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Non-Existence Results for Stable Solutions to Weighted Elliptic Systems Including Advection Terms

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Abstract: In this paper, we study a non-linear weighted Grushin system including advection terms. We prove some Liouville-type theorems for stable solutions of the system, based on the comparison property and the bootstrap iteration. Our results generalise and improve upon some previous works.

Keywords: Liouville-type theorem; stable solutions; weighted Grushin system

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1. Introduction

Consider the following weighted system including advection terms:

$$-\Delta_{\mathbf{z}} u + \nu \cdot \nabla_{\mathbf{z}} u = \varrho(\mu) v^{q_1}, \quad -\Delta_{\mathbf{z}} v + \nu \cdot \nabla_{\mathbf{z}} u = \varrho(\mu) u^{q_2}, \qquad v, u > 0 \quad \text{in } \mathbb{R}^N := \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, \tag{1}$$

where $\varrho(\mu) := \left(1 + \|\mu\|^{2(\mathbf{z}+1)}\right)^{\frac{\rho}{2(\mathbf{z}+1)}}$, and the scalar equation:

$$-\Delta_{\mathbf{z}} u + \nu \cdot \nabla_{\mathbf{z}} u = \varrho(\mu) u^{q_1}, \qquad u > 0 \quad \text{in } \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, \tag{2}$$

where $\Delta_{\mathbf{z}}$ is the Grushin operator defined by

$$\Delta_{\mathbf{z}} := \nabla_{\mathbf{z}} \cdot \nabla_{\mathbf{z}} = \sum_{i=1}^{N_1} X_i^2 + \sum_{j=1}^{N_2} Y_j^2 = \Delta_x + |x|^{2\mathbf{z}} \Delta_y, \quad \text{with} \quad \nabla_{\mathbf{z}} := (\nabla_x, |x|^{\mathbf{z}} \nabla_y).$$

Δ_x and Δ_y are Laplace operators in the variables $x \in \mathbb{R}^{N_1}$ and $y \in \mathbb{R}^{N_2}$, respectively.

Here, we always assume that $\rho \geq 0$, $\mathbf{z} \geq 0$, $q_1 \geq q_2 > 1$ and ν is a smooth divergence-free vector field:

$$\mathrm{div}_{\mathbf{z}} \nu = 0, \quad \text{and} \quad |\nu(\mu)| \leq \frac{\kappa}{1 + \|\mu\|} \quad \text{for all } \mu := (x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, \qquad \kappa \text{ small enough.} \tag{3}$$

$\mathrm{div}_{\mathbf{z}} = \mathrm{div}_x + |x|^{\mathbf{z}} \mathrm{div}_y$, and $\|\mu\| = \left(|x|^{2(\mathbf{z}+1)} + |y|^2\right)^{\frac{1}{2(\mathbf{z}+1)}}$, is the norm corresponding to the Grushin distance, where $|x|$ and $|y|$ are the usual Euclidean norms in $\mathbb{R}^{N_1}$ and $\mathbb{R}^{N_2}$, respectively. It is easy to check that the $\|\mu\|$-norm is 1-homogeneous for the group of anisotropic dilations related to $\Delta_{\mathbf{z}}$. It is defined by

$$\sigma_\eta(\mu) = (\eta x, \eta^{1+\mathbf{z}} y), \quad \eta > 0 \quad \text{and} \quad \mu := (x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}.$$

The change of variable formula for the Lebesgue measure gives that

$$d\sigma_\eta(\mu) = \eta^{N_1 + (1+\mathbf{z})N_2} dxdy = \eta^{N_{\mathbf{z}}} d\mu, \quad \text{where} \quad \mathbf{G}^*_{\mathbf{z}} := N_1 + (1 + \mathbf{z})N_2,$$

is the homogeneous dimension with respect to dilation σ_η and $dxdy = d\mu$ denotes the Lebesgue measure on $\mathbb{R}^N$.

Recall that the Grushin operator is elliptic for $|x| \neq 0$ and degenerates on the manifold $\{0\} \times \mathbb{R}^{N_2}$. This operator was introduced in [1]. Problems involving the Grushin operator have been extensively studied over the years. Recall, in adition the papers [2–4]. In an appropriate context, the results on Grushin's operator were obtained in the framework of Heisenberg groups [5]. The study of PDEs involving the Grushin operator has become more and more attractive in the last decades since it can serve to describe nonhomogeneous phenomena, which can occur in different branches of science as physics and astrophysics.

Recently, much attention has been focused on proving Liouville-type theorems for solutions to nonlinear degenerate elliptic systems involving advection terms such as Equations (1) and (2). This result allows us to describe qualitative properties of solutions such as existence, regularity, oscillation, asymptotic or even universal behaviour, pointwise a priori estimates of local solutions, universal and singularity estimates, decay estimates, blow-up rate of solutions of nonstationary problems, etc.; see [6–14] and references therein.

Firstly, we mention that, for the autonomous case, i.e., when $\varrho(\mu) \equiv 1$ and $\nu = \mathbf{z} = 0$, much attention has been focused on obtaining Liouville-type theorems for stable solutions of

$$-\Delta u = v^{q_1}, \quad -\Delta v = u^{q_2}, \quad \text{in } \mathbb{R}^N. \tag{4}$$

We refer to [6,15,16]. The author in [6] has first explored the nonexistence of stable solutions of (4) if $N \leq 10$, for any $q_1 \geq q_2 > 2$. Hu extended this result in [16], for the following systems with positive weights $\varrho(\tau) := (1+|\tau|^2)^{\frac{\rho}{2}}$ with $\rho > 0$:

$$-\Delta u = \varrho(\tau) v^{q_1}, \quad -\Delta v = \varrho(\tau) u^{q_2}, \quad \text{in } \mathbb{R}^N,$$

with $N \leq 10 + \rho$ and $q_1 \geq q_2 > \frac{4}{3}$. We also mention that the previous works [6,16] were improved in [15], where the authors proved a new comparison property for $1 < q_1 \leq \frac{4}{3}$. Among other things, in [9], Liouville-type results for stable solutions of (4) were established, $\forall\, q_1, q_2 > 0$, verifying

$$N < 2 + \alpha + \beta \quad \text{where} \quad \alpha = \frac{2(q_1+1)}{q_1 q_2 - 1}, \quad \text{and} \quad \beta = \frac{2(q_2+1)}{q_1 q_2 - 1}, \quad q_1 q_2 > 1.$$

In the other direction, inspired by the ideas in [6,15,16], Duong [17] proved the nonexistence of stable solutions for the following system with advection:

$$-\Delta u + \nu \cdot \nabla u = v^{q_1}, \quad -\Delta v + \nu \cdot \nabla v = u^{q_2}, \quad v, u > 0 \quad \text{in } \mathbb{R}^N. \tag{5}$$

In particular, Duong [17] proved the following theorem:

Theorem 1.

1. *If* $q_1 \geq q_2 > \frac{4}{3}$ *and*

$$N < 2 + 2\beta k_0^+, \quad \textit{where} \quad k_0^{\pm} = \sqrt{\varpi} \pm \sqrt{\varpi - \sqrt{\varpi}}, \quad \textit{with} \quad \varpi = \frac{q_1 q_2 (q_1+1)}{q_2+1}.$$

There is no stable, positive solution to (5)*. In particular, there is no stable positive solution to* (5) *provided* $N \leq 10$.

2. *If* $1 < q_1 \leq \min(\frac{4}{3}, q_2)$ *and*

$$N < 2 + \left[2 + \alpha + \frac{4(2-q_1)}{q_2+q_1-2}\right] k_0^+,$$

(5) *has no bounded stable positive solution. In particular,* (5) *does not admit bounded stable positive solution provided* $N \leq 6$.

It should be noticed that when $q_1 = q_2$, the result is a natural extension of that in [18], for the following equation with advection:

$$-\Delta u + \nu \cdot \nabla u = u^{q_1}, \quad u > 0 \quad \text{in} \quad \mathbb{R}^N.$$

In the special case $\varrho(\mu) \equiv 1$ and $\nu = 0$, the system (1) becomes

$$-\Delta_{\mathbf{z}} u = v^{q_1}, \quad -\Delta_{\mathbf{z}} v = u^{q_2}, \quad \text{where } q_1 \geq q_2 > 1. \tag{6}$$

The main difficulty is due to the fact that $\Delta_{\mathbf{z}}$ is not symmetric and it is generated on the manifold $\{0\} \times \mathbb{R}^{N_2}$, which causes some mathematical problems. Very recently, in [11], the authors have proved that if $\mathbf{G}_{\mathbf{z}}^* := N_1 + (1+\mathbf{z})N_2 < 2 + \alpha + \beta$, Equation (1) has no stable solution for any $q_1, q_2 > 0$.

Furthermore, adopting the new approach of Cowan [6], the author in [19] established the nonexistence of stable solutions of (6) when $q_1 \geq q_2 > \frac{4}{3}$, and $\mathbf{G}_{\mathbf{z}}^*$ satisfies

$$\mathbf{G}_{\mathbf{z}}^* := N_1 + (1+\mathbf{z})N_2 < 2 + 2\beta_0 t_0^+.$$

This result was then generalized in [10], for the system (1), i.e., $\nu = 0$.

Inspired by the mentioned previous works, we classify stable positive solutions of (1) under condition (3). First of all, we need to recall the following:

Definition 1. *We say that a smooth solution $(u, v) \in C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N)$ of (1) is stable if there exist positive smooth functions φ, χ verifying*

$$-\Delta_{\mathbf{z}} \varphi + \nu \cdot \nabla_{\mathbf{z}} \varphi = q_1 \varrho(\mu) v^{q_1 - 1} \psi, \quad -\Delta_{\mathbf{z}} \psi + \nu \cdot \nabla_{\mathbf{z}} \psi = q_2 \varrho(\mu) u^{q_2 - 1} \varphi \quad \text{in } \mathbb{R}^N.$$

This definition is motivated by [6,19,20]. Our first result concerns stable solutions:

Theorem 2. *Let ρ be positive and m_0 be the largest root of the polynomial*

$$Q(m) = m^4 - q_1 q_2 \alpha \beta \Big[4m^2 - 2(q_1 + q_2)m + 1\Big]. \tag{7}$$

1. *If $\frac{4}{3} < q_1 \leq q_2$ and $\mathbf{G}_{\mathbf{z}}^* < (2+\rho)m_0 + 2$, then (1) does not admit any positive stable solution. Consequently, if $\mathbf{G}_{\mathbf{z}}^* \leq 10 + 4\rho$, then (1) has no stable solution for all $\frac{4}{3} < q_1 \leq q_2$.*
2. *If $1 < q_1 \leq \min(\frac{4}{3}, q_2)$ and*

$$\mathbf{G}_{\mathbf{z}}^* < 2 + \frac{1}{2}\left[q_1 + \frac{4(2-q_1)}{\beta(q_1+q_2-2)}\right](\rho+2)m_0,$$

then (1) has no bounded stable solution.

If in addition, $\mathbf{G}_{\mathbf{z}}^* \leq 6 + 2\rho$, then (1) does not admit any bounded stable solution for all $q_2 \geq q_1 > 1$.

If $q_1 = q_2$, by means of the comparison property (see Lemmas 1 and 2 below), we get the following result.

Proposition 1.

1. *If $\frac{4}{3} < q_1$, then (2) has no stable solution if*

$$\mathbf{G}_{\mathbf{z}}^* < 2 + \frac{2(2+\rho)}{q_1 - 1}\left(q_1 + \sqrt{q_1^2 - q_1}\right). \tag{8}$$

In particular, if $\mathbf{G}_{\mathbf{z}}^* \leq 10 + 4\rho$, then (2) does not admit any stable solution for all $\frac{4}{3} < q_1$.

2. *If $1 < q_1 \leq \frac{4}{3}$ and $\mathbf{G}_{\mathbf{z}}^*$ verifies (8) then (2) has no bounded stable solution.*

Then, (2) does not admit any bounded stable solution for all $q_1 > 1$ if $\mathbf{G}_{\mathbf{z}}^* \leq 10 + 4\rho$.

As in [19], the key techniques in proving Theorem 2 and Proposition 1 deal with the property of comparison and nonlinear integral estimates. Nevertheless, previous tools used in proving the comparison relation (see, e.g., [10,17,19]) do not seem to be applicable for the system (1), since the operator becomes non-symmetric and it degenerates on the manifold $\{0\} \times \mathbb{R}^{N_2}$ due to the advection term. This causes some principal problems in proving Theorem 2. So, we need to use other techniques motivated by [10,17]. We may also use the idea in [15] to establish the "inverse" comparison relation, which is important to treat the case $1 < p \leq \frac{4}{3}$. In addition, the L^1-estimate to the boostrap iteration in [17] does not work in the case of Grushin operator; we instead switch to the L^2-estimate in the boostrap argument.

Remark 1.

- *Let m_0 be the largest root of the polynomial Q defined in (7). It should be noticed that*

$$m_0 > \beta k_0^+ > 4, \qquad \text{for any} \quad q_2 \geq q_1 > 1,$$

  (see Remark 2.1 below). Therefore, Theorem 2 enhances the bound given by Theorem 1 with $\varrho(\mu) \equiv 1$. Consequently, the range in Theorem 2, is larger than that in [17] (see Theorem 1).
- *Our results can be applied also to the general class of degenerate operators (see [7,8,21,22]), namely*

$$\Delta_{\mathbf{z}} := \sum_{j=1}^{N} \partial x_j \left(\mathbf{z}_j^2 \partial x_i \right) \quad \mathbf{z} := (\mathbf{z}_1, \ldots, \mathbf{z}_N) : \mathbb{R}^N \to \mathbb{R}^{\mathrm{N}}.$$

  Here $\mathbf{z}_j : \mathbb{R}^N \to \mathbb{R}, \quad j = 1, \ldots, N$, are nonnegative functions that are continuous and verify some properties as the homogenity of $\Delta_{\mathbf{z}}$ of degree two with respect to a group dilation in $\mathbb{R}^N$.

To our knowledge, all results presented here are new. This paper proceeds as follows: In Section 2, we give some basic results. In Section 3, we prove Theorem 2 and Proposition 1.

2. Main Technical Tool

In this section, we define the following parameters: For $\alpha' > 0$, set

$$\Omega_{\alpha' R} := B_{\alpha' R} \times B_{(\alpha' R)^{1+\mathbf{z}}}, \quad \varrho(\mu) := \left(1 + \|\mu\|^{2(\mathbf{z}+1)}\right)^{\frac{\rho}{2(\mathbf{z}+1)}}, \quad \text{and} \quad dxdy := d\mu.$$

In the following, C always denotes a generic positive constant, which could be changed from one line to another.

Our proofs necessitate some technical lemmas.

2.1. Comparison Principle

Here, we establish the comparison property for system (1).

Lemma 1. *Assume that (u, v) is a bounded positive solution of (1). Set $q_2 \geq q_1 > 1$ and (3) holds. Then*

$$u^{q_2+1} \leq \frac{q_2+1}{q_1+1} v^{q_1+1}. \tag{9}$$

Proof. Let $\sigma = \frac{q_1+1}{q_2+1} \in (0,1]$, $\lambda = \sigma^{\frac{-1}{q_2+1}}$; we conclude then that

$$\text{Equation 9} \quad \Leftrightarrow \quad u \leq \lambda v^{\sigma}, \tag{10}$$

we put $w = u - \lambda v^{\sigma}$. A simple calculation gives

$$\begin{aligned}\Delta_{\mathbf{z}} w = \Delta_{\mathbf{z}} u - \lambda\sigma v^{\sigma-1}\Delta_{\mathbf{z}} v - \lambda\sigma(\sigma-1)|\nabla_{\mathbf{z}} v|^2 v^{\sigma-2} &\geq \Delta_{\mathbf{z}} u - \lambda\sigma v^{\sigma-1}\Delta_{\mathbf{z}} v \\ &= \varrho(\mu)\Big[-v^{q_1} + \lambda\sigma v^{\sigma-1}u^{q_2}\Big] \\ &\quad + \nu\cdot\nabla_{\mathbf{z}} u - \lambda\sigma v^{\sigma-1} a\cdot\nabla_{\mathbf{z}} v \\ &= \nu\cdot\nabla_{\mathbf{z}} w + v^{\sigma-1}\Big[-v^{q_1-\sigma+1} + \lambda\sigma u^{q_2}\Big] \\ &= \nu\cdot\nabla_{\mathbf{z}} w + \varrho(\mu)v^{\sigma-1}\big[\lambda^{-q_2}u^{q_2} - v^{q_2\sigma}\big].\end{aligned}$$

Then

$$Cv^{\sigma-1}\left[u^{q_2} - \left(\lambda v^{\sigma}\right)^{q_2}\right] \leq \varrho(\mu)v^{\sigma-1}\left[\frac{u^{q_2} - \left(\lambda v^{\sigma}\right)^{q_2}}{\lambda^{q_2}}\right] \leq \Delta_{\mathbf{z}} w - \nu\cdot\nabla_{\mathbf{z}} w. \tag{11}$$

Now, we use a contradiction argument to prove (10). Suppose that

$$M = \sup_{\mathbb{R}^N} w > 0 \quad (M \leq \infty). \tag{12}$$

Next, the proof splits into two cases:

Case 1: If there is μ^* verifying $\sup_{\mathbb{R}^N} w = w(\mu^*) = u(\mu^*) - \lambda v^{\sigma}(\mu^*) > 0$, we have

$$\frac{\partial w}{\partial \mu_i}(\mu^*) = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial \mu_i^2}(\mu^*) \leq 0 \quad i = 1, \dots, n.$$

This gives

$$\nabla_{\mathbf{z}} w(\mu^*) = 0 \quad \text{and} \quad \Delta_{\mathbf{z}} w(\mu^*) \leq 0.$$

In addition, the left-hand side of (11) at μ^* is positive, which is a contradiction.

Case 2: The supremum of w is attained at infinity.

Choose now $\phi_R(x,y) = \psi^t(\frac{x}{R}, \frac{y}{R^{1+\mathbf{z}}})$, where $t > 0$, ψ is a cut-off function in $C_c^{\infty}(\mathbb{R}^N)$, with

$$\psi = 1 \quad \text{on} \quad B_1 \times B_1, \quad \text{and} \quad \psi = 0 \quad \text{outside} \quad B_2 \times B_{2^{1+\mathbf{z}}}.$$

So, we get

$$\frac{|\nabla_{\mathbf{z}}\phi_R|^2}{\phi_R} \leq \frac{C}{R^2}\phi_R^{\frac{t-2}{t}} \quad \text{and} \quad |\Delta_{\mathbf{z}}(\phi_R)| \leq \frac{C}{R^2}\phi_R^{\frac{t-2}{t}}. \tag{13}$$

Let $w_R = \phi_R w$, a compactly supported function. Therefore, there is $\mu_R = (x_R, y_R) \in \Omega_{2R}$, with

$$w_R(\mu_R) = \max_{\mathbb{R}^N} w_R(\mu) \to \sup_{\mathbb{R}^N} w(\mu) \quad \text{as} \quad R \to \infty.$$

Then

$$\nabla_{\mathbf{z}} w_R(\mu_R) = 0 \quad \text{and} \quad \Delta_{\mathbf{z}} w_R(\mu_R) \leq 0.$$

Next, we take all the estimates at the point μ_R. Now, using the fact that $\nabla_{\mathbf{z}} w_R(\mu_R) = 0$, we get

$$\nabla_{\mathbf{z}} w = -\phi_R^{-1}\nabla_{\mathbf{z}}\phi_R w. \tag{14}$$

Since $\Delta_{\mathbf{z}} w_R(\mu_R) \leq 0$, we obtain

$$\phi_R \Delta_{\mathbf{z}} w \leq 2w\phi_R^{-1}|\nabla_{\mathbf{z}}\phi_R|^2 - w\Delta_{\mathbf{z}}\phi_R. \tag{15}$$

From (13) and (15), one concludes

$$\phi_R \Delta_{\mathbf{z}} w \leq \frac{C}{R^2}\phi_R^{\frac{t-2}{t}} w. \tag{16}$$

Using (13) and (15), and the fact that

$$|\nu(\mu)| \leq \frac{\kappa}{1+\|\mu\|},$$

we can deduce that for any $\kappa > 0$, there exists a positive constant C depending only on κ such that

$$|\nu \cdot \nabla_{\mathbf{z}} w\phi_R| \leq \frac{C_\kappa}{R^2}\phi_R^{\frac{t-1}{t}} w. \tag{17}$$

Recalling now that $w = u - \lambda v^\sigma \geq 0$, and at μ_R, it is shown that

$$\frac{u^{q_2}}{w^{q_2}} - \frac{(\lambda v^\sigma)^{q_2}}{w^{q_2}} \geq 1, \quad \text{or equivalently} \quad \lambda^{-q_2}u^{q_2} - v^{q_2\sigma} \geq \lambda^{-q_2}w^{q_2}. \tag{18}$$

Multiplying (11) by ϕ_R, and using (16)–(18), there holds

$$v^{\sigma-1}w^{q_2}\phi_R^{\frac{t+2}{2}} \leq \frac{C}{R^2}w\phi_R.$$

The sequence $v(\mu_R)$ is bounded, as $\sigma \leq 1$. We choose

$$q_2 - 1 = \frac{2}{t} \quad \text{so that } \ t = \frac{2}{q_2-1},$$

we then get

$$w_R^{q_2-1}(\mu_R) \leq \frac{C}{R^2}.$$

Letting $R \to \infty$, we get $\sup_{\mathbb{R}^N} w = 0$, which is a contradiction with (12). The proof is completed. □

We proceed, like for the proof of the above lemma, to establish an inverse comparison property.

Lemma 2. *Let $q_2 \geq q_1 > 1$. Assume that (u,v) is a bounded positive solution of (1), which satisfies Equation (3); we have*

$$v \leq \|u\|_\infty^{\frac{q_2-q_1}{q_1+1}} u. \tag{19}$$

Proof. Put $ł = \|u\|_\infty^{\frac{q_2-q_1}{q_1+1}}$ and $w = v - łu$, . As $q_2 \geq q_1$, we get

$$\begin{aligned}
\Delta_s w - \nu \cdot \nabla_s w = \varrho(\mu)(łv^{q_1} - u^{q_2}) &= \varrho(\mu)\left[łv^{q_1} - \left(\frac{u}{\|u\|_\infty}\right)^{q_2}\|u\|_\infty^{q_2}\right] \\
&\geq \varrho(\mu)\left[łv^{q_1} - \left(\frac{u}{\|u\|_\infty}\right)^{q_1}\|u\|_\infty^{q_2}\right] \\
&= \varrho(\mu)\|u\|_\infty^{q_2-q_1}\left(\frac{łv^{q_1}}{\|u\|_\infty^{q_2-q_1}} - u^{q_1}\right) \\
&= \varrho(\mu)\|u\|_\infty^{q_2-q_1}(ł^{-q_1}v^{q_1} - u^{q_1}).
\end{aligned}$$

The rest of the proof is then obtained by Lemma 1. We then omit the details. □

2.2. Integral Estimates

We now point out the following useful lemma.

Lemma 3. *Assume that* (u, v) *is a positive stable solution of* (1) *with* (3) *verified. Then, for* $\gamma \in C_c^1(\mathbb{R}^N)$*, we obtain*

$$\sqrt{q_1 q_2} \int_{\mathbb{R}^N} \varrho(\mu) u^{\frac{q_2-1}{2}} v^{\frac{q_1-1}{2}} \gamma^2 dxdy \leq \frac{1}{4} \int_{\mathbb{R}^N} \left|\nu\gamma + 2\nabla_{\mathbf{z}}\gamma\right|^2 dxdy. \tag{20}$$

Proof. Let $\gamma, \varphi \in C_c^1(\mathbb{R}^N)$. We multiply the first equation in Definition 1 of stability by $\frac{\gamma^2}{\varphi}$ and integrate over $\mathbb{R}^N$, we get

$$q_1 \int_{\mathbb{R}^N} \varrho(\mu) v^{q_1-1} \frac{\psi}{\varphi} \gamma^2 dxdy = -\int_{\mathbb{R}^N} \Big(\frac{\Delta_{\mathbf{z}}\varphi}{\varphi}\gamma^2 + \nu \cdot \nabla_{\mathbf{z}}\varphi \frac{\gamma^2}{\varphi} \Big) dxdy.$$

A direct calculation gives

$$\begin{aligned}
\int_{\mathbb{R}^N} \Big(-\frac{\Delta_{\mathbf{z}}\varphi}{\varphi}\gamma^2 + \nu \cdot \nabla_{\mathbf{z}}\varphi \frac{\gamma^2}{\varphi} \Big) dxdy &= \int_{\mathbb{R}^N} \Big(\nabla_{\mathbf{z}}\varphi \cdot \nabla_{\mathbf{z}}(\gamma^2\varphi^{-1}) + \nu \cdot \nabla_{\mathbf{z}}\varphi \frac{\gamma^2}{\varphi} \Big) dxdy \\
&= \int_{\mathbb{R}^N} \Big(-\varphi^{-2}|\nabla_{\mathbf{z}}\varphi|^2\gamma^2 + 2\varphi^{-1}\gamma\nabla_{\mathbf{z}}\varphi \cdot \nabla_{\mathbf{z}}\gamma + \nu \cdot \nabla_{\mathbf{z}}\varphi \frac{\gamma^2}{\varphi} \Big) dxdy \\
&= \int_{\mathbb{R}^N} \Bigg(+\varphi^{-1}\gamma\nabla_{\mathbf{z}}\varphi \Big(\nu\gamma + 2\nabla_{\mathbf{z}}\gamma\Big) - \varphi^{-2}|\nabla_{\mathbf{z}}\varphi|^2\gamma^2 \Bigg) dxdy.
\end{aligned}$$

Using the inequality $2ab - a^2 \leq b^2$, we deduce that

$$4q_1 \int_{\mathbb{R}^N} \varrho(\mu) v^{q_1-1} \frac{\psi}{\varphi} \gamma^2 dxdy \leq \int_{\mathbb{R}^N} \left|\nu\gamma + 2\nabla_{\mathbf{z}}\gamma\right|^2 dxdy. \tag{21}$$

By the same argument, we also have

$$4q_2 \int_{\mathbb{R}^N} \varrho(\mu) u^{q_2-1} \frac{\varphi}{\psi} \gamma^2 dxdy \leq \int_{\mathbb{R}^N} \left|\nu\gamma + 2\nabla_{\mathbf{z}}\gamma\right|^2 dxdy. \tag{22}$$

Adding the inequalities (21) and (22), we obtain

$$\int_{\mathbb{R}^N} \varrho(\mu) \Big(q_1 v^{q_1-1} \frac{\psi}{\varphi}\gamma^2 + q_2 u^{q_2-1} \frac{\varphi}{\psi}\gamma^2 \Big) dxdy \leq \frac{1}{2} \int_{\mathbb{R}^N} \left|\nu\gamma + 2\nabla_{\mathbf{z}}\gamma\right|^2 dxdy. \tag{23}$$

Denote

$$2\varrho(\mu)\sqrt{q_1 q_2 v^{q_1-1} u^{q_2-1} \gamma^4} \leq \varrho(\mu) \Big(q_1 v^{q_1-1} \frac{\psi}{\varphi}\gamma^2 + q_2 u^{q_2-1} \frac{\varphi}{\psi}\gamma^2 \Big). \tag{24}$$

Combining (24) with (23), we readily get the estimate (20). □

We will use also the following integral estimates for all solutions of (1), where (3) is satisfied.

Lemma 4. *Assume that* $q_2 \geq q_1 > 1$ *and* (3) *holds. There exists* $C > 0$ *for any solution* (u, v) *of* (1) *such that*

$$\int_{\Omega_R} \varrho(\mu) v^{q_1} dxdy \leq C R^{\gamma_1}, \quad \text{where} \quad \gamma_1 := \mathbf{G}_{\mathbf{z}}^* - q_1\beta - \frac{\alpha}{2}\rho \quad \text{and} \quad R \geq 1. \tag{25}$$

$$\int_{\Omega_R} \varrho(\mu) u^{q_2} dxdy \leq C R^{\gamma_2}, \quad \text{where} \quad \gamma_2 := \mathbf{G}_{\mathbf{z}}^* - q_2\alpha - \frac{\beta}{2}\rho, \tag{26}$$

where

$$\Omega_R := B_R \times B_{R^{1+\mathbf{z}}}, \quad \varrho(\mu) := \Big(1 + \|\mu\|^{2(\mathbf{z}+1)}\Big)^{\frac{\rho}{2(\mathbf{z}+1)}}, \quad \textit{and} \quad \alpha = \frac{2(q_1+1)}{q_1q_2-1}, \quad \beta = \frac{2(q_2+1)}{q_1q_2-1}, \quad q_1q_2 > 1.$$

Proof. We use the cut-off function $0 \le \xi_k \in C_c^\infty(\mathbb{R}) \le 1$, satisfying

$\xi_k = 1$ on $[-1,1]$, and $\xi_k = 0$ outside $[-2^{1+(k-1)\mathbf{z}}, 2^{1+(k-1)\mathbf{z}}]$ where $k = 1,2$.

For $R \ge 1$, put $\psi_R(x,y) = \xi_1(\frac{x}{R})\xi_2(\frac{y}{R^{1+\mathbf{z}}})$, we can easily see that

$$|\nabla_x \psi_R| \le \frac{C}{R} \quad \text{and} \quad |\nabla_y \psi_R| \le \frac{C}{R^{1+\mathbf{z}}},$$

$$|\Delta_x \psi_R| \le \frac{C}{R^2} \quad \text{and} \quad |\Delta_y \psi_R| \le \frac{C}{R^{2(1+\mathbf{z})}}.$$

Multiplying

$$-\Delta_{\mathbf{z}} u + \nu \cdot \nabla_{\mathbf{z}} u = \varrho(\mu) v^{q_1},$$

By ψ_R^m, and integrating by parts, there holds

$$\int_{\Omega_{2R}} \varrho(\mu) v^{q_1} \psi_R^m dxdy = -\int_{\Omega_{2R}} u\Big(\Delta_{\mathbf{z}}(\psi_R^m) + \nu \cdot \nabla_{\mathbf{z}}(\psi_R^m)\Big)dxdy \le \frac{C}{R^2}\int_{\Omega_{2R}} u\psi_R^{m-2} dxdy,$$

where

$$\Omega_{2R} := B_{2R} \times B_{(2R)^{1+\mathbf{z}}}.$$

Let $\frac{1}{q_2} + \frac{1}{q_2'} = 1$. Applying Hölder's inequality, we get then

$$\begin{aligned}\int_{\Omega_{2R}} \varrho(\mu) v^{q_1} \psi_R^m dxdy &\le \frac{C}{R^2}\left[\int_{\Omega_{2R}} \Big(\varrho(\mu)\Big)^{-\frac{q_2'}{q_2}} dxdy\right]^{\frac{1}{q_2'}} \times \left(\int_{\Omega_{2R}} \varrho(\mu) u^{q_2} \psi_R^{(m-2)q_2} dxdy\right)^{\frac{1}{q_2}} \\ &\le CR^{\frac{N_1+(1+\mathbf{z})N_2}{q_2'} - \frac{\rho}{q_2} - 2}\left(\int_{\Omega_{2R}} \varrho(\mu) u^{q_2} \psi_R^{(m-2)q_2} dxdy\right)^{\frac{1}{q_2}}.\end{aligned}$$

Now, we multiply

$$-\Delta_{\mathbf{z}} v + \nu \cdot \nabla_{\mathbf{z}} u = \varrho(\mu) u^{q_2},$$

by ψ_R^k with $k \ge 2$ and we integrate by parts. By Hölder's inequality, we have

$$\int_{\Omega_{2R}} \varrho(\mu) u^{q_2} \psi_R^k dxdy \le CR^{\frac{N_1+(1+\mathbf{z})N_2}{q_1'} - \frac{\rho}{q_1} - 2}\left(\int_{\Omega_{2R}} \varrho(\mu) v^{q_1} \psi_R^{(k-2)q_1} dxdy\right)^{\frac{1}{q_1}},$$

where $\frac{1}{q_1} + \frac{1}{q_1'} = 1$. Taking large k and m such that $m \le (k-2)q_1$ and $k \le (m-2)q_2$, in view of the two above inequalities, we get

$$\begin{aligned}&\int_{\Omega_{2R}} \varrho(\mu) v^{q_1} \psi_R^m dxdy \\ &\le CR^{\frac{N_1+(1+\mathbf{z})N_2}{q_2'} - \frac{\rho}{q_2} - 2} R^{\left(\frac{N_1+(1+\mathbf{z})N_2}{q_1'} - \frac{\rho}{q_1} - 2\right)\frac{1}{q_2}}\left(\int_{\Omega_{2R}} \varrho(\mu) v^{q_1} \psi_R^{(k-2)q_1} dxdy\right)^{\frac{1}{q_1q_2}} \\ &\le CR^{N_1+(1+\mathbf{z})N_2 - \frac{N_1+(1+\mathbf{z})N_2}{q_1q_2} - \frac{\rho(q_1+1)}{q_1q_2} - \frac{2(q_2+1)}{q_2}}\left(\int_{\Omega_{2R}} \varrho(\mu) v^{q_1} \psi_R^m dxdy\right)^{\frac{1}{q_1q_2}}.\end{aligned}$$

So, we obtain

$$\int_{\Omega_{2R}} \varrho(\mu) v^{q_1} dxdy \le \int_{\Omega_{2R}} \varrho(\mu) v^{q_1} \psi_R^m dxdy \le CR^{\mathbf{G}_{\mathbf{z}}^* - q_1\beta - \frac{\alpha}{2}\rho}.$$

Finally, using the same argument as above, we obtain the estimate (26). □

We need the following integral estimate for u, which is crucial to deal with the case $1 < q_1 \leq \frac{4}{3}$.

Lemma 5. *Let (u, v) be a stable solution of (1), $1 < q_1 \leq \min(\frac{4}{3}, q_2)$, and (3) is satisfied. Assume that u is bounded; we have*

$$\int_{\Omega_R} \varrho(\mu) v^2 dxdy \leq CR^{\gamma_3}, \tag{27}$$

where

$$\Omega_R := B_R \times B_{R^{1+\mathbf{z}}}, \quad \varrho(\mu) := \left(1 + \|\mu\|^{2(\mathbf{z}+1)}\right)^{\frac{\rho}{2(\mathbf{z}+1)}}, \quad and \quad \gamma_3 := \gamma_1 - \frac{2(2+\rho)(2-q_1)}{q_1+q_2-2}, \quad \gamma_1 := \mathbf{G}^*_{\mathbf{z}} - q_1\beta - \frac{\alpha}{2}\rho.$$

Proof. Take $\eta_R(x,y) = \xi(\frac{x}{R}, \frac{y}{R^{1+\mathbf{z}}})$, where $\xi \in C_c^\infty(\mathbb{R}^N)$ is a cut-off function satisfying

$$\xi = 1 \quad \text{on} \quad B_1 \times B_1, \quad \text{and} \quad \xi = 0 \quad \text{outside} \quad B_2 \times B_{2^{1+\mathbf{z}}}.$$

Multiplying

$$-\Delta_{\mathbf{z}} v + \nu \cdot \nabla_{\mathbf{z}} v = \varrho(\mu) u^{q_2},$$

by $v\eta_R^2$ and integrating parts, we get

$$\int_{\mathbb{R}^N} |\nabla_{\mathbf{z}} v|^2 \eta_R^2 dxdy = \int_{\mathbb{R}^N} \varrho(\mu) u^{q_2} vs.\eta_R^2 dxdy + \frac{1}{2}\int_{\mathbb{R}^N} v^2 \Big(\Delta_{\mathbf{z}}(\eta_R^2) + \nu \cdot \nabla_{\mathbf{z}}(\eta_R^2)\Big) dxdy.$$

Using Lemma 1, we get

$$\int_{\mathbb{R}^N} |\nabla_{\mathbf{z}} v|^2 \eta_R^2 dxdy \leq \sqrt{\frac{q_2+1}{q_1+1}} \int_{\mathbb{R}^N} \varrho(\mu) u^{\frac{q_2-1}{2}} v^{\frac{q_1+1}{2}} v\eta_R^2 dxdy + \frac{1}{2}\int_{\mathbb{R}^N} v^2 \Big(\Delta_{\mathbf{z}}(\eta_R^2) + \nu \cdot \nabla_{\mathbf{z}}(\eta_R^2)\Big) dxdy.$$

Set $\gamma = vs.\eta_R$ in (20) and integrating by parts; we obtain

$$\begin{aligned}
&\sqrt{q_2 q_1} \int_{\mathbb{R}^N} \varrho(\mu) u^{\frac{q_2-1}{2}} v^{\frac{q_1-1}{2}} v^2 \eta_R^2 dxdy \\
&\leq \frac{1}{4}\int_{\mathbb{R}^N} \Big|\nu vs.\eta_R + 2\nabla_{\mathbf{z}}(v\eta_R)\Big|^2 dxdy \\
&= \frac{1}{4}\left(\int_{\mathbb{R}^N} \Big|\nu^2 v^2 \eta_R^2 + 4(\nabla_{\mathbf{z}}(v\eta_R))^2\Big|\right) dxdy + \int_{\mathbb{R}^N} \Big|\nu \cdot \nabla_{\mathbf{z}}(v\eta_R) v\eta_R\Big| dxdy \\
&\leq \int_{\mathbb{R}^N} |\nabla_{\mathbf{z}} v|^2 \eta_R^2 dxdy + \int_{\mathbb{R}^N} v^2 \Big(\frac{1}{4}\nu^2 v^2 \eta_R^2 + |\nabla_{\mathbf{z}} \eta_R|^2\Big) dxdy - \frac{1}{2}\int_{\mathbb{R}^N} v^2 \Delta_{\mathbf{z}}(\eta_R^2) dxdy \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} \Big|\nu \cdot \nabla_{\mathbf{z}}(v^2\eta_R^2)\Big| dxdy.
\end{aligned}$$

Combining the two last inequalities, we obtain

$$\left(\sqrt{q_2 q_1} - \sqrt{\frac{q_2+1}{q_1+1}}\right) \int_{\mathbb{R}^N} \varrho(\mu) u^{\frac{q_2-1}{2}} v^{\frac{q_1+3}{2}} \eta_R^2 dxdy \leq \frac{1}{2}\int_{\mathbb{R}^N} v^2 \left(\frac{1}{2}\nu^2 v^2 \eta_R^2 + 2|\nabla_{\mathbf{z}} \eta_R|^2 + \nu \cdot \nabla_{\mathbf{z}}(\eta_R^2)\right) dxdy.$$

We assume that $\eta_R = \varphi_R^m$ with $m > 2$. Using Lemma 2, we get

$$\int_{\mathbb{R}^N} \varrho(\mu) v^{q_0} \varphi_R^{2m} dxdy \leq \frac{C}{R^{2+\rho}} \int_{\mathbb{R}^N} \varrho(\mu) v^2 \varphi_R^{2m-2} dxdy, \tag{28}$$

where $q_0 = \frac{q_2+q_1+2}{2}$. Denote

$$J_1 := \int_{\mathbb{R}^N} \varrho(\mu) v^{q_0} \varphi_R^{2m} dxdy, \quad J_2 := \int_{\mathbb{R}^N} \varrho(\mu) v^2 \varphi_R^{2m-2} dxdy.$$

As $q_1 \geq q_2$, we observe that $q_1 < 2 < q_0$ for $1 < q_1 \leq \frac{4}{3}$. A simple calculation yields

$$2 = q_1\lambda + q_0(1-\lambda) \quad \text{with } \lambda = \frac{q_2 + q_1 - 2}{q_2 - q_1 + 2} \in (0,1).$$

Since we assume that m large with $m\lambda > 1$, from Hölder's inequality, Lemma 4, and according to inequality (28), we have

$$\begin{aligned} J_2 \leq J_1^{1-\lambda} \left(\int_{\mathbb{R}^N} \varrho(\mu) v^{q_1} \varphi_R^{2m\lambda - 2} dxdy \right)^{\lambda} &\leq \left(\frac{CJ_2}{R^{2+\rho}} \right)^{1-\lambda} \left(\int_{\Omega_{2R}} \varrho(\mu) v^{q_1} dxdy \right)^{\lambda} \\ &\leq C' J_2^{1-\lambda} R^{-(2+\rho)(1-\lambda)} R^{\lambda\gamma_1}, \end{aligned}$$

which gives

$$J_2 \leq CR^{\gamma_1 - \frac{2(2+\rho)(2-q_1)}{q_1+q_2-2}},$$

where γ_1 is given in Lemma 4. So we are done. $\square$

We need also the following technical lemma, which plays a crucial role in establishing Theorem 2 and Proposition 1.

Lemma 6. *Assume that (u, v) is a stable solution to (1) with $\rho \geq 0$, and (3) is satisfied. There exists $C < \infty$ such that for any $n > \frac{q_1+1}{2}$ verifying $P(n) < 0$, we have*

$$\int_{B_R \times B_{R^{1+\mathbf{z}}}} \varrho(\mu) u^{q_2} v^{n-1} dxdy \leq \frac{C}{R^2} \int_{B_{2R} \times B_{(2R)^{1+\mathbf{z}}}} v^n dxdy, \tag{29}$$

where

$$P(n) := n^4 - 16 q_1 q_2 \frac{\alpha}{\beta} \left(n^2 + \frac{q_1 + q_2 + 2}{q_2 + 1} n - \frac{\alpha}{\beta} \right). \tag{30}$$

Proof. Set $\phi \in C_0^2(\mathbb{R}^N)$. Let (u, v) be a stable solution of (1); we integrate by parts to get

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla_{\mathbf{z}} u^{\frac{s+1}{2}}|^2 \phi^2 dxdy &= \frac{(s+1)^2}{4} \int_{\mathbb{R}^N} u^{s-1} |\nabla_{\mathbf{z}} u|^2 \phi^2 dxdy \\ &= \frac{(s+1)^2}{4s} \int_{\mathbb{R}^N} \phi^2 \nabla_{\mathbf{z}}(u^s) \nabla_{\mathbf{z}} u dxdy \\ &= \frac{(s+1)^2}{4s} \int_{\mathbb{R}^N} \varrho(\mu) u^s v^{q_1} \phi^2 dxdy \\ &\quad + \frac{s+1}{4s} \int_{\mathbb{R}^N} u^{s+1} \Big(\Delta_{\mathbf{z}}(\phi^2) + \nu \cdot \nabla_{\mathbf{z}}(\phi^2) \Big) dxdy, \end{aligned}$$

$$(s+1) \int_{\mathbb{R}^N} u^s \phi \nabla_{\mathbf{z}} u \nabla \phi dxdy = \frac{1}{2} \int_{\mathbb{R}^N} \nabla(u^{s+1}) \nabla_{\mathbf{z}}(\phi^2) dxdy = -\frac{1}{2} \int_{\mathbb{R}^N} u^{s+1} \Delta_{\mathbf{z}}(\phi^2) dxdy,$$

and

$$(s+1) \int_{\mathbb{R}^N} \nu \cdot \nabla_{\mathbf{z}} u u^s \phi^2 dxdy = \int_{\mathbb{R}^N} \nu \cdot \nabla_{\mathbf{z}}(u^{s+1}) \phi^2 dxdy = -\int_{\mathbb{R}^N} u^{s+1} \nu \cdot \nabla_{\mathbf{z}}(\phi^2) dxdy.$$

We now apply the stability inequality (20) for $\gamma = u^{\frac{s+1}{2}} \phi$ with $s > 0$ and $\phi \in C_c^\infty(\mathbb{R}^N, [0,1])$. In view of the above equalities, we deduce that

$$
\begin{aligned}
&\sqrt{q_1 q_2}\int_{\mathbb{R}^N} \varrho(\mu) u^{\frac{q_2-1}{2}} v^{\frac{q_1-1}{2}} u^{s+1}\phi^2 dxdy \\
&\leq \frac{1}{4}\int_{\mathbb{R}^N} \left| \nu u^{\frac{s+1}{2}}\phi + 2\nabla_{\mathbf{z}}(u^{\frac{s+1}{2}}\phi)\right|^2 dxdy \\
&\leq \frac{1}{4}\int_{\mathbb{R}^N} \nu^2 u^{s+1}\phi^2 dxdy + \int_{\mathbb{R}^N} |\nabla_{\mathbf{z}}(u^{\frac{s+1}{2}}\phi)|^2 dxdy + \frac{1}{2}\int_{\mathbb{R}^N}\left|\nu\cdot\nabla_{\mathbf{z}}(u^{s+1}\phi^2)\right| dxdy \\
&\leq \frac{(s+1)^2}{4s}\int_{\mathbb{R}^N} \varrho(\mu) u^s v^{q_1}\phi^2 dxdy + C\int_{\mathbb{R}^N} u^{s+1}\left[\nu^2\phi^2 + |\nabla_{\mathbf{z}}\phi|^2 + \Delta_{\mathbf{z}}(\phi^2) + \nu\cdot\nabla_{\mathbf{z}}(\phi^2)\right] dxdy.
\end{aligned}
$$

So we get

$$
\begin{aligned}
&b_1\int_{\mathbb{R}^N} \varrho(\mu) u^{\frac{q_2-1}{2}} v^{\frac{q_1-1}{2}} u^{s+1}\phi^2 dxdy \\
&\leq \int_{\mathbb{R}^N} \varrho(\mu) u^s v^{q_1}\phi^2 dxdy + C\int_{\mathbb{R}^N} u^{s+1}\left[\nu^2\phi^2 + |\nabla_{\mathbf{z}}\phi|^2 + \Delta_{\mathbf{z}}(\phi^2) + \nu\cdot\nabla_{\mathbf{z}}(\phi^2)\right] dxdy,
\end{aligned}
$$

where $b_1 = \frac{4s\sqrt{q_1 q_2}}{(s+1)^2}$. We choose $\phi(x,y) = \xi(\frac{x}{R}, \frac{y}{R^{1+\mathbf{z}}})$, where $0 \leq \xi \in C_c^\infty(\mathbb{R}^{\mathbb{N}}) \leq 1$, is a cut-off function satisfying

$$
\xi = 1 \quad \text{on} \quad B_1 \times B_1, \quad \text{and} \quad \xi = 0 \quad \text{outside} \quad B_2 \times B_{2^{1+\mathbf{z}}}.
$$

A direct calculation gives

$$
|\nabla_{\mathbf{z}}\phi| \leq \frac{C}{R} \quad \text{and} \quad |\Delta_{\mathbf{z}}(\phi^2)| \leq \frac{C}{R^2}.
$$

Hence,

$$
I_1 := \int_{\mathbb{R}^N} \varrho(\mu) u^{\frac{q_2-1}{2}} v^{\frac{q_1-1}{2}} u^{s+1}\phi^2 dxdy \leq \frac{1}{b_1}\int_{\mathbb{R}^N} \varrho(\mu) u^s v^{q_1}\phi^2 dxdy + \frac{C}{R^2}\int_{B_{2R}\times B_{(2R)^{1+\mathbf{z}}}} u^{s+1} dxdy.
$$

If we now invoke (20) with $\gamma = v^{\frac{t+1}{2}}\phi$, $t > 0$, it follows from I_1 that

$$
I_2 := \int_{\mathbb{R}^N} \varrho(\mu) u^{\frac{q_2-1}{2}} v^{\frac{q_1-1}{2}} v^{t+1}\phi^2 dxdy \leq \frac{1}{b_2}\int_{\mathbb{R}^N} \varrho(\mu) u^{q_2} v^t\phi^2 dxdy + \frac{C}{R^2}\int_{B_{2R}\times B_{(2R)^{1+\mathbf{z}}}} v^{t+1} dxdy,
$$

with $b_2 = \frac{4t\sqrt{q_1 q_2}}{(t+1)^2}$. Combining the two last inequalities, we have

$$
\begin{aligned}
&I_1 + b_2^{\frac{2(t+1)}{q_1+1}} I_2 \\
&\leq \frac{1}{b_1}\int_{\mathbb{R}^N} \varrho(\mu) u^s v^{q_1}\phi^2 dxdy + b_2^{\frac{2t+1-q_1}{q_1+1}}\int_{\mathbb{R}^N} \varrho(\mu) u^{q_2} v^t\phi^2 dxdy \\
&\quad + \frac{C}{R^2}\int_{B_{2R}\times B_{(2R)^{1+\mathbf{z}}}} \left(u^{s+1} + v^{t+1}\right) dxdy.
\end{aligned} \tag{31}
$$

Fix

$$
s = \frac{(q_2+1)p}{q_1+1} + \frac{q_2-q_1}{q_1+1}, \quad \text{or equivalently} \quad s+1 = \frac{(q_2+1)(p+1)}{q_1+1}. \tag{32}
$$

Let $t > \frac{q_1-1}{2}$. We apply Young's inequality, and from (32), we get then

$$\begin{aligned}
\frac{1}{b_1}\int_{\mathbb{R}^N}\varrho(\mu)u^s v^{q_1}\phi^2 dxdy &= \frac{1}{b_1}\int_{\mathbb{R}^N}\varrho(\mu)u^{\frac{q_2-1}{2}}v^{\frac{q_1-1}{2}}u^{\frac{(q_2+1)t}{q_1+1}+\frac{q_2+1}{q_1+1}\left(\frac{1-q_1}{2}\right)}v^{\frac{q_1+1}{2}}\phi^2\,dxdy\\
&= \frac{1}{b_1}\int_{\mathbb{R}^N}\varrho(\mu)u^{\frac{q_2-1}{2}}v^{\frac{q_1-1}{2}}u^{(s+1)\frac{2t+1-q_1}{2(t+1)}}v^{\frac{q_1+1}{2}}\phi^2 dxdy\\
&\le \frac{2t+1-q_1}{2(t+1)}\int_{\mathbb{R}^N}\varrho(\mu)u^{\frac{q_2-1}{2}}v^{\frac{q_1-1}{2}}u^{s+1}\phi^2 dxdy\\
&\quad+\frac{q_1+1}{2(t+1)}b_1^{-\frac{2(t+1)}{q_1+1}}\int_{\mathbb{R}^N}\varrho(\mu)u^{\frac{q_2-1}{2}}v^{\frac{q_1-1}{2}}v^{t+1}\phi^2 dxdy\\
&= \frac{2t+1-q_1}{2(t+1)}I_1+\frac{q_1+1}{2(t+1)}b_1^{-\frac{2(t+1)}{q_1+1}}I_2,
\end{aligned}$$

and similarly

$$b_2^{\frac{2t+1-q_1}{q_1+1}}\int_{\mathbb{R}^N}\varrho(\mu)u^{q_2}v^t\phi^2 dxdy \le \frac{q_1+1}{2(t+1)}I_1+\frac{2t+1-q_1}{2(t+1)}b_2^{\frac{2(t+1)}{q_1+1}}I_2.$$

We insert the two above estimates in (31), and we get

$$b_2^{\frac{2(t+1)}{q_1+1}}I_2 \le \left[\frac{2t+1-q_1}{2(t+1)}b_2^{\frac{2(t+1)}{q_1+1}}+\frac{q_1+1}{2(t+1)}b_1^{\frac{-2(t+1)}{q_1+1}}\right]I_2+\frac{C}{R^2}\int_{B_{2R}\times B_{(2R)^{1+\mathbf{z}}}}\left(u^{s+1}+v^{t+1}\right)dxdy.$$

Combining (32) and (9), we obtain

$$u^{s+1}\le Cv^{t+1}\quad\text{and}\quad u^{\frac{q_2-1}{2}}v^{\frac{q_1-1}{2}}v^{t+1}\ge u^{q_2}v^t.$$

We get then

$$\frac{q_1+1}{2(t+1)}\left[(b_1b_2)^{\frac{2(t+1)}{q_1+1}}-1\right]\int_{\mathbb{R}^N}u^{q_2}v^t\phi^2 dxdy \le CR^{-2}b_1^{\frac{2(t+1)}{q_1+1}}\int_{B_{2R}\times B_{(2R)^{1+\mathbf{z}}}}v^{t+1}dxdy.$$

Hence, if $b_1b_2>1$, we conclude then

$$\int_{B_R\times B_{(R)^{1+\mathbf{z}}}}\varrho(\mu)u^{q_2}v^t dxdy \le \int_{\mathbb{R}^N}u^{q_2}v^t\phi^2 dxdy \le \frac{C}{R^2}\int_{B_{2R}\times B_{(2R)^{1+\mathbf{z}}}}v^{t+1}dxdy.$$

Denote $n-1=t$; we deduce that if $b_1b_2>1$ and $n>\frac{q_1+1}{2}$,

$$\int_{B_R\times B_{(R)^{1+\mathbf{z}}}}\varrho(\mu)u^{q_2}v^{n-1}dxdy \le \frac{C}{R^2}\int_{B_{2R}\times B_{(2R)^{1+\mathbf{z}}}}v^n dxdy.$$

Furthermore, we can verify the equivalence between $b_1b_2>1$ and $P(n)<0$. So we are done. □

We change the variables $m=\left(\frac{\beta}{2}\right)n$ in (7); a direct calculation gives

$$Q(m)=\left(\frac{\beta}{2}\right)^4 P(n),\quad\text{where}\quad \beta=\frac{2(q_2+1)}{q_1q_2-1},$$

and P is given by (30). Hence $Q(m)<0$ if and only if $P(n)<0$. Moreover, using Lemma 6 in [15], we have the following remark.

Remark 2.

- *Let $1<q_1\le q_2$, then $P(2)<0$, and P has a unique root n_0 in $(2,\infty)$ and $2k_0^+<n_0$.*
- *If $q_1>\frac{4}{3}$, then $P(q_1)<0$ and n_0 is the unique root of P in (q_1,∞), hence $m_0=\left(\frac{\beta}{2}\right)n_0$.*

- *From Remark 3 in [6], we get*

$$m_0 > \beta k_0^+ > 4, \quad \forall\, q_2 \ge q_1 > 1.$$

- *Obviously* $2k_0^- < q_1$ *if* $q_1 > \frac{4}{3}$. *Indeed, if* $q_1 > \frac{4}{3}$ *then* $q_2 \ge q_1 > \frac{4}{3}$ *and* $\varpi := \frac{q_1 q_2 (q_1+1)}{q_2+1} > \frac{16}{9}$. *Since* $g(\varpi) := \sqrt{\varpi} - \sqrt{\varpi - \sqrt{\varpi}}$ *is decreasing in* ϖ*; there holds* $2k_0^- = 2g(\varpi) < 2f(\frac{16}{9}) = \frac{4}{3} < q_1$.

3. Proofs of Main Results

Let $0 \le \eta \in C_c^\infty(\mathbb{R}^{\mathbb{N}}) \le 1$ be a cut-off function satisfying

$$\eta = 1 \quad \text{on} \quad B_1 \times B_1, \quad \text{and} \quad \eta = 0 \quad \text{outside} \quad B_2 \times B_{2^{1+\mathbf{z}}}. \tag{33}$$

The proof may be divided in three parts.

Step 1. There exists a constant $C > 0$ such that for any smooth function $h > 0$, and $\theta_{\mathbf{z}} = \frac{\mathbf{G}_{\mathbf{z}}^*}{\mathbf{G}_{\mathbf{z}}^* - 2}$, we have

$$\left(\int_{\Omega_R} h^{2\theta_{\mathbf{z}}} dxdy \right)^{\frac{1}{\theta_{\mathbf{z}}}} \le C R^{\mathbf{G}_{\mathbf{z}}^* \left(\frac{1}{\theta_{\mathbf{z}}} - 1 \right) + 2} \int_{\Omega_{2R}} |\nabla_{\mathbf{z}} h|^2 dxdy + C R^{\mathbf{G}_{\mathbf{z}}^* \left(\frac{1}{\theta_{\mathbf{z}}} - 1 \right)} \int_{\Omega_{2R}} h^2 dxdy, \tag{34}$$

where

$$\Omega_{\alpha' R} := B_{\alpha' R} \times B_{(\alpha' R)^{1+\mathbf{z}}} \quad \text{with,} \quad \alpha' > 0.$$

Indeed, employing Sobolev inequality [14] and integrating by parts, we get

$$\begin{aligned} \left(\int_{B_1 \times B_1} h^{2\theta_{\mathbf{z}}} dxdy \right)^{\frac{1}{2\theta_{\mathbf{z}}}} &\le \left(\int_{B_2 \times B_{2^{1+\mathbf{z}}}} (h\eta)^{2\theta_{\mathbf{z}}} dxdy \right)^{\frac{1}{2\theta_{\mathbf{z}}}} \\ &\le C \left(\int_{B_2 \times B_{2^{1+\mathbf{z}}}} |\nabla_{\mathbf{z}} (h\eta)|^2 dxdy \right)^{\frac{1}{2}} \\ &\le C \left[\int_{B_2 \times B_{2^{1+\mathbf{z}}}} \left(|\nabla_{\mathbf{z}} h|^2 \eta^2 + h^2 |\nabla_{\mathbf{z}} \eta|^2 - \frac{h^2}{2} \Delta_{\mathbf{z}}(\eta) \right) dxdy \right]^{\frac{1}{2}}. \end{aligned}$$

So, we obtain

$$\left(\int_{B_1 \times B_1} h^{2\theta_{\mathbf{z}}} dxdy \right)^{\frac{1}{\theta_{\mathbf{z}}}} \le C \int_{B_2 \times B_{2^{1+\mathbf{z}}}} \left(|\nabla_{\mathbf{z}} h|^2 + h^2 \right) dxdy.$$

Making use of scaling argument, we get the inequality (34).

Step 2. There exists a positive constant $C > 0$ such that for any $\theta_{\mathbf{z}} = \frac{\mathbf{G}_{\mathbf{z}}^*}{\mathbf{G}_{\mathbf{z}}^* - 2}$ and $2k_0^- < n_0$, there holds

$$\left(\int_{\Omega_R} v^{n_0 \theta_{\mathbf{z}}} dxdy \right)^{\frac{1}{\theta_{\mathbf{z}}}} \le C R^{\mathbf{G}_{\mathbf{z}}^* \left(\frac{1}{\theta_{\mathbf{z}}} - 1 \right)} \int_{\Omega_{2R}} v^{n_0} dxdy. \tag{35}$$

In order to prove this, for $2k_0^- < n_0$, in what follows, taking

$$h = v^{\frac{n_0}{2}}.$$

Set $\eta_R(x,y) = \eta(\frac{x}{R}, \frac{y}{R^{1+\mathbf{z}}})$, where η is given in (33). By a simple calculation, we obtain readily

$$\int_{\Omega_R} |\nabla_{\mathbf{z}} h|^2 dxdy \leq C \int_{\Omega_{2R}} v^{n_0-2} |\nabla_{\mathbf{z}} v|^2 \eta_R^2 dxdy. \tag{36}$$

Multiplying

$$-\Delta_{\mathbf{z}} v + \nu \cdot \nabla_{\mathbf{z}} v = \varrho(\mu) u^{q_2},$$

by $v^{n_0-1}\eta_R^2$ and integrating by parts, we derive

$$\begin{aligned} &(n_0-1)\int_{\Omega_{2R}} v^{n_0-2}|\nabla_{\mathbf{z}} v|^2 \eta_R^2 dxdy \\ &= \int_{\Omega_{2R}} \varrho(\mu) v^{n_0-1} u^{q_2} \eta_R^2 dxdy - \frac{1}{n_0}\int_{\Omega_{2R}} \nu \cdot \nabla_{\mathbf{z}}(v^{n_0})\eta_R^2 - 2\int_{\Omega_{2R}} \eta_R v^{n_0-1} \nabla_{\mathbf{z}} v \cdot \nabla_{\mathbf{z}} \eta_R dxdy. \end{aligned} \tag{37}$$

Using Young's inequality, we obtain

$$2\int_{\Omega_{2R}} v^{n_0-1}|\nabla_{\mathbf{z}} v||\nabla_{\mathbf{z}}\eta_R|\eta_R dxdy \leq \frac{n_0-1}{2}\int_{\Omega_{2R}} v^{n_0-2}|\nabla_{\mathbf{z}} v|^2\eta_R^2 dxdy + C\int_{\Omega_{2R}} v^{n_0}|\nabla_{\mathbf{z}}\eta_R|^2 dxdy.$$

Substituting this in (37), and by inequality (36), we get

$$\int_{\Omega_R} |\nabla_{\mathbf{z}} h|^2 dxdy \leq \int_{\Omega_{2R}} v^{n_0-2}|\nabla_{\mathbf{z}} v|^2 \eta_R^2 dxdy \leq C\int_{\Omega_{2R}} \varrho(\mu) v^{n_0-1} u^{q_2}\eta_R^2 dxdy + \frac{C}{R^2}\int_{\Omega_{2R}} v^{n_0} dxdy.$$

The inequality (34) and Lemma 6 give the estimate (35).

Step 3. Let $q \in (2k_0^-, n_0)$, then for any $\theta_{\mathbf{z}} = \frac{\mathbf{G}_{\mathbf{z}}^*}{\mathbf{G}_{\mathbf{z}}^*-2}$ and $q < n_m\theta_{\mathbf{z}}$, we deduce that there exists a positive constant $C > 0$ with

$$\left(\int_{\Omega_R} v^{n_m\theta_{\mathbf{z}}} dxdy\right)^{\frac{1}{n_m\theta_{\mathbf{z}}}} \leq CR^{\mathbf{G}_{\mathbf{z}}^*\left(\frac{1}{n_m\theta_{\mathbf{z}}} - \frac{1}{q}\right)} \left(\int_{\Omega_{R_m}} v^q dxdy\right)^{\frac{1}{q}}, \tag{38}$$

where

$$\Omega_{R_m} := B_{R_m} \times B_{(R_m)^{1+\mathbf{z}}} \quad \text{with,} \quad m > 0.$$

We know that $2k_0^- < q_1$, from Remark 2. We choose a real number $q > 0$ such as

$$2k_0^- < q < q_1.$$

Let m be a non-negative integer satisfying

$$q\theta_{\mathbf{z}}^{m-1} < n_0 < q\theta_{\mathbf{z}}^m.$$

We construct an increasing geometric sequence given by

$$n_1 = qk, \quad n_2 = qk\theta_{\mathbf{z}}, ..., n_m = qk\theta_{\mathbf{z}}^{m-1},$$

where

$$2k_0^- < n_1 < n_2 <, ..., < n_m < n_0.$$

Here we choose the constant $k \in [1, \theta_{\mathbf{z}}]$, such that n_m is arbitrarily close to n_0. Set $R_n = 2^n R$. From the inequality (35) and by using an induction argument, we get then

$$\begin{aligned}\left(\int_{\Omega_R} v^{n_m\theta_{\mathbf{z}}}dxdy\right)^{\frac{1}{n_m\theta_{\mathbf{z}}}} &\le CR^{\mathbf{G}_{\mathbf{z}}^*\left(\frac{1}{n_m\theta_{\mathbf{z}}}-\frac{1}{n_m}\right)}\left(\int_{\Omega_1} v^{n_m}dxdy\right)^{\frac{1}{n_m}}\\ &= CR^{\mathbf{G}_{\mathbf{z}}^*\left(\frac{1}{n_m\theta_{\mathbf{z}}}-\frac{1}{n_m}\right)}\left(\int_{\Omega_1} v^{n_{m-1}\theta_{\mathbf{z}}}dxdy\right)^{\frac{1}{n_{m-1}\theta_{\mathbf{z}}}}\\ &\le CR^{\mathbf{G}_{\mathbf{z}}^*\left(\frac{1}{n_m\theta_{\mathbf{z}}}-\frac{1}{n_1}\right)}\left(\int_{\Omega_{R_m}} v^{n_1}dxdy\right)^{\frac{1}{n_1}}\\ &\le CR^{\mathbf{G}_{\mathbf{z}}^*\left(\frac{1}{n_m\theta_{\mathbf{z}}}-\frac{1}{qk}\right)}\left(\int_{\Omega_{R_m}} v^{qk}dxdy\right)^{\frac{1}{qk}},\end{aligned} \tag{39}$$

where $\Omega_1 := B_1 \times B_{(1)^{1+\mathbf{z}}}$. Now, using Hölder's inequality, there holds

$$\begin{aligned}\left(\int_{\Omega_{R_m}} v^{qk}dxdy\right)^{\frac{1}{qk}} &\le \left[\left(\int_{\Omega_{R_m}} v^{q\theta_{\mathbf{z}}}dxdy\right)^{\frac{k}{\theta_{\mathbf{z}}}}\left(\int_{B_{R_m}\times B_{(R_m)^{1+s}}} dxdy\right)^{1-\frac{k}{\theta_{\mathbf{z}}}}\right]^{\frac{1}{qk}}\\ &\le C\left[\left(\int_{\Omega_{R_m}} v^{q\theta_{\mathbf{z}}}dxdy\right)^{\frac{k}{\theta_{\mathbf{z}}}} CR^{\mathbf{G}_{\mathbf{z}}^*\left(1-\frac{k}{\theta_{\mathbf{z}}}\right)}\right]^{\frac{1}{qk}}\\ &\le CR^{\mathbf{G}_{\mathbf{z}}^*\left(\frac{1}{kq}-\frac{1}{q\theta_{\mathbf{z}}}\right)}\left(\int_{\Omega_{R_m}} v^{q\theta_{\mathbf{z}}}dxdy\right)^{\frac{1}{q\theta_{\mathbf{z}}}}\\ &\le CR^{\mathbf{G}_{\mathbf{z}}^*\left(\frac{1}{kq}-\frac{1}{q\theta_{\mathbf{z}}}\right)}R^{\mathbf{G}_{\mathbf{z}}^*\left(\frac{1}{q\theta_{\mathbf{z}}}-\frac{1}{q}\right)}\left(\int_{\Omega_{R_m}} v^{q}dxdy\right)^{\frac{1}{q}}.\end{aligned} \tag{40}$$

Then, we combine the last tow inequalities to get the result.

We are now ready to prove Theorem 2.

3.1. Proof of Theorem 2

Let $1 < q_1 \le q_2$ and (u, v) be a stable solution to (1), where (3) is satisfied. We divide the proof into two cases:

Case 1: $q_1 > \frac{4}{3}$**.** Let $q_1 > q > 0$. From (25), Hölder's inequality implies

$$\begin{aligned}\int_{\Omega_R} v^q dxdy &\le \left(\int_{\Omega_R} \varrho(\mu)v^{q_1}dxdy\right)^{\frac{q}{q_1}} \times \left(\int_{\Omega_R}\left(1+\|\mu\|^{2(\mathbf{z}+1)}\right)^{-\frac{\rho q_1}{2(\mathbf{z}+1)(q_1-q)}}dxdy\right)^{\frac{q_1-q}{q_1}}\\ &\le CR^{\left[\mathbf{G}_{\mathbf{z}}^*-\frac{2(q_2+1)q_1}{q_1q_2-1}-\frac{(q_1+1)\rho}{q_1q_2-1}\right]\frac{q}{q_1}+\left(\mathbf{G}_{\mathbf{z}}^*-\frac{\rho q}{q_1-q}\right)\frac{q_1-q}{q_1}} = CR^{\mathbf{G}_{\mathbf{z}}^*-\frac{(2+\rho)(q_2+1)}{q_1q_2-1}q}.\end{aligned} \tag{41}$$

Substituting this in (38), we then get

$$\left(\int_{\Omega_R} v^{n_m\theta_{\mathbf{z}}}dxdy\right)^{\frac{1}{n_m\theta_{\mathbf{z}}}} \le CR^{\frac{\mathbf{G}_{\mathbf{z}}^*}{n_m\theta_{\mathbf{z}}}-\frac{\beta(2+\rho)}{2}}. \tag{42}$$

Recall that $\theta_{\mathbf{z}} = \frac{\mathbf{G}_{\mathbf{z}}^*}{\mathbf{G}_{\mathbf{z}}^*-2}$. Since

$$\mathbf{G}_{\mathbf{z}}^* < 2 + \left(\frac{\beta}{2}\right)(2+\rho)n_0,$$

We choose $k \in [1, \theta_{\mathbf{z}}]$, such that n_m is close to n_0. Then, (42) implies that

$$\|v\|_{L^{n_m\theta_{\mathbf{z}}}(\mathbb{R}^N)} = 0, \quad \text{as}, \quad R \to \infty$$

i.e., $v \equiv 0$ in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. This is a contradiction. Then, we deduce that (1) does not admit any stable solution if

$$\mathbf{G}_{\mathbf{z}}^{*} < 2 + (2+\rho)m_0 \quad \text{where,} \quad m_0 = \left(\frac{\beta}{2}\right)n_0 \quad \text{and} \quad \beta = \frac{2(q_2+1)}{q_1q_2-1}.$$

Finally, Remark 2 implies that if $\mathbf{G}_{\mathbf{z}}^{*} \leq 10 + 4\rho$, (1) has no stable solution for any $q_2 \geq q_1 > \frac{4}{3}$.

Case 2: $1 < q_1 \leq \frac{4}{3}$ and u is bounded. Let $0 < q < 2$ and from (27), we derive

$$\begin{aligned}\int_{\Omega_R} v^q dxdy \ &\leq \left(\int_{\Omega_R} \varrho(\mu)v^2 dxdy\right)^{\frac{q}{2}} \times \left(\int_{\Omega_R} \left(1+\|\mu\|^{2(\mathbf{z}+1)}\right)^{-\frac{\rho q}{2(\mathbf{z}+1)(2-q)}} dxdy\right)^{\frac{2-q}{2}} \\ &\leq CR^{\gamma_3\frac{q}{2}+\left(\mathbf{G}_{\mathbf{z}}^{*}-\frac{\rho q}{2-q}\right)\frac{2-q}{2}} = CR^{\mathbf{G}_{\mathbf{z}}^{*}-\left[\left(\frac{\beta}{2}\right)q_1+\frac{(2+\rho)(2-q_1)}{q_2+q_1-2}+\left(\frac{\beta}{4}\right)q_1\rho\right]q}.\end{aligned}$$

Substituting this in (38), we get

$$\left(\int_{\Omega_R} v^{n_m\theta_{\mathbf{z}}} dxdy\right)^{\frac{1}{n_m\theta_{\mathbf{z}}}} \leq CR^{\frac{\mathbf{G}_{\mathbf{z}}^{*}}{n_m\theta_{\mathbf{z}}}-\left[\left(\frac{\beta}{2}\right)q_1+\frac{(2+\rho)(2-q_1)}{q_2+q_1-2}+\left(\frac{\beta}{4}\right)q_1\rho\right]}.$$

Proceeding as Case 1, we get the desired result. □

3.2. *Proof of Proposition 1*

Let u be a stable solution of (2) with $q_1 = q_2 > 1$. We can proceed like for the proof of Theorem 2. By Remark 2, we can easily show that if $k_0^+ > q_1 > 1$ then $2k_0^+$ is the largest root of

$$P(n) = n^4 - 16q_1^2n^2 + 32q_1^2n - 16q_1^2 = (n^2 + 4q_1(n-1))(n-2k_0^-)(n-2k_0^+),$$

with $k_0^{\pm} = q_1 \pm \sqrt{q_1^2 - q_1}$.

So, we get

$$m_0 = \frac{2q_1 + 2\sqrt{q_1^2 - q_1}}{q_1 - 1} > 4 \quad \text{for all } q_1 > 1.$$

The result follows directly by relying on Theorem 2. □

4. Conclusions

In this paper, we consider a class of weighted Grushin system involving the advection term. Relying on Mtiri's approach [10] and using the techniques developed in [17,19], we gave a Liouville-type theorem for the class of stable positive solution under some assumptions. Therefore, our conclusion of Theorem 2 and Proposition 1 can be viewed as an expansion of previous works, which is therefore interesting and meaningful. For future works, giving attention to [23], we believe that Theorem 2 can be generalised for systems including advection terms with negative exponents and for fractional Grushin systems involving advection term with exponential nonlinearity.

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