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On the Classification of Telescopic Numerical Semigroups of Some Fixed Multiplicity

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Abstract: The telescopic numerical semigroups are a subclass of symmetric numerical semigroups widely used in algebraic geometric codes. Suer and Ilhan gave the classification of triply generated telescopic numerical semigroups up to multiplicity 10 and by using this classification they computed some important invariants in terms of the minimal system of generators. In this article, we extend the results of Suer and Ilhan for telescopic numerical semigroups of multiplicities 8 and 12 with embedding dimension four. Furthermore, we compute two important invariants namely the Frobenius number and genus for these classes in terms of the minimal system of generators.

Keywords: telescopic numerical semigroup; embedding dimension; multiplicity; genus; Frobenius number

MSC: 20M14; 20M30

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1. Introduction

In the beginning, the numerical semigroup theory was utilized in elementary number theory. Currently, it interacts in many fields such as commutative algebra, graph theory, algebraic geometry, combinatorics, coding theory, etc. The numerical semigroup is related to the problem of determining nonnegative integers $\mathbb{N}$ that can be expressed in the form $x_1a_1 + x_2a_2 + \ldots + x_ra_r$ for a given set $\{a_1, a_2, \ldots, a_r\}$ of positive integers and for arbitrary nonnegative integers $x_1, x_2, \ldots, x_r$. This problem was studied by many mathematicians such as Frobenius and Sylvester [1] at the end of the 19th century. Modern studies on the Frobenius problem started with Brauers and continues until today. During the second half of the twentieth century, interest in the study of numerical semigroups resurfaced because of their applications in algebraic geometry.

Assi et al. [2] discussed some important applications of a numerical semigroup in the solution of linear Diophantine equations, algebraic geometry, and the factorization of monoids. Bras-Amoros [3] presented some results on one-point codes related to numerical semigroups. In [4], Bras-Amoros proved that the sequence (ν_i) and the binary operation $\bigoplus$ uniquely determined the corresponding numerical semigroup. He used the concept of the (ν_i) sequence to improve the dimension of existing codes and drive bounds on the minimum distance. Bras-Amoros [5] proved that the $\bigoplus$ operation of the semigroups was important to define other classes of improved codes. Delgado et al. [6] introduced a GAP package for computations related to the numerical semigroup theory. In [7], Feng et al. presented a simple approach to constructing codes. Hoholdt et al. [8] provided a survey of the existing literature on the decoding of algebraic geometric codes. To study different concepts related to numerical semigroups and their applications in coding theory, the readers can see [9].

Let $\mathbb{N}$ be a set of nonnegative integers. A set $\mathrm{Y} \subset \mathbb{N}$ is said to be a numerical semigroup if it is closed under addition, $0 \in \mathbb{N}$, and $\mathbb{N} \setminus \mathrm{Y}$ is finite. The smallest positive integer that belongs to the set Y denoted by $m(\mathrm{Y})$ is called the multiplicity of Y. The elements of the set $\mathbb{N} \setminus \mathrm{Y}$ are called gaps. We denote the set $\mathbb{N} \setminus \mathrm{Y}$ by $G(\mathrm{Y})$ and call it a gap set of Y. The largest integer that belongs to $G(\mathrm{Y})$ is called the Frobenius number of Y and it is denoted by $\mathfrak{F}(\mathrm{Y})$. Frobenius asked how to find the largest b such that the Diophantine equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, where $a_1, a_2, \ldots, a_n, b \in \mathbb{N}$ has no solution over nonnegative integers. Since then, this problem is known as the Frobenius problem. More explicitly Frobenius's problem asks for a formula in terms of the minimal generating set for the largest element of the complement $\mathbb{N} \setminus \mathrm{Y}$. It is well-known that every numerical semigroup is finitely generated. We say that Y is generated by a set $S = \{s_1, s_2, \ldots, s_n\}$ if every $a \in \mathrm{Y}$ can be written as a linear combination of elements of S. In other words $a = a_1s_1 + a_2s_2 + \cdots + a_ns_n$, where $a_1, a_2, \ldots, a_n$ are nonnegative integers. We use the notation $\mathrm{Y} = \langle S \rangle = \langle s_1, s_2, \ldots, s_n \rangle$ if Y is generated by $s_1, s_2, \ldots, s_n$. If no proper subset of S generates Y, then we say that S is the minimal system of the generator of Y. Since the cancellation law holds in S, there always exists a unique minimal system of generators of Y. If S is the minimal system of the generator of Y, then the number of elements in set S denoted by $e(\mathrm{Y})$ is called the embedding dimension of Y. It is an easy observation that $e(\mathrm{Y}) \leq m(\mathrm{Y})$. For more details related to numerical semigroup, the readers can see the book by [10]. Let $s_1 < s_2 < \ldots < s_l$ be a sequence of positive integers such that $\gcd(s_1, s_2, \ldots, s_l) = 1$. Let $d_j = \gcd(s_1, s_2, \ldots, s_j)$ and $S_j = \{\frac{s_1}{d_j}, \frac{s_2}{d_j}, \ldots, \frac{s_j}{d_j}\}$ for $j = 1, 2, \ldots, l$. Assume that $d_0 = 0$ and Y_i is a semigroup generated by S_i. If $\frac{s_j}{d_j} \in S_{j-1}$ for $j = 1, 2, \ldots, l$, then the sequence $(s_1, s_2, \ldots, s_l)$ is called telescopic. The semigroup generated by a telescopic sequence is called the telescopic semigroup [11]. Let $\mathrm{Y} = \langle s_1, s_2, s_3 \rangle$ with $\gcd(s_1, s_2, s_3) = 1$. Then, Y is a triply generated telescopic semigroup if $s_3 \in \langle \frac{s_1}{d}, \frac{s_2}{d} \rangle$ where $d = \gcd(s_1, s_2)$ [12].

Kirfel and Pellikaan [11] showed that a proper subclass of a symmetric numerical semigroup was a class of telescopic numerical semigroup and they worked on the Feng–Rao distance. Garcia-Sanchez et al. [13] established a relationship between the second Feng–Rao number and the multiplicity of the telescopic numerical semigroup. Currently, telescopic numerical semigroups continue to be updated with applications in algebraic error-correcting codes. Sedat Ilhan [14] showed that a triply generated numerical semigroup $A = \langle a, a + 2, 2a + 1 \rangle$ with $a > 2$ an even integer was a telescopic numerical semigroup. In [15–17], Suer and Ilhan provided some classes of telescopic numerical semigroups with embedding dimension three and multiplicities 4, 6, 8, 9, and 10. They also calculated the Genus, Frobenius number, and Sylvester number in these cases. In this work, we characterize all numerical semigroups of embedding dimension four with multiplicities 8 and 12. Furthermore, explicit expressions are obtained to compute the Genus and Frobenius number by using the following Lemma.

Lemma 1. *Let* $\mathrm{Y} = \langle s_1, s_2, \ldots, s_n \rangle$ *be a numerical semigroup and* $d = \gcd(s_1, s_2, \ldots, s_{n-1})$. *Let* $S = \langle \frac{s_1}{d}, \frac{s_2}{d}, \ldots, \frac{s_{n-1}}{s}, s_n \rangle$, *then*

1. $\mathfrak{F}(\mathrm{Y}) = d\mathfrak{F}(S) + s_n(d-1)$.
2. $\mathfrak{g}(\mathrm{Y}) = s\mathfrak{F}(S) + \frac{(d-1)(s_n-1)}{2}$.

The paper is organized as follows: In the second section, we prove that if a telescopic numerical semigroup has embedding dimension four, then its multiplicity is at least the product of three primes. In light of this result, we give a complete characterization of telescopic numerical semigroups having embedding dimension four and multiplicity eight. Section 3 deals with the classification of telescopic numerical semigroups having embedding dimension four and multiplicity 12. In both cases, we give explicit expressions for the Frobenius number and Genus. In the end, the conclusion contains some open problems related to the study.

2. Telescopic Numerical Semigroup with Multiplicity Eight and Embedding Dimension Four

In this section, we give a characterization of numerical semigroups with embedding dimension four and multiplicity eight. In the following lemma, we give a condition on the multiplicity of a telescopic numerical semigroup with embedding dimension four.

Lemma 2. *Let* Y *be a telescopic numerical semigroup with embedding dimension four. Then, following conditions hold:*

1. $1 < d_2, d_3 < m(\mathrm{Y})$.
2. $d_2 > d_3$.
3. *The multiplicity of* Y *is the product of at least three prime numbers.*

Proof. We may assume that $\mathrm{Y} = \langle s_1, s_2, s_3, s_4 \rangle$. To prove (1), we need to show that $d_2, d_3 \notin \{1, m(\mathrm{Y})\}$. If $d_2 = 1$, then by the definition of a telescopic numerical semigroup $d_3 = 1$ and therefore $s_3 \in S_2 = \langle s_1, s_2 \rangle$. This implies an embedding dimension of Y is less than four, a contradiction. This gives $d_2 > 1$. Similar arguments give $d_3 > 1$. Now, if $d_2 = m(\mathrm{Y})$ (or $d_3 = m(\mathrm{Y})$) then s_2 is a multiple of s_1 (or s_3 is a multiple of s_1). This implies the embedding dimension of Y is less than four, which is again not possible.

To prove (2), we only need to show that $d_2 \neq d_3$, then, from definition of telescopic numerical semigroups, it follows that $d_2 > d_3$. Thus, if $d_2 = d_3$, then

$$\frac{s_3}{d_3} \in \langle \frac{s_1}{d_2}, \frac{s_2}{d_2} \rangle,$$

that is

$$\frac{s_3}{d_2} \in \langle \frac{s_1}{d_2}, \frac{s_2}{d_2} \rangle.$$

This implies $s_3 \in \langle s_1, s_2 \rangle$, which is not possible as the embedding dimension of Y is four.

To prove (3), we need to show that s_1 is neither a prime nor a product of two primes. If s_1 is prime, then clearly the embedding dimension of Y cannot be four. So let $s_1 = p_1 p_2$, where p_1 and p_2 are two prime numbers. From (1), we have $d_2 = p_1$ or $d_2 = p_2$. In both cases, we get $d_3 = 1$, which is a contradiction as $d_3 > 1$ (see (1)). Hence, s_1 must be the product of at least three prime numbers. □

Now, we give a classification of telescopic numerical semigroups with embedding dimension four and multiplicity eight. Furthermore, we compute the genus and the Frobenius number in terms of minimal set of generators.

Theorem 1. *Let* Y *be a numerical semigroup with embedding dimension four and multiplicity eight. Then,* Y *is telescopic if and only if* Y *is a member of one of the following families:*

1. $\alpha = \{\langle 8, 8e_1 + 4, 8e_2 + 2, 8e_2 + 2 + j \rangle : e_1, e_2 \in \mathbb{Z}^+, e_1 < e_2\}$, *where* j *is an odd integer.*
2. $\beta = \{\langle 8, 8e_1 + 4, 8e_2 + 6, 8e_2 + 6 + j \rangle : e_1, e_2 \in \mathbb{Z}^+, e_1 \leq e_2\}$, *where* j *is an odd integer.*

Proof. ($\Rightarrow$) Let $\mathrm{Y} = \langle 8, A, B, C \rangle$ be a telescopic numerical semigroup of embedding dimension four, then $d_2 = \gcd(8, A)$, $d_3 = \gcd(8, A, B)$. From (1) of Lemma 2, we have $d_2, d_3 \in \{2, 4\}$. Moreover, (2) of Lemma 2 gives $d_2 = 4$ and $d_3 = 2$. This implies $A = 8e_1 + 4$ and $B = 8e_2 + 2$ or $8e_2 + 6$, where $e_1, e_2 \in \mathbb{Z}^+$. If $A = 8e_1 + 4$ and $B = 8e_2 + 2$, then we may assume that $\mathrm{Y} = \langle 8, 8e_1 + 4, 8e_2 + 2, 8e_2 + 2 + j \rangle$ with $e_1 < e_2$. Since $\gcd(8, A, B, C) = 1$, $2 \nmid j$. As Y is telescopic and

$$\mathfrak{F}(\langle 4, 4e_1 + 2, 4e_2 + 1 \rangle) = 4e_1 + 4e_2 - 1 < 8e_2 + 2 + j,$$

this implies $8e_2 + 2 + j \in \langle 4, 4e_1 + 2, 4e_2 + 1 \rangle$ for all odd values of j and therefore, $\mathrm{Y} \in \alpha$. Now, if $A = 8e_1 + 4$ and $B = 8e_2 + 6$, then similar arguments imply $\mathrm{Y} \in \beta$.

($\Leftarrow$) Let $\mathrm{Y} \in \alpha$, then $\gcd(8, 8e_1+4) = 4$ and $\gcd(8, 8e_1+4, 8e_2+2) = 2$. Note that

$$\langle \frac{8}{4}, \frac{8e_1+4}{4} \rangle = \langle 2, 2e_1+1 \rangle.$$

Since $\mathfrak{F}(\langle 2, 2e_1+1 \rangle) = 2e_1 - 1 < 4e_2 + 1$ for some $e_1 < e_2$,

$$4e_2 + 1 \in \langle 2, 2e_1+1 \rangle.$$

Furthermore, $\langle \frac{8}{2}, \frac{8e_1+4}{2}, \frac{8e_2+2}{2} \rangle = \langle 4, 4e_1+2, 4e_2+1 \rangle$. Then, $\mathfrak{F}(\langle 4, 4e_1+2, 4e_2+1 \rangle) = 4e_2 + 4e_1 - 1 < 8e_2 + 2 + j$. This implies

$$8e_2 + 2 + j \in \langle 4, 4e_1+2, 4e_2+1 \rangle.$$

Hence, Y is telescopic.
Now, if $\mathrm{Y} \in \beta$ then $\gcd(8, 8e_1+4) = 4$ and $\gcd(8, 8e_1+4, 8e_2+6) = 2$. Since

$$\langle \frac{8}{4}, \frac{8e_1+4}{4} \rangle = \langle 2, 2e_1+1 \rangle,$$

$\mathfrak{F}(\langle 2, 2e_1+1 \rangle) = 2e_1 - 1 < 4e_2 + 3$ for some $e_1 \leq e_2$. This implies

$$4e_2 + 3 \in \langle 2, 2e_1+1 \rangle.$$

Furthermore, $\langle \frac{8}{2}, \frac{8e_1+4}{2}, \frac{8e_2+6}{2} \rangle = \langle 4, 4e_1+2, 4e_2+3 \rangle$. Then, $\mathfrak{F}(\langle 4, 4e_1+2, 4e_2+3 \rangle) = 4e_1 + 4e_2 + 1 < 8e_2 + 6 + j$. This implies

$$8e_2 + 6 + j \in \langle 4, 4e_1+2, 4e_2+3 \rangle.$$

Hence, Y is telescopic. □

Corollary 1. *Let* Y *be a telescopic numerical semigroup of multiplicity eight.*

1. *If* $\mathrm{Y} \in \alpha$, *then* $\mathfrak{F}(\mathrm{Y}) = 8e_1 + 8e_2 - 2 + x$ *and* $\mathfrak{g}(\mathrm{Y}) = 4e_1 + 8e_2 + \frac{j+1}{2}$.
2. *If* $\mathrm{Y} \in \beta$, *then* $\mathfrak{F}(\mathrm{Y}) = 8e_1 + 8e_3 + 2 + y$ *and* $\mathfrak{g}(\mathrm{Y}) = 4e_1 + 8e_2 + \frac{j+9}{2}$.

Proof. Let $\mathrm{Y} \in \alpha$, then $\gcd(8, 8e_1+4, 8e_2+2) = 2$ for $e_1 < e_2$. Consider

$$T = \langle \frac{8}{2}, \frac{8e_1+4}{2}, \frac{8e_2+2}{2} \rangle = \langle 4, 4e_1+2, 4e_2+1 \rangle.$$

By using Lemma 1, we get

$$\mathfrak{F}(T) = 2(2e_1 - 1) + (2-1)(4e_2+1) = 4e_1 + 4e_2 - 1.$$

This implies

$$\mathfrak{F}(\mathrm{Y}) = 2(4e_1 + 4e_2 - 1) + (2-1)(8e_2 + 2 + j) = 8(e_1 + 2e_2) + j.$$

Since $\mathfrak{g}(\mathrm{Y}) = \frac{\mathfrak{F}(\mathrm{Y})+1}{2}$, $\mathfrak{g}(\mathrm{Y}) = 4(e_1 + 2e_2) + \frac{j+1}{2}$. The remaining cases can be proved in a similar way. □

3. Telescopic Numerical Semigroup with Multiplicity 12 and Embedding Dimension Four

In this section, we classify all telescopic numerical semigroups with embedding dimension four and multiplicity 12. Furthermore, we compute the genus and Frobenius number for these classes in terms of the minimal set of generators.

Theorem 2. *Let* Y *be a numerical semigroup with embedding dimension four and multiplicity* 12. *Then,* Y *is telescopic if and only if* Y *is a member of one of the following families:*

1. $\alpha_1 = \{\langle 12, 12e_1+4, 12e_2+2, 12e_2+2+j\rangle : e_1, e_2 \in \mathbb{Z}^+, e_1 < e_2, j$ *is a positive odd integer and* $j \neq 3, 9, 15, \ldots, 12e_1 - 6e_2 - 3\}$.
2. $\alpha_2 = \{\langle 12, 12e_1+4, 12e_2+6, 12e_2+6+j\rangle : e_1, e_2 \in \mathbb{Z}^+, e_1 \leq e_2, j$ *is a positive odd integer and* $j \neq 1, 7, 13, \ldots, 12e_1 - 6e_2 - 5\}$.
3. $\alpha_3 = \{\langle 12, 12e_1+4, 12e_2+10, 12e_2+10+j\rangle : e_1, e_2 \in \mathbb{Z}^+, e_1 \leq e_2$, j *is a positive odd integer and* $j \neq 5, 11, 17, \ldots, 12e_1 - 6e_2 - 7\}$.
4. $\alpha_4 = \{\langle 12, 12e_1+8, 12e_2+2, 12e_2+2+j\rangle : e_1, e_2 \in \mathbb{Z}^+, e_1 < e_2, j$ *is a positive odd integer and* $j \neq 1, 7, 13, \ldots, 12e_1 - 6e_2 - 5\}$.
5. $\alpha_5 = \{\langle 12, 12e_1+8, 12e_2+6, 12e_2+6+j\rangle : e_1, e_2 \in \mathbb{Z}^+, e_1 < e_2, j$ *is a positive odd integer and* $j \neq 5, 11, 17, \ldots, 12e_1 - 6e_2 - 1\}$.
6. $\alpha_6 = \{\langle 12, 12e_1+8, 12e_3+10, 12e_3+10+j\rangle : e_1, e_2 \in \mathbb{Z}^+, e_1 \leq e_2, j$ *is a positive odd integer and* $j \neq 3, 9, 15, \ldots, 12e_1 - 6e_2 - 3\}$.
7. $\alpha_7 = \{\langle 12, 12e_1+6, 12e_2+2, 12e_2+2+j\rangle : e_1, e_2 \in \mathbb{Z}^+, e_1 < e_2, j$ *is a positive odd integer and* $j \neq 3, 9, 15, \ldots, 6e_1 - 3\}$.
8. $\alpha_8 = \{\langle 12, 12e_1+6, 12e_2+3, 12e_2+3+j\rangle : e_1, e_2 \in \mathbb{Z}^+, e_1 < e_2, j$ *is a positive odd integer and* $3 \nmid j\}$.
9. $\alpha_9 = \{\langle 12, 12e_1+6, 12e_2+4, 12e_2+4+j\rangle : e_1, e_2 \in \mathbb{Z}^+, e_1 < e_2, j$ *is a positive odd integer and* $j \neq 3, 9, 12, \ldots, 6e_1 - 3\}$.
10. $\alpha_{10} = \{\langle 12, 12e_1+6, 12e_3+8, 12e_3+8+j\rangle : e_1, e_2 \in \mathbb{Z}^+, e_1 \leq e_2$, j *is a positive odd integer and* $j \neq 3, 9, 12, \ldots, 6e_1 - 3\}$.
11. $\alpha_{11} = \{\langle 12, 12e_1+6, 12e_3+9, 12e_3+9+j\rangle : e_1, e_2 \in \mathbb{Z}^+, e_1 \leq e_2$, j *is a positive odd integer and* $3 \nmid j\}$.
12. $\alpha_{12} = \{\langle 12, 12e_1+6, 12e_2+10, 12e_2+10+j\rangle : e_1, e_2 \in \mathbb{Z}^+, e_1 \leq e_2$, j *is a positive odd integer and* $j \neq 3, 9, 12, \ldots, 6e_1 - 3\}$.

Proof. ($\Rightarrow$) Let $Y = \langle 12, A, B, C\rangle$ be a telescopic numerical semigroup of embedding dimension four, then $d_2 = \gcd(12, A)$, $d_3 = \gcd(12, A, B)$. From (1) of Lemma 2, we have $d_2, d_3 \in \{2, 3, 4, 6\}$. Moreover, (2) of Lemma 2 gives the following possibilities:

- $d_2 = 4$ and $d_3 = 2$.
- $d_2 = 6$ and $d_3 = 2$.
- $d_2 = 6$ and $d_3 = 3$.

If $d_2 = 4$ and $d_3 = 2$, then $A \in \{12e_1 + 4, 12e_1 + 8\}$ and $B \in \{12e_2 + 2, 12e_2 + 6, 12e_2 + 10\}$. Now, if $A = 12e_1 + 4$ and $B = 12e_2 + 2$, then we may assume that $Y = \{\langle 12, 12e_1 + 4, 12e_2 + 2, 12e_2 + 2 + j\rangle$ with $e_1 < e_2$. Since $\gcd(12, A, B, C) = 1$, j must be an odd integer. Note that

$$\mathfrak{F}(\langle 6, 6e_1 + 2, 6e_2 + 1\rangle) = 12e_1 + 6e_2 - 1.$$

If $j > 12e_1 - 6e_2 - 3$, then $12e_2 + 2 + j > \mathfrak{F}(\langle 6, 6e_1 + 2, 6e_2 + 1\rangle)$. This implies $12e_2 + 2 + j \in \langle 6, 6e_1 + 2, 6e_2 + 1\rangle$ for all odd values of j. Now, if $j < 12e_1 - 6e_2 - 3$, then $12e_2 + 2 + j < \mathfrak{F}(\langle 6, 6e_1 + 2, 6e_2 + 1\rangle)$. Since Y is telescopic, we can write

$$12e_2 + 2 + j = 6a + (6e_1 + 2)b + (6e_1 + 1)c,$$

where $a, b, c \geq 0$. Since j is odd, c is also odd. We have the following possible solutions:

(i) $e_2 < a \leq 2e_1 - 1$, $b = 0$, and $c = 1$.
(ii) $2e_2 - 2e_1 \leq a \leq e_1 - 1$, $b = 1$, and $c = 1$.

If $e_2 < a \leq 2e_1 - 1$, $b = 0$, and $c = 1$, then $j = 6(a - e_2) - 1$, and if $2e_2 - 2e_1 \leq a \leq e_1 - 1$, $b = 1$, and $c = 1$, then $j = 6(a + e_1 - e_2) + 1$. Both solutions give j as an odd integer that cannot be the multiple of three, i.e., if $12e_2 + 2 + j < \mathfrak{F}(\langle 6, 6e_1 + 2, 6e_2 + 1\rangle)$, then $12e_2 + 2 + j \in \langle 6, 6e_1 + 2, 6e_2 + 1\rangle$, when $j \neq 3, 6, 9, 12, 15, \ldots, 12e_1 - 6e_2 - 3$. This

implies $Y \in \alpha_1$. In a similar way, we can show that $Y \in \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ for the remaining possibilities of this case.

If $d_2 = 6$ and $d_3 = 2$, then $A = 12e_1 + 6$ and $B \in \{12e_2 + 2, 12e_2 + 4, 12e_2 + 8, 12e_2 + 10\}$. Now, if $A = 12e_1 + 6$ and $B = 12e_2 + 2$, then we may assume that $Y = \langle 12, 12e_1 + 6, 12e_2 + 2, 12e_2 + 2 + j\rangle$ with $e_1 < e_2$. Since $\gcd(12, A, B, C) = 1$, $2 \nmid j$. If $j > 6e_1 - 3$, then $12e_2 + 2 + j > \mathfrak{F}(\langle 6, 6e_1 + 3, 6e_2 + 1\rangle)$ and for $j < 6e_1 - 3$, we have $12e_2 + 2 + j < \mathfrak{F}(\langle 6, 6e_1 + 3, 6e_2 + 1\rangle)$. Since Y is telescopic, we can assume that

$$12e_2 + 2 + j = 6a + (6e_1 + 3)b + (6e_2 + 1)c,$$

and both b and c cannot be even or odd at the same time. As $12e_2 + 2 + j < \mathfrak{F}(\langle 6, 6e_1 + 3, 6e_2 + 1\rangle)$, we have the following:

(i) $a \leq e_1 + e_2 - 1, b = 0$, and $c = 1$.
(ii) $a \leq 2e_2 - 1, b = 1$, and $c = 0$.
(iii) $a \leq 2(e_2 - e_1 - 1), b = 3$, and $c = 0$.

All three solutions above imply $j \neq 3, 9, \ldots, 6e_1 - 3$ and therefore $Y \in \alpha_7$. The remaining possibilities give $Y \in \alpha_9$ or $Y \in \alpha_{10}$ or $Y \in \alpha_{12}$.

Now if $d_2 = 6$ and $d_3 = 3$, then $A = 12e_1 + 6$ and $B = 12e_2 + 3$ or $B = 12e_2 + 9$. If $A = 12e_1 + 6$ and $B = 12e_2 + 3$, then we may assume that $Y = \langle 12, 12e_1 + 6, 12e_2 + 3, 12e_2 + 3 + j\rangle$ with $e_1 < e_2$. Since $\gcd(12, A, B, C) = 1$, $3 \nmid j$. Note that

$$\mathfrak{F}(\langle 4, 4e_1 + 2, 4e_2 + 1\rangle) = 4e_1 + 4e_2 - 1 < 12e_2 + 3 + j.$$

Since Y is telescopic, $12e_2 + 3 + j \in \langle 4, 4e_1 + 2, 4e_2 + 1\rangle$ for all values of j except when $3 \nmid j$, therefore $Y \in \alpha_8$. Similarly if $A = 12e_1 + 6$ and $B = 12e_2 + 9$, then $Y \in \alpha_{11}$.

($\Leftarrow$) Let $Y \in \alpha_1$, then $\gcd(12, 12e_1 + 4) = 4$ and $\gcd(12, 12e_1 + 4, 12e_2 + 2) = 2$. Note that

$$\langle \frac{12}{4}, \frac{12e_1 + 4}{4}\rangle = \langle 3, 3e_1 + 1\rangle.$$

Since $\mathfrak{F}(\langle 3, 3e_1 + 1\rangle) = 6e_1 - 1 < 6e_2 + 1$ for some $e_1 < e_2$, therefore

$$6e_2 + 1 \in \langle 3, 3e_1 + 1\rangle.$$

Let $x = 12e_2 + 2 + j$, where j is an odd integer and $j \neq 3, 9, 12, \ldots, 12e_1 - 6e_2 - 3$. If $j > 12e_1 - 6e_2 - 3$ then we can write $j = 12e_1 - 6e_2 - 3 + 2k$ for $k \geq 1$. This gives

$$x = 12e_2 + 2 + 12e_1 - 6e_2 - 3 + 2k = 12e_1 + 6e_2 - 1 + 2k > \mathfrak{F}(\langle 6, 6e_1 + 3, 6e_2 + 1\rangle),$$

therefore $x \in \langle 6, 6e_1 + 3, 6e_2 + 1\rangle$. Now, if $j < 12e_1 - 6e_2 - 3$ then we can write $j = 12e_2 + 2 + 12e_1 - 3 - 2k$, where $k = 1 + 3q_1$ or $k = 2 + 3q_2$ for some integers q_1, q_2. If $k = 1 + 3q_1$, then $j = 6(2e_1 - e_2 - q_1 - 1) + 1$. Since $j > 0$, $2e_1 - e_2 - q_1 - 1 \geq 0$. This implies $2e_1 \geq e_2 + q_1 + 1$. As $e_2 > e_1$, $e_1 > q_1 + 1$. Therefore, $x = 6(e_1 - q_1 - 1) + (6e_1 + 2) + (6e_2 + 1)$. Since $e_1 > q_1 + 1$, $e_1 - q_1 - 1 > 0$. This gives $x \in \langle 6, 6e_1 + 2, 6e_2 + 1\rangle$. Now, if $k = 2 + 3q_2$, then $j = 6(2e_1 - e_2 - q_2 - 1) - 1$. Since $j > 0$, $2e_1 - e_2 - q_2 - 1 > 0$. So $x = 6(2e_1 - q_2 - 1) + (6e_2 + 1)$. Since $2e_1 - e_2 - q_2 - 1 > 0$, $2e_1 - q_2 - 1 > 0$. This gives $x \in \langle 6, 6e_1 + 2, 6e_2 + 1\rangle$. Consequently Y is telescopic. Cases (2) to (6) can be proved in a similar way.

Now, let $Y \in \alpha_7$; then, $\gcd(12, 12e_1 + 6) = 6$ and $\gcd(12, 12e_1 + 6, 12e_2 + 2) = 2$. Note that

$$\langle \frac{12}{6}, \frac{12e_1 + 6}{6}\rangle = \langle 2, 2e_1 + 1\rangle.$$

Since $\mathfrak{F}(\langle 2, 2e_1 + 1\rangle) = 2e_1 - 1 < 6e_2 + 1$ for some $e_1 < e_2$,

$$6e_2 + 1 \in \langle 2, 2e_1 + 1\rangle.$$

Let $x = 12e_2 + 2 + j$, where j is an odd integer and $j \neq 3, 9, 12, \ldots, 6e_1 - 3$. If $j > 6e_1 - 3$, then we can write $j = 6e_1 - 3 + 2k$ for $k \geq 1$. This gives

$$x = 12e_2 + 2 + 6e_1 - 3 + 2k = 12e_2 + 6e_1 - 1 + 2k > \mathfrak{F}(\langle 6, 6e_1 + 3, 6e_2 + 1\rangle),$$

therefore $x \in \langle 6, 6e_1 + 3, 6e_2 + 1\rangle$. Now, if $j < 6e_1 - 3$, then we can write $j = 6e_1 - 3 - 2k$, where either $k = 1 + 3q_1$ or $k = 2 + 3q_2$ for some integers q_1, q_2. If $k = 1 + 3q_1$, then $j = 6e_1 - 6q_1 - 6 + 1$. Since $j > 0$, $6e_1 - 6q_1 - 6 \geq 0$. This implies $e_1 \geq q_1 + 1$. As $e_2 > e_1$, $e_2 > q_1 + 1$. Now, $x = 12e_2 + 2 + 6e_1 - 6q_1 - 5 = 6(2e_2 - q_1 - 1) + (6e_1 + 3)$. Since $e_2 > q_1 + 1$, $2e_2 - q_1 - 1 > 0$. This gives $x \in \langle 6, 6e_1 + 3, 6e_2 + 1\rangle$. Now, if $k = 2 + 3q_2$, then $j = 6(e_1 - q_2 - 1) - 1$. Since $j > 0$, $e_1 - q_2 - 1 > 0$. This implies $e_1 + e_2 - q_2 - 1 > 0$. Now, we can write $x = 6(e_1 + e_2 - q_2 - 1) + (6e_2 + 1)$. This gives $x \in \langle 6, 6e_1 + 3, 6e_2 + 1\rangle$. Consequently Y is telescopic. Cases (9), (10), and (12) can be proved in a similar way.

Let $Y \in \alpha_8$; then, $\gcd(12, 12e_1 + 6) = 6$ and $\gcd(12, 12e_1 + 6, 12e_2 + 3) = 3$. Note that

$$\langle \frac{12}{6}, \frac{12e_1 + 6}{6} \rangle = \langle 2, 2e_1 + 1\rangle.$$

Since $\mathfrak{F}(\langle 2, 2e_1 + 1\rangle) = 2e_1 - 1 < 4e_2 + 1$ for some $e_1 < e_2$,

$$4e_2 + 1 \in \langle 2, 2e_1 + 1\rangle.$$

Furthermore, $\langle \frac{12}{3}, \frac{12e_1+6}{3}, \frac{12e_2+3}{3} \rangle = \langle 4, 4e_1 + 2, 4e_2 + 1\rangle$. Then, $\mathfrak{F}(\langle 4, 4e_1 + 2, 4e_2 + 1\rangle) = 4e_1 + 4e_2 - 1 < 12e_2 + 3 + j$. This implies

$$12e_2 + 2 + j \in \langle 4, 4e_1 + 2, 4e_2 + 1\rangle.$$

Hence, Y is telescopic. (11) can be proved in a similar way as we proved (8). □

Corollary 2. *Let* Y *be a telescopic numerical semigroup of multiplicity eight.*

1. *If* $Y \in \alpha_1$, *then* $\mathfrak{F}(Y) = 24(e_1 + e_2) + j$ *and* $\mathfrak{g}(Y) = 12(e_1 + e_2) + \frac{j+1}{2}$.
2. *If* $Y \in \alpha_2$, *then* $\mathfrak{F}(Y) = 24(e_1 + e_2) + j + 8$ *and* $\mathfrak{g}(Y) = 12(e_1 + e_2) + \frac{j+9}{2}$.
3. *If* $Y \in \alpha_3$, *then* $\mathfrak{F}(Y) = 24(e_1 + e_2) + j + 16$ *and* $\mathfrak{g}(Y) = 12(e_1 + e_2) + \frac{j+17}{2}$.
4. *If* $Y \in \alpha_4$, *then* $\mathfrak{F}(Y) = 24(e_1 + e_2) + j + 8$ *and* $\mathfrak{g}(Y) = 12(e_1 + e_2) + \frac{j+9}{2}$.
5. *If* $Y \in \alpha_5$, *then* $\mathfrak{F}(Y) = 24(e_1 + e_2) + j + 16$ *and* $\mathfrak{g}(Y) = 12(e_1 + e_2) + \frac{j+17}{2}$.
6. *If* $Y \in \alpha_6$, *then* $\mathfrak{F}(Y) = 24(e_1 + e_2) + j + 24$ *and* $\mathfrak{g}(Y) = 12(e_1 + e_2) + \frac{j+25}{2}$.
7. *If* $Y \in \alpha_7$, *then* $\mathfrak{F}(Y) = 12(e_1 + 3e_2) + j$ *and* $\mathfrak{g}(Y) = 6(e_1 + 3e_2) + \frac{j+1}{2}$.
8. *If* $Y \in \alpha_8$, *then* $\mathfrak{F}(Y) = 12e_1 + 18e_2 + j + 1$ *and* $\mathfrak{g}(Y) = 6e_1 + 9e_2 + \frac{j+1}{2}$.
9. *If* $Y \in \alpha_9$, *then* $\mathfrak{F}(Y) = 12(e_1 + 3e_2) + j + 8$ *and* $\mathfrak{g}(Y) = 6(e_1 + 3e_2) + \frac{j+9}{2}$.
10. *If* $Y \in \alpha_{10}$, *then* $\mathfrak{F}(Y) = 12(e_1 + 3e_2) + j + 18$ *and* $\mathfrak{g}(Y) = 6(e_1 + 3e_2) + \frac{j+19}{2}$.
11. *If* $Y \in \alpha_{11}$, *then* $\mathfrak{F}(Y) = 8e_1 + 20e_2 + j + 11$ *and* $\mathfrak{g}(Y) = 4e_1 + 10e_2 + \frac{j+12}{2}$.
12. *If* $Y \in \alpha_{12}$, *then* $\mathfrak{F}(Y) = 12(e_1 + 3e_2) + j + 24$ *and* $\mathfrak{g}(Y) = 6(e_1 + 3e_2) + \frac{j+25}{2}$.

Proof. Let $Y \in \alpha_1$; then, $\gcd(12, 12e_1 + 4, 12e_2 + 2) = 2$ for $e_1 < e_2$. Consider

$$T = \langle \frac{12}{2}, \frac{12e_1 + 4}{2}, \frac{12e_2 + 2}{2} \rangle = \langle 6, 6e_1 + 2, 6e_2 + 1\rangle.$$

By using Lemma 1, we get

$$\mathfrak{F}(T) = 2(6e_1 - 1) + (2 - 1)(6e_2 + 1) = 12e_1 + 6e_2 - 1.$$

This implies

$$\mathfrak{F}(Y) = 2(12e_1 + 6e_2 - 1) + (2 - 1)(12e_2 + 2 + j) = 24(e_1 + e_2) + j.$$

Since $\mathfrak{g}(Y) = \frac{\mathfrak{F}(Y)+1}{2}$, $\mathfrak{g}(Y) = 12(e_1 + e_2) + \frac{j+1}{2}$. The remaining cases can be proved in a similar way. □

Example 1. *Let* $Y = \langle 12, 40, 46, 46 + j\rangle$. *We want to find the values of j for which this numerical semigroup is telescopic. Since* $\gcd(12, 40) = 4$ *and* $\mathfrak{F}(\langle \frac{12}{4}, \frac{40}{4}\rangle) = 17 < \frac{46}{2} = 23$, *it follows that* $23 \in \langle 3, 10\rangle$. *Furthermore,* $\mathfrak{F}(\langle \frac{12}{2}, \frac{40}{2}, \frac{46}{2}\rangle) = 57$. *Now, we need to check for which values of j, the expression* $46 + j \in \langle \frac{12}{2}, \frac{40}{2}, \frac{46}{2}\rangle$ *holds. By definition of a numerical semigroup,* $\gcd(12, 40, 46, 46 + j) = 1$, *therefore j must be a positive odd integer. Note that* $46 + j \in \langle 6, 20, 23\rangle$ *for all* $j > 11$. *For* $j = 1$, $47 \in \langle 6, 20, 23\rangle$. *For* $j = 3$, $49 \in \langle 6, 20, 23\rangle$. *For* $j = 5$, $51 \notin \langle 6, 20, 23\rangle$. *For* $j = 7$, $53 \in \langle 6, 20, 23\rangle$. *For* $j = 9$, $55 \in \langle 6, 20, 23\rangle$. *For* $j = 11$, $57 \notin \langle 6, 20, 23\rangle$. *This shows that* Y *is a telescopic numerical semigroup for all positive odd values of j except* $j = 5, 11$. *Moreover,* $\mathfrak{F}(Y) = 160 + j$ *and* $\mathfrak{g}(Y) = 72 + \frac{j+17}{2}$.

4. Conclusions

Numerical semigroups are among the simplest objects to study, but they are involved in very hard problems. They have applications in many applied fields including cryptography, error-correcting codes, and combinatorial structures for privacy applications. In this work, we studied a couple of classes of the telescopic numerical semigroup of embedding dimension four. We proved that if a telescopic numerical semigroup had embedding dimension four, then its multiplicity was at least a multiple of three primes. The first two classes among them were the telescopic numerical semigroups with embedding dimension four and multiplicities 8 and 12. We gave a complete classification of telescopic numerical semigroups for these two classes. In the future, one can characterize the numerical semigroups of embedding dimension four and multiplicities 16 and 18.

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