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Some New Existence Results for Positive Periodic Solutions to First-Order Neutral Differential Equations with Variable Coefficients

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Abstract: In this article, we deal with some new existence results for positive periodic solutions for a class of neutral functional differential equations by employing Krasnoselskii's fixed-point theorem and the properties of a neutral operator. Our results generalize corresponding works from the past. An example is given to show the feasibility and application of the obtained results.

Keywords: first-order; fixed point theorem; existence; positive periodic solution

MSC: 34A33

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1. Introduction

In [1], the authors studied the existence of positive periodic solutions to the following first-order neutral differential equation:

$$\frac{d}{dt}[x(t) - cx(t-\tau(t))] = -a(t)x(t) + f(t, x(t-\tau(t))), \tag{1}$$

where $a \in C(\mathbb{R}, (0, \infty))$, $\tau \in C(\mathbb{R}, \mathbb{R})$, $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $|c| < 1$ is a constant. We will now list the main results for Equation (1):

Theorem 1. *Assume that $c \in [0, 1)$ and that there exist nonnegative constants m and M such that*

$$(1-c)m \le F(t, x) \le (1-c)M \text{ for } \forall t \in [0, \omega], x \in [m, M],$$

where $F(t, x) = \frac{f(t,x)}{a(t)} - cx$. Then, Equation (1) has at least one positive ω-periodic solution $x(t) \in (m, M]$.

Theorem 2. *Assume that $c \in (-1, 0)$ and that there exist nonnegative constants m and M such that*

$$m - cM \le F(t, x) \le M - cm \text{ for } \forall t \in [0, \omega], x \in [m, M],$$

where $F(t, x) = \frac{f(t,x)}{a(t)} - cx$. Then, Equation (1) has at least one positive ω-periodic solution $x(t) \in (m, M]$.

After that, Candan [2] considered the following first-order neutral differential equation:

$$\frac{d}{dt}[x(t) - P(t)x(t-\tau)] = -Q(t)x(t) + f(t, x(t-\tau)), \tag{2}$$

where $Q \in C(\mathbb{R}, (0, \infty))$, $P \in C^1(\mathbb{R}, \mathbb{R})$, $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\tau > 0$ is a constant. The main contribution of [2] is that Equation (2) has at least one positive ω-periodic solution when $|c| \neq 1$, which generalizes the results of [1]. We will now list the main results for Equation (2):

Theorem 3. *Assume that $1 < p_0 \leq P(t) \leq p_1 < \infty$ and that there exist positive constants m and M such that*

$$(p_1 - 1)m \leq \left[P(t)x - \frac{f(t,x)}{Q(t)}\right] \leq (p_0 - 1)M \text{ for } \forall t \in [0,\omega], x \in [m,M].$$

Then, Equation (2) has at least one positive ω-periodic solution $x(t) \in (m,M)$.

Theorem 4. *Assume that $-\infty < p_0 \leq P(t) \leq p_1 < -1$ and that there exist positive constants m and M such that*

$$M - p_0 m \leq \left[-P(t)x + \frac{f(t,x)}{Q(t)}\right] \leq m - p_1 M \text{ for } \forall t \in [0,\omega], x \in [m,M].$$

Then, Equation (2) has at least one positive ω-periodic solution $x(t) \in (m,M)$.

The first-order neutral differential equation is widely used in many natural and social phenomena, such as Hematopoiesis models [3–5], Nicholson's blowflies models [6–10] and blood cell production models [11–13]. In recent years, there have been many results for first-order neutral differential equations and first-order differential equations. Lobo and Valaulikar [14] obtained a Lie-type invariance condition for first-order neutral differential equations using Taylor's theorem for a function of several variables. Berezansky and Braverman [15] investigated solution estimates and stability tests for linear neutral differential equations. In [16], the authors considered a control problem governed by an iterative differential inclusion. Ngoc and Long [17] studied a first-order differential system with initial and nonlocal boundary conditions. In 2009, we obtained the properties of neutral operators in [18], which can be found in Lemma 1 below. In the present paper, using the above properties and Krasnoselskii's fixed point theorem, we give new sufficient conditions for the existence of positive periodic solutions to the following first-order neutral differential equation:

$$\frac{d}{dt}[x(t) - c(t)x(t-\gamma)] = -a(t)x(t) + f(t, x(t-\tau(t))), \tag{3}$$

where $a \in C(\mathbb{R}, (0,\infty))$, $\tau \in C(\mathbb{R}, \mathbb{R})$, $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $c \in C(\mathbb{R}, \mathbb{R})$ with $|c| \neq 1, \gamma > 0$ as a constant, $a(t)$, $\tau(t)$ and $c(t)$ are ω-periodic functions and f is ω-periodic with respect to the first variable.

The main contributions of our study lie on two sides:

(1) We introduce a new method for studying Equations (1)–(3) which is different from the methods in existing papers (see [1,2,5,8]).

(2) Our conditions for the existence of positive periodic solutions obtained by us are simpler and easier to verify than those in [1,2]. Therefore, our results are more widely applicable.

The following sections are organized as follows. Section 2 gives some of the main lemmas. In Section 3, some sufficient conditions for the existence of positive periodic solutions to Equation (3) are obtained. In Section 4, an example is given to show the feasibility of our results. Finally, Section 5 concludes the paper.

2. Preliminaries

In [19], Hale introduced stable a D-operator for studying neutral differential equations. However, when the operator D is not stable, there exist few results for the existence of neutral differential equations. In [20], Zhang considered a neutral differential equation and relieved the stability restriction. When the D-operator was stable or unstable, we [18] generalized the conclusion of [20] and gave new results for the D-operator. Let C_ω be a ω-periodic continuous function space with the norm $||\phi|| = \max_{t\in[0,\omega]} |\phi(t)|$ for all $\phi \in C_\omega$. Let $A : C_\omega \to C_\omega$ be defined by

$$(Ax)(t) = x(t) - c(t)x(t-\gamma).$$

Lemma 1 ([18])**.** *If $|c(t)| \neq 1$, then operator A has a continuous inverse A^{-1} on C_ω, satisfying the following:*

(1)

$$[A^{-1}f](t) = \begin{cases} f(t) + \sum\limits_{j=1}^{\infty} \prod\limits_{i=1}^{j} c(t-(i-1)\gamma)f(t-j\gamma), \ c_\infty < 1, \ \forall f \in C_\omega, \\ -\frac{f(t+\gamma)}{c(t+\gamma)} - \sum\limits_{j=1}^{\infty} \prod\limits_{i=1}^{j+1} \frac{1}{c(t+i\gamma)} f(t+j\gamma+\gamma), \ c_0 > 1, \ \forall f \in C_\omega, \end{cases}$$

(2)

$$||A^{-1}f|| \leq \begin{cases} \frac{1}{1-c_\infty}||f||, \ c_\infty < 1, \ \forall f \in C_\omega, \\ \frac{1}{c_0-1}||f||, \ c_0 > 1, \ \forall f \in C_\omega, \end{cases}$$

where $c_\infty = \max_{t\in[0,\omega]_{\mathbb{T}}} |c(t)|$ and $c_0 = \min_{t\in[0,\omega]_{\mathbb{T}}} |c(t)|$.

Remark 1. *Lemma 1 generalizes the results of [20] as follows:*

(1) *If $c(t)$ is a constant c with $c \neq \pm 1$, then A has a continuous inverse A^{-1} on C_ω, satisfying*

$$[A^{-1}f](t) = \begin{cases} \sum\limits_{j\geq 0} c^j f(t-j\gamma), \ \text{if } |c| < 1, \ \forall f \in C_\omega, \\ -\sum\limits_{j\geq 1} c^{-j} f(t+j\gamma), \ \text{if } |c| > 1, \ \forall f \in C_\omega; \end{cases}$$

(2) $||A^{-1}f|| \leq \frac{||f||}{|1-|c||}, \forall f \in C_\omega$.

Lemma 2 (Krasnoselskii's fixed point theorem [21])**.** *Let B be a Banach space and Ω be a bounded, closed and convex subset in B. Let S_1 and S_2 be maps of Ω into B such that $S_1x + S_1y \in \Omega$ for every pair $x, y \in \Omega$. If S_1 is a contractive operator, and S_2 is completely continuous operator, then the equation $S_1x + S_2x = x$ has a solution in Ω.*

3. Main Results

In this section, we need to assume the following:

(H_1) $f(t,x)$ satisfies the Lipschitz condition about x (i.e., for all $x, y \in \mathbb{R}$, there exists a constant $L > 0$ such that $|f(t,x) - f(t,y)| \leq L|x-y|$).

(H_2) $f(t,x)$ satisfies the Lipschitz condition about x (i.e., for all $x, y \in \mathbb{R}$, there exists a constant $L > 0$ such that $|f(t,x) - f(t,y)| \leq L|x-y|$ with $\frac{L}{c_0-1} < 1$).

Theorem 5. *Suppose that $c_\infty < \frac{1}{3}$, assumption (H_1) holds, and there exist nonnegative constants m and M such that*

$$a(t)\left(m + \frac{c_\infty M}{1-c_\infty}\right) \leq f(t, A^{-1}x) \leq a(t)\left(M - \frac{c_\infty M}{1-c_\infty}\right) \text{ for } t \in [0,\omega], \ x \in [m, M]. \quad (4)$$

Then, Equation (3) has at least one positive ω-periodic solution.

Proof. If $(Ax)(t) = x(t) - c(t)x(t-\gamma) = u(t)$, then $x(t) = (A^{-1}u)(t)$. Equation (3) can be rewritten as

$$u'(t) = -a(t)u(t) - H(u(t)) + f(t, (A^{-1}u)(t-\tau(t))), \tag{5}$$

where $H(u(t)) = a(t)c(t)(A^{-1}u)(t-\gamma)$. Equation (5) has the following equivalent equation:

$$u(t) = \int_t^{t+\omega} G(t,s)[f(s,(A^{-1}u)(s-\tau(s))) - H(u(s))]ds, \tag{6}$$

where $G(t,s) = \frac{\exp(\int_t^s a(r)dr)}{\exp(\int_0^\omega a(r)dr)-1}$. It is well known that to find an ω-periodic solution to Equation (3) is equivalent to finding an ω-periodic solution to Equation (6). Let

$$\Omega = \{u \in C_\omega : m \le u(t) \le M,\ t \in [0,\omega],\ M > m > 0\}.$$

Obviously, Ω is a bounded, closed and convex subset of C_ω. We define the operators $T, S : \Omega \to C_\omega$ as follows:

$$(Tu)(t) = \int_t^{t+\omega} G(t,s)f(s,(A^{-1}u)(s-\tau(s)))ds, \tag{7}$$

$$(Su)(t) = \int_t^{t+\omega} -G(t,s)H(u(s))ds. \tag{8}$$

For any $u \in \Omega$ and $t \in \mathbb{R}$, it follows by Equations (7) and (8) that

$$\begin{aligned}
(Tu)(t+\omega) &= \int_{t+\omega}^{t+2\omega} G(t+\omega,s)f(s,(A^{-1}u)(s-\tau(s)))ds \\
&= \int_t^{t+\omega} G(t+\omega,r+\omega)f(r,(A^{-1}u)(r-\tau(r)))dr \\
&= \int_t^{t+\omega} G(t,r)f(r,(A^{-1}u)(r-\tau(r)))dr \\
&= (Tu)(t)
\end{aligned}$$

and

$$\begin{aligned}
(Su)(t+\omega) &= \int_{t+\omega}^{t+2\omega} -G(t+\omega,s)H(u(s))ds \\
&= \int_t^{t+\omega} -G(t+\omega,r+\omega)H(u(r))dr \\
&= \int_t^{t+\omega} -G(t,r)H(u(r))dr \\
&= (Su)(t).
\end{aligned}$$

Thus, $T(\Omega) \subset C_\omega$ and $S(\Omega) \subset C_\omega$. For each $x, y \in \Omega$, by Equation (4) and Lemma 1, we have

$$\begin{aligned}
(Tx)(t) + (Sy)(t) &= \int_t^{t+\omega} G(t,s)[f(s,(A^{-1}x)(s-\tau(s))) - a(s)c(s)(A^{-1}y)(s-\gamma)]ds \\
&\le \int_t^{t+\omega} G(t,s)a(s)\left(M - \frac{c_\infty M}{1-c_\infty} + \frac{c_\infty M}{1-c_\infty}\right)ds \\
&= M.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(Tx)(t)+(Sy)(t)&=\int_t^{t+\omega}G(t,s)[f(s,(A^{-1}x)(s-\tau(s)))-a(s)c(s)(A^{-1}y)(s-\gamma)]ds\\&\geq\int_t^{t+\omega}G(t,s)a(s)\left(m+\frac{c_\infty M}{1-c_\infty}-\frac{c_\infty M}{1-c_\infty}\right)\\&=m.\end{aligned}$$

Hence, for all $x,y\in\Omega$ and $t\in\mathbb{R}$, we have $(Tx)(t)+(Sy)(t)\in\Omega$. For each $x,y\in\Omega$, by Lemma 1, we have

$$\begin{aligned}|S(x)-S(y)|&=\left|\int_t^{t+\omega}G(t,s)[a(s)c(s)(A^{-1}x)(s-\gamma)-a(s)c(s)(A^{-1}y)(s-\gamma)]ds\right|\\&\leq\int_t^{t+\omega}G(t,s)a(s)\frac{c_\infty}{1-c_\infty}|x(s-\gamma)-y(s-\gamma)|ds.\end{aligned}$$

By taking the norm of both sides, we see that

$$||S(x)-S(y)||\leq\frac{c_\infty}{1-c_\infty}||x-y||$$

and S is a contraction mapping. We show that T is completely continuous on Ω. First, we shall show that T is continuous. Let $x_k\in\Omega$ be a convergent sequence with $x_k(t)\to x(t)$ as $k\to\infty$. For $t\in[0,\omega]$, by assumption (H_1), we have

$$\begin{aligned}|(Tx_k)(t)-(Tx)(t)|&=\left|\int_t^{t+\omega}G(t,s)[f(s,(A^{-1}x_k)(s-\tau(s)))-f(s,(A^{-1}x)(s-\tau(s)))]ds\right|\\&\leq\frac{L\omega\exp(\int_0^\omega a(r)dr)}{\exp(\int_0^\omega a(r)dr)-1}|(A^{-1}x_k)(s-\tau(s))-(A^{-1}x)(s-\tau(s))|\\&\leq\frac{L\omega\exp(\int_0^\omega a(r)dr)}{\exp(\int_0^\omega a(r)dr)-1}\frac{1}{1-c_\infty}|x_k(t)-x(t)|,\end{aligned}$$

which results in

$$\lim_{t\to\infty}||(Tx_k)(t)-(Tx)(t)||=0.$$

Thus, T is continuous. Second, we prove that $T(\Omega)$ is relatively compact. For each $x\in\Omega$, by Equations (4) and (7), we have

$$\begin{aligned}|(Tx)(t)|&=\left|\int_t^{t+\omega}G(t,s)f(s,(A^{-1}x)(s-\tau(s)))ds\right|\\&\leq\int_t^{t+\omega}G(t,s)a(s)\left((M-\frac{c_\infty M}{1-c_\infty}\right)\\&=\frac{(1-2c_\infty)M}{1-c_\infty}\end{aligned}$$

In addition, it follows that

$$||Tx||\leq\frac{(1-2c_\infty)M}{1-c_\infty}.$$

On the other hand, for each $x \in \Omega$, by Equation (3.4), we have

$$\begin{aligned}|(Tx)'(t)| &\le |f(t,(A^{-1}x)(t-\tau(t)))| \\ &\le ||a||\left((M - \frac{c_\infty M}{1-c_\infty}\right) \\ &\le ||a||\frac{(1-2c_\infty)M}{1-c_\infty}.\end{aligned}$$

Hence, $T(\Omega)$ is equi-continuous. By Lemma 2, there is $x \in \Omega$ such that $Tx + Sx = x$. Thus, $x(t)$ a positive ω-periodic solution to Equation (3). $\square$

Remark 2. *To ensure the establishment of Equation (4), the inequality $M - \frac{2c_\infty M}{1-c_\infty} > m$ must hold. Hence, the condition $c_\infty < \frac{1}{3}$ is necessary for the above inequality.*

Theorem 6. *Suppose that $c_0 > 1$, assumption (H_2) holds, and there exist nonnegative constants m and M such that*

$$a(t)\left(m - \frac{c_0 m}{c_\infty - 1}\right) \le f(t, A^{-1}x) \le a(t)\left(M - \frac{c_\infty M}{c_0 - 1}\right) \text{ for } t \in [0,\omega],\ x \in [m, M]. \tag{9}$$

Then, Equation (3) has at least one positive ω-periodic solution.

Proof. Let $T, S, G(t,s)$ and Ω be the same as in the proof of Theorem 5. Obviously, $T(\Omega) \subset C_\omega$ and $S(\Omega) \subset C_\omega$. For each $x, y \in \Omega$, by Equation (9) and Lemma 1, we have

$$\begin{aligned}(Tx)(t) + (Sy)(t) &= \int_t^{t+\omega} G(t,s)[f(s,(A^{-1}x)(s-\tau(s))) - a(s)c(s)(A^{-1}y)(s-\gamma)]ds \\ &\le \int_t^{t+\omega} G(t,s)a(s)\left(M - \frac{c_\infty M}{c_0-1} + \frac{c_\infty M}{c_0-1}\right)ds \\ &= M.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(Tx)(t) + (Sy)(t) &= \int_t^{t+\omega} G(t,s)[f(s,(A^{-1}x)(s-\tau(s))) - a(s)c(s)(A^{-1}y)(s-\gamma)]ds \\ &\ge \int_t^{t+\omega} G(t,s)a(s)\left(m - \frac{c_0 m}{c_\infty-1} + \frac{c_0 m}{c_\infty-1}\right) \\ &= m.\end{aligned}$$

For each $x, y \in \Omega$, we have

$$\begin{aligned}|T(x) - T(y)| &= \left|\int_t^{t+\omega} G(t,s)[f(s,(A^{-1}x)(s-\tau(s))) - f(s,(A^{-1}y)(s-\tau(s)))]ds\right| \\ &\le \int_t^{t+\omega} G(t,s)a(s)\frac{L}{c_0-1}|x(s-\gamma) - y(s-\gamma)|ds.\end{aligned}$$

By taking the norm of both sides, we see that

$$||T(x) - T(y)|| \le \frac{L}{c_0-1}||x-y||.$$

Hence, by (H_2), T is a contraction mapping. We show that S is completely continuous on Ω. First, we shall show that S is continuous. Let $x_k \in \Omega$ be a convergent sequence with $x_k(t) \to x(t)$ as $k \to \infty$. For $t \in [0, \omega]$, we have

$$\begin{aligned} |(Sx_k)(t) - (Sx)(t)| &= \left| \int_t^{t+\omega} G(t,s)[a(s)c(s)(A^{-1}x)(s-\gamma) - a(s)c(s)(A^{-1}y)(s-\gamma)]ds \right| \\ &\leq \int_t^{t+\omega} G(t,s)a(s)\frac{c_\infty}{c_0 - 1}|x_k(s) - x(s)|ds, \end{aligned}$$

which results in

$$\lim_{t\to\infty} ||(Sx_k)(t) - (Sx)(t)|| = 0.$$

Thus, S is continuous. Second, we prove that $S(\Omega)$ is relatively compact. For each $x \in \Omega$, by Lemma 1, we have

$$\begin{aligned} |(Sx)(t)| &= \left| \int_t^{t+\omega} G(t,s)a(s)c(s)(A^{-1}x)(s-\gamma)ds \right| \\ &\leq \int_t^{t+\omega} G(t,s)a(s)\frac{c_\infty M}{c_0 - 1} \\ &= \frac{c_\infty M}{c_0 - 1} \end{aligned}$$

Additionally, it follows that

$$||Sx|| \leq \frac{c_\infty M}{c_0 - 1}.$$

On the other hand, for each $x \in \Omega$, by Lemma 1, we have

$$\begin{aligned} |(Sx)'(t)| &\leq |a(t)c(t)(A^{-1}x)(t-\gamma)| \\ &\leq ||a||\frac{c_\infty M}{c_0 - 1}. \end{aligned}$$

Hence, $S(\Omega)$ is equi-continuous. By Lemma 2, there is $x \in \Omega$ such that $Tx + Sx = x$. Thus, $x(t)$ is a positive ω-periodic solution to Equation (3). $\square$

Remark 3. *In the proofs of Theorems 1–4, in order to obtain the existence of positive periodic solutions, it is necessary to discuss the parameter $c(t)$ between partitions. However, we can also obtain the existence of positive periodic solutions by using the properties of neutral operators without the above partitions about $c(t)$.*

Remark 4. *In [19], Hale pointed out that the operator A is stable when $|c| < 1$ and the operator A is not stable when $|c| > 1$. When A is stable or unstable, Lemma 1 gives sufficient conditions for the existence of the inverse operator A^{-1} and some inequality properties. In this paper, when the operator A is stable or unstable, we obtain the existence results of positive periodic solutions.*

4. Example

Consider the following first-order neutral differential equation:

$$\frac{d}{dt}[x(t) - \frac{1}{5}x(t-\pi)] = -(1 + \frac{1}{5}\sin t)x(t) + \exp(\cos t) + \sin x(t-\pi), \tag{10}$$

where $c(t) = \frac{1}{5}$, $\gamma = \pi$, $a(t) = 1 + \frac{1}{5}\sin t$ and $f(t,x) = \exp(\cos t) + \sin x(t-\pi)$. It is easy to verify that the conditions of Theorem 5 are satisfied with $M = 4$ and $m = 0.1$. Thus, Equation (10) has at least one positive ω-periodic solution. The corresponding numerical simulation is presented in Figure 1.

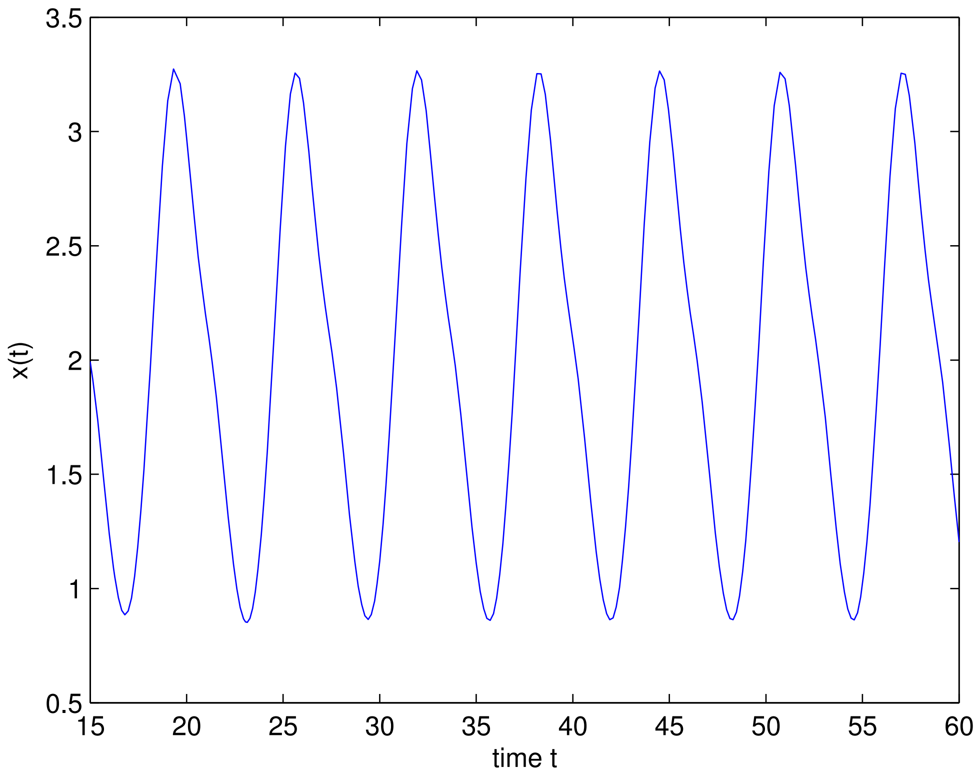

Figure 1. Positive periodic solution $x(t)$ to Equation (10).

5. Conclusions and Discussions

In this paper, some results for the existence of positive periodic solutions to a first-order neutral equation were obtained by the use of Krasnoselskii's fixed-point theorem and mathematical analysis technology. Since there exists a neutral-type term in the considered system, and the existing methods rely too heavily on mathematical skills, we developed a new technique based on the properties of the neutral operator which is markedly different from the existing methods. It is noteworthy that the properties of neutral operators are important for estimating the bounds of solutions. Finally, an example is given to illustrate the effectiveness and feasibility of the proposed criterion.

The methods in this article can also be used to deal with other types of neutral systems and differential equations, such as neural-type equations with stochastic disturbance and parameter uncertainties, neural-type dynamic systems with mixed delays and so on.

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