} \Bigg\{ \Bigg[\sum_{r=0}^{i-1} \frac{\mathcal{P}_i(r+1)}{\rho_r^{\frac{\alpha_r}{\rho_r} + 1 - \gamma_r}} \int_{t_r}^{t_{r+1}} \Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r} - \gamma_r}(t_{r+1}, s) \mathbb{E}_{\frac{\alpha_r}{\rho_r}, \frac{\alpha_r}{\rho_r} + 1 - \gamma_r}\big(\lambda_r \big(\rho_r^{-1} \Phi_{\phi_r}(t_{r+1}, s)\big)^{\frac{\alpha_r}{\rho_r}}\big) h(s) \phi_r'(s) ds \\
& + \sum_{r=1}^{i} \varphi_r(u(t_r)) \mathcal{P}_i(r) \Bigg] \frac{\kappa_i \Phi_{\phi_i}^{\gamma_i - 1}(\eta_i, t_i)}{\rho_i^{\gamma_i - 1}} \mathbb{E}_{\frac{\alpha_i}{\rho_i}, \gamma_i}\big(\lambda_i \big(\rho_i^{-1} \Phi_{\phi_i}(\eta_i, t_i)\big)^{\frac{\alpha_i}{\rho_i}}\big) \Bigg\} \Bigg) \\
& \times \frac{\mathcal{P}_k(0) \Phi_{\phi_k}^{\gamma_k - 1}(t, t_k)}{\rho_k^{\gamma_k - 1}} \mathbb{E}_{\frac{\alpha_k}{\rho_k}, \gamma_k}\big(\lambda_k \big(\rho_k^{-1} \Phi_{\phi_k}(t, t_k)\big)^{\frac{\alpha_k}{\rho_k}}\big).
\end{aligned} \tag{16}$$

Proof. Let $u \in \mathcal{PC}_{\phi_k}^{1-\gamma_k}$ be a solution of the impulsive (ρ_k, ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (4). We consider the following several cases.

For $t_1 \in [t_0, t_1]$, we obtain

$$
\begin{aligned}
u(t) &= \frac{1}{\rho_0^{\frac{\alpha_0}{\rho_0}}} \int_{t_0}^{t} \Phi_{\phi_0}^{\frac{\alpha_0}{\rho_0}-1}(t,s) \mathbb{E}_{\frac{\alpha_0}{\rho_0},\frac{\alpha_0}{\rho_0}}\big(\lambda_0 \big(\rho_0^{-1}\Phi_{\phi_0}(t,s)\big)^{\frac{\alpha_0}{\rho_0}}\big) h(s)\phi_0'(s)ds \\
&\quad + u_0 \frac{\Phi_{\phi_0}^{\gamma_0-1}(t,t_0)}{\rho_0^{\gamma_0-1}} \mathbb{E}_{\frac{\alpha_0}{\rho_0},\gamma_0}\big(\lambda_0 \big(\rho_0^{-1}\Phi_{\phi_0}(t,t_0)\big)^{\frac{\alpha_0}{\rho_0}}\big).
\end{aligned}
$$

Taking the operator ${}_{\rho_0}\mathcal{I}_{t_0^+}^{\rho_0(1-\gamma_0);\phi_0}$ into the above equation with Lemmas 4 and 5, which implies that

$$
\begin{aligned}
{}_{\rho_0}\mathcal{I}_{t_0^+}^{\rho_0(1-\gamma_0);\phi_0} u(t) &= \frac{1}{\rho_0^{\frac{\alpha_0}{\rho_0}+1-\gamma_0}} \int_{t_0}^{t} \Phi_{\phi_0}^{\frac{\alpha_0}{\rho_0}-\gamma_0}(t,s) \mathbb{E}_{\frac{\alpha_0}{\rho_0},\frac{\alpha_0}{\rho_0}+1-\gamma_0}\big(\lambda_0 \big(\rho_0^{-1}\Phi_{\phi_0}(t,s)\big)^{\frac{\alpha_0}{\rho_0}}\big) h(s)\phi_0'(s)ds \\
&\quad + u_0 \mathbb{E}_{\frac{\alpha_0}{\rho_0}}\big(\lambda_0 \big(\rho_0^{-1}\Phi_{\phi_0}(t,t_0)\big)^{\frac{\alpha_0}{\rho_0}}\big)
\end{aligned}
$$

In particular, for $t = t_1$, it follows that

$$
\begin{aligned}
&{}_{\rho_0}\mathcal{I}_{t_0^+}^{\rho_0(1-\gamma_0);\phi_0} u(t_1) \\
&= \frac{1}{\rho_0^{\frac{\alpha_0}{\rho_0}+1-\gamma_0}} \int_{t_0}^{t_1} \Phi_{\phi_0}^{\frac{\alpha_0}{\rho_0}-\gamma_0}(t_1,s) \mathbb{E}_{\frac{\alpha_0}{\rho_0},\frac{\alpha_0}{\rho_0}+1-\gamma_0}\big(\lambda_0 \big(\rho_0^{-1}\Phi_{\phi_0}(t_1,s)\big)^{\frac{\alpha_0}{\rho_0}}\big) h(s)\phi_0'(s)ds \\
&\quad + u_0 \mathbb{E}_{\frac{\alpha_0}{\rho_0}}\big(\lambda_0 \big(\rho_0^{-1}\Phi_{\phi_0}(t_1,t_0)\big)^{\frac{\alpha_0}{\rho_0}}\big)
\end{aligned}
$$

For $t \in (t_1, t_2]$, we obtain

$$
\begin{aligned}
u(t) &= \frac{1}{\rho_1^{\frac{\alpha_1}{\rho_1}}} \int_{t_1}^{t} \Phi_{\phi_1}^{\frac{\alpha_1}{\rho_1}-1}(t,s) \mathbb{E}_{\frac{\alpha_1}{\rho_1},\frac{\alpha_1}{\rho_1}}\big(\lambda_1 \big(\rho_1^{-1}\Phi_{\phi_1}(t,s)\big)^{\frac{\alpha_1}{\rho_1}}\big) h(s)\phi_1'(s)ds \\
&\quad + {}_{\rho_1}\mathcal{I}_{t_1^+}^{\rho(1-\gamma_1);\phi_1} u(t_1^+) \frac{\Phi_{\phi_1}^{\gamma_1-1}(t,t_1)}{\rho_1^{\gamma_1-1}} \mathbb{E}_{\frac{\alpha_1}{\rho_1},\gamma_1}\big(\lambda_1 \big(\rho_1^{-1}\Phi_{\phi_1}(t,t_1)\big)^{\frac{\alpha_1}{\rho_1}}\big).
\end{aligned}
$$

By using the impulsive condition, that is ${}_{\rho_1}\mathcal{I}_{t_1^+}^{\rho_1(1-\gamma_1);\phi_1} u(t_1^+) = {}_{\rho_0}\mathcal{I}_{t_0^+}^{\rho_0(1-\gamma_0);\phi_0} u(t_1^-) + \varphi_1(u(t_1))$, we obtain

$$
\begin{aligned}
u(t) &= \frac{1}{\rho_1^{\frac{\alpha_1}{\rho_1}}} \int_{t_1}^{t} \Phi_{\phi_1}^{\frac{\alpha_1}{\rho_1}-1}(t,s) \mathbb{E}_{\frac{\alpha_1}{\rho_1},\frac{\alpha_1}{\rho_1}}\big(\lambda_1 \big(\rho_1^{-1}\Phi_{\phi_1}(t,s)\big)^{\frac{\alpha_1}{\rho_1}}\big) h(s)\phi_1'(s)ds \\
&\quad + \Big[{}_{\rho_0}\mathcal{I}_{t_0^+}^{\rho_0(1-\gamma_0);\phi_0} u(t_1^-) + \varphi_1(u(t_1))\Big] \frac{\Phi_{\phi_1}^{\gamma_1-1}(t,t_1)}{\rho_1^{\gamma_1-1}} \mathbb{E}_{\frac{\alpha_1}{\rho_1},\gamma_1}\big(\lambda_1 \big(\rho_1^{-1}\Phi_{\phi_1}(t,t_1)\big)^{\frac{\alpha_1}{\rho_1}}\big) \\
&= \frac{1}{\rho_1^{\frac{\alpha_1}{\rho_1}}} \int_{t_1}^{t} \Phi_{\phi_1}^{\frac{\alpha_1}{\rho_1}-1}(t,s) \mathbb{E}_{\frac{\alpha_1}{\rho_1},\frac{\alpha_1}{\rho_1}}\big(\lambda_1 \big(\rho_1^{-1}\Phi_{\phi_1}(t,s)\big)^{\frac{\alpha_1}{\rho_1}}\big) h(s)\phi_1'(s)ds \\
&\quad + \Bigg[\frac{1}{\rho_0^{\frac{\alpha_0}{\rho_0}+1-\gamma_0}} \int_{t_0}^{t_1} \Phi_{\phi_0}^{\frac{\alpha_0}{\rho_0}-\gamma_0}(t_1,s) \mathbb{E}_{\frac{\alpha_0}{\rho_0},\frac{\alpha_0}{\rho_0}+1-\gamma_0}\big(\lambda_0 \big(\rho_0^{-1}\Phi_{\phi_0}(t_1,s)\big)^{\frac{\alpha_0}{\rho_0}}\big) h(s)\phi_0'(s)ds \\
&\quad + u_0 \mathbb{E}_{\frac{\alpha_0}{\rho_0}}\big(\lambda_0 \big(\rho_0^{-1}\Phi_{\phi_0}(t_1,t_0)\big)^{\frac{\alpha_0}{\rho_0}}\big) + \varphi_1(u(t_1))\Bigg] \frac{\Phi_{\phi_1}^{\gamma_1-1}(t,t_1)}{\rho_1^{\gamma_1-1}} \mathbb{E}_{\frac{\alpha_1}{\rho_1},\gamma_1}\big(\lambda_1 \big(\rho_1^{-1}\Phi_{\phi_1}(t,t_1)\big)^{\frac{\alpha_1}{\rho_1}}\big).
\end{aligned}
$$

Taking ${}_{\rho_1}\mathcal{I}_{t_1^+}^{\rho_1(1-\gamma_1);\phi_1}$ into the above equation with Lemmas 4 and 5, one has

$$\begin{aligned}
&{}_{\rho_1}\mathcal{I}_{t_1^+}^{\rho_1(1-\gamma_1);\phi_1}u(t)\\
=\;& \frac{1}{\rho_1^{\frac{\alpha_1}{\rho_1}+1-\gamma_1}}\int_{t_1}^{t}\Phi_{\phi_1}^{\frac{\alpha_1}{\rho_1}-\gamma_1}(t,s)\mathbb{E}_{\frac{\alpha_1}{\rho_1},\frac{\alpha_1}{\rho_1}+1-\gamma_1}\big(\lambda_1(\rho_1^{-1}\Phi_{\phi_1}(t,s))^{\frac{\alpha_1}{\rho_1}}\big)h(s)\phi_1'(s)ds\\
&+\Bigg[\frac{1}{\rho_0^{\frac{\alpha_0}{\rho_0}+1-\gamma_0}}\int_{t_0}^{t_1}\Phi_{\phi_0}^{\frac{\alpha_0}{\rho_0}-\gamma_0}(t_1,s)\mathbb{E}_{\frac{\alpha_0}{\rho_0},\frac{\alpha_0}{\rho_0}+1-\gamma_0}\big(\lambda_0(\rho_0^{-1}\Phi_{\phi_0}(t_1,s))^{\frac{\alpha_0}{\rho_0}}\big)h(s)\phi_0'(s)ds\\
&+u_0\mathbb{E}_{\frac{\alpha_0}{\rho_0}}\big(\lambda_0(\rho_0^{-1}\Phi_{\phi_0}(t_1,t_0))^{\frac{\alpha_0}{\rho_0}}\big)+\varphi_1(u(t_1))\Bigg]\mathbb{E}_{\frac{\alpha_1}{\rho_1}}\big(\lambda_1(\rho_1^{-1}\Phi_{\phi_1}(t,t_1))^{\frac{\alpha_1}{\rho_1}}\big).
\end{aligned}$$

In particular, for $t=t_2$, we have

$$\begin{aligned}
&{}_{\rho_1}I_{t_1^+}^{\rho_1(1-\gamma_1);\phi_1}u(t_2)\\
=\;& \frac{1}{\rho_1^{\frac{\alpha_1}{\rho_1}+1-\gamma_1}}\int_{t_1}^{t_2}\Phi_{\phi_1}^{\frac{\alpha_1}{\rho_1}-\gamma_1}(t_2,s)\mathbb{E}_{\frac{\alpha_1}{\rho_1},\frac{\alpha_1}{\rho_1}+1-\gamma_1}\big(\lambda_1(\rho_1^{-1}\Phi_{\phi_1}(t_2,s))^{\frac{\alpha_1}{\rho_1}}\big)h(s)\phi_1'(s)ds\\
&+\Bigg[\frac{1}{\rho_0^{\frac{\alpha_0}{\rho_0}+1-\gamma_0}}\int_{t_0}^{t_1}\Phi_{\phi_0}^{\frac{\alpha_0}{\rho_0}-\gamma_0}(t_1,s)\mathbb{E}_{\frac{\alpha_0}{\rho_0},\frac{\alpha_0}{\rho_0}+1-\gamma_0}\big(\lambda_0(\rho_0^{-1}\Phi_{\phi_0}(t_1,s))^{\frac{\alpha_0}{\rho_0}}\big)h(s)\phi_0'(s)ds\\
&+u_0\mathbb{E}_{\frac{\alpha_0}{\rho_0}}\big(\lambda_0(\rho_0^{-1}\Phi_{\phi_0}(t_1,t_0))^{\frac{\alpha_0}{\rho_0}}\big)+\varphi_1(u(t_1))\Bigg]\mathbb{E}_{\frac{\alpha_1}{\rho_1}}\big(\lambda_1(\rho_1^{-1}\Phi_{\phi_1}(t_2,t_1))^{\frac{\alpha_1}{\rho_1}}\big).
\end{aligned}$$

For $t\in(t_2,t_3]$ with ${}_{\rho_2}\mathcal{I}_{t_2^+}^{\rho_2(1-\gamma_2);\phi_2}u(t_2^+)={}_{\rho_1}\mathcal{I}_{t_1^+}^{\rho_1(1-\gamma_1);\phi_1}u(t_2^-)+\varphi_2(u(t_2))$, we obtain

$$\begin{aligned}
u(t)\;=\;& \frac{1}{\rho_2^{\frac{\alpha_2}{\rho_2}}}\int_{t_2}^{t}\Phi_{\phi_2}^{\frac{\alpha_2}{\rho_2}-1}(t,s)\mathbb{E}_{\frac{\alpha_2}{\rho_2},\frac{\alpha_2}{\rho_2}}\big(\lambda_2(\rho_2^{-1}\Phi_{\phi_2}(t,s))^{\frac{\alpha_2}{\rho_2}}\big)h(s)\phi_2'(s)ds\\
&+\Bigg\{\frac{1}{\rho_1^{\frac{\alpha_1}{\rho_1}+1-\gamma_1}}\int_{t_1}^{t_2}\Phi_{\phi_1}^{\frac{\alpha_1}{\rho_1}-\gamma_1}(t_2,s)\mathbb{E}_{\frac{\alpha_1}{\rho_1},\frac{\alpha_1}{\rho_1}+1-\gamma_1}\big(\lambda_1(\rho_1^{-1}\Phi_{\phi_1}(t_2,s))^{\frac{\alpha_1}{\rho_1}}\big)h(s)\phi_1'(s)ds\\
&+\int_{t_0}^{t_1}\Phi_{\phi_0}^{\frac{\alpha_0}{\rho_0}-\gamma_0}(t_1,s)\mathbb{E}_{\frac{\alpha_0}{\rho_0},\frac{\alpha_0}{\rho_0}+1-\gamma_0}\big(\lambda_0(\rho_0^{-1}\Phi_{\phi_0}(t_1,s))^{\frac{\alpha_0}{\rho_0}}\big)h(s)\phi_0'(s)ds\\
&\times\frac{\mathbb{E}_{\frac{\alpha_1}{\rho_1}}\big(\lambda_1(\rho_1^{-1}\Phi_{\phi_1}(t_2,t_1))^{\frac{\alpha_1}{\rho_1}}\big)}{\rho_0^{\frac{\alpha_0}{\rho_0}+1-\gamma_0}}+u_0\mathbb{E}_{\frac{\alpha_0}{\rho_0}}\big(\lambda_0(\rho_0^{-1}\Phi_{\phi_0}(t_1,t_0))^{\frac{\alpha_0}{\rho_0}}\big)\\
&\times\mathbb{E}_{\frac{\alpha_1}{\rho_1}}\big(\lambda_1(\rho_1^{-1}\Phi_{\phi_1}(t_2,t_1))^{\frac{\alpha_1}{\rho_1}}\big)+\varphi_1(u(t_1))\mathbb{E}_{\frac{\alpha_1}{\rho_1}}\big(\lambda_1(\rho_1^{-1}\Phi_{\phi_1}(t_2,t_1))^{\frac{\alpha_1}{\rho_1}}\big)\\
&+\varphi_2(u(t_2))\Bigg\}\frac{\Phi_{\phi_2}^{\gamma_2-1}(t,t_2)}{\rho_2^{\gamma_2-1}}\mathbb{E}_{\frac{\alpha_2}{\rho_2},\gamma_2}\big(\lambda_2(\rho_2^{-1}\Phi_{\phi_2}(t,t_2))^{\frac{\alpha_2}{\rho_2}}\big).
\end{aligned}$$

Repeating the previous procedure, for $t \in \mathcal{J}_k$, $k = 0, 1, \ldots, m$, it follows form

$$
\begin{aligned}
u(t) \;=\; & \frac{1}{\rho_k^{\frac{\alpha_k}{\rho_k}}} \int_{t_k}^{t} \Phi_{\phi_k}^{\frac{\alpha_k}{\rho_k}-1}(t,s) \mathbb{E}_{\frac{\alpha_k}{\rho_k},\frac{\alpha_k}{\rho_k}}\left(\lambda_k \left(\rho_k^{-1}\Phi_{\phi_k}(t,s)\right)^{\frac{\alpha_k}{\rho_k}}\right) h(s)\phi_k'(s)ds \\
& + \left[\sum_{r=0}^{k-1} \frac{1}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}} \prod_{i=r+1}^{k-1} \mathbb{E}_{\frac{\alpha_i}{\rho_i},1}\left(\lambda_i \left(\rho_i^{-1}\Phi_{\phi_i}(t_{i+1},t_i)\right)^{\frac{\alpha_i}{\rho_i}}\right) \right. \\
& \times \int_{t_r}^{t_{r+1}} \Phi_{\phi_r}^{\frac{\alpha_j}{\rho_r}-\gamma_r}(t_{r+1},s) \mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r}+1-\gamma_r}\left(\lambda_r \left(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s)\right)^{\frac{\alpha_r}{\rho_r}}\right) h(s)\phi_r'(s)ds \\
& \left. + \sum_{r=1}^{k} \varphi_r(u(t_r)) \prod_{i=r}^{k-1} \mathbb{E}_{\frac{\alpha_i}{\rho_i},1}\left(\lambda_i \left(\rho_i^{-1}\Phi_{\phi_i}(t_{i+1},t_i)\right)^{\frac{\alpha_i}{\rho_i}}\right) \right] \\
& \times \frac{\Phi_{\phi_k}^{\gamma_k-1}(t,t_k)}{\rho_k^{\gamma_k-1}} \mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\left(\lambda_k \left(\rho_k^{-1}\Phi_{\phi_k}(t,t_k)\right)^{\frac{\alpha_k}{\rho_k}}\right) + u_0 \prod_{i=0}^{k-1} \mathbb{E}_{\frac{\alpha_i}{\rho_i},1}\left(\lambda_i \left(\rho_i^{-1}\Phi_{\phi_i}(t_{i+1},t_i)\right)^{\frac{\alpha_i}{\rho_i}}\right) \\
& \times \frac{\Phi_{\phi_k}^{\gamma_k-1}(t,t_k)}{\rho_k^{\gamma_k-1}} \mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\left(\lambda_k \left(\rho_k^{-1}\Phi_{\phi_k}(t,t_k)\right)^{\frac{\alpha_k}{\rho_k}}\right).
\end{aligned}
\tag{17}
$$

By using the symbol (13), the form (17) can be rewritten

$$
\begin{aligned}
u(t) \;=\; & \frac{1}{\rho_k^{\frac{\alpha_k}{\rho_k}}} \int_{t_k}^{t} \Phi_{\phi_k}^{\frac{\alpha_k}{\rho_k}-1}(t,s) \mathbb{E}_{\frac{\alpha_k}{\rho_k},\frac{\alpha_k}{\rho_k}}\left(\lambda_k \left(\rho_k^{-1}\Phi_{\phi_k}(t,s)\right)^{\frac{\alpha_k}{\rho_k}}\right) h(s)\phi_k'(s)ds \\
& + \left[\sum_{r=0}^{k-1} \frac{\mathcal{P}_k(r+1)}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}} \int_{t_r}^{t_{r+1}} \Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s) \mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r}+1-\gamma_r}\left(\lambda_r \left(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s)\right)^{\frac{\alpha_r}{\rho_r}}\right) \right. \\
& \left. \times h(s)\phi_r'(s)ds + \sum_{r=1}^{k} \varphi_r(u(t_r))\mathcal{P}_k(r) \right] \frac{\Phi_{\phi_k}^{\gamma_k-1}(t,t_k)}{\rho_k^{\gamma_k-1}} \mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\left(\lambda_k \left(\rho_k^{-1}\Phi_{\phi_k}(t,t_k)\right)^{\frac{\alpha_k}{\rho_k}}\right) \\
& + u_0 \frac{\mathcal{P}_k(0)\Phi_{\phi_k}^{\gamma_k-1}(t,t_k)}{\rho_k^{\gamma_k-1}} \mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\left(\lambda_k \left(\rho_k^{-1}\Phi_{\phi_k}(t,t_k)\right)^{\frac{\alpha_k}{\rho_k}}\right).
\end{aligned}
\tag{18}
$$

From the non-local condition, $\sum_{i=0}^{m} \kappa_i u(\eta_i) = \sum_{j=0}^{n} \omega_{j\rho_j} \mathcal{I}_{t_j^+}^{\mu_j;\phi_j} u(\xi_j) + A$, we obtain

$$
\begin{aligned}
& \sum_{i=0}^{m} \kappa_i u(\eta_i) \\
= \; & \sum_{i=0}^{m} \frac{\kappa_i}{\rho_i^{\frac{\alpha_i}{\rho_i}}} \int_{t_i}^{\eta_i} \Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}-1}(\eta_i,s) \mathbb{E}_{\frac{\alpha_i}{\rho_i},\frac{\alpha_i}{\rho_i}}\left(\lambda_i \left(\rho_i^{-1}\Phi_{\phi_i}(\eta_i,s)\right)^{\frac{\alpha_i}{\rho_i}}\right) h(s)\phi_r'(s)ds \\
& + \sum_{i=0}^{m} \left\{ \left[\sum_{r=0}^{i-1} \frac{\mathcal{P}_i(r+1)}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}} \int_{t_r}^{t_{r+1}} \Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s) \mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r}+1-\gamma_r}\left(\lambda_r \left(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s)\right)^{\frac{\alpha_r}{\rho_r}}\right) \right. \right. \\
& \left. \left. \times h(s)\phi_r'(s)ds + \sum_{r=1}^{i} \varphi_r(u(t_r))\mathcal{P}_i(r) \right] \frac{\kappa_i \Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\rho_i^{\gamma_i-1}} \mathbb{E}_{\frac{\alpha_i}{\rho_i},\gamma_i}\left(\lambda_i \left(\rho_i^{-1}\Phi_{\phi_i}(\eta_i,t_i)\right)^{\frac{\alpha_i}{\rho_i}}\right) \right\} \\
& + u_0 \sum_{i=0}^{m} \frac{\kappa_i \mathcal{P}_i(0)\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\rho_i^{\gamma_i-1}} \mathbb{E}_{\frac{\alpha_i}{\rho_i},\gamma_i}\left(\lambda_i \left(\rho_i^{-1}\Phi_{\phi_i}(\eta_i,t_i)\right)^{\frac{\alpha_i}{\rho_i}}\right).
\end{aligned}
\tag{19}
$$

and

$$\begin{aligned}
&\sum_{j=0}^{n}\omega_{j\rho_j}\mathcal{I}_{t_j^+}^{\mu_j;\phi_j}u(\xi_j)\\
=\ &\sum_{j=0}^{n}\frac{\omega_j}{\rho_j^{\frac{\alpha_j+\mu_j}{\rho_j}}}\int_{t_j}^{\xi_j}\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,s)\mathbb{E}_{\frac{\alpha_j}{\rho_j},\frac{\alpha_j+\mu_j}{\rho_j}}\big(\lambda_j(\rho_j^{-1}\Phi_{\phi_j}(\xi_j,s))^{\frac{\alpha_j}{\rho_j}}\big)h(s)\phi_j'(s)ds\\
&+\sum_{j=0}^{n}\Bigg\{\Bigg[\sum_{r=0}^{j-1}\frac{\mathcal{P}_j(r+1)}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)\mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r}+1-\gamma_r}\big(\lambda_r(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s))^{\frac{\alpha_r}{\rho_r}}\big)\\
&\times h(s)\phi_r'(s)ds+\sum_{r=1}^{j}\varphi_r(u(t_r))\mathcal{P}_j(r)\Bigg]\frac{\omega_j\Phi_{\phi_j}^{\gamma_j+\frac{\mu_j}{\rho_j}-1}(\xi_j,t_j)}{\rho_j^{\frac{\mu_j}{\rho_j}+\gamma_j-1}}\mathbb{E}_{\frac{\alpha_j}{\rho_j},\gamma_j+\frac{\mu_j}{\rho_j}}\big(\lambda_j(\rho_j^{-1}\Phi_{\phi_j}(\xi_j,t_j))^{\frac{\alpha_j}{\rho_j}}\big)\Bigg\}\\
&+u_0\sum_{j=0}^{n}\frac{\omega_j\mathcal{P}_j(0)\Phi_{\phi_j}^{\gamma_j+\frac{\mu_j}{\rho_j}-1}(\xi_j,t_j)}{\rho_j^{\frac{\mu_j}{\rho_j}+\gamma_j-1}}\mathbb{E}_{\frac{\alpha_j}{\rho_j},\gamma_j+\frac{\mu_j}{\rho_j}}\big(\lambda_j(\rho_j^{-1}\Phi_{\phi_j}(\xi_j,t_j))^{\frac{\alpha_j}{\rho_j}}\big).
\end{aligned}\tag{20}$$

Solving the above system (19) and (20), we obtain

$$\begin{aligned}
&u_0\\
=\ &\frac{1}{\Xi}\Bigg(\sum_{j=0}^{n}\frac{\omega_j}{\rho_j^{\frac{\alpha_j+\mu_j}{\rho_j}}}\int_{t_j}^{\xi_j}\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,s)\mathbb{E}_{\frac{\alpha_j}{\rho_j},\frac{\alpha_j+\mu_j}{\rho_j}}\big(\lambda_j(\rho_j^{-1}\Phi_{\phi_j}(\xi_j,s))^{\frac{\alpha_j}{\rho_j}}\big)h(s)\phi_j'(s)ds\\
&+\sum_{j=0}^{n}\Bigg\{\Bigg[\sum_{r=0}^{j-1}\frac{\mathcal{P}_j(r+1)}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)\mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r}+1-\gamma_r}\big(\lambda_r(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s))^{\frac{\alpha_r}{\rho_r}}\big)\\
&\times h(s)\phi_r'(s)ds+\sum_{r=1}^{j}\varphi_r(u(t_r))\mathcal{P}_j(r)\Bigg]\frac{\omega_j\Phi_{\phi_j}^{\gamma_j+\frac{\mu_j}{\rho_j}-1}(\xi_j,t_j)}{\rho_j^{\frac{\mu_j}{\rho_j}+\gamma_j-1}}\mathbb{E}_{\frac{\alpha_j}{\rho_j},\gamma_j+\frac{\mu_j}{\rho_j}}\big(\lambda_j(\rho_j^{-1}\Phi_{\phi_j}(\xi_j,t_j))^{\frac{\alpha_j}{\rho_j}}\big)\Bigg\}\\
&+A-\sum_{i=0}^{m}\frac{\kappa_i}{\rho_i^{\frac{\alpha_i}{\rho_i}}}\int_{t_i}^{\eta_i}\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}-1}(\eta_i,s)\mathbb{E}_{\frac{\alpha_i}{\rho_i},\frac{\alpha_i}{\rho_i}}\big(\lambda_i(\rho_i^{-1}\Phi_{\phi_i}(\eta_i,s))^{\frac{\alpha_i}{\rho_i}}\big)h(s)\phi_r'(s)ds\\
&-\sum_{i=0}^{m}\Bigg\{\Bigg[\sum_{r=0}^{i-1}\frac{\mathcal{P}_i(r+1)}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)\mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r}+1-\gamma_r}\big(\lambda_r(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s))^{\frac{\alpha_r}{\rho_r}}\big)\\
&\times h(s)\phi_r'(s)ds+\sum_{r=1}^{i}\varphi_r(u(t_r))\mathcal{P}_i(r)\Bigg]\frac{\kappa_i\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\rho_i^{\gamma_i-1}}\mathbb{E}_{\frac{\alpha_i}{\rho_i},\gamma_i}\big(\lambda_i(\rho_i^{-1}\Phi_{\phi_i}(\eta_i,t_i))^{\frac{\alpha_i}{\rho_i}}\big)\Bigg\}\Bigg),
\end{aligned}$$

where Ξ is given by (14). Inserting u_0 into (18), we obtain the solution (16).

Conversely, it is easy to show by direct calculation that the solution $u(t)$ is defined by (16) satisfies the impulsive (ρ_k,ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (15). □

3. Existence Results

By Lemma 7, we define an operator $\mathcal{Q}:\mathcal{PC}_{\phi_k}^{1-\gamma_k}\to\mathcal{PC}_{\phi_k}^{1-\gamma_k}$ as

$$
\begin{aligned}
(\mathcal{Q}u)(t) \;=\;& \frac{1}{\rho_k^{\frac{\alpha_k}{\rho_k}}}\int_{t_k}^{t}\Phi_{\phi_k}^{\frac{\alpha_k}{\rho_k}-1}(t,s)\mathbb{E}_{\frac{\alpha_k}{\rho_k},\frac{\alpha_k}{\rho_k}}\big(\lambda_k\big(\rho_k^{-1}\Phi_{\phi_k}(t,s)\big)^{\frac{\alpha_k}{\rho_k}}\big)\mathcal{F}_u(s)\phi_k'(s)ds \\
&+\Bigg[\sum_{r=0}^{k-1}\frac{\mathcal{P}_k(r+1)}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)\mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r}+1-\gamma_r}\big(\lambda_r\big(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s)\big)^{\frac{\alpha_r}{\rho_r}}\big)\mathcal{F}_u(s)\phi_r'(s)ds \\
&+\sum_{r=1}^{k}\varphi_r(u(t_r))\mathcal{P}_k(r)\Bigg]\frac{\Phi_{\phi_k}^{\gamma_k-1}(t,t_k)}{\rho_k^{\gamma_k-1}}\mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\big(\lambda_k\big(\rho_k^{-1}\Phi_{\phi_k}(t,t_k)\big)^{\frac{\alpha_k}{\rho_k}}\big) \\
&+\frac{1}{\Xi}\Bigg(\sum_{j=0}^{n}\frac{\omega_j}{\rho_j^{\frac{\alpha_j+\mu_j}{\rho_j}}}\int_{t_j}^{\xi_j}\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,s)\mathbb{E}_{\frac{\alpha_j}{\rho_j},\frac{\alpha_j+\mu_j}{\rho_j}}\big(\lambda_j\big(\rho_j^{-1}\Phi_{\phi_j}(\xi_j,s)\big)^{\frac{\alpha_j}{\rho_j}}\big)\mathcal{F}_u(s)\phi_j'(s)ds \\
&+\sum_{j=0}^{n}\Bigg\{\Bigg[\sum_{r=0}^{j-1}\frac{\mathcal{P}_j(r+1)}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)\mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r}+1-\gamma_r}\big(\lambda_r\big(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s)\big)^{\frac{\alpha_r}{\rho_r}}\big)\mathcal{F}_u(s)\phi_r'(s)ds \\
&+\sum_{r=1}^{j}\varphi_r(u(t_r))\mathcal{P}_j(r)\Bigg]\frac{\omega_j\Phi_{\phi_j}^{\gamma_j+\frac{\mu_j}{\rho_j}-1}(\xi_j,t_j)}{\rho_j^{\frac{\mu_j}{\rho_j}+\gamma_j-1}}\mathbb{E}_{\frac{\alpha_j}{\rho_j},\gamma_j+\frac{\mu_j}{\rho_j}}\big(\lambda_j\big(\rho_j^{-1}\Phi_{\phi_j}(\xi_j,t_j)\big)^{\frac{\alpha_j}{\rho_j}}\big)\Bigg\} \\
&+A-\sum_{i=0}^{m}\frac{\kappa_i}{\rho_i^{\frac{\alpha_i}{\rho_i}}}\int_{t_i}^{\eta_i}\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}-1}(\eta_i,s)\mathbb{E}_{\frac{\alpha_i}{\rho_i},\frac{\alpha_i}{\rho_i}}\big(\lambda_i\big(\rho_i^{-1}\Phi_{\phi_i}(\eta_i,s)\big)^{\frac{\alpha_i}{\rho_i}}\big)\mathcal{F}_u(s)\phi_r'(s)ds \\
&-\sum_{i=0}^{m}\Bigg\{\Bigg[\sum_{r=0}^{i-1}\frac{\mathcal{P}_i(r+1)}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)\mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r}+1-\gamma_r}\big(\lambda_r\big(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s)\big)^{\frac{\alpha_r}{\rho_r}}\big)\mathcal{F}_u(s)\phi_r'(s)ds \\
&+\sum_{r=1}^{i}\varphi_r(u(t_r))\mathcal{P}_i(r)\Bigg]\frac{\kappa_i\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\rho_i^{\gamma_i-1}}\mathbb{E}_{\frac{\alpha_i}{\rho_i},\gamma_i}\big(\lambda_i\big(\rho_i^{-1}\Phi_{\phi_i}(\eta_i,t_i)\big)^{\frac{\alpha_i}{\rho_i}}\big)\Bigg\}\Bigg) \\
&\times\frac{\mathcal{P}_k(0)\Phi_{\phi_k}^{\gamma_k-1}(t,t_k)}{\rho_k^{\gamma_k-1}}\mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\big(\lambda_k\big(\rho_k^{-1}\Phi_{\phi_k}(t,t_k)\big)^{\frac{\alpha_k}{\rho_k}}\big). \qquad (21)
\end{aligned}
$$

where $\mathcal{F}_u(t) = f(t,u(t),{}_{\rho_k}\mathcal{I}_{t_k}^{\delta_k;\phi_k}u(t),{}_{\rho_k}\mathcal{I}_{t_k}^{\theta_k;\phi_k}u(t))$. It should be noted that $\mathcal{Q}$ has fixed points if and only if the impulsive (ρ_k,ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (4) has solutions.

We define the notations of constants that will be used throughout this paper.

$$
\begin{aligned}
\Delta_1 \;=\;& \frac{\Phi_{\phi_m}^{\frac{\alpha_m}{\rho_m}-\gamma_m+1}(T,t_m)}{\Gamma_{\rho_m}(\alpha_m+\rho_m)}+\frac{1}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\Bigg[\sum_{r=0}^{m-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r+1}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))} \\
&+\frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n}\frac{\big|\omega_j\big|\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}}(\xi_j,t_j)}{\Gamma_{\rho_j}\big(\alpha_j+\mu_j+\rho_j\big)}+\sum_{j=0}^{n}\frac{\big|\omega_j\big|\Phi_{\phi_j}^{\frac{\mu_j}{\rho_j}+\gamma_j-1}(\xi_j,t_j)}{\Gamma_{\rho_j}\big(\rho_j\gamma_j+\mu_j\big)}\sum_{r=0}^{j-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r+1}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))} \\
&+\sum_{i=0}^{m}\frac{|\kappa_i|\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}}(\eta_i,t_i)}{\Gamma_{\rho_i}(\alpha_i+\rho_i)}+\sum_{i=0}^{m}\frac{|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\sum_{r=0}^{i-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r+1}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))}\Bigg)\Bigg], \qquad (22)
\end{aligned}
$$

$$
\Delta_2 = \frac{\Phi_{\phi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\Bigg[m+\frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n}\frac{j\big|\omega_j\big|\Phi_{\phi_j}^{\frac{\mu_j}{\rho_j}+\gamma_j-1}(\xi_j,t_j)}{\Gamma_{\rho_j}\big(\rho_j\gamma_j+\mu_j\big)}+\sum_{i=0}^{m}\frac{i|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\Bigg)\Bigg], \qquad (23)
$$

$$
\Delta_3 = 1+\frac{1}{|\Xi|\Gamma_{\rho_m}(\rho_m\gamma_m)}\Bigg(\sum_{j=0}^{n}\big|\omega_j\big|+\sum_{i=0}^{m}|\kappa_i|\Bigg). \qquad (24)
$$

3.1. Uniqueness Result via Banach's Fixed Point Theorem

Lemma 8 (Banach's fixed point theorem [53]). *Assume that $\mathcal{B}$ is a non-empty closed subset of $\mathcal{X}$ where $\mathcal{X}$ is a Banach. Then, any contraction mapping $\mathcal{Q}$ from $\mathcal{B}$ into itself has a unique fixed point.*

Theorem 1. *Let $f \in \mathcal{C}(\mathcal{J} \times \mathbb{R}^3, \mathbb{R})$ and $\varphi_k \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ for $k = 1, 2 \ldots, m$. Suppose that*

$(\mathbb{H}_1)$ There exist constants $\mathbb{L}_1$, $\mathbb{L}_2$, $\mathbb{L}_3 > 0$ so that

$$\begin{aligned}&|f(t,u_1,v_1,w_1) - f(t,u_2,v_2,w_2)|\\&\leq \Phi_{\rho_k}^{\gamma_k-1}(t,t_k)\Big[\mathbb{L}_1\|u_1-u_2\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}} + \mathbb{L}_2\|v_1-v_2\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}} + \mathbb{L}_3\|w_1-w_2\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}\Big],\end{aligned}$$

for all $t \in \mathcal{J}$ and u_i, v_i, $w_i \in \mathbb{R}$, $i = 1, 2$.

$(\mathbb{H}_2)$ There exists a constant $\mathbb{N}_1 > 0$ so that

$$|\varphi_k(u) - \varphi_k(v)| \leq \mathbb{N}_1 \Phi_{\rho_k}^{\gamma_k-1}(t,t_k)\|u-v\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}, \quad u, v \in \mathbb{R},\ k = 1, 2, \ldots, m.$$

Then, the impulsive (ρ_k, ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (4) has a unique solution if

$$\Delta_1(\mathbb{L}_1 + \mathbb{L}_2\Phi_1^* + \mathbb{L}_3\Phi_2^*) + \Delta_2\mathbb{N}_1 < 1, \tag{25}$$

where Φ_1^* and Φ_2^* are given by

$$\Phi_1^* := \frac{\Gamma_{\rho_m}(\rho_m\gamma_m)\Phi_{\rho_m}^{\frac{\delta_m}{\rho_m}+\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m+\delta_m)}, \quad \Phi_2^* := \frac{\Gamma_{\rho_m}(\rho_m\gamma_m)\Phi_{\rho_m}^{\frac{\theta_m}{\rho_m}+\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m+\theta_m)}. \tag{26}$$

Proof. Transformation the problem (4) into a fixed point problem, $u = \mathcal{Q}u$, where $\mathcal{Q}$ is define by (21). We know that the fixed points of $\mathcal{Q}$ are solutions to the problem (4). We separate the procedure into two steps.

Step 1: We show that $\mathcal{Q}\mathcal{B}_{\Upsilon_1} \subset \mathcal{B}_{\Upsilon_1}$.

Let $\sup_{t\in\mathcal{J}}|f(t,0,0,0)| := \mathbb{F}_1 < \infty$ and $\mathbb{I}_1 := \max\{|\varphi_k(0)| : k = 1, 2, \ldots, m\}$. Define $\mathcal{B}_{\Upsilon_1} := \{u \in \mathcal{PC}_{\phi_k}^{1-\gamma_k} : \|u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}} \leq \Upsilon_1\}$ with the radius

$$\Upsilon_1 \geq \frac{\Delta_1\mathbb{F}_1 + \Delta_2\mathbb{I}_1 + \frac{|A|}{|\Xi|\Gamma_{\rho_m}(\rho_m\gamma_m)}}{1 - \left[\Delta_1(\mathbb{L}_1 + \mathbb{L}_2\Phi_1^* + \mathbb{L}_3\Phi_2^*) + \Delta_2\mathbb{N}_1\right]}.$$

Obviously, the set $\mathcal{B}_{\Upsilon_1}$ is a bounded, closed, and convex subset of $\mathcal{PC}_{\phi_k}^{1-\gamma_k}$.

For every $u \in \mathcal{B}_{\Upsilon_1}$, we obtain

$$\begin{aligned}&\left|\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)(\mathcal{Q}u)(t)\right|\\ \leq\ & \frac{\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)}{\rho_k^{\frac{\alpha_k}{\rho_k}}}\int_{t_k}^{t}\Phi_{\phi_k}^{\frac{\alpha_k}{\rho_k}-1}(t,s)\left|\mathbb{E}_{\frac{\alpha_k}{\rho_k},\frac{\alpha_k}{\rho_k}}\left(\lambda_k\left(\rho_k^{-1}\Phi_{\phi_k}(t,s)\right)^{\frac{\alpha_k}{\rho_k}}\right)\right||\mathcal{F}_u(s)|\phi_k'(s)ds\\ &+\left[\sum_{r=0}^{k-1}\frac{|\mathcal{P}_k(r+1)|}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)\left|\mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r}+1-\gamma_r}\left(\lambda_r\left(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s)\right)^{\frac{\alpha_r}{\rho_r}}\right)\right|\right.\\ &\left.\times|\mathcal{F}_u(s)|\phi_r'(s)ds + \sum_{r=1}^{k}|\varphi_r(u(t_r))||\mathcal{P}_k(r)|\right]\frac{1}{\rho_k^{\gamma_k-1}}\left|\mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\left(\lambda_k\left(\rho_k^{-1}\Phi_{\phi_k}(t,t_k)\right)^{\frac{\alpha_k}{\rho_k}}\right)\right|\end{aligned}$$

$$+\frac{1}{|\Xi|}\left(\sum_{j=0}^{n}\frac{|\omega_j|}{\rho_j^{\frac{\alpha_j+\mu_j}{\rho_j}}}\int_{t_j}^{\xi_j}\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,s)\left|\mathbb{E}_{\frac{\alpha_j}{\rho_j},\frac{\alpha_j+\mu_j}{\rho_j}}\left(\lambda_j\left(\rho_j^{-1}\Phi_{\phi_j}(\xi_j,s)\right)^{\frac{\alpha_j}{\rho_j}}\right)\right||\mathcal{F}_u(s)|\phi_j'(s)ds\right.$$
$$+\sum_{j=0}^{n}\left\{\left[\sum_{r=0}^{j-1}\frac{|\mathcal{P}_j(r+1)|}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)\left|\mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r}+1-\gamma_r}\left(\lambda_r\left(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s)\right)^{\frac{\alpha_r}{\rho_r}}\right)\right|\right.\right.$$
$$\left.\times|\mathcal{F}_u(s)|\phi_r'(s)ds+\sum_{r=1}^{j}|\varphi_r(u(t_r))||\mathcal{P}_j(r)|\right]\frac{|\omega_j|\Phi_{\phi_j}^{\gamma_j+\frac{\mu_j}{\rho_j}-1}(\xi_j,t_j)}{\rho_j^{\frac{\mu_j}{\rho_j}+\gamma_j-1}}$$
$$\left.\times\left|\mathbb{E}_{\frac{\alpha_j}{\rho_j},\gamma_j+\frac{\mu_j}{\rho_j}}\left(\lambda_j\left(\rho_j^{-1}\Phi_{\phi_j}(\xi_j,t_j)\right)^{\frac{\alpha_j}{\rho_j}}\right)\right|\right\}+|A|+\sum_{i=0}^{m}\frac{|\kappa_i|}{\rho_i^{\frac{\alpha_i}{\rho_i}}}\int_{t_i}^{\eta_i}\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}-1}(\eta_i,s)$$
$$\times\left|\mathbb{E}_{\frac{\alpha_i}{\rho_i},\frac{\alpha_i}{\rho_i}}\left(\lambda_i\left(\rho_i^{-1}\Phi_{\phi_i}(\eta_i,s)\right)^{\frac{\alpha_i}{\rho_i}}\right)\right||\mathcal{F}_u(s)|\phi_r'(s)ds+\sum_{i=0}^{m}\left\{\left[\sum_{r=0}^{i-1}\frac{|\mathcal{P}_i(r+1)|}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}}\right.\right.$$
$$\times\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)\left|\mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r}+1-\gamma_r}\left(\lambda_r\left(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s)\right)^{\frac{\alpha_r}{\rho_r}}\right)\right||\mathcal{F}_u(s)|\phi_r'(s)ds$$
$$\left.\left.\left.+\sum_{r=1}^{i}|\varphi_r(u(t_r))||\mathcal{P}_i(r)|\right]\frac{|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\rho_i^{\gamma_i-1}}\left|\mathbb{E}_{\frac{\alpha_i}{\rho_i},\gamma_i}\left(\lambda_i\left(\rho_i^{-1}\Phi_{\phi_i}(\eta_i,t_i)\right)^{\frac{\alpha_i}{\rho_i}}\right)\right|\right\}\right)$$
$$\times\frac{|\mathcal{P}_k(0)|}{\rho_k^{\gamma_k-1}}\left|\mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\left(\lambda_k\left(\rho_k^{-1}\Phi_{\phi_k}(t,t_k)\right)^{\frac{\alpha_k}{\rho_k}}\right)\right|.$$

Since,

$$\begin{aligned}\left|{}_{\rho_k}\mathcal{I}_{t_k}^{\delta_k;\phi_k}u(t)\right| &= \frac{1}{\rho_k\Gamma_{\rho_k}(\delta_k)}\int_{t_k}^{t}\Phi_{\phi_k}^{\frac{\delta_k}{\rho_k}-1}(t,s)\phi_k'(s)|u(s)|ds\\ &\le \frac{\Gamma_{\rho_k}(\rho_k\gamma_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k+\delta_k)}\Phi_{\rho_k}^{\frac{\delta_k}{\rho_k}+\gamma_k-1}(t,t_k)\|u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}},\\ \left|{}_{\rho_k}\mathcal{I}_{t_k}^{\theta_k;\phi_k}u(t)\right| &\le \frac{\Gamma_{\rho_k}(\rho_k\gamma_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k+\theta_k)}\Phi_{\rho_k}^{\frac{\theta_k}{\rho_k}+\gamma_k-1}(t,t_k)\|u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}.\end{aligned}$$

By applying ($\mathbb{H}_1$)–($\mathbb{H}_2$), we have the following inequalities

$$\begin{aligned}|\mathcal{F}_u(t)| &\le \left|f(t,u(t),{}_{\rho_k}\mathcal{I}_{t_k}^{\delta_k;\phi_k}u(t),{}_{\rho_k}\mathcal{I}_{t_k}^{\theta_k;\phi_k}u(t))-f(t,0,0,0)\right|+|f(t,0,0,0)|\\ &\le \mathbb{L}_1|u(t)|+\mathbb{L}_2\left|{}_{\rho_k}\mathcal{I}_{t_k}^{\delta_k;\phi_k}u(t)\right|+\mathbb{L}_3\left|{}_{\rho_k}\mathcal{I}_{t_k}^{\theta_k;\phi_k}u(t)\right|+|f(t,0,0,0)|\\ &\le (\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*)\|u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}+\mathbb{F}_1, \qquad (27)\\ |\varphi_k(u(t_k))| &\le |\varphi_k(u(t_k))-\varphi_k(0)|+|\varphi_k(0)|\le\mathbb{N}_1\Phi_{\phi_m}^{\gamma_m-1}(T,t_m)\|u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}+\mathbb{I}_1. \qquad (28)\end{aligned}$$

By using the properties $\mathbb{E}_{\alpha}(z)\le 1$ and $\mathbb{E}_{\alpha,\beta}(z)\le 1/\Gamma(\beta)$ with (27) and (28), we obtain that

$$\begin{aligned}&\left|\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)(\mathcal{Q}u)(t)\right|\\ \le\ &\frac{(\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*)\|u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}+\mathbb{F}_1}{\rho_k^{\frac{\alpha_k}{\rho_k}}\Gamma\left(\frac{\alpha_k}{\rho_k}\right)}\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)\int_{t_k}^{t}\Phi_{\phi_k}^{\frac{\alpha_k}{\rho_k}-1}(t,s)\phi_k'(s)ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\rho_k^{\gamma_k-1}\Gamma(\gamma_k)}\Bigg[\sum_{r=0}^{k-1}\frac{(\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*)\|u\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}}+\mathbb{F}_1}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}\Gamma\Big(\frac{\alpha_r}{\rho_r}+1-\gamma_r\Big)}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)\phi_r'(s)ds\\
&+k\Big[\mathbb{N}_1\Phi_{\phi_m}^{\gamma_m-1}(T,t_m)\|u\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}}+\mathbb{I}_1\Big]\Bigg]\\
&+\frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n}\frac{|\omega_j|\Big[(\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*)\|u\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}}+\mathbb{F}_1\Big]}{\rho_j^{\frac{\alpha_j+\mu_j}{\rho_j}}\Gamma\Big(\frac{\alpha_j+\mu_j}{\rho_j}\Big)}\int_{t_j}^{\xi_j}\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,s)\phi_j'(s)ds\\
&+\sum_{j=0}^{n}\Bigg\{\Bigg[\sum_{r=0}^{j-1}\frac{(\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*)\|u\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}}+\mathbb{F}_1}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}\Gamma\Big(\frac{\alpha_r}{\rho_r}+1-\gamma_r\Big)}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)\phi_r'(s)ds\\
&+j\Big[\mathbb{N}_1\Phi_{\phi_m}^{\gamma_m-1}(T,t_m)\|u\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}}+\mathbb{I}_1\Big]\Bigg]\frac{|\omega_j|\Phi_{\phi_j}^{\gamma_j+\frac{\mu_j}{\rho_j}-1}(\xi_j,t_j)}{\rho_j^{\frac{\mu_j}{\rho_j}+\gamma_j-1}\Gamma\Big(\gamma_j+\frac{\mu_j}{\rho_j}\Big)}\Bigg\}\\
&+|A|+\sum_{i=0}^{m}\frac{|\kappa_i|\Big[(\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*)\|u\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}}+\mathbb{F}_1\Big]}{\rho_i^{\frac{\alpha_i}{\rho_i}}\Gamma\Big(\frac{\alpha_i}{\rho_i}\Big)}\int_{t_i}^{\eta_i}\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}-1}(\eta_i,s)\phi_r'(s)ds\\
&+\sum_{i=0}^{m}\Bigg\{\Bigg[\sum_{r=0}^{i-1}\frac{(\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*)\|u\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}}+\mathbb{F}_1}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}\Gamma\Big(\frac{\alpha_r}{\rho_r}+1-\gamma_r\Big)}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)\phi_r'(s)ds\\
&+i\Big[\mathbb{N}_1\Phi_{\phi_m}^{\gamma_m-1}(T,t_m)\|u\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}}+\mathbb{I}_1\Big]\Bigg]\frac{|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\rho_i^{\gamma_i-1}\Gamma(\gamma_i)}\Bigg\}\Bigg)\frac{1}{\rho_k^{\gamma_k-1}\Gamma(\gamma_k)}.
\end{aligned}
$$

By using (6) and the property (i) in Lemma 1, we have

$$
\begin{aligned}
&\Big|\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)(\mathcal{Q}u)(t)\Big|\\
\le\;&\frac{1}{\rho_k\Gamma_{\rho_k}(\alpha_k)}\int_{t_k}^{t}\Phi_{\phi_k}^{\frac{\alpha_k}{\rho_k}-1}(t,s)\phi_k'(s)ds\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)\Big[(\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*)\|u\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}}+\mathbb{F}_1\Big]\\
&+\frac{1}{\Gamma_{\rho_k}(\beta_k(\rho_k-\alpha_k)+\alpha_k)}\Bigg[\sum_{r=0}^{k-1}\frac{1}{\rho_r\Gamma_{\rho_r}(\alpha_r+\rho_r(1-\gamma_r))}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r+\rho_r(1-\gamma_r)}{\rho_r}-1}(t_{r+1},s)\phi_r'(s)ds\\
&\times\Big[(\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*)\|u\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}}+\mathbb{F}_1\Big]+k\Big[\mathbb{N}_1\Phi_{\phi_m}^{\gamma_m-1}(T,t_m)\|u\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}}+\mathbb{I}_1\Big]\Bigg]\\
&+\frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n}\frac{|\omega_j|}{\rho_j\Gamma_{\rho_j}\Big(\alpha_j+\mu_j\Big)}\int_{t_j}^{\xi_j}\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,s)\phi_j'(s)ds\Big[(\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*)\|u\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}}\\
&+\mathbb{F}_1\Big]+\sum_{j=0}^{n}\Bigg\{\frac{|\omega_j|\Phi_{\phi_j}^{\frac{\beta_j(\rho_j-\alpha_j)+\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,t_j)}{\Gamma_{\rho_j}\Big(\beta_j(\rho_j-\alpha_j)+\alpha_j+\mu_j\Big)}\Bigg[\sum_{r=0}^{j-1}\frac{1}{\rho_r\Gamma_{\rho_r}(\alpha_r+\rho_r(1-\gamma_r))}\\
&\times\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r+\rho_r(1-\gamma_r)}{\rho_r}-1}(t_{r+1},s)\phi_r'(s)ds\Big[(\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*)\|u\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}}+\mathbb{F}_1\Big]\\
&+j\Big[\mathbb{N}_1\Phi_{\phi_m}^{\gamma_m-1}(T,t_m)\|u\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}}+\mathbb{I}_1\Big]\Bigg]\Bigg\}+|A|\\
&+\frac{1}{|\Xi|}\sum_{i=0}^{m}\frac{|\kappa_i|}{\rho_i\Gamma_{\rho_i}(\alpha_i)}\int_{t_i}^{\eta_i}\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}-1}(\eta_i,s)\phi_r'(s)ds\Big[(\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*)\|u\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}}+\mathbb{F}_1\Big]
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{i=0}^{m}\left\{\frac{|\kappa_i|\Phi_{\phi_i}^{\frac{\beta_i(\rho_i-\alpha_i)+\alpha_i}{\rho_i}-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\beta_i(\rho_i-\alpha_i)+\alpha_i)}\left[\sum_{r=0}^{i-1}\frac{1}{\rho_r\Gamma_{\rho_r}(\alpha_r+\rho_r(1-\gamma_r))}\right.\right.\\
& \times\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r+\rho_r(1-\gamma_r)}{\rho_r}-1}(t_{r+1},s)\phi_r'(s)ds\Big[(\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*)\|u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}+\mathbb{F}_1\Big]\\
& \left.\left.+i\Big[\mathbb{N}_1\Phi_{\phi_m}^{\gamma_m-1}(T,t_m)\|u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}+\mathbb{I}_1\Big]\right]\right\}\Bigg)\frac{1}{\Gamma_{\rho_k}(\beta_k(\rho_k-\alpha_k)+\alpha_k)}\\
\leq\ & \left\{\frac{\Phi_{\phi_m}^{\frac{\alpha_m}{\rho_m}-\gamma_m+1}(T,t_m)}{\Gamma_{\rho_m}(\alpha_m+\rho_m)}+\frac{1}{\Gamma_{\rho_m}(\beta_m(\rho_m-\alpha_m)+\alpha_m)}\left[\sum_{r=0}^{m-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r+\rho_r(1-\gamma_r)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))}\right.\right.\\
& +\frac{1}{|\Xi|}\left(\sum_{j=0}^{n}\frac{\left|\omega_j\right|\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}}(\xi_j,t_j)}{\Gamma_{\rho_j}\left(\alpha_j+\mu_j+\rho_j\right)}+\sum_{j=0}^{n}\frac{\left|\omega_j\right|\Phi_{\phi_j}^{\frac{\beta_j(\rho_j-\alpha_j)+\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,t_j)}{\Gamma_{\rho_j}\left(\beta_j(\rho_j-\alpha_j)+\alpha_j+\mu_j\right)}\sum_{r=0}^{j-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r+\rho_r(1-\gamma_r)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))}\right.\\
& \left.\left.\left.+\sum_{i=0}^{m}\frac{|\kappa_i|\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}}(\eta_i,t_i)}{\Gamma_{\rho_i}(\alpha_i+\rho_i)}+\sum_{i=0}^{m}\frac{|\kappa_i|\Phi_{\phi_i}^{\frac{\beta_i(\rho_i-\alpha_i)+\alpha_i}{\rho_i}-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\beta_i(\rho_i-\alpha_i)+\alpha_i)}\sum_{r=0}^{i-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r+\rho_r(1-\gamma_r)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))}\right)\right]\right\}\\
& \times\Big[(\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*)\|u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}+\mathbb{F}_1\Big]+\frac{1}{\Gamma_{\rho_m}(\beta_m(\rho_m-\alpha_m)+\alpha_m)}\Bigg\{m\\
& +\frac{1}{|\Xi|}\left(\sum_{j=0}^{n}\frac{j\left|\omega_j\right|\Phi_{\phi_j}^{\frac{\beta_j(\rho_j-\alpha_j)+\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,t_j)}{\Gamma_{\rho_j}\left(\beta_j(\rho_j-\alpha_j)+\alpha_j+\mu_j\right)}+\sum_{i=0}^{m}\frac{i|\kappa_i|\Phi_{\phi_i}^{\frac{\beta_i(\rho_i-\alpha_i)+\alpha_i}{\rho_i}-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\beta_i(\rho_i-\alpha_i)+\alpha_i)}\right)\Bigg\}\\
& \times\Big[\mathbb{N}_1\Phi_{\phi_m}^{\gamma_m-1}(T,t_m)\|u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}+\mathbb{I}_1\Big]+\frac{|A|}{|\Xi|\Gamma_{\rho_m}(\beta_m(\rho_m-\alpha_m)+\alpha_m)}\\
=\ & \left\{\frac{\Phi_{\phi_m}^{\frac{\alpha_m}{\rho_m}-\gamma_m+1}(T,t_m)}{\Gamma_{\rho_m}(\alpha_m+\rho_m)}+\frac{1}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\left[\sum_{r=0}^{m-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r+1}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))}\right.\right.\\
& +\frac{1}{|\Xi|}\left(\sum_{j=0}^{n}\frac{\left|\omega_j\right|\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}}(\xi_j,t_j)}{\Gamma_{\rho_j}\left(\alpha_j+\mu_j+\rho_j\right)}+\sum_{j=0}^{n}\frac{\left|\omega_j\right|\Phi_{\phi_j}^{\frac{\mu_j}{\rho_j}+\gamma_j-1}(\xi_j,t_j)}{\Gamma_{\rho_j}\left(\rho_j\gamma_j+\mu_j\right)}\sum_{r=0}^{j-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r+1}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))}\right.\\
& \left.\left.\left.+\sum_{i=0}^{m}\frac{|\kappa_i|\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}}(\eta_i,t_i)}{\Gamma_{\rho_i}(\alpha_i+\rho_i)}+\sum_{i=0}^{m}\frac{|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\sum_{r=0}^{i-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r+1}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))}\right)\right]\right\}\\
& \times\Big[(\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*)\|u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}+\mathbb{F}_1\Big]+\frac{1}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\Bigg[m\\
& +\frac{1}{|\Xi|}\left(\sum_{j=0}^{n}\frac{j\left|\omega_j\right|\Phi_{\phi_j}^{\frac{\mu_j}{\rho_j}+\gamma_j-1}(\xi_j,t_j)}{\Gamma_{\rho_j}\left(\rho_j\gamma_j+\mu_j\right)}+\sum_{i=0}^{m}\frac{i|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\right)\Bigg]\\
& \times\Big[\mathbb{N}_1\Phi_{\phi_m}^{\gamma_m-1}(T,t_m)\|u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}+\mathbb{I}_1\Big]+\frac{|A|}{|\Xi|\Gamma_{\rho_m}(\rho_m\gamma_m)}\\
=\ & \Big[\Delta_1(\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*)+\Delta_2\mathbb{N}_1\Big]\|u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}+\Delta_1\mathbb{F}_1+\Delta_2\mathbb{I}_1+\frac{|A|}{|\Xi|\Gamma_{\rho_m}(\rho_m\gamma_m)}\\
\leq\ & \mathrm{Y}_1.
\end{aligned}
$$

Hence, $\|\mathcal{Q}u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}\leq\mathrm{Y}_1$ which implies that $\mathcal{Q}\mathcal{B}_{\mathrm{Y}_1}\subset\mathcal{B}_{\mathrm{Y}_1}$.

Step 2: We show that $\mathcal{Q}$ is a contraction.
For each $u, v \in \mathcal{B}_{\Upsilon_1}$ and for any $t \in \mathcal{J}$, we obtain that

$$\begin{aligned}
&\left|\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)\big((\mathcal{Q}u)(t)-(\mathcal{Q}v)(t)\big)\right| \\
\leq\ & \frac{\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)}{\rho_k^{\frac{\alpha_k}{\rho_k}}}\int_{t_k}^{t}\Phi_{\phi_k}^{\frac{\alpha_k}{\rho_k}-1}(t,s)\left|\mathbb{E}_{\frac{\alpha_k}{\rho_k},\frac{\alpha_k}{\rho_k}}\left(\lambda_k\big(\rho_k^{-1}\Phi_{\phi_k}(t,s)\big)^{\frac{\alpha_k}{\rho_k}}\right)\right||\mathcal{F}_u(s)-\mathcal{F}_v(s)|\phi_k'(s)ds \\
&+\left[\sum_{r=0}^{k-1}\frac{|\mathcal{P}_k(r+1)|}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)\left|\mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r}+1-\gamma_r}\left(\lambda_r\big(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s)\big)^{\frac{\alpha_r}{\rho_r}}\right)\right|\right. \\
&\left.\times|\mathcal{F}_u(s)-\mathcal{F}_v(s)|\phi_r'(s)ds+\sum_{r=1}^{k}|\varphi_r(u(t_r))-\varphi_r(v(t_r))||\mathcal{P}_k(r)|\right] \\
&\times\frac{1}{\rho_k^{\gamma_k-1}}\left|\mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\left(\lambda_k\big(\rho_k^{-1}\Phi_{\phi_k}(t,t_k)\big)^{\frac{\alpha_k}{\rho_k}}\right)\right|+\frac{1}{|\Xi|}\left(\sum_{j=0}^{n}\frac{\left|\omega_j\right|}{\rho_j^{\frac{\alpha_j+\mu_j}{\rho_j}}}\int_{t_j}^{\xi_j}\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,s)\right. \\
&\times\left|\mathbb{E}_{\frac{\alpha_j}{\rho_j},\frac{\alpha_j+\mu_j}{\rho_j}}\left(\lambda_j\big(\rho_j^{-1}\Phi_{\phi_j}(\xi_j,s)\big)^{\frac{\alpha_j}{\rho_j}}\right)\right||\mathcal{F}_u(s)-\mathcal{F}_v(s)|\phi_j'(s)ds+\sum_{j=0}^{n}\left\{\left[\sum_{r=0}^{j-1}\frac{\left|\mathcal{P}_j(r+1)\right|}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}}\right.\right. \\
&\times\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)\left|\mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r}+1-\gamma_r}\left(\lambda_r\big(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s)\big)^{\frac{\alpha_r}{\rho_r}}\right)\right||\mathcal{F}_u(s)-\mathcal{F}_v(s)|\phi_r'(s)ds \\
&\left.+\sum_{r=1}^{j}|\varphi_r(u(t_r))-\varphi_r(v(t_r))|\left|\mathcal{P}_j(r)\right|\right]\frac{\left|\omega_j\right|\Phi_{\phi_j}^{\gamma_j+\frac{\mu_j}{\rho_j}-1}(\xi_j,t_j)}{\rho_j^{\frac{\mu_j}{\rho_j}+\gamma_j-1}} \\
&\left.\times\left|\mathbb{E}_{\frac{\alpha_j}{\rho_j},\gamma_j+\frac{\mu_j}{\rho_j}}\left(\lambda_j\big(\rho_j^{-1}\Phi_{\phi_j}(\xi_j,t_j)\big)^{\frac{\alpha_j}{\rho_j}}\right)\right|\right\}+\sum_{i=0}^{m}\frac{|\kappa_i|}{\rho_i^{\frac{\alpha_i}{\rho_i}}}\int_{t_i}^{\eta_i}\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}-1}(\eta_i,s) \\
&\times\left|\mathbb{E}_{\frac{\alpha_i}{\rho_i},\frac{\alpha_i}{\rho_i}}\left(\lambda_i\big(\rho_i^{-1}\Phi_{\phi_i}(\eta_i,s)\big)^{\frac{\alpha_i}{\rho_i}}\right)\right||\mathcal{F}_u(s)-\mathcal{F}_v(s)|\phi_r'(s)ds+\sum_{i=0}^{m}\left\{\left[\sum_{r=0}^{i-1}\frac{|\mathcal{P}_i(r+1)|}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}}\right.\right. \\
&\times\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)\left|\mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r}+1-\gamma_r}\left(\lambda_r\big(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s)\big)^{\frac{\alpha_r}{\rho_r}}\right)\right| \\
&\left.\times|\mathcal{F}_u(s)-\mathcal{F}_v(s)|\phi_r'(s)ds+\sum_{r=1}^{i}|\varphi_r(u(t_r))-\varphi_r(v(t_r))||\mathcal{P}_i(r)|\right]\frac{|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\rho_i^{\gamma_i-1}} \\
&\left.\left.\times\left|\mathbb{E}_{\frac{\alpha_i}{\rho_i},\gamma_i}\left(\lambda_i\big(\rho_i^{-1}\Phi_{\phi_i}(\eta_i,t_i)\big)^{\frac{\alpha_i}{\rho_i}}\right)\right|\right\}\right)\frac{|\mathcal{P}_k(0)|}{\rho_k^{\gamma_k-1}}\left|\mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\left(\lambda_k\big(\rho_k^{-1}\Phi_{\phi_k}(t,t_k)\big)^{\frac{\alpha_k}{\rho_k}}\right)\right|.
\end{aligned}$$

By using $(\mathbb{H}_1)$–$(\mathbb{H}_2)$, we have

$$\begin{aligned}
|\mathcal{F}_u(t)-\mathcal{F}_v(t)| &\leq \Big|f\big(t,u(t),{}_{\rho_k}\mathcal{I}_{t_k}^{\delta_k;\phi_k}u(t),{}_{\rho_k}\mathcal{I}_{t_k}^{\theta_k;\phi_k}u(t)\big) \\
&\quad -f\big(t,v(t),{}_{\rho_k}\mathcal{I}_{t_k}^{\delta_k;\phi_k}v(t),{}_{\rho_k}\mathcal{I}_{t_k}^{\theta_k;\phi_k}v(t)\big)\Big| \\
&\leq \left(\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*\right)\|u-v\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}, \qquad (29) \\
|\varphi_k(u(t_k))-\varphi_k(v(t_k))| &\leq \mathbb{N}_1\Phi_{\phi_m}^{\gamma_m-1}(T,t_m)\|u-v\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}. \qquad (30)
\end{aligned}$$

By using Lemma 1, Lemma 3 and (29) and (30), implies that

$$
\begin{aligned}
&\left|\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)\big((\mathcal{Q}u)(t)-(\mathcal{Q}v)(t)\big)\right| \\
\leq\ & \Bigg\{\frac{\Phi_{\phi_m}^{\frac{\alpha_m}{\rho_m}-\gamma_m+1}(T,t_m)}{\Gamma_{\rho_m}(\alpha_m+\rho_m)}+\frac{1}{\Gamma_{\rho_m}(\beta_m(\rho_m-\alpha_m)+\alpha_m)}\Bigg[\sum_{r=0}^{m-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r+\rho_r(1-\gamma_r)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))} \\
&+\frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n}\frac{\left|\omega_j\right|\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}}(\xi_j,t_j)}{\Gamma_{\rho_j}\left(\alpha_j+\mu_j+\rho_j\right)}+\sum_{i=0}^{m}\frac{|\kappa_i|\Phi_{\phi_i}^{\frac{\beta_i(\rho_i-\alpha_i)+\alpha_i}{\rho_i}-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\beta_i(\rho_i-\alpha_i)+\alpha_i)}\sum_{r=0}^{i-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r+\rho_r(1-\gamma_r)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))}\Bigg)\Bigg]\Bigg\} \\
&\times(\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*)\|u-v\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}+\frac{\mathbb{N}_1\Phi_{\phi_m}^{\gamma_m-1}(T,t_m)\|u-v\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}}{\Gamma_{\rho_m}(\beta_m(\rho_m-\alpha_m)+\alpha_m)}\Bigg\{m \\
&+\frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n}\frac{j\left|\omega_j\right|\Phi_{\phi_j}^{\frac{\beta_j(\rho_j-\alpha_j)+\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,t_j)}{\Gamma_{\rho_j}\left(\beta_j(\rho_j-\alpha_j)+\alpha_j+\mu_j\right)}+\sum_{i=0}^{m}\frac{i|\kappa_i|\Phi_{\phi_i}^{\frac{\beta_i(\rho_i-\alpha_i)+\alpha_i}{\rho_i}-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\beta_i(\rho_i-\alpha_i)+\alpha_i)}\Bigg)\Bigg\} \\
=\ & \Bigg\{\frac{\Phi_{\phi_m}^{\frac{\alpha_m}{\rho_m}-\gamma_m+1}(T,t_m)}{\Gamma_{\rho_m}(\alpha_m+\rho_m)}+\frac{1}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\Bigg[\sum_{r=0}^{m-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r+1}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))} \\
&+\frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n}\frac{\left|\omega_j\right|\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}}(\xi_j,t_j)}{\Gamma_{\rho_j}\left(\alpha_j+\mu_j+\rho_j\right)}+\sum_{j=0}^{n}\frac{\left|\omega_j\right|\Phi_{\phi_j}^{\frac{\mu_j}{\rho_j}+\gamma_j-1}(\xi_j,t_j)}{\Gamma_{\rho_j}\left(\rho_j\gamma_j+\mu_j\right)}\sum_{r=0}^{j-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r+1}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))} \\
&+\sum_{i=0}^{m}\frac{|\kappa_i|\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}}(\eta_i,t_i)}{\Gamma_{\rho_i}(\alpha_i+\rho_i)}+\sum_{i=0}^{m}\frac{|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\sum_{r=0}^{i-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r+1}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))}\Bigg)\Bigg]\Bigg\} \\
&\times(\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*)\|u-v\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}+\frac{\Phi_{\phi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\Bigg[m \\
&+\frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n}\frac{j\left|\omega_j\right|\Phi_{\phi_j}^{\frac{\mu_j}{\rho_j}+\gamma_j-1}(\xi_j,t_j)}{\Gamma_{\rho_j}\left(\rho_j\gamma_j+\mu_j\right)}+\sum_{i=0}^{m}\frac{i|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\Bigg)\Bigg]\mathbb{N}_1\|u-v\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}} \\
=\ & \Big[\Delta_1(\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*)+\Delta_2\mathbb{N}_1\Big]\|u-v\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}},
\end{aligned}
$$

this yields that $\|\mathcal{Q}u-\mathcal{Q}v\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}\leq[\Delta_1(\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*)+\Delta_2\mathbb{N}_1]\|u-v\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}$. Since condition (25) holds, then $\mathcal{Q}$ is a contraction map. Hence, by Lemma 8, the impulsive (ρ_k,ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (4) has a unique solution. □

3.2. Existence Result via O'Regan's Fixed Point Theorem

Lemma 9 (O'Regan's fixed point theorem [54]). *Let $\mathcal{K}$ be an open set in a closed, convex set $\mathcal{B}$ of a Banach space $\mathcal{X}$, with $\overline{\mathcal{K}}$ and $\partial\mathcal{K}$ representing the closure and boundary of $\mathcal{K}$, respectively. Moreover, it is assumed that $0\in\mathcal{K}$ and $\mathcal{Q}:\overline{\mathcal{K}}\to\mathcal{B}$ is such that $\mathcal{Q}(\overline{\mathcal{K}})$ is bounded and that $\mathcal{Q}=\mathcal{Q}_1+\mathcal{Q}_2$, where $\mathcal{Q}_1:\mathcal{K}\to\mathcal{B}$ is continuous and completely continuous and $\mathcal{Q}_2:\mathcal{K}\to\mathcal{B}$ is non-linear contraction, that is, there exists a non-negative non-decreasing function $\psi:[0,\infty)\to[0,\infty)$, such that $\psi(z)<z$ for $z>0$, and $\|\mathcal{Q}_2u-\mathcal{Q}_2v\|\leq\psi(\|u-v\|)$ for all $u,v\in\mathcal{K}$. Then, either (C_1) $\mathcal{Q}$ has a fixed point $u\in\overline{\mathcal{K}}$; or ($C_2$) there exist a point $u\in\partial\mathcal{K}$ and $\sigma\in(0,1)$, such that $u=\sigma\mathcal{Q}(u)$.*

Theorem 2. *Let $f\in\mathcal{C}(\mathcal{J}\times\mathbb{R}^3,\mathbb{R})$ and $\varphi_k\in\mathcal{C}(\mathbb{R},\mathbb{R})$, $k=1,2\dots,m$. Suppose that*

($\mathbb{H}_3$) *There exists $\mathbb{M}_1>0$ such that $|\varphi_k(u)|\leq\mathbb{M}_1$ for all $u\in\mathbb{R}$, $k=1,2,\dots,m$.*

($\mathbb{H}_4$) *There exist a continuous non-decreasing function* $\psi : [0, \infty) \to (0, \infty)$ *and* p_1, p_2, $p_3 \in \mathcal{C}([0, T], \mathbb{R}^+)$, *such that*

$$|f(t,u,v,w)| \leq p_1(t)\psi\left(\Phi_{\rho_k}^{\gamma_k - 1}(t,t_k)|u|\right) + \Phi_{\rho_k}^{\gamma_k - 1}(t,t_k)\left[p_2(t)|v| + p_3(t)|w|\right], \tag{31}$$

for any $(t,u,v,w) \in \mathcal{J} \times \mathbb{R}^3$.

($\mathbb{H}_5$) *There exist a continuous non-decreasing function* $q_1 : [0, \infty) \to [0, \infty)$ *and* $b_1 > 0$, *such that*

$$\begin{aligned} |\varphi_k(u) - \varphi_k(v)| &\leq q_1\left(\Phi_{\rho_k}^{\gamma_k - 1}(t,t_k)|u - v|\right) && (32) \\ q_1\left(\Phi_{\rho_k}^{\gamma_k - 1}(t,t_k)|u|\right) &\leq b_1\Phi_{\rho_k}^{\gamma_k - 1}(t,t_k)|u|, && (33) \end{aligned}$$

for any u, $v \in \mathbb{R}$, $k = 1, 2, \ldots, m$ *satisfying* $\Delta_2\Phi_{\phi_m}^{\gamma_m - 1}(T, t_m)b_1 < 1$ *where* Δ_2 *is defined by* (23).

($\mathbb{H}_6$) *The following condition holds*

$$\sup_{\Upsilon_2 \in (0,\infty)} \frac{\Upsilon_2}{\Delta_1 p_1^* \psi(\Upsilon_2) + \Delta_2\mathbb{M}_1 + \frac{|A|}{|\Xi|\Gamma_{\rho_m}(\rho_m\gamma_m)}} > \frac{1}{1 - \left[\Delta_1(p_2^*\Phi_1^* + p_3^*\Phi_2^*)\right]}, \tag{34}$$

with $\left[\Delta_1(p_2^*\Phi_1^* + p_3^*\Phi_2^*)\right] < 1$ *and* Δ_i, $(i = 1, 2)$, *are given by* (22) *and* (23), *respectively,* $p_1^* = \sup_{t \in \mathcal{J}} |p_1(t)|$, $p_2^* = \sup_{t \in \mathcal{J}} |p_2(t)|$, $p_3^* = \sup_{t \in \mathcal{J}} |p_3(t)|$.

Then, the impulsive (ρ_k, ϕ_k)*-Hilfer* $\mathbb{FIDE}$*-*$\mathbb{NMP}$*-*$\mathbb{FIBC}$*s* (4) *has at least one solution.*

Proof. We separate $\mathcal{Q} : \mathcal{PC}_{\phi_k}^{1-\gamma_k} \to \mathcal{PC}_{\phi_k}^{1-\gamma_k}$ by (21) into two operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ as $(\mathcal{Q}u)(t) = (\mathcal{Q}_1u)(t) + (\mathcal{Q}_2u)(t)$ for $t \in \mathcal{J}$, where

$$\begin{aligned} (\mathcal{Q}_1u)(t) = {} & \frac{1}{\rho_k^{\frac{\alpha_k}{\rho_k}}} \int_{t_k}^{t} \Phi_{\phi_k}^{\frac{\alpha_k}{\rho_k} - 1}(t,s)\mathbb{E}_{\frac{\alpha_k}{\rho_k},\frac{\alpha_k}{\rho_k}}\left(\lambda_k\left(\rho_k^{-1}\Phi_{\phi_k}(t,s)\right)^{\frac{\alpha_k}{\rho_k}}\right)\mathcal{F}_u(s)\phi_k'(s)ds \\ & + \sum_{r=0}^{k-1} \frac{\mathcal{P}_k(r+1)}{\rho_r^{\frac{\alpha_r}{\rho_r} + 1 - \gamma_r}} \int_{t_r}^{t_{r+1}} \Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r} - \gamma_r}(t_{r+1},s)\mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r} + 1 - \gamma_r}\left(\lambda_r\left(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s)\right)^{\frac{\alpha_r}{\rho_r}}\right)\mathcal{F}_u(s)\phi_r'(s)ds \frac{\Phi_{\phi_k}^{\gamma_k - 1}(t,t_k)}{\rho_k^{\gamma_k - 1}} \\ & \times\mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\left(\lambda_k\left(\rho_k^{-1}\Phi_{\phi_k}(t,t_k)\right)^{\frac{\alpha_k}{\rho_k}}\right) + \left(\sum_{j=0}^{n} \frac{\omega_j}{\rho_j^{\frac{\alpha_j + \mu_j}{\rho_j}}} \int_{t_j}^{\xi_j} \Phi_{\phi_j}^{\frac{\alpha_j + \mu_j}{\rho_j} - 1}(\xi_j,s)\mathbb{E}_{\frac{\alpha_j}{\rho_j},\frac{\alpha_j + \mu_j}{\rho_j}}\left(\lambda_j\left(\rho_j^{-1}\Phi_{\phi_j}(\xi_j,s)\right)^{\frac{\alpha_j}{\rho_j}}\right)\mathcal{F}_u(s)\phi_j'(s)ds\right. \\ & + \sum_{j=0}^{n}\left\{\sum_{r=0}^{j-1} \frac{\mathcal{P}_j(r+1)}{\rho_r^{\frac{\alpha_r}{\rho_r} + 1 - \gamma_r}} \int_{t_r}^{t_{r+1}} \Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r} - \gamma_r}(t_{r+1},s)\mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r} + 1 - \gamma_r}\left(\lambda_r\left(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s)\right)^{\frac{\alpha_r}{\rho_r}}\right)\mathcal{F}_u(s)\phi_r'(s)ds\right. \\ & \times\left.\frac{\omega_j\Phi_{\phi_j}^{\gamma_j + \frac{\mu_j}{\rho_j} - 1}(\xi_j,t_j)}{\rho_j^{\frac{\mu_j}{\rho_j} + \gamma_j - 1}}\mathbb{E}_{\frac{\alpha_j}{\rho_j},\gamma_j + \frac{\mu_j}{\rho_j}}\left(\lambda_j\left(\rho_j^{-1}\Phi_{\phi_j}(\xi_j,t_j)\right)^{\frac{\alpha_j}{\rho_j}}\right)\right\} - \sum_{i=0}^{m} \frac{\kappa_i}{\rho_i^{\frac{\alpha_i}{\rho_i}}} \int_{t_i}^{\eta_i} \Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i} - 1}(\eta_i,s)\mathbb{E}_{\frac{\alpha_i}{\rho_i},\frac{\alpha_i}{\rho_i}}\left(\lambda_i\left(\rho_i^{-1}\Phi_{\phi_i}(\eta_i,s)\right)^{\frac{\alpha_i}{\rho_i}}\right) \\ & \times\mathcal{F}_u(s)\phi_r'(s)ds - \sum_{i=0}^{m} \frac{\kappa_i\Phi_{\phi_i}^{\gamma_i - 1}(\eta_i,t_i)}{\rho_i^{\gamma_i - 1}}\mathbb{E}_{\frac{\alpha_i}{\rho_i},\gamma_i}\left(\lambda_i\left(\rho_i^{-1}\Phi_{\phi_i}(\eta_i,t_i)\right)^{\frac{\alpha_i}{\rho_i}}\right)\left\{\sum_{r=0}^{i-1} \frac{\mathcal{P}_i(r+1)}{\rho_r^{\frac{\alpha_r}{\rho_r} + 1 - \gamma_r}} \int_{t_r}^{t_{r+1}} \Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r} - \gamma_r}(t_{r+1},s)\right. \\ & \times\left.\left.\mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r} + 1 - \gamma_r}\left(\lambda_r\left(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s)\right)^{\frac{\alpha_r}{\rho_r}}\right)\mathcal{F}_u(s)\phi_r'(s)ds\right\}\right) \frac{\mathcal{P}_k(0)\Phi_{\phi_k}^{\gamma_k - 1}(t,t_k)}{\Xi\rho_k^{\gamma_k - 1}}\mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\left(\lambda_k\left(\rho_k^{-1}\Phi_{\phi_k}(t,t_k)\right)^{\frac{\alpha_k}{\rho_k}}\right), \end{aligned} \tag{35}$$

$$\begin{aligned}(\mathcal{Q}_2u)(t) \quad = \quad & \frac{\Phi_{\phi_k}^{\gamma_k-1}(t,t_k)}{\rho_k^{\gamma_k-1}}\mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\big(\lambda_k\big(\rho_k^{-1}\Phi_{\phi_k}(t,t_k)\big)^{\frac{\alpha_k}{\rho_k}}\big)\sum_{r=1}^{k}\varphi_r(u(t_r))\mathcal{P}_k(r) \\ & +\frac{1}{\Xi}\Bigg(\sum_{j=0}^{n}\Bigg[\frac{\omega_j\Phi_{\phi_j}^{\gamma_j+\frac{\mu_j}{\rho_j}-1}(\xi_j,t_j)}{\rho_j^{\frac{\mu_j}{\rho_j}+\gamma_j-1}}\mathbb{E}_{\frac{\alpha_j}{\rho_j},\gamma_j+\frac{\mu_j}{\rho_j}}\big(\lambda_j\big(\rho_j^{-1}\Phi_{\phi_j}(\xi_j,t_j)\big)^{\frac{\alpha_j}{\rho_j}}\big)\sum_{r=1}^{j}\varphi_r(u(t_r))\mathcal{P}_j(r)\Bigg] \\ & +A-\sum_{i=0}^{m}\Bigg[\frac{\kappa_i\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\rho_i^{\gamma_i-1}}\mathbb{E}_{\frac{\alpha_i}{\rho_i},\gamma_i}\big(\lambda_i\big(\rho_i^{-1}\Phi_{\phi_i}(\eta_i,t_i)\big)^{\frac{\alpha_i}{\rho_i}}\big)\sum_{r=1}^{i}\varphi_r(u(t_r))\mathcal{P}_i(r)\Bigg]\Bigg) \\ & \times\frac{\mathcal{P}_k(0)\Phi_{\phi_k}^{\gamma_k-1}(t,t_k)}{\rho_k^{\gamma_k-1}}\mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\big(\lambda_k\big(\rho_k^{-1}\Phi_{\phi_k}(t,t_k)\big)^{\frac{\alpha_k}{\rho_k}}\big). \end{aligned} \tag{36}$$

Let $\mathcal{B}_{\Upsilon_2}=\{u\in\mathcal{PC}_{\phi_k}^{1-\gamma_k}:\|u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}\leq\Upsilon_2\}$ satisfying

$$\frac{\Upsilon_2}{\Delta_1p_1^*\psi(\Upsilon_2)+\Delta_2\mathbb{M}_1+\frac{|A|}{|\Xi|\Gamma_{\rho_m}(\rho_m\gamma_m)}}>\frac{1}{1-\big[\Delta_1(p_2^*\Phi_1^*+p_3^*\Phi_2^*)\big]}.$$

From Theorem 1, we can prove that $\mathcal{Q}_1$ is continuous. By using $(\mathbb{H}_4)$, we show that $\mathcal{Q}_1(\mathcal{B}_{\Upsilon_2})$ is bounded. For any $u\in\mathcal{B}_{\Upsilon_2}$, we obtain

$$\begin{aligned} & \Big|\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)(\mathcal{Q}_1u)(t)\Big| \\ \leq \quad & \Big[p_1^*\psi(\Upsilon_2)+(p_2^*\Phi_1^*+p_3^*\Phi_2^*)\Upsilon_2\Big]\Bigg\{\frac{\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)}{\rho_k\Gamma_{\rho_k}(\alpha_k)}\int_{t_k}^{t}\Phi_{\phi_k}^{\frac{\alpha_k}{\rho_k}-1}(t,s)\phi_k'(s)ds \\ & +\frac{1}{\Gamma_{\rho_k}(\beta_k(\rho_k-\alpha_k)+\alpha_k)}\Bigg[\sum_{r=0}^{k-1}\frac{1}{\rho_r\Gamma_{\rho_r}(\alpha_r+\rho_r(1-\gamma_r))}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r+\rho_r(1-\gamma_r)}{\rho_r}-1}(t_{r+1},s)\phi_r'(s)ds \\ & +\frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n}\frac{|\omega_j|}{\rho_j^{\frac{\alpha_j+\mu_j}{\rho_j}}\Gamma\Big(\frac{\alpha_j+\mu_j}{\rho_j}\Big)}\int_{t_j}^{\xi_j}\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,s)\mathcal{F}_u(s)\phi_j'(s)ds \\ & +\sum_{j=0}^{n}\frac{|\omega_j|\Phi_{\phi_j}^{\frac{\beta_j(\rho_j-\alpha_j)+\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,t_j)}{\Gamma_{\rho_j}(\beta_j(\rho_j-\alpha_j)+\alpha_j+\mu_j)}\sum_{r=0}^{j-1}\frac{1}{\rho_r\Gamma_{\rho_r}(\alpha_r+\rho_r(1-\gamma_r))} \\ & \times\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r+\rho_r(1-\gamma_r)}{\rho_r}-1}(t_{r+1},s)\phi_r'(s)ds \\ & +\sum_{i=0}^{m}\frac{|\kappa_i|}{\rho_i\Gamma_{\rho_i}(\alpha_i)}\int_{t_i}^{\eta_i}\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}-1}(\eta_i,s)\phi_r'(s)ds+\sum_{i=0}^{m}\frac{|\kappa_i|\Phi_{\phi_i}^{\frac{\beta_i(\rho_i-\alpha_i)+\alpha_i}{\rho_i}-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\beta_i(\rho_i-\alpha_i)+\alpha_i)} \\ & \times\sum_{r=0}^{i-1}\frac{1}{\rho_r\Gamma_{\rho_r}(\alpha_r+\rho_r(1-\gamma_r))}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)\phi_r'(s)ds\Bigg)\Bigg]\Bigg\} \\ \leq \quad & \Delta_1\Big[p_1^*\psi(\Upsilon_2)+(p_2^*\Phi_1^*+p_3^*\Phi_2^*)\Upsilon_2\Big], \end{aligned}$$

which implies that $\mathcal{Q}_1(\mathcal{B}_{\Upsilon_2})\leq\Delta_1[p_1^*\psi(\Upsilon_2)+(p_2^*\Phi_1^*+p_3^*\Phi_2^*)\Upsilon_2]$.

Next, we will show that $\mathcal{Q}_1$ maps bounded set $\mathcal{B}_{\Upsilon_2}$ into equicontinuous set of $\mathcal{PC}^{1-\gamma_k}_{\phi_k}$. Let $\tau_1, \tau_2 \in \mathcal{J}_k$, for $k = 0, 1, \ldots, m$ with $\tau_1 < \tau_2$ and $u \in \mathcal{B}_{\Upsilon_2}$. Then, we obtain

$$
\begin{aligned}
& |(\mathcal{Q}_1 u)(\tau_2) - (\mathcal{Q}_1 u)(\tau_1)| \\
\leq\ & \Big[p_1^* \psi(\Upsilon_2) + (p_2^*\Phi_1^* + p_3^*\Phi_2^*)\Upsilon_2\Big] \Bigg\{ \frac{\left|\Phi_{\phi_m}^{\frac{\alpha_m}{\rho_m}}(\tau_2, t_m) - \Phi_{\phi_m}^{\frac{\alpha_m}{\rho_m}}(\tau_1, t_m)\right|}{\Gamma_{\rho_m}(\alpha_m + \rho_m)} \\
& + \frac{\left|\Phi_{\phi_m}^{\frac{\beta_m(\rho_m-\alpha_m)+\alpha_m}{\rho_m}-1}(\tau_2, t_m) - \Phi_{\phi_m}^{\frac{\beta_m(\rho_m-\alpha_m)+\alpha_m}{\rho_m}-1}(\tau_1, t_m)\right|}{\Gamma_{\rho_m}(\beta_m(\rho_m - \alpha_m) + \alpha_m)} \Bigg[\sum_{r=0}^{k-1} \frac{\Phi_{\phi_r}^{\frac{\alpha_r+\rho_r(1-\gamma_r)}{\rho_r}}(t_{r+1}, t_r)}{\Gamma_{\rho_r}(\alpha_r + \rho_r(2-\gamma_r))} \\
& + \frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n} \frac{|\omega_j|\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}}(\xi_j, t_j)}{\Gamma_{\rho_j}(\alpha_j + \mu_j + \rho_j)} + \sum_{j=0}^{n} \frac{|\omega_j|\Phi_{\phi_j}^{\frac{\beta_j(\rho_j-\alpha_j)+\alpha_j+\mu_j}{\rho_j}-1}(\xi_j, t_j)}{\Gamma_{\rho_j}(\beta_j(\rho_j - \alpha_j) + \alpha_j + \mu_j)} \sum_{r=0}^{j-1} \frac{\Phi_{\phi_r}^{\frac{\alpha_r+\rho_r(1-\gamma_r)}{\rho_r}}(t_{r+1}, t_r)}{\Gamma_{\rho_r}(\alpha_r + \rho_r(2-\gamma_r))} \\
& + \sum_{i=0}^{m} \frac{|\kappa_i|\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}}(\eta_i, t_i)}{\Gamma_{\rho_i}(\alpha_i + \rho_i)} + \sum_{i=0}^{m} \frac{|\kappa_i|\Phi_{\phi_i}^{\frac{\beta_i(\rho_i-\alpha_i)+\alpha_i}{\rho_i}-1}(\eta_i, t_i)}{\Gamma_{\rho_i}(\beta_i(\rho_i - \alpha_i) + \alpha_i)} \sum_{r=0}^{i-1} \frac{\Phi_{\phi_r}^{\frac{\alpha_r+\rho_r(1-\gamma_r)}{\rho_r}}(t_{r+1}, t_r)}{\rho_r\Gamma_{\rho_r}(\alpha_r + \rho_r(1-\gamma_r))} \Bigg) \Bigg] \Bigg\}.
\end{aligned}
$$

It is easy to see that the above result is independent of variable $u \in \mathcal{B}_{\Upsilon_2}$, which implies that $|(\mathcal{Q}_1 u)(\tau_2) - (\mathcal{Q}_1 u)(\tau_1)| \to 0$ as $\tau_2 \to \tau_1$. Since, $\mathcal{Q}_1$ maps bounded set $\mathcal{B}_{\Upsilon_2}$ into equicontinuous set of $\mathcal{PC}^{1-\gamma_k}_{\phi_k}$. Thus, by the Arzelá-Ascoli theorem, we get that $\mathcal{Q}_1$ is completely continuous.

Next, we will show that $\mathcal{Q}_2$ is a non-linear contraction. Define a continuous non-decreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(\epsilon) = \Delta_2 b_1 \epsilon$, for all $\epsilon \geq 0$. Clearly, $\psi(t)$ satisfies $\psi(0) = 0$ and by applying $\Delta_2 b_1 < 1$, we obtain $\psi(\epsilon) < \epsilon$ for all $\epsilon > 0$. For every $u, v \in \mathcal{B}_{\Upsilon_2}$, we have

$$
\begin{aligned}
& \left|\Phi_{\phi_k}^{1-\gamma_k}(t, t_k)\big((\mathcal{Q}_2 u)(t) - (\mathcal{Q}_2 v)(t)\big)\right| \\
\leq\ & \frac{\Phi_{\phi_k}^{\gamma_k-1}(t, t_k)}{\rho_k^{\gamma_k-1}} \left|\mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\Big(\lambda_k\big(\rho_k^{-1}\Phi_{\phi_k}(t, t_k)\big)^{\frac{\alpha_k}{\rho_k}}\Big)\right| \sum_{r=1}^{k} \left|\Phi_{\phi_k}^{1-\gamma_k}(t, t_k)\big(\varphi_r(u(t_r)) - \varphi_r(v(t_r))\big)\right| \\
& \times|\mathcal{P}_k(r)| + \frac{\Phi_{\phi_k}^{\gamma_k-1}(t, t_k)}{|\Xi|}\Bigg(\sum_{j=0}^{n} \Bigg[\frac{|\omega_j|\Phi_{\phi_j}^{\gamma_j+\frac{\mu_j}{\rho_j}-1}(\xi_j, t_j)}{\rho_j^{\frac{\mu_j}{\rho_j}+\gamma_j-1}} \left|\mathbb{E}_{\frac{\alpha_j}{\rho_j},\gamma_j+\frac{\mu_j}{\rho_j}}\Big(\lambda_j\big(\rho_j^{-1}\Phi_{\phi_j}(\xi_j, t_j)\big)^{\frac{\alpha_j}{\rho_j}}\Big)\right| \\
& \times \sum_{r=1}^{j} \left|\Phi_{\phi_k}^{1-\gamma_k}(t, t_k)\big(\varphi_r(u(t_r)) - \varphi_r(v(t_r))\big)\right||\mathcal{P}_j(r)| \Bigg] + \sum_{i=0}^{m} \Bigg[\frac{|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i, t_i)}{\rho_i^{\gamma_i-1}} \\
& \times\left|\mathbb{E}_{\frac{\alpha_i}{\rho_i},\gamma_i}\Big(\lambda_i\big(\rho_i^{-1}\Phi_{\phi_i}(\eta_i, t_i)\big)^{\frac{\alpha_i}{\rho_i}}\Big)\right| \sum_{r=1}^{i} \left|\Phi_{\phi_k}^{1-\gamma_k}(t, t_k)\big(\varphi_r(u(t_r)) - \varphi_r(v(t_r))\big)\right||\mathcal{P}_i(r)| \Bigg]\Bigg) \\
& \times\frac{|\mathcal{P}_k(0)|}{\rho_k^{\gamma_k-1}}\left|\mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\Big(\lambda_k\big(\rho_k^{-1}\Phi_{\phi_k}(t, t_k)\big)^{\frac{\alpha_k}{\rho_k}}\Big)\right| \\
\leq\ & \frac{\Phi_{\phi_m}^{\gamma_m-1}(T, t_m)}{\rho_m^{\gamma_m-1}\Gamma(\gamma_m)} \sum_{r=1}^{m} q_1(\|u - v\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}}) + \frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n} \Bigg[\frac{|\omega_j|\Phi_{\phi_j}^{\gamma_j+\frac{\mu_j}{\rho_j}-1}(\xi_j, t_j)}{\rho_j^{\frac{\mu_j}{\rho_j}+\gamma_j-1}\Gamma\Big(\gamma_j + \frac{\mu_j}{\rho_j}\Big)} \\
& \times \sum_{r=1}^{j} q_1(\|u - v\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}}) \Bigg] + \sum_{i=0}^{m} \Bigg[\frac{|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i, t_i)}{\rho_i^{\gamma_i-1}\Gamma(\gamma_i)} \sum_{r=1}^{i} q_1(\|u - v\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}}) \Bigg]\Bigg) \frac{\Phi_{\phi_m}^{\gamma_m-1}(T, t_m)}{\rho_m^{\gamma_m-1}\Gamma(\gamma_m)} \\
\leq\ & \frac{\Phi_{\phi_m}^{\gamma_m-1}(T, t_m)}{\rho_m^{\gamma_m-1}\Gamma(\gamma_m)} \Bigg[m + \frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n} \frac{j|\omega_j|\Phi_{\phi_j}^{\gamma_j+\frac{\mu_j}{\rho_j}-1}(\xi_j, t_j)}{\rho_j^{\frac{\mu_j}{\rho_j}+\gamma_j-1}\Gamma\Big(\gamma_j + \frac{\mu_j}{\rho_j}\Big)} + \sum_{i=0}^{m} \frac{i|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i, t_i)}{\rho_i^{\gamma_i-1}\Gamma(\gamma_i)} \Bigg)\Bigg] b_1\|u - v\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}} \\
=\ & \Delta_2 b_1\|u - v\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}}.
\end{aligned}
$$

By setting $\psi(\epsilon) = \Delta_2 b_1 \epsilon$, note that $\psi(0) = 0$ and $\psi(\epsilon) < \epsilon$ for all $\epsilon > 0$. So,

$$\|\mathcal{Q}_2 u - \mathcal{Q}_2 v\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}} \leq \psi(\|u - v\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}}),$$

which yields that $\mathcal{Q}_2$ is non-linear contraction.

Next, we will show that $\mathcal{Q}(\mathcal{B}_{\Upsilon_2})$ is bounded. By using $(\mathbb{H}_3)$, for any $u \in \mathcal{B}_{\Upsilon_2}$, we obtain

$$\begin{aligned}
&\left|\Phi^{1-\gamma_k}_{\phi_k}(t,t_k)(\mathcal{Q}_2 u)(t)\right| \\
\leq\ & \frac{\Phi^{\gamma_k-1}_{\phi_k}(t,t_k)}{\rho_k^{\gamma_k-1}}\left|\mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\left(\lambda_k\left(\rho_k^{-1}\Phi_{\phi_k}(t,t_k)\right)^{\frac{\alpha_k}{\rho_k}}\right)\right|\sum_{r=1}^{k}\left|\Phi^{1-\gamma_k}_{\phi_k}(t,t_k)\varphi_r(u(t_r))\right||\mathcal{P}_k(r)| \\
&+\frac{1}{|\Xi|}\left(\sum_{j=0}^{n}\left[\frac{|\omega_j|\Phi_{\phi_j}^{\gamma_j+\frac{|\mu_j|}{\rho_j}-1}(\xi_j,t_j)}{\rho_j^{\frac{\mu_j}{\rho_j}+\gamma_j-1}}\left|\mathbb{E}_{\frac{\alpha_j}{\rho_j},\gamma_j+\frac{\mu_j}{\rho_j}}\left(\lambda_j\left(\rho_j^{-1}\Phi_{\phi_j}(\xi_j,t_j)\right)^{\frac{\alpha_j}{\rho_j}}\right)\right|\Phi^{\gamma_k-1}_{\phi_k}(t,t_k)\right.\right. \\
&\times\left.\sum_{r=1}^{j}\left|\Phi^{1-\gamma_k}_{\phi_k}(t,t_k)\varphi_r(u(t_r))\right||\mathcal{P}_j(r)|\right]+|A|+\sum_{i=0}^{m}\left[\frac{|\kappa_i|\Phi^{\gamma_i-1}_{\phi_i}(\eta_i,t_i)}{\rho_i^{\gamma_i-1}}\right. \\
&\times\left.\left.\left|\mathbb{E}_{\frac{\alpha_i}{\rho_i},\gamma_i}\left(\lambda_i\left(\rho_i^{-1}\Phi_{\phi_i}(\eta_i,t_i)\right)^{\frac{\alpha_i}{\rho_i}}\right)\right|\Phi^{\gamma_k-1}_{\phi_k}(t,t_k)\sum_{r=1}^{i}\left|\Phi^{1-\gamma_k}_{\phi_k}(t,t_k)\varphi_r(u(t_r))\right||\mathcal{P}_i(r)|\right]\right) \\
&\times\frac{|\mathcal{P}_k(0)|}{\rho_k^{\gamma_k-1}}\left|\mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\left(\lambda_k\left(\rho_k^{-1}\Phi_{\phi_k}(t,t_k)\right)^{\frac{\alpha_k}{\rho_k}}\right)\right| \\
\leq\ & \left\{\frac{\Phi^{\gamma_m-1}_{\phi_k}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\left[m+\frac{1}{|\Xi|}\left(\sum_{j=0}^{n}\frac{j|\omega_j|\Phi_{\phi_j}^{\frac{\mu_j}{\rho_j}+\gamma_j-1}(\xi_j,t_j)}{\Gamma_{\rho_j}(\rho_j\gamma_j+\mu_j)}+\sum_{i=0}^{m}\frac{i|\kappa_i|\Phi^{\gamma_i-1}_{\phi_i}(\eta_i,t_i)}{\rho_i\Gamma_{\rho_i}(\rho_i\gamma_i)}\right)\right]\right. \\
&\left.+|A|\right\}\mathbb{M}_1 \\
=\ & \left\{\Delta_2+\frac{|A|}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\right\}\mathbb{M}_1.
\end{aligned}$$

This yields that $\mathcal{Q}(\mathcal{B}_{\Upsilon_2})$ is bounded with the boundedness of the set $\mathcal{Q}_1(\mathcal{B}_{\Upsilon_2})$.

Finally, we will show that (C_2) of Theorem 2 does not true. On the contrary, assume that (C_2) holds. Then, there exists $\sigma \in (0,1)$ and for every $u \in \mathcal{B}_{\Upsilon_2}$ so that $u = \sigma\mathcal{Q}u$. Hence, we have $\|u\|_{\mathcal{PC}^{1-\gamma_k}_{\phi_k}} \leq \Upsilon_2$ and

$$\begin{aligned}
&\left|\Phi^{1-\gamma_k}_{\phi_k}(t,t_k)u(t)\right| \\
=\ & \sigma\left|\Phi^{1-\gamma_k}_{\phi_k}(t,t_k)(\mathcal{Q}u)(t)\right| \\
\leq\ & \left|\Phi^{1-\gamma_k}_{\phi_k}(t,t_k)\big((\mathcal{Q}_1 u)(t)+(\mathcal{Q}_2 u)(t)\big)\right| \\
\leq\ & \frac{p_1^*\psi(\Upsilon_2)+(p_2^*\Phi_1^*+p_3^*\Phi_2^*)\Upsilon_2}{\rho_m\Gamma_{\rho_m}(\alpha_m)}\int_{t_m}^{T}\Phi_{\phi_m}^{\frac{\alpha_m}{\rho_m}-1}(T,s)\phi_m'(s)ds\,\Phi^{1-\gamma_m}_{\phi_m}(T,t_m) \\
&+\frac{p_1^*\psi(\Upsilon_2)+(p_2^*\Phi_1^*+p_3^*\Phi_2^*)\Upsilon_2}{\Gamma_{\rho_m}(\beta_m(\rho_m-\alpha_m)+\alpha_m)}\sum_{r=0}^{m-1}\frac{1}{\rho_r\Gamma_{\rho_r}(\alpha_r+\rho_r(1-\gamma_r))} \\
&\times\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r+\rho_r(1-\gamma_r)}{\rho_r}-1}(t_{r+1},s)\phi_r'(s)ds+\left(\sum_{j=0}^{n}\frac{|\omega_j|}{\rho_j\Gamma_{\rho_j}(\alpha_j+\mu_j)}\int_{t_j}^{\xi_j}\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,s)\phi_j'(s)ds\right.
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=0}^{n}\frac{|\omega_j|\Phi_{\phi_j}^{\frac{\beta_j(\rho_j-\alpha_j)+\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,t_j)}{\Gamma_{\rho_j}(\beta_j(\rho_j-\alpha_j)+\alpha_j+\mu_j)}\sum_{r=0}^{j-1}\frac{1}{\rho_r\Gamma_{\rho_r}(\alpha_r+\rho_r(1-\gamma_r))}\\
&\times\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r+\rho_r(1-\gamma_r)}{\rho_r}-1}(t_{r+1},s)\phi_r'(s)ds+\sum_{i=0}^{m}\frac{|\kappa_i|}{\rho_i\Gamma_{\rho_i}(\alpha_i)}\int_{t_i}^{\eta_i}\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}-1}(\eta_i,s)\phi_r'(s)ds\\
&+\sum_{i=0}^{m}\frac{|\kappa_i|\Phi_{\phi_i}^{\frac{\beta_i(\rho_i-\alpha_i)+\alpha_i}{\rho_i}-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\beta_i(\rho_i-\alpha_i)+\alpha_i)}\sum_{r=0}^{i-1}\frac{1}{\rho_r\Gamma_{\rho_r}(\alpha_r+\rho_r(1-\gamma_r))}\\
&\times\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r+\rho_r(1-\gamma_r)}{\rho_r}-1}(t_{r+1},s)\phi_r'(s)ds\Bigg)\frac{p_1^*\psi(\Upsilon_2)+(p_2^*\Phi_1^*+p_3^*\Phi_2^*)\Upsilon_2}{|\Xi|\Gamma_{\rho_m}(\beta_m(\rho_m-\alpha_m)+\alpha_m)}\\
&+\frac{\Phi_{\phi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\beta_m(\rho_m-\alpha_m)+\alpha_m)}\Bigg[m+\frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n}\frac{j|\omega_j|\Phi_{\phi_j}^{\frac{\beta_j(\rho_j-\alpha_j)+\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,t_j)}{\Gamma_{\rho_j}(\beta_j(\rho_j-\alpha_j)+\alpha_j+\mu_j)}\\
&+\sum_{i=0}^{m}\frac{i|\kappa_i|\Phi_{\phi_i}^{\frac{\beta_i(\rho_i-\alpha_i)+\alpha_i}{\rho_i}-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\beta_i(\rho_i-\alpha_i)+\alpha_i)}\Bigg)\Bigg]\mathbb{M}_1+\frac{|A|}{|\Xi|\Gamma_{\rho_m}(\beta_m(\rho_m-\alpha_m)+\alpha_m)}\\
\le\ &\Big[p_1^*\psi(\Upsilon_2)+(p_2^*\Phi_1^*+p_3^*\Phi_2^*)\Upsilon_2\Big]\Bigg\{\frac{\Phi_{\phi_m}^{\frac{\alpha_m}{\rho_m}-\gamma_m+1}(T,t_m)}{\Gamma_{\rho_m}(\alpha_m+\rho_m)}\\
&+\frac{1}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\Bigg[\sum_{r=0}^{m-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r+1}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))}+\frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n}\frac{|\omega_j|\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}}(\xi_j,t_j)}{\Gamma_{\rho_j}(\alpha_j+\mu_j+\rho_j)}\\
&+\sum_{j=0}^{n}\frac{|\omega_j|\Phi_{\phi_j}^{\frac{\mu_j}{\rho_j}+\gamma_j-1}(\xi_j,t_j)}{\Gamma_{\rho_j}(\rho_j\gamma_j+\mu_j)}\sum_{r=0}^{j-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r+1}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))}+\sum_{i=0}^{m}\frac{|\kappa_i|\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}}(\eta_i,t_i)}{\Gamma_{\rho_i}(\alpha_i+\rho_i)}\\
&+\sum_{i=0}^{m}\frac{|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\sum_{r=0}^{i-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r+1}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))}\Bigg)\Bigg]\Bigg\}+\frac{\Phi_{\phi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\Bigg[m\\
&+\frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n}\frac{j|\omega_j|\Phi_{\phi_j}^{\frac{\mu_j}{\rho_j}+\gamma_j-1}(\xi_j,t_j)}{\Gamma_{\rho_j}(\rho_j\gamma_j+\mu_j)}+\sum_{i=0}^{m}\frac{i|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\Bigg)\Bigg]\mathbb{M}_1+\frac{|A|}{|\Xi|\Gamma_{\rho_m}(\rho_m\gamma_m)}\\
=\ &\Delta_1\Big[p_1^*\psi(\Upsilon_2)+(p_2^*\Phi_1^*+p_3^*\Phi_2^*)\Upsilon_2\Big]+\Delta_2\mathbb{M}_1+\frac{|A|}{|\Xi|\Gamma_{\rho_m}(\rho_m\gamma_m)},
\end{aligned}
$$

which yields that

$$\Upsilon_2\le\Delta_1\Big[p_1^*\psi(\Upsilon_2)+(p_2^*\Phi_1^*+p_3^*\Phi_2^*)\Upsilon_2\Big]+\Delta_2\mathbb{M}_1+\frac{|A|}{|\Xi|\Gamma_{\rho_m}(\rho_m\gamma_m)}.$$

Then, we have

$$\frac{\Upsilon_2}{\Delta_1p_1^*\psi(\Upsilon_2)+\Delta_2\mathbb{M}_1+\frac{|A|}{|\Xi|\Gamma_{\rho_m}(\rho_m\gamma_m)}}\le\frac{1}{1-\big[\Delta_1(p_2^*\Phi_1^*+p_3^*\Phi_2^*)\big]},$$

which contradicts the condition $(\mathbb{H}_6)$. Hence, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ fulfill all the conditions of Lemma 9. Therefore, the impulsive (ρ_k,ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (4) has at least one solution. □

4. Stability Results

First of all, we give the following inequalities for analyzing Ulam's stability of the impulsive (ρ_k,ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (4). Let $\mathcal{T}\in\mathcal{C}(\mathcal{J},\mathbb{R}^+)$ be a non-decreasing function, $\epsilon > 0$, $\tau \geq 0$, $z \in \mathcal{B}$, such that for $t \in \mathcal{J}_k$, $k = 1, 2, \ldots, m$, the following inequalities are fulfilled:

$$\begin{cases} \left| {}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\phi_k}_{t_k^+} z(t) - \lambda_k z(t) - f(t, z(t), {}_{\rho_k}\mathcal{I}^{\delta_k;\phi_k}_{t_k} z(t), {}_{\rho_k}\mathcal{I}^{\theta_k;\phi_k}_{t_k} z(t)) \right| \leq \epsilon, \\ \left| {}_{\rho_k}\mathcal{I}^{\rho(1-\gamma_k);\phi_k}_{t_k^+} z(t_k^+) - {}_{\rho_{k-1}}\mathcal{I}^{\rho(1-\gamma_{k-1});\phi_{k-1}}_{t_{k-1}^+} z(t_k^-) - \varphi_k(z(t_k)) \right| \leq \epsilon, \end{cases} \tag{37}$$

$$\begin{cases} \left| {}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\phi_k}_{t_k^+} z(t) - \lambda_k z(t) - f(t, z(t), {}_{\rho_k}\mathcal{I}^{\delta_k;\phi_k}_{t_k} z(t), {}_{\rho_k}\mathcal{I}^{\theta_k;\phi_k}_{t_k} z(t)) \right| \leq \mathcal{T}(t), \\ \left| {}_{\rho_k}\mathcal{I}^{\rho(1-\gamma_k);\phi_k}_{t_k^+} z(t_k^+) - {}_{\rho_{k-1}}\mathcal{I}^{\rho(1-\gamma_{k-1});\phi_{k-1}}_{t_{k-1}^+} z(t_k^-) - \varphi_k(z(t_k)) \right| \leq \tau, \end{cases} \tag{38}$$

$$\begin{cases} \left| {}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\phi_k}_{t_k^+} z(t) - \lambda_k z(t) - f(t, z(t), {}_{\rho_k}\mathcal{I}^{\delta_k;\phi_k}_{t_k} z(t), {}_{\rho_k}\mathcal{I}^{\theta_k;\phi_k}_{t_k} z(t)) \right| \leq \epsilon\mathcal{T}(t), \\ \left| {}_{\rho_k}\mathcal{I}^{\rho(1-\gamma_k);\phi_k}_{t_k^+} z(t_k^+) - {}_{\rho_{k-1}}\mathcal{I}^{\rho(1-\gamma_{k-1});\phi_{k-1}}_{t_{k-1}^+} z(t_k^-) - \varphi_k(z(t_k)) \right| \leq \epsilon\tau. \end{cases} \tag{39}$$

Definition 3. *The impulsive (ρ_k, ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (4) is called Ulam–Hyers ($\mathcal{UH}$) stable, if there exists $\mathfrak{C}_{\mathcal{F}} > 0$, such that for any $\epsilon > 0$ and for each $z \in \mathcal{B}$ of (37) there is $u \in \mathcal{B}$ of (4) that satisfies*

$$|z(t) - u(t)| \leq \mathfrak{C}_{\mathcal{F}}\epsilon, \quad t \in \mathcal{J}. \tag{40}$$

Definition 4. *The impulsive (ρ_k, ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (4) is called generalized Ulam-Hyers ($\mathcal{GUH}$) stable, if there exists $\mathcal{T} \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ with $\mathcal{T}(0) = 0$, such that for any $\epsilon > 0$ and for each $z \in \mathcal{B}$ of (38) there is $u \in \mathcal{B}$ of (4) that satisfies*

$$|z(t) - u(t)| \leq \mathcal{T}(\epsilon), \quad t \in \mathcal{J}. \tag{41}$$

Definition 5. *The impulsive (ρ_k, ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (4) is called Ulam–Hyers–Rassias $\mathcal{UHR}$ stable with respect to $(\tau, \mathcal{T})$, if there exists $\mathfrak{C}_{\mathcal{F},\mathfrak{C}_{\mathcal{F}}} > 0$ such that for any $\epsilon > 0$ and for each $z \in \mathcal{B}$ of (39) there is $u \in \mathcal{B}$ of (4) that satisfies*

$$|z(t) - u(t)| \leq \mathfrak{C}_{\mathcal{F},\mathfrak{C}_{\mathcal{F}}}\epsilon(\tau + \mathcal{T}(t)), \quad t \in \mathcal{J}. \tag{42}$$

Definition 6. *The impulsive (ρ_k, ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (4) is called generalized Ulam–Hyers–Rassias $\mathcal{GUHR}$ stable with respect to $(\tau, \mathcal{T})$, if there exists $\mathfrak{C}_{\mathcal{F},\mathfrak{C}_{\mathcal{F}}} > 0$ such that for any $z \in \mathcal{B}$ of (38) there is $u \in \mathcal{B}$ of (4) that satisfies*

$$|z(t) - u(t)| \leq \mathfrak{C}_{\mathcal{F},\mathfrak{C}_{\mathcal{F}}}(\tau + \mathcal{T}(t)), \quad t \in \mathcal{J}. \tag{43}$$

Remark 3. *It is easy to see that: (i) Definition 3 $\Rightarrow$ Definition 4, (ii) Definition 5 $\Rightarrow$ Definition 6, and (iii) Definition 5 with $\tau + \mathcal{T}(t) = 1 \Rightarrow$ Definition 3.*

Remark 4. *$z \in \mathcal{B}$ is a solution of (37) if there is $x \in \mathcal{B}$ and x_k, $k = 1, 2, \ldots, m$, (which depends on z), such that*

(i) $|x(t)| \leq \epsilon$, $|x_k| \leq \epsilon$, $t \in \mathcal{J}$,

(ii) ${}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\phi_k}_{t_k^+} z(t) = \lambda_k z(t) + f(t, z(t), {}_{\rho_k}\mathcal{I}^{\delta_k;\phi_k}_{t_k} z(t), {}_{\rho_k}\mathcal{I}^{\theta_k;\phi_k}_{t_k} z(t)) + x(t)$, $t \in \mathcal{J}$,

(iii) ${}_{\rho_k}\mathcal{I}^{\rho(1-\gamma_k);\phi_k}_{t_k^+} z(t_k^+) - {}_{\rho_{k-1}}\mathcal{I}^{\rho(1-\gamma_{k-1});\phi_{k-1}}_{t_{k-1}^+} z(t_k^-) = \varphi_k(z(t_k)) + x_k$.

Remark 5. *$z \in \mathcal{B}$ is a solution of (38) if there is $x \in \mathcal{B}$ and x_k, $k = 1, 2, \ldots, m$, (which depends on z), such that*

(i) $|x(t)| \leq \mathcal{T}(t)$, $|x_k| \leq \tau$, $t \in \mathcal{J}$,

(ii) ${}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\phi_k}_{t_k^+} z(t) = \lambda_k z(t) + f(t, z(t), {}_{\rho_k}\mathcal{I}^{\delta_k;\phi_k}_{t_k} z(t), {}_{\rho_k}\mathcal{I}^{\theta_k;\phi_k}_{t_k} z(t)) + x(t)$, $t \in \mathcal{J}$,

(*iii*) $\;{}_{\rho_k}\mathcal{I}^{\rho(1-\gamma_k);\phi_k}_{t_k^+} z(t_k^+) - {}_{\rho_{k-1}}\mathcal{I}^{\rho(1-\gamma_{k-1});\phi_{k-1}}_{t_{k-1}^+} z(t_k^-) = \varphi_k(z(t_k)) + x_k.$

Remark 6. *$z \in \mathcal{B}$ is a solution of* (39) *if there is $x \in \mathcal{B}$ and x_k, $k = 1, 2, \ldots, m$, (which depends on z), such that*

(*i*) $\;|x(t)| \leq \epsilon\mathcal{T}(t)$, $|x_k| \leq \epsilon\tau$, $t \in \mathcal{J}$,

(*ii*) $\;{}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\phi_k}_{t_k^+} z(t) = \lambda_k z(t) + f(t, z(t), {}_{\rho_k}\mathcal{I}^{\delta_k;\phi_k}_{t_k} z(t), {}_{\rho_k}\mathcal{I}^{\theta_k;\phi_k}_{t_k} z(t)) + x(t)$, $t \in \mathcal{J}$,

(*iii*) $\;{}_{\rho_k}\mathcal{I}^{\rho(1-\gamma_k);\phi_k}_{t_k^+} z(t_k^+) - {}_{\rho_{k-1}}\mathcal{I}^{\rho(1-\gamma_{k-1});\phi_{k-1}}_{t_{k-1}^+} z(t_k^-) = \varphi_k(z(t_k)) + x_k.$

4.1. Ulam–Hyers Stability Results

We construct the proof of the following lemma, which gives a base for obtaining a solution to the impulsive (ρ_k, ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (4).

Theorem 3. *Let $\alpha_k \in (0,1)$, $\beta_k \in [0,1]$, $\rho_k > 0$, $\gamma_k = (\beta_k(\rho_k - \alpha_k) + \alpha_k)/\rho_k$, $\lambda_k \in \mathbb{R}$, $\phi_k \in \mathcal{C}(\mathcal{J}, \mathbb{R})$ with $\phi_k' > 0$ for $k = 1, 2, \ldots, m$. Assume that $f \in \mathcal{C}(\mathcal{J} \times \mathbb{R}^3, \mathbb{R})$ and $\varphi_k \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ $(k = 1, 2 \ldots, m)$. Suppose that $(\mathbb{H}_1)$-$(\mathbb{H}_2)$ hold. Then, the impulsive (ρ_k, ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s* (4) *is $\mathcal{UH}$ stable if*

$$\Big(\Delta_1\big[\mathbb{L}_1 + \mathbb{L}_2\Phi_1^* + \mathbb{L}_3\Phi_2^*\big] + \Delta_2\mathbb{N}_1\Big) < 1.$$

Proof. Suppose that $x \in \mathcal{PC}^{1-\gamma_k}_{\phi_k}$ is a solution of the problem (37). From Lemma (15) with Remark 4 (*ii*)–(*iii*), we obtain

$$\begin{cases} {}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\phi_k}_{t_k^+} z(t) = \lambda_k z(t) + \mathcal{F}_z(t) + x(t), \quad t \in \mathcal{J}_k \subseteq \mathcal{J}, t \neq t_k, \\ \Delta_{\rho_k}\mathcal{I}^{\rho_k(1-\gamma_k);\phi_k}_{t_k^+} z(t_k) = \varphi_k(z(t_k)) + x_k, \quad k = 1, 2, \ldots, m, \\ \displaystyle\sum_{i=0}^{m} \kappa_i z(\eta_i) = \sum_{j=0}^{n} \omega_{j\rho_j}\mathcal{I}^{\mu_j;\phi_j}_{t_j^+} z(\xi_j) + A, \quad \eta_i \in (t_i, t_{i+1}], \xi_j \in (t_j, t_{j+1}], \end{cases} \tag{44}$$

then the solution of (44) can be rewritten as

$$\begin{aligned} z(t) &= \frac{1}{\rho_k^{\frac{\alpha_k}{\rho_k}}}\int_{t_k}^{t} \Phi_{\phi_k}^{\frac{\alpha_k}{\rho_k}-1}(t,s)\mathbb{E}_{\frac{\alpha_k}{\rho_k},\frac{\alpha_k}{\rho_k}}\big(\lambda_k(\rho_k^{-1}\Phi_{\phi_k}(t,s))^{\frac{\alpha_k}{\rho_k}}\big)\mathcal{F}_z(s)\phi_k'(s)ds + \Bigg[\sum_{r=0}^{k-1}\frac{\mathcal{P}_k(r+1)}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s) \\ &\quad\times\mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r}+1-\gamma_r}\big(\lambda_r(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s))^{\frac{\alpha_r}{\rho_r}}\big)\mathcal{F}_z(s)\phi_r'(s)ds + \sum_{r=1}^{k}\varphi_r(z(t_r))\mathcal{P}_k(r)\Bigg]\frac{\Phi_{\phi_k}^{\gamma_k-1}(t,t_k)}{\rho_k^{\gamma_k-1}}\mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\big(\lambda_k(\rho_k^{-1}\Phi_{\phi_k}(t,t_k))^{\frac{\alpha_k}{\rho_k}}\big) \\ &\quad+\frac{1}{\Xi}\Bigg(\sum_{j=0}^{n}\frac{\omega_j}{\rho_j^{\frac{\alpha_j+\mu_j}{\rho_j}}}\int_{t_j}^{\xi_j}\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,s)\mathbb{E}_{\frac{\alpha_j}{\rho_j},\frac{\alpha_j+\mu_j}{\rho_j}}\big(\lambda_j(\rho_j^{-1}\Phi_{\phi_j}(\xi_j,s))^{\frac{\alpha_j}{\rho_j}}\big)\mathcal{F}_z(s)\phi_j'(s)ds + \sum_{j=0}^{n}\Bigg\{\Bigg[\sum_{r=0}^{j-1}\frac{\mathcal{P}_j(r+1)}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}} \\ &\quad\times\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)\mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r}+1-\gamma_r}\big(\lambda_r(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s))^{\frac{\alpha_r}{\rho_r}}\big)\mathcal{F}_z(s)\phi_r'(s)ds + \sum_{r=1}^{j}\varphi_r(z(t_r))\mathcal{P}_j(r)\Bigg] \\ &\quad\times\frac{\omega_j\Phi_{\phi_j}^{\gamma_j+\frac{\mu_j}{\rho_j}-1}(\xi_j,t_j)}{\rho_j^{\frac{\mu_j}{\rho_j}+\gamma_j-1}}\mathbb{E}_{\frac{\alpha_j}{\rho_j},\gamma_j+\frac{\mu_j}{\rho_j}}\big(\lambda_j(\rho_j^{-1}\Phi_{\phi_j}(\xi_j,t_j))^{\frac{\alpha_j}{\rho_j}}\big)\Bigg\} + A - \sum_{i=0}^{m}\frac{\kappa_i}{\rho_i^{\frac{\alpha_i}{\rho_i}}}\int_{t_i}^{\eta_i}\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}-1}(\eta_i,s)\mathbb{E}_{\frac{\alpha_i}{\rho_i},\frac{\alpha_i}{\rho_i}}\big(\lambda_i(\rho_i^{-1}\Phi_{\phi_i}(\eta_i,s))^{\frac{\alpha_i}{\rho_i}}\big) \\ &\quad\times\mathcal{F}_z(s)\phi_r'(s)ds - \sum_{i=0}^{m}\Bigg\{\Bigg[\sum_{r=0}^{i-1}\frac{\mathcal{P}_i(r+1)}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)\mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r}+1-\gamma_r}\big(\lambda_r(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s))^{\frac{\alpha_r}{\rho_r}}\big)\mathcal{F}_z(s)\phi_r'(s)ds \\ &\quad+\sum_{r=1}^{i}\varphi_r(z(t_r))\mathcal{P}_i(r)\Bigg]\frac{\kappa_i\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\rho_i^{\gamma_i-1}}\mathbb{E}_{\frac{\alpha_i}{\rho_i},\gamma_i}\big(\lambda_i(\rho_i^{-1}\Phi_{\phi_i}(\eta_i,t_i))^{\frac{\alpha_i}{\rho_i}}\big)\Bigg\}\Bigg)\frac{\mathcal{P}_k(0)\Phi_{\phi_k}^{\gamma_k-1}(t,t_k)}{\rho_k^{\gamma_k-1}}\mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\big(\lambda_k(\rho_k^{-1}\Phi_{\phi_k}(t,t_k))^{\frac{\alpha_k}{\rho_k}}\big) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\rho_k^{\frac{\alpha_k}{\rho_k}}}\int_{t_k}^{t}\Phi_{\phi_k}^{\frac{\alpha_k}{\rho_k}-1}(t,s)\mathbb{E}_{\frac{\alpha_k}{\rho_k},\frac{\alpha_k}{\rho_k}}\big(\lambda_k\big(\rho_k^{-1}\Phi_{\phi_k}(t,s)\big)^{\frac{\alpha_k}{\rho_k}}\big)x(s)\phi_k'(s)ds+\Bigg[\sum_{r=0}^{k-1}\frac{\mathcal{P}_k(r+1)}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)\\
&\times\mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r}+1-\gamma_r}\big(\lambda_r\big(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s)\big)^{\frac{\alpha_r}{\rho_r}}\big)x(s)\phi_r'(s)ds+\sum_{r=1}^{k}x_r\mathcal{P}_k(r)\Bigg]\frac{\Phi_{\phi_k}^{\gamma_k-1}(t,t_k)}{\rho_k^{\gamma_k-1}}\mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\big(\lambda_k\big(\rho_k^{-1}\Phi_{\phi_k}(t,t_k)\big)^{\frac{\alpha_k}{\rho_k}}\big)\\
&+\frac{1}{\Xi}\Bigg(\sum_{j=0}^{n}\frac{\omega_j}{\rho_j^{\frac{\alpha_j+\mu_j}{\rho_j}}}\int_{t_j}^{\xi_j}\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,s)\mathbb{E}_{\frac{\alpha_j}{\rho_j},\frac{\alpha_j+\mu_j}{\rho_j}}\big(\lambda_j\big(\rho_j^{-1}\Phi_{\phi_j}(\xi_j,s)\big)^{\frac{\alpha_j}{\rho_j}}\big)x(s)\phi_j'(s)ds+\sum_{j=0}^{n}\Bigg\{\Bigg[\sum_{r=0}^{j-1}\frac{\mathcal{P}_j(r+1)}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}}\\
&\times\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)\mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r}+1-\gamma_r}\big(\lambda_r\big(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s)\big)^{\frac{\alpha_r}{\rho_r}}\big)x(s)\phi_r'(s)ds+\sum_{r=1}^{j}x_r\mathcal{P}_j(r)\Bigg]\frac{\omega_j\Phi_{\phi_j}^{\gamma_j+\frac{\mu_j}{\rho_j}-1}(\xi_j,t_j)}{\rho_j^{\frac{\mu_j}{\rho_j}+\gamma_j-1}}\\
&\times\mathbb{E}_{\frac{\alpha_j}{\rho_j},\gamma_j+\frac{\mu_j}{\rho_j}}\big(\lambda_j\big(\rho_j^{-1}\Phi_{\phi_j}(\xi_j,t_j)\big)^{\frac{\alpha_j}{\rho_j}}\big)\Bigg\}-\sum_{i=0}^{m}\frac{\kappa_i}{\rho_i^{\frac{\alpha_i}{\rho_i}}}\int_{t_i}^{\eta_i}\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}-1}(\eta_i,s)\mathbb{E}_{\frac{\alpha_i}{\rho_i},\frac{\alpha_i}{\rho_i}}\big(\lambda_i\big(\rho_i^{-1}\Phi_{\phi_i}(\eta_i,s)\big)^{\frac{\alpha_i}{\rho_i}}\big)x(s)\phi_r'(s)ds\\
&-\sum_{i=0}^{m}\Bigg\{\Bigg[\sum_{r=0}^{i-1}\frac{\mathcal{P}_i(r+1)}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)\mathbb{E}_{\frac{\alpha_r}{\rho_r},\frac{\alpha_r}{\rho_r}+1-\gamma_r}\big(\lambda_r\big(\rho_r^{-1}\Phi_{\phi_r}(t_{r+1},s)\big)^{\frac{\alpha_r}{\rho_r}}\big)x(s)\phi_r'(s)ds+\sum_{r=1}^{i}x_r\mathcal{P}_i(r)\Bigg]\\
&\times\frac{\kappa_i\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\rho_i^{\gamma_i-1}}\mathbb{E}_{\frac{\alpha_i}{\rho_i},\gamma_i}\big(\lambda_i\big(\rho_i^{-1}\Phi_{\phi_i}(\eta_i,t_i)\big)^{\frac{\alpha_i}{\rho_i}}\big)\Bigg\}\Bigg)\frac{\mathcal{P}_k(0)\Phi_{\phi_k}^{\gamma_k-1}(t,t_k)}{\rho_k^{\gamma_k-1}}\mathbb{E}_{\frac{\alpha_k}{\rho_k},\gamma_k}\big(\lambda_k\big(\rho_k^{-1}\Phi_{\phi_k}(t,t_k)\big)^{\frac{\alpha_k}{\rho_k}}\big).
\end{aligned}
\tag{45}
$$

By applying Lemma 3 with $|\mathcal{P}_a(b)|\le 1$, for any $t\in\mathcal{J}$, we obtain that

$$
\begin{aligned}
&\left|\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)\big(z(t)-u(t)\big)\right|\\
\le\;&\frac{\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)}{\rho_k^{\frac{\alpha_k}{\rho_k}}\Gamma\left(\frac{\alpha_k}{\rho_k}\right)}\int_{t_k}^{t}\Phi_{\phi_k}^{\frac{\alpha_k}{\rho_k}-1}(t,s)|\mathcal{F}_z(s)-\mathcal{F}_u(s)|\phi_k'(s)ds+\frac{1}{\rho_k^{\gamma_k-1}\Gamma(\gamma_k)}\\
&\times\Bigg[\sum_{r=0}^{k-1}\frac{1}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}\Gamma\left(\frac{\alpha_r}{\rho_r}+1-\gamma_r\right)}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)|\mathcal{F}_z(s)-\mathcal{F}_u(s)|\phi_r'(s)ds\\
&+\sum_{r=1}^{k}|\varphi_r(z(t_r))-\varphi_r(z(t_r))|\Bigg]+\frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n}\frac{|\omega_j|}{\rho_j^{\frac{\alpha_j+\mu_j}{\rho_j}}\Gamma\left(\frac{\alpha_j+\mu_j}{\rho_j}\right)}\int_{t_j}^{\xi_j}\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,s)\\
&\times|\mathcal{F}_z(s)-\mathcal{F}_u(s)|\phi_j'(s)ds+\sum_{j=0}^{n}\frac{|\omega_j|\Phi_{\phi_j}^{\gamma_j+\frac{\mu_j}{\rho_j}-1}(\xi_j,t_j)}{\rho_j^{\frac{\mu_j}{\rho_j}+\gamma_j-1}\Gamma\left(\gamma_j+\frac{\mu_j}{\rho_j}\right)}\Bigg[\sum_{r=0}^{j-1}\frac{1}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}\Gamma\left(\frac{\alpha_r}{\rho_r}+1-\gamma_r\right)}\\
&\times\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)|\mathcal{F}_z(s)-\mathcal{F}_u(s)|\phi_r'(s)ds+\sum_{r=1}^{j}|\varphi_r(z(t_r))-\varphi_r(u(t_r))|\Bigg]\\
&+\sum_{i=0}^{m}\frac{|\kappa_i|}{\rho_i^{\frac{\alpha_i}{\rho_i}}\Gamma\left(\frac{\alpha_i}{\rho_i}\right)}\int_{t_i}^{\eta_i}\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}-1}(\eta_i,s)|\mathcal{F}_z(s)-\mathcal{F}_u(s)|\phi_r'(s)ds\\
&+\sum_{i=0}^{m}\Bigg\{\Bigg[\sum_{r=0}^{i-1}\frac{1}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}\Gamma\left(\frac{\alpha_r}{\rho_r}+1-\gamma_r\right)}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)|\mathcal{F}_z(s)-\mathcal{F}_u(s)|\phi_r'(s)ds\\
&+\sum_{r=1}^{i}|\varphi_r(z(t_r))-\varphi_r(u(t_r))|\Bigg]\frac{|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\rho_i^{\gamma_i-1}\Gamma(\gamma_i)}\Bigg\}\Bigg)\frac{1}{\rho_k^{\gamma_k-1}\Gamma(\gamma_k)}\\
&+\frac{1}{\rho_k^{\frac{\alpha_k}{\rho_k}}\Gamma\left(\frac{\alpha_k}{\rho_k}\right)}\int_{t_k}^{t}\Phi_{\phi_k}^{\frac{\alpha_k}{\rho_k}-1}(t,s)|x(s)|\phi_k'(s)ds+\frac{1}{\rho_k^{\gamma_k-1}\Gamma(\gamma_k)}\Bigg[\sum_{r=0}^{k-1}\frac{1}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}\Gamma\left(\frac{\alpha_r}{\rho_r}+1-\gamma_r\right)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_{t_r}^{t_{r+1}} \Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)|x(s)|\phi_r'(s)ds + \sum_{r=1}^{k}|x_r|\Bigg] + \frac{1}{|\Xi|\rho_k^{\gamma_k-1}\Gamma(\gamma_k)}\Bigg(\sum_{j=0}^{n}\frac{|\omega_j|}{\rho_j^{\frac{\alpha_j+\mu_j}{\rho_j}}\Gamma\Big(\frac{\alpha_j+\mu_j}{\rho_j}\Big)} \\
&\times \int_{t_j}^{\xi_j}\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,s)|x(s)|\phi_j'(s)ds + \sum_{j=0}^{n}\frac{|\omega_j|\Phi_{\phi_j}^{\gamma_j+\frac{\mu_j}{\rho_j}-1}(\xi_j,t_j)}{\rho_j^{\frac{\mu_j}{\rho_j}+\gamma_j-1}\Gamma\Big(\gamma_j+\frac{\mu_j}{\rho_j}\Big)} \\
&\times\Bigg[\sum_{r=0}^{j-1}\frac{1}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}\Gamma\Big(\frac{\alpha_r}{\rho_r}+1-\gamma_r\Big)}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)|x(s)|\phi_r'(s)ds+\sum_{r=1}^{j}|x_r|\Bigg] \\
&+\sum_{i=0}^{m}\frac{|\kappa_i|}{\rho_i^{\frac{\alpha_i}{\rho_i}}\Gamma\Big(\frac{\alpha_i}{\rho_i}\Big)}\int_{t_i}^{\eta_i}\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}-1}(\eta_i,s)|x(s)|\phi_r'(s)ds+\sum_{i=0}^{m}\frac{|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\rho_i^{\gamma_i-1}\Gamma(\gamma_i)} \\
&\times\Bigg[\sum_{r=0}^{i-1}\frac{1}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}\Gamma\Big(\frac{\alpha_r}{\rho_r}+1-\gamma_r\Big)}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)|x(s)|\phi_r'(s)ds+\sum_{r=1}^{i}|x_r|\Bigg]\Bigg]\Bigg).
\end{aligned}
$$

Thanks to (i) of Remark 4 with ($\mathbb{H}_1$)–($\mathbb{H}_2$), we obtain the following result

$$
\begin{aligned}
&\Big|\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)\big(z(t)-u(t)\big)\Big| \\
\leq\ &\Bigg\{\Big[\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*\Big]\Bigg(\frac{\Phi_{\phi_m}^{\frac{\alpha_m}{\rho_m}-\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\alpha_m+\rho_m)}+\frac{1}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\Bigg[\sum_{r=0}^{m-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r+1}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))} \\
&+\frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n}\frac{|\omega_j|\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}}(\xi_j,t_j)}{\Gamma_{\rho_j}(\alpha_j+\mu_j+\rho_j)}+\sum_{j=0}^{n}\frac{|\omega_j|\Phi_{\phi_j}^{\frac{\mu_j}{\rho_j}+\gamma_j-1}(\xi_j,t_j)}{\Gamma_{\rho_j}(\rho_j\gamma_j+\mu_j)}\sum_{r=0}^{j-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r+1}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))} \\
&+\sum_{i=0}^{m}\frac{|\kappa_i|\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}}(\eta_i,t_i)}{\Gamma_{\rho_i}(\alpha_i+\rho_i)}+\sum_{i=0}^{m}\frac{|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\sum_{r=0}^{i-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r+1}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))}\Bigg)\Bigg]\Bigg) \\
&+\frac{1}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\Bigg[m+\frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n}\frac{j|\omega_j|\Phi_{\phi_j}^{\frac{\mu_j}{\rho_j}+\gamma_j-1}(\xi_j,t_j)}{\Gamma_{\rho_j}(\rho_j\gamma_j+\mu_j)}+\sum_{i=0}^{m}\frac{i|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\Bigg)\Bigg]\mathbb{N}_1\Bigg\} \\
&\times\|z-u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}+\Bigg\{\frac{\Phi_{\phi_m}^{\frac{\alpha_m}{\rho_m}-\gamma_m+1}(T,t_m)}{\Gamma_{\rho_m}(\alpha_m+\rho_m)}+\frac{1}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\Bigg[\sum_{r=0}^{m-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r+1}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))} \\
&+\frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n}\frac{|\omega_j|\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}}(\xi_j,t_j)}{\Gamma_{\rho_j}(\alpha_j+\mu_j+\rho_j)}+\sum_{j=0}^{n}\frac{|\omega_j|\Phi_{\phi_j}^{\frac{\mu_j}{\rho_j}+\gamma_j-1}(\xi_j,t_j)}{\Gamma_{\rho_j}(\rho_j\gamma_j+\mu_j)}\sum_{r=0}^{j-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r+1}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))} \\
&+\sum_{i=0}^{m}\frac{|\kappa_i|\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}}(\eta_i,t_i)}{\Gamma_{\rho_i}(\alpha_i+\rho_i)}+\sum_{i=0}^{m}\frac{|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\sum_{r=0}^{i-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r+1}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))}\Bigg)\Bigg] \\
&+\frac{\Phi_{\phi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\Bigg[m+\frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n}\frac{j|\omega_j|\Phi_{\phi_j}^{\frac{\mu_j}{\rho_j}+\gamma_j-1}(\xi_j,t_j)}{\Gamma_{\rho_j}(\rho_j\gamma_j+\mu_j)}+\sum_{i=0}^{m}\frac{i|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\Bigg)\Bigg]\Bigg\}\epsilon,
\end{aligned}
$$

which implies that

$$
\|z-u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}\leq\Big(\Delta_1\big[\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*\big]+\Delta_2\mathbb{N}_1\Big)\|z-u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}+\Big\{\Delta_1+\Delta_2\Big\}\epsilon.
$$

Then $\|z-u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}} \leq \mathfrak{C}_{\mathcal{F}}\epsilon$, where

$$\mathfrak{C}_{\mathcal{F}} := \frac{\Delta_1+\Delta_2}{1-\left(\Delta_1\left[\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*\right]+\Delta_2\mathbb{N}_1\right)}. \tag{46}$$

Therefore, the impulsive (ρ_k,ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (4) is $\mathcal{UH}$ stable in $\mathcal{B}$. □

Corollary 1. *Under conditions in Theorem 3, if $\mathcal{T}(\epsilon)=\mathfrak{C}_{\mathcal{F}}\epsilon$ so that $\mathcal{T}(0)=0$, then we have the impulsive (ρ_k,ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (4) becomes $\mathcal{GUH}$ stable.*

4.2. Ulam–Hyers–Rassias Stability Results

To analyze $\mathcal{UHR}$ stability results, we will need the following condition as follows:

($\mathbb{H}_7$) There exists a non-decreasing function $\mathcal{T}\in\mathcal{C}(\mathcal{J},\mathbb{R})$ and there is $\mathfrak{C}_{\mathcal{T}}>0$, for each $\epsilon>0$, such that the following inequality

$${}_{\rho_k}\mathcal{I}_{t_k^+}^{\alpha_k;\phi_k}\mathcal{T}(t)\leq \mathfrak{C}_{\mathcal{T}}\mathcal{T}(t). \tag{47}$$

Theorem 4. *Let $f\in(\mathcal{J}\times\mathbb{R}^3,\mathbb{R})$ and $\varphi_k\in\mathcal{C}(\mathbb{R},\mathbb{R})$, $(k=1,2\ldots,m)$. If ($\mathbb{H}_1$), ($\mathbb{H}_2$), ($\mathbb{H}_7$), and*

$$\left(\Delta_1\left[\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*\right]+\Delta_2\mathbb{N}_1\right)<1$$

are fulfilled. Then the impulsive (ρ_k,ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (4) is $\mathcal{UHR}$ stable with respect to $(\tau,\mathcal{T})$.

Proof. Let $z\in\mathcal{B}$ be any solution of (39) and $u\in\mathcal{B}$ be the solution of the problem (4). By the same process in Theorem 3, we have

$$\begin{aligned}
&\left|\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)\big(z(t)-u(t)\big)\right| \\
\leq\;& \frac{\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)}{\rho_k^{\frac{\alpha_k}{\rho_k}}\Gamma\left(\frac{\alpha_k}{\rho_k}\right)}\int_{t_k}^{t}\Phi_{\phi_k}^{\frac{\alpha_k}{\rho_k}-1}(t,s)|\mathcal{F}_z(s)-\mathcal{F}_u(s)|\phi_k'(s)ds+\frac{1}{\rho_k^{\gamma_k-1}\Gamma(\gamma_k)} \\
&\times\left[\sum_{r=0}^{k-1}\frac{1}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}\Gamma\left(\frac{\alpha_r}{\rho_r}+1-\gamma_r\right)}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)|\mathcal{F}_z(s)-\mathcal{F}_u(s)|\phi_r'(s)ds\right. \\
&\left.+\sum_{r=1}^{k}|\varphi_r(z(t_r))-\varphi_r(z(t_r))|\right]+\frac{1}{|\Xi|}\left(\sum_{j=0}^{n}\frac{|\omega_j|}{\rho_j^{\frac{\alpha_j+\mu_j}{\rho_j}}\Gamma\left(\frac{\alpha_j+\mu_j}{\rho_j}\right)}\int_{t_j}^{\xi_j}\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,s)\right. \\
&\times|\mathcal{F}_z(s)-\mathcal{F}_u(s)|\phi_j'(s)ds+\sum_{j=0}^{n}\frac{|\omega_j|\Phi_{\phi_j}^{\gamma_j+\frac{\mu_j}{\rho_j}-1}(\xi_j,t_j)}{\rho_j^{\frac{\mu_j}{\rho_j}+\gamma_j-1}\Gamma\left(\gamma_j+\frac{\mu_j}{\rho_j}\right)}\left[\sum_{r=0}^{j-1}\frac{1}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}\Gamma\left(\frac{\alpha_r}{\rho_r}+1-\gamma_r\right)}\right. \\
&\left.\times\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)|\mathcal{F}_z(s)-\mathcal{F}_u(s)|\phi_r'(s)ds+\sum_{r=1}^{j}|\varphi_r(z(t_r))-\varphi_r(u(t_r))|\right] \\
&+\sum_{i=0}^{m}\frac{|\kappa_i|}{\rho_i^{\frac{\alpha_i}{\rho_i}}\Gamma\left(\frac{\alpha_i}{\rho_i}\right)}\int_{t_i}^{\eta_i}\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}-1}(\eta_i,s)|\mathcal{F}_z(s)-\mathcal{F}_u(s)|\phi_r'(s)ds
\end{aligned}$$

$$+\sum_{i=0}^{m}\Bigg\{\Bigg[\sum_{r=0}^{i-1}\frac{1}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}\Gamma\Big(\frac{\alpha_r}{\rho_r}+1-\gamma_r\Big)}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)|\mathcal{F}_z(s)-\mathcal{F}_u(s)|\phi_r'(s)ds$$

$$+\sum_{r=1}^{i}|\varphi_r(z(t_r))-\varphi_r(u(t_r))|\Bigg]\frac{|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\rho_i^{\gamma_i-1}\Gamma(\gamma_i)}\Bigg\}\Bigg)\frac{1}{\rho_k^{\gamma_k-1}\Gamma(\gamma_k)}$$

$$+\frac{\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)}{\rho_k^{\frac{\alpha_k}{\rho_k}}\Gamma\Big(\frac{\alpha_k}{\rho_k}\Big)}\int_{t_k}^{t}\Phi_{\phi_k}^{\frac{\alpha_k}{\rho_k}-1}(t,s)|x(s)|\phi_k'(s)ds+\frac{1}{\rho_k^{\gamma_k-1}\Gamma(\gamma_k)}$$

$$\times\Bigg[\sum_{r=0}^{k-1}\frac{1}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}\Gamma\Big(\frac{\alpha_r}{\rho_r}+1-\gamma_r\Big)}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)|x(s)|\phi_r'(s)ds+\sum_{r=1}^{k}|x_r|\Bigg]$$

$$+\frac{1}{|\Xi|\rho_k^{\gamma_k-1}\Gamma(\gamma_k)}\Bigg(\sum_{j=0}^{n}\frac{|\omega_j|}{\rho_j^{\frac{\alpha_j+\mu_j}{\rho_j}}\Gamma\Big(\frac{\alpha_j+\mu_j}{\rho_j}\Big)}\int_{t_j}^{\xi_j}\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}-1}(\xi_j,s)|x(s)|\phi_j'(s)ds$$

$$+\sum_{j=0}^{n}\frac{|\omega_j|\Phi_{\phi_j}^{\gamma_j+\frac{\mu_j}{\rho_j}-1}(\xi_j,t_j)}{\rho_j^{\frac{\mu_j}{\rho_j}+\gamma_j-1}\Gamma\Big(\gamma_j+\frac{\mu_j}{\rho_j}\Big)}\Bigg[\sum_{r=0}^{j-1}\frac{1}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}\Gamma\Big(\frac{\alpha_r}{\rho_r}+1-\gamma_r\Big)}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)$$

$$\times|x(s)|\phi_r'(s)ds+\sum_{r=1}^{j}|x_r|\Bigg]+\sum_{i=0}^{m}\frac{|\kappa_i|}{\rho_i^{\frac{\alpha_i}{\rho_i}}\Gamma\Big(\frac{\alpha_i}{\rho_i}\Big)}\int_{t_i}^{\eta_i}\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}-1}(\eta_i,s)|x(s)|\phi_r'(s)ds$$

$$+\sum_{i=0}^{m}\frac{|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\rho_i^{\gamma_i-1}\Gamma(\gamma_i)}\Bigg[\sum_{r=0}^{i-1}\frac{1}{\rho_r^{\frac{\alpha_r}{\rho_r}+1-\gamma_r}\Gamma\Big(\frac{\alpha_r}{\rho_r}+1-\gamma_r\Big)}\int_{t_r}^{t_{r+1}}\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r}(t_{r+1},s)|x(s)|\phi_r'(s)ds$$

$$+\sum_{r=1}^{i}|x_r|\Bigg]\Bigg).$$

Thanks to (i) of Remark 6 with ($\mathbb{H}_1$), ($\mathbb{H}_2$), and ($\mathbb{H}_7$), we have the following result

$$\begin{aligned}
&\Big|\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)\big(z(t)-u(t)\big)\Big| \\
\le\ &\Bigg\{\Big[\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*\Big]\Bigg(\frac{\Phi_{\phi_m}^{\frac{\alpha_m}{\rho_m}-\gamma_m+1}(T,t_m)}{\Gamma_{\rho_m}(\alpha_m+\rho_m)}+\frac{1}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\Bigg[\sum_{r=0}^{m-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r+1}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))} \\
&+\frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n}\frac{|\omega_j|\Phi_{\phi_j}^{\frac{\alpha_j+\mu_j}{\rho_j}}(\xi_j,t_j)}{\Gamma_{\rho_j}(\alpha_j+\mu_j+\rho_j)}+\sum_{j=0}^{n}\frac{|\omega_j|\Phi_{\phi_j}^{\frac{\mu_j}{\rho_j}+\gamma_j-1}(\xi_j,t_j)}{\Gamma_{\rho_j}(\rho_j\gamma_j+\mu_j)}\sum_{r=0}^{j-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r+1}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))} \\
&+\sum_{i=0}^{m}\frac{|\kappa_i|\Phi_{\phi_i}^{\frac{\alpha_i}{\rho_i}}(\eta_i,t_i)}{\Gamma_{\rho_i}(\alpha_i+\rho_i)}+\sum_{i=0}^{m}\frac{|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\sum_{r=0}^{i-1}\frac{\Phi_{\phi_r}^{\frac{\alpha_r}{\rho_r}-\gamma_r+1}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r+\rho_r(2-\gamma_r))}\Bigg)\Bigg]\Bigg) \\
&+\frac{\Phi_{\phi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\Bigg[m+\frac{1}{|\Xi|}\Bigg(\sum_{j=0}^{n}\frac{j|\omega_j|\Phi_{\phi_j}^{\frac{\mu_j}{\rho_j}+\gamma_j-1}(\xi_j,t_j)}{\Gamma_{\rho_j}(\rho_j\gamma_j+\mu_j)}+\sum_{i=0}^{m}\frac{i|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\Bigg)\Bigg]\mathbb{N}_1\Bigg\} \\
&\times\|z-u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}+\Bigg\{\mathfrak{E}_{\mathcal{T}}\mathcal{T}(t)\Bigg[1+\frac{1}{|\Xi|\Gamma_{\rho_m}(\rho_m\gamma_m)}\Bigg(\sum_{j=0}^{n}|\omega_j|+\sum_{i=0}^{m}|\kappa_i|\Bigg)\Bigg]
\end{aligned}$$

$$+\frac{\Phi_{\phi_m}^{\gamma_m-1}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\left[m+\frac{1}{|\Xi|}\left(\sum_{j=0}^{n}\frac{j|\omega_j|\Phi_{\phi_j}^{\frac{\mu_j}{\rho_j}+\gamma_j-1}(\xi_j,t_j)}{\Gamma_{\rho_j}(\rho_j\gamma_j+\mu_j)}+\sum_{i=0}^{m}\frac{i|\kappa_i|\Phi_{\phi_i}^{\gamma_i-1}(\eta_i,t_i)}{\rho_i^{\gamma_i-1}\Gamma(\gamma_i)}\right)\right]$$
$$\times\left[\mathfrak{E}_{\mathcal{T}}\mathcal{T}(t)+\tau\right]\Bigg\}\epsilon,$$

which implies that

$$\begin{aligned}
&\|z-u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}} \\
\leq\;& \left(\Delta_1\left[\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*\right]+\Delta_2\mathbb{N}_1\right)\|z-u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}+\Big([\Delta_2+\Delta_3]\mathfrak{E}_{\mathcal{T}}\mathcal{T}(t)+\Delta_2\tau\Big)\epsilon \\
\leq\;& \left(\Delta_1\left[\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*\right]+\Delta_2\mathbb{N}_1\right)\|z-u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}+\Big([\Delta_2+\Delta_3]\mathfrak{E}_{\mathcal{T}}+\Delta_2\Big)\epsilon(\tau+\mathcal{T}(t)).
\end{aligned}$$

Then $\|z-u\|_{\mathcal{PC}_{\phi_k}^{1-\gamma_k}}\leq\mathfrak{C}_{\mathcal{F},\mathfrak{E}_{\mathcal{F}}}\epsilon(\tau+\mathcal{T}(t))$, with

$$\mathfrak{C}_{\mathcal{F},\mathfrak{E}_{\mathcal{F}}}:=\frac{[\Delta_2+\Delta_3]\mathfrak{E}_{\mathcal{T}}+\Delta_2}{1-\left(\Delta_1\left[\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*\right]+\Delta_2\mathbb{N}_1\right)}. \tag{48}$$

Hence, the impulsive (ρ_k,ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (4) is $\mathcal{UHR}$ stable with respect to $(\tau,\mathcal{T})$ in $\mathcal{B}$. □

Corollary 2. *Under conditions in Theorem 4, if $\epsilon=1$ so that $\mathcal{T}(0)=0$, then we have the impulsive (ρ_k,ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (4) becomes $\mathcal{GUHR}$ stable.*

5. Numerical Examples

This section provides some illustrative examples of the exactness and applicability of our main results.

Example 1. *Consider the following impulsive (ρ_k,ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s of the form:*

$$\begin{cases}
{}^{H}_{\frac{k+2}{2}}\mathfrak{D}_{t_k^+}^{\frac{7-3k}{9-3k},\frac{3k+2}{10};\phi_k}u(t)=-\dfrac{\sqrt{k^2+1}}{k+1}u(t)+f\Big(t,u(t),{}_{\frac{k+2}{2}}\mathcal{I}_{t_k}^{\frac{e^{-k}}{3};\phi_k}u(t),{}_{\frac{k+2}{2}}\mathcal{I}_{t_k}^{\sin^2\frac{\pi}{k+2};\phi_k}u(t)\Big),\\
\Delta\,{}_{\frac{k+2}{2}}\mathcal{I}_{t_k^+}^{\frac{k+2}{2}(1-\gamma_k);\phi_k}u(t_k)=\varphi_k(u(t_k)),\quad k=1,2\\
\displaystyle\sum_{i=0}^{2}\left(\frac{i+1}{3}\right)u\left(\frac{2i+1}{5}\right)=\sum_{j=0}^{1}\left(\frac{j+2}{2}\right){}_{\frac{j+2}{2}}\mathcal{I}_{t_j^+}^{\frac{8j+8}{10};\phi_j}u\left(\frac{2j+2}{9}\right)+2.
\end{cases} \tag{49}$$

Form the problem (49), we obtain that $\alpha_k=(7-3k)/(9-3k)$, $\beta_k=(3k+2)/10$, $\rho_k=(k+2)/2$, $\lambda_k=-\sqrt{k^2+1}/(k+1)$, $\delta_k=\exp(-k)/3$, $\theta_k=\sin^2(\pi/(k+2))$, $\phi_k(t)=\ln(t+k+2)/(2k+2)$, $t_k=2k/5$, $k=0,1,2$, $T=6/5$, $k=0,1,2$, $\kappa_i=(i+2)/3$, $\eta_i=(2i+1)/5$, $\omega_j=(j+2)/2$, $\mu_j=(8j+8)/10$, $\xi_j=(2j+2)/9$, $i=0,1,2$, $j=0,1$, and $A=2$. From the given all data, we can find that $\Xi\approx 3.350618628$, $\Delta_1\approx 0.7268674525$, $\Delta_2\approx 1.676600970$ and $\Delta_3\approx 0.8516099910$. The following functions will be considered for theoretical confirmation:

(i). Consider the functions

$$
\begin{aligned}
f(t,u(t),v(t),w(t)) &= \frac{5t^2-3}{2t^2+4}+\frac{(3-t)\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)}{7-2\sin^2\pi t}\cdot\frac{|u(t)|}{5+|u(t)|}\\
&\quad+\frac{3e^{-2t-1}\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)}{6-\cos^2\pi t}\cdot\frac{|v(t)|}{4+|v(t)|}\\
&\quad+\frac{4\sin 2\pi t\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)}{8-3e^{-t}}\cdot\frac{|w(t)|}{4+|w(t)|},\\
\varphi_k(u(t_k)) &= \frac{4\Phi_{\phi_k}^{1-\gamma_k}(t_{k+1},t_k)u(t_k)}{27-t_k}+3t_k.
\end{aligned}
$$

For u_i, v_i, $w_i\in\mathbb{R}$, $i=1,2$, and $t\in[0,1]$, we can find that

$$
\begin{aligned}
|f(t,u_1,v_1,w_1)-f(t,u_2,v_2,w_2)| &\le \frac{3}{25}|u_1-u_2|+\frac{3}{20}|v_1-v_2|+\frac{1}{5}|v_1-v_2|,\\
|\varphi_k(u_1)-\varphi_k(u_2)| &\le \frac{4}{25}|u_1-u_2|.
\end{aligned}
$$

The assumption $(\mathbb{H}_1)$–$(\mathbb{H}_2)$ are satisfied with $\mathbb{L}_1=3/5$, $\mathbb{L}_2=3/20$, $\mathbb{L}_3=1/5$ and $\mathbb{N}_1=4/25$. Hence,

$$\Delta_1(\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*)+\Delta_2\mathbb{N}_1\approx 0.6598729041<1.$$

Since, all the conditions of Theorem 1 are fulfilled. Then, the impulsive (ρ_k,ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (49) has a unique solution on $[0,1]$. Moreover, by Theorem 3, we also find that

$$\mathfrak{C}_{\mathcal{F}}:=\frac{\Delta_1+\Delta_2}{1-\Big(\Delta_1\big[\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*\big]+\Delta_2\mathbb{N}_1\Big)}\approx 7.066383275>0$$

Then, the impulsive (ρ_k,ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (49) is $\mathcal{UH}$ stable on $[0,1]$. Taking $\mathcal{T}(\epsilon)=\mathfrak{C}_{\mathcal{F}}\epsilon$ via $\mathcal{T}(0)=0$, then, by Corollary 1, the following impulsive (ρ_k,ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (49) is $\mathcal{GUH}$ stable on $[0,1]$. Taking $\mathcal{T}(t)=\Phi_{\phi_k}(t,t_k)$ with $\tau=1$, we obtain

$${}_{\rho_k}\mathcal{I}_{t_k^+}^{\alpha_k;\phi_k}\mathcal{T}(t)=\frac{\Phi_{\phi_k}^{\frac{\alpha_k}{\rho_k}}(t,t_k)}{\rho_k\Gamma_{\rho_k}(\alpha_k)}\Phi_{\phi_k}(t,t_k)\le\frac{\Phi_{\phi_m}^{\frac{\alpha_m}{\rho_m}}(T,t_m)}{\rho_m\Gamma_{\rho_m}(\alpha_m)}\mathcal{T}(t)$$

From $(\mathbb{H}_7)$, we obtain

$$\mathfrak{C}_{\mathcal{T}}=\frac{\Phi_{\phi_m}^{\frac{\alpha_m}{\rho_m}}(T,t_m)}{\rho_m\Gamma_{\rho_m}(\alpha_m)}\approx 0.07794511398>0.$$

Then,

$$\mathfrak{C}_{\mathcal{F},\mathfrak{C}_{\mathcal{F}}}:=\frac{[\Delta_2+\Delta_3]\mathfrak{C}_{\mathcal{T}}+\Delta_2}{1-\Big(\Delta_1\Big[\mathbb{L}_1+\mathbb{L}_2\Phi_1^*+\mathbb{L}_3\Phi_2^*\Big]+\Delta_2\mathbb{N}_1\Big)}\approx 5.508713313>0.$$

Hence, by all conditions in Theorem 4, the following impulsive (ρ_k,ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (49) is $\mathcal{UHR}$ stable on $[0,1]$. Moreover, if $\mathcal{T}(\epsilon)=\mathfrak{C}_{\mathcal{F},\mathfrak{C}_{\mathcal{F}}}\epsilon$ with $\mathcal{T}(0)=0$, then, by Corollary 2, the impulsive (ρ_k,ϕ_k)-Hilfer $\mathbb{FIDE}$-$\mathbb{NMP}$-$\mathbb{FIBC}$s (49) is $\mathcal{GUHR}$ stable with respect to $(\tau,\mathcal{T})$.

(ii). Consider the functions

$$
\begin{aligned}
f(t,u(t),v(t),w(t)) &= \frac{(2t-1)\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)}{12-2\cos\pi t}\left[\sin u(t)+\frac{2|u(t)|}{3+|u(t)|}\right] \\
&\quad +\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)\left[\frac{3\cos\pi t}{4+e^t}\cdot\frac{|v(t)|}{5+|v(t)|}+\frac{6e^{-t}}{10\ln(t+e)}\cdot\frac{|w(t)|}{4+|w(t)|}\right], \\
\varphi_k(u(t_k)) &= \frac{(6-t^2)\Phi_{\phi_k}^{1-\gamma_k}(t,t_k)}{5e^t}\cdot\frac{|u(t_k)|}{5+4|u(t_k)|}.
\end{aligned}
$$

For u, v, $w\in\mathbb{R}$, $i=1,2$, and $t\in[0,1]$, we can find that

$$
\begin{aligned}
|f(t,u,v,w)| &\le \frac{2t-1}{12-2\cos\pi t}(|u|+2)+\frac{3\cos\pi t}{20+5e^t}|v|+\frac{6e^{-t}}{40\ln(t+e)}|w|, \\
|\varphi_k(u)-\varphi_k(v)| &\le \frac{6}{25}|u-v|, \qquad |\varphi_k(u)|\le\frac{6}{20}.
\end{aligned}
$$

The assumption $(\mathbb{H}_3)$–$(\mathbb{H}_6)$ are satisfied with $\psi(|u|)=|u|+2$, $p_1^*=0.2$, $p_2^*=0.12$, $p_3^*=0.15$, $\mathbb{M}_1=0.3$ and $b_1=0.24$. Hence, $\Delta_2\Phi_{\phi_m}^{\gamma_m-1}(T,t_m)b_1\approx 0.8262520217<1$, $(1-[\Delta_1(p_2^*\Phi_1^*+p_3^*\Phi_2^*)])^{-1}\approx 1.31318002$, and

$$
\sup_{\Upsilon_2\in(0,\infty)}\frac{\Upsilon_2}{\Delta_1 p_1^*\psi(\Upsilon_2)+\Delta_2\mathbb{M}_1+\frac{|A|}{|\Xi|\Gamma_{\rho_m}(\rho_m\gamma_m)}}\approx 6.878833250.
$$

Since, all the problem (49) has at least one solution on $[0,1]$.

(iii). Consider the functions $f(t,u(t),v(t),w(t))=0$ and $\varphi_k(u(t_k))=7-4k$. By using (14), (22), (23), (24), the numerical values of Ξ, Δ_i, $i=1,2,3$, for $\alpha_k\in\{0.2,0.4,0.6,0.8,1.0\}$, as shown in Table 1. From Lemma 7, we obtain the implicit solutions of the problem (49), as shown in Figure 1, via fixed values of $\phi_k(t)=\frac{\ln(t+k+2)}{2k+2}$, $\beta_k=\frac{3k+2}{10}$, and $\rho_k=\frac{k+2}{2}$ with vary $\alpha_k\in\{0.2,0.4,0.6,0.8,1.0\}$ for $k=0,1,2$. By using (14), (22), (23), (24), the numerical values of Ξ, Δ_i, $i=1,2,3$, for $\alpha_k\in\{0.2,0.4,0.6,0.8,1.0\}$, as shown in Table 2. From Lemma 7, we obtain the implicit solutions of the problem (49) as shown in Figure 2 via fixed values of $\phi_k(t)=\frac{\ln(t+k+2)}{2k+2}$, $\beta_k=\frac{3k+2}{10}$, and $\rho_k=\frac{k+2}{2}$ with vary $\alpha_k\in\{0.2,0.4,0.6,0.8,1.0\}$ for $k=0,1,2$. Later, by using (14), (22), (23), (24), the numerical values of Ξ, Δ_i, $i=1,2,3$, for $\alpha_k\in\{0.2,0.4,0.6,0.8,1.0\}$, as shown in Table 3. From Lemma 7, we obtain the implicit solutions of the problem (49) as shown in Figure 3 via fixed values of $\phi_k(t)=\sin(\frac{\pi t+k+1}{(2+k)t+3k+3})$, $\beta_k=\cos(\frac{\pi}{5-k})$, and $\rho_k=(k+2)\tan(\frac{\pi}{6-k})$ with vary $\alpha_k\in\{0.2,0.4,0.6,0.8,1.0\}$ for $k=0,1,2$. In addition, we will show the implicit solutions of the problem (49) as shown in Figure 4 for each values of $\phi_k(t)$, α_k, β_k, ρ_k are given as in Table 4.

Table 1. Numerical values of Ξ and Δ_i, $i=1,2,3$, for $\alpha_k\in\{0.2,0.4,0.6,0.8,1.0\}$ when $\phi_k(t)=\frac{\ln(t+k+2)}{2k+2}$, $\beta_k=\frac{3k+2}{10}$, and $\rho_k=\frac{k+2}{2}$ for $k=0,1,2$.

α_k	Ξ	Δ_1	Δ_2	Δ_3
0.2	2.892662425	1.3437981140	2.058239791	0.8286028011
0.4	3.213267392	0.8843819924	1.946083874	0.8450705102
0.6	3.328008050	0.6154263942	1.917611370	0.8499260482
0.8	3.199108790	0.4512723574	1.962021647	0.8435009608
1.0	2.903673651	0.3448765974	2.059944134	0.8272976029

Table 2. Numerical values of Ξ and Δ_i, $i = 1, 2, 3$, for $\alpha_k \in \{0.2, 0.4, 0.6, 0.8, 1.0\}$ when $\phi_k(t) = t^{\sqrt{(k+1)^2+3}}$, $\beta_k = \frac{1}{\sqrt{k+2}}$, and $\rho_k = 3e^{-\sqrt{k+1}}$ for $k = 0, 1, 2$.

α_k	Ξ	Δ_1	Δ_2	Δ_3
0.2	0.979991112	4.377072500	3.467001554	0.6667371782
0.4	1.086173111	4.286940343	3.644959141	0.6463248146
0.6	1.192145946	4.039684022	3.713433921	0.6338233539
0.8	1.288330209	3.702297047	3.704677442	0.6253964583
1.0	1.366630096	3.320188403	3.641434555	0.6183465441

Table 3. Numerical values of Ξ and Δ_i, $i = 1, 2, 3$, for $\alpha_k \in \{0.2, 0.4, 0.6, 0.8, 1.0\}$ when $\phi_k(t) = \sin(\frac{\pi t+k+1}{(2+k)t+3k+3})$, $\beta_k = \cos(\frac{\pi}{5-k})$, and $\rho_k = (k+2)\tan(\frac{\pi}{6-k})$ for $k = 0, 1, 2$.

α_k	Ξ	Δ_1	Δ_2	Δ_3
0.2	2.706972884	5.693274732	1.322584648	0.7888666540
0.4	3.075302870	3.950732086	1.271388814	0.8122697082
0.6	3.448104601	2.748016737	1.239777125	0.8313127993
0.8	3.800110763	1.924362576	1.227618946	0.8461678966
1.0	4.087413705	1.363408014	1.236212063	0.8565872889

Table 4. The values of $\phi_k(t)$, α_k, β_k, ρ_k, and Ξ.

Case	$\phi_k(t)$	α_k	β_k	ρ_k	Ξ
I	$\frac{\ln(t+k+2)}{2k+2}$	$\frac{7-3k}{9-3k}$	$\frac{3k+2}{10}$	$\frac{k+2}{2}$	3.350618629
II	$t^{\sqrt{(k+1)^2+3}}$	$\frac{\sqrt{k+1}}{2}$	$\frac{1}{\sqrt{k+2}}$	$3\,e^{-\sqrt{k+1}}$	1.221677115
III	$\sin\left(\frac{\pi t+k+1}{(2+k)t+3k+3}\right)$	$\sin\left(\frac{\pi}{4-k}\right)$	$\cos\left(\frac{\pi}{5-k}\right)$	$(k+2)\tan\left(\frac{\pi}{6-k}\right)$	3.593850012

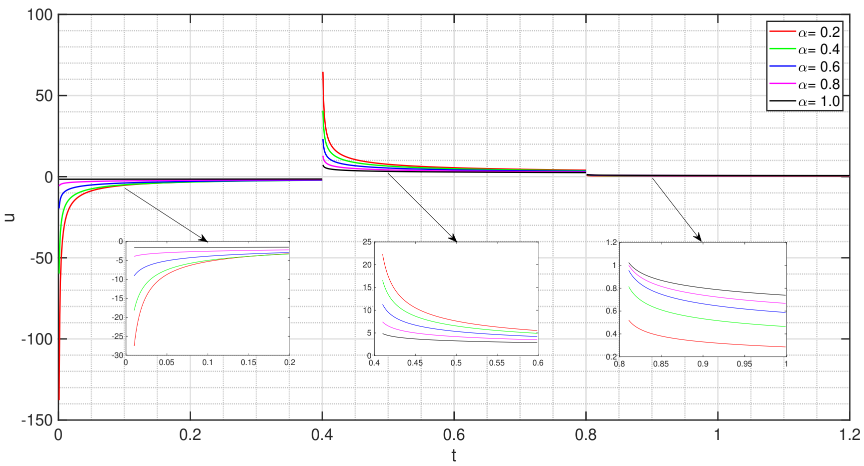

Figure 1. The implicit solutions of Example (49) via $\alpha_k \in \{0.2, 0.4, 0.6, 0.8, 1.0\}$ when $\phi_k(t) = \frac{\ln(t+k+2)}{2k+2}$, $\beta_k = \frac{3k+2}{10}$, and $\rho_k = \frac{k+2}{2}$ for $k = 0, 1, 2$.

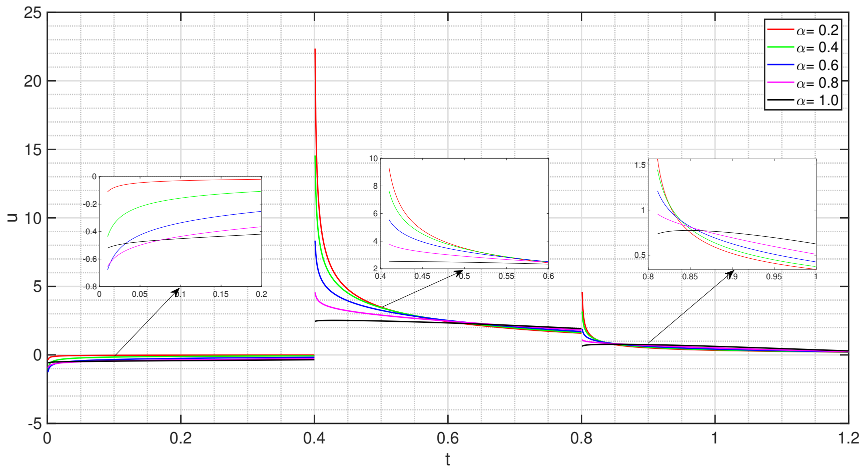

Figure 2. The implicit solutions of Example (49) via $\alpha_k \in \{0.2, 0.4, 0.6, 0.8, 1.0\}$ when $\phi_k(t) = t^{\sqrt{(k+1)^2+3}}$, $\beta_k = \frac{1}{\sqrt{k+2}}$, and $\rho_k = 3e^{-\sqrt{k+1}}$ for $k = 0, 1, 2$.

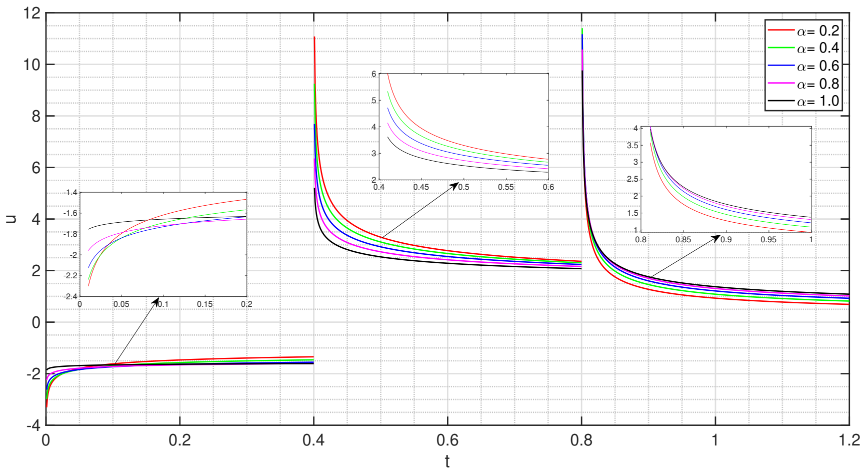

Figure 3. The implicit solutions of Example (49) via $\alpha_k \in \{0.2, 0.4, 0.6, 0.8, 1.0\}$ when $\phi_k(t) = \sin(\frac{\pi t+k+1}{(2+k)t+3k+3})$, $\beta_k = \cos(\frac{\pi}{5-k})$, and $\rho_k = (k+2)\tan(\frac{\pi}{6-k})$ for $k = 0, 1, 2$.

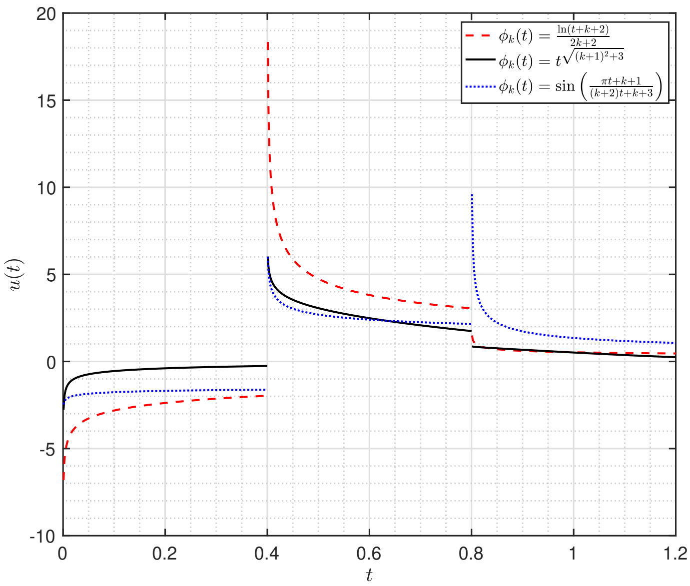

Figure 4. The implicit solutions of Example (49) via α_k, $\phi_k(t)$, β_k, and ρ_k for $k = 0, 1, 2$ as in Table 4.

6. Conclusions

In this paper, we studied the impulsive (ρ_k, ϕ_k)-Hilfer $\mathbb{FIDE}$s with a constant coefficient involving $\mathbb{NMP}$-$\mathbb{FIBC}$s. Firstly, we created some essential properties to apply to our main results. The formula of the solution to the linear (ρ, ϕ)-Hilfer fractional Cauchy problem was constructed in the form of the Mittag–Leffler kernel. The non-linear impulsive (ρ_k, ϕ_k)-Hilfer fractional Cauchy $\mathbb{BVP}$ was converted into a fixed-point problem via an auxiliary lemma regarding a linear variant of the problem. The uniqueness result was investigated by Banach's fixed point theorem, while the existence result was proved by a fixed point theorem due to O'Regan. In addition, by applying non-linear functional analysis methods and qualitative theory, a variety of $\mathcal{UH}$ stability, $\mathcal{UHR}$ stability, and their generalization are also examined. To confirm all the achieved theoretical results, numerical examples were given to present the application of our main results in the recent past. Apart from that, our main results are not only novel in the context of the impulsive problem at hand, but they also show some new special situations by adjusting the parameters involved. They have enriched the qualitative theory literature on non-linear impulsive (ρ_k, ϕ_k)-Hilfer $\mathbb{FIDE}$s of order in $(0, 1]$ equipped with $\mathbb{NMP}$-$\mathbb{FIBC}$s. In future work areas, we recommend working on the qualitative theory literature on non-linear fractional integro-differential equations/inclusions involving a special function, such as the linear Cauchy-type problem with variable coefficients, stability, or the algorithms to solve the (ρ, ϕ)-Hilfer fractional differential equations in mathematical software.

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Abbreviations

The following abbreviations are used in this manuscript:

$\mathbb{FC}$	Fractional calculus
$\mathbb{RL}$	Riemann–Liouville
$\mathbb{NMP}$	Non-local multi-point
$\mathbb{BVP}$	Boundary value problem
$\mathbb{FIO}$	Fractional integral operator
$\mathbb{FDO}$	Fractional derivative operator
$\mathbb{FDE}$	Fractional differential equation
$\mathbb{FIBC}$	Fractional integral boundary condition
$\mathbb{FIDE}$	Fractional integro-differential equation

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