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Minimum Number of Colours to Avoid k -Term Monochromatic Arithmetic Progressions

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Abstract: By recalling van der Waerden theorem, there exists a least a positive integer $w = w(k; r)$ such that for any $n \geq w$, every r -colouring of $[1, n]$ admits a monochromatic k -term arithmetic progression. Let $k \geq 2$ and $r_k(n)$ denote the minimum number of colour required so that there exists a $r_k(n)$ -colouring of $[1, n]$ that avoids any monochromatic k -term arithmetic progression. In this paper, we give necessary and sufficient conditions for $r_k(n+1) = r_k(n)$. We also show that $r_k(n) = 2$ for all $k \leq n \leq 2(k-1)^2$ and give an upper bound for $r_p(p^m)$ for any prime $p \geq 3$ and integer $m \geq 2$.

Keywords: van der Waerden theorem; monochromatic arithmetic progression

1. Introduction

Let $\mathbb{N}$ be the set of positive integer, i.e., $\mathbb{N} = \{1, 2, 3, \dots\}$. A subset $A \subseteq \mathbb{N}$ is said to have an arithmetic progression (a.p.) of length k , if there exist $a, d \in \mathbb{N}$ such that

$$a, a + d, a + 2d, \dots, a + (k-1)d \in A.$$

The arithmetic progression of length k is also called a k -term a.p. with difference d , or we write $\{a + ld\}_{0 \leq l \leq k-1}$.

There are many problems related to arithmetic progressions. One of them is to find an arrangement of $[n] = \{1, 2, \dots, n\}$ that avoids k -term arithmetic progressions. An arrangement of $[n]$ is a sequence $a_1, a_2, \dots, a_n$ such that $\{a_1, a_2, \dots, a_n\} = [n]$. Davis et al. [1] discovered an arrangement of $[n] = \{1, 2, \dots, n\}$ that avoids three-term monotone arithmetic progressions. By using the construction of certain magic square, Sim and Wong [2] proposed an arrangement of $[n]$ that avoids a k -term monotone arithmetic progression, but contains a $(k-1)$ -term monotone arithmetic progression. For the infinite case, Davis et al. [1] and Sidorenko [3] showed that there is arrangement of $\mathbb{N}$ that avoids three-term monotone arithmetic progressions. Here, an arrangement of $\mathbb{N}$ is a sequence $a_1, a_2, \dots, a_n, \dots$, such that $\{a_1, a_2, \dots, a_n, \dots\} = \mathbb{N}$. They [1] also showed that there exists an arrangement of $\mathbb{N}$ that avoid five-term monotone arithmetic progressions. Recently, Geneson [4] constructed an arrangement of the set of integers that avoids six-term monotone arithmetic progressions. Up to now, it is still unknown whether there exists an arrangement of positive integers that avoids four-term monotone arithmetic progressions [5].

Another interesting problem is the colouring problem, which we will describe shortly. For a positive integer t , we denote the set $\{1, 2, 3, \dots, t\}$ by $[1, t]$. A r -colouring of a set S is a function $\chi : S \rightarrow [1, r]$. A monochromatic k -term a.p. refers to a k -term a.p. such that all of its elements are of the same colour.

We now state the van der Waerden's theorem, which was proved in 1927, see [6].

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Theorem 1. (Van der Waerden's Theorem) Let $k, r \geq 2$ be integers. There exists a least positive integer $w = w(k; r)$ such that for any $n \geq w$, every r -colouring of $[1, n]$ admits a monochromatic k -term arithmetic progression.

In an effort to investigate the van der Waerden theorem, Erdős and Turán [7] approached the problem in the opposite direction: Given a positive integer n , what is the maximum size of a subset of $[1, n]$ that does not contain an arithmetic progression of length k ? They define the following.

Definition 1. For $k \geq 2$ and $n \geq 3$, let $\mathcal{S}$ be the collection of sets $S \subseteq [1, n]$ such that S does not contain an arithmetic progression of length k . Then $v_k(n) = \max(\{|S| : S \in \mathcal{S}\})$.

Note that $v_k(n)$ is monotone increasing in $k \geq 3$. Suppose $v_k(n) = w$, then there is a subset of $w + 1$ integers in $[1, n]$ which may contain only k -term a.p. but definitely does not contain a $k + 1$ a.p. Hence $v_{k+1}(n) > v_k(n)$. They [7] showed that $v_3(2n) \leq n$ for $n \geq 8$ and conjectured $v_3(n) = o(n)$. The conjecture was solved by Roth [8] and has been strengthened (see [9,10]). Studying $v_k(n)$ and having an upper bound on these numbers would lead to an upper bound on $w(k; r)$. In 2018, Blankenship, Cummings, and Taranchuk [11] showed that for $p \geq 5$ and q be primes, then $w(p + 1; q) \geq p(q^p - 1) + 1$. The current best known general lower bound of this type is due to Kozik and Shabanov [12], who in 2016 proved that, for all $r \geq 2$

$$w(k; r) \geq cr^{k-1} \quad (1)$$

for some constant $c > 0$.

Given two positive integers n and k , the *anti-van der Waerden number* $aw([n], k)$ is the smallest r such that every exact r -colouring of $[n]$ contains a rainbow k -a.p., where an exact r -colouring is a colouring using all the r colours. For more results related to rainbow arithmetic progressions, see [13–16]. Recently, Li et al. [17] investigated the integer colourings with no rainbow 3-term arithmetic progression. They obtained the asymptotic number of r -colourings of $[n]$ without rainbow 3-term arithmetic progressions, and showed that the typical colourings with this property are 2-colourings.

Theorem 2 ([5]). Let $k \geq 2, r \geq 2$ and assume $v_k(m) \leq f(k, m)$, with $f(k, m) \leq \frac{m-1}{r}$. Then $w(k; r) \leq rf(k, m) + 1$.

Let $k \geq 2$ and $r_k(n)$ denote the minimum number of colour required so that there exists a $r_k(n)$ -colouring of $[1, n]$ that avoids any monochromatic k -term a.p. Since $[1, n]$ is coloured by $r_k(n)$ colours, clearly

$$v_k(n) \geq \left\lceil \frac{n}{r_k(n)} \right\rceil.$$

From [18], $r_k(n) < \frac{2n \log n}{v_k(n)}(1 + o(1))$ is obtained from the consequence of Symmetric Hypergraph Theorem. For sufficiently large n , $r_k(n) < \frac{3n \log n}{v_k(n)}$, see [19]. In this paper, we are focusing on the the minimum number of colour in $[1, n]$ to avoid a monochromatic k -term a.p. From the known results of $w(3; 2) = 9$, $w(3; 3) = 27$, $w(3; 4) = 76$ and $w(3; 5) \geq 171$, see [20–22], we have $r_3(9) = 3$, $r_3(27) = 4$, $r_3(76) \geq 5$. Additionally, if $1 < n_1 < n_2$, then $r_k(n_1) \leq r_k(n_2)$.

In this paper, we give necessary and sufficient conditions for $r_k(n + 1) = r_k(n)$ (Theorem 3). We also show that $r_k(n) = 2$ for all $k \leq n \leq 2(k - 1)^2$ (Theorem 4). Finally, we give an upper bound for $r_p(p^m)$ for any prime $p \geq 3$ and integer $m \geq 2$ (Theorem 5).

2. Main Results

Note that $r_2(n) = n$. Therefore, we shall assume that $k \geq 3$.

Lemma 1.

$$r_k(n_1 + n_2) \leq r_k(n_1) + r_k(n_2).$$

Proof. Colour $[1, n_1]$ with $r_k(n_1)$ colours that avoids any monochromatic k -term a.p. and colour $[n_1 + 1, n_1 + n_2]$ with different $r_k(n_2)$ colours that avoids any monochromatic k -term a.p.. Note that this colouring avoids any monochromatic k -term a.p. in $[1, n_1 + n_2]$. Hence, the lemma follows. $\square$

The following corollary is a consequence of Lemma 1 by noting that $r_k(i) = 1$ for $1 \leq i \leq k - 1$.

Corollary 1. For $1 \leq i \leq k - 1$,

$$r_k(n + i) \leq r_k(n) + 1.$$

Since $r_k(n) \leq r_k(n + 1)$, we have the following corollary.

Corollary 2. $r_k(n + 1) = r_k(n)$ or $r_k(n) + 1$.

Corollary 3. $r_k(w(k; l) - 1) = l$ and $r_k(n) = l + 1$ for $w(k; l) \leq n \leq w(k; l + 1) - 1$.

Proof. There is a l -colouring of $[1, w(k; l) - 1]$ that avoids any k -term a.p.. Therefore, $r_k(w(k; l) - 1) \leq l$. If $r_k(w(k; l) - 1) \leq l - 1$, then by Corollary 2, $r_k(w(k; l)) \leq l$. This is not possible as any l -colouring of $[1, w(k; l)]$ has a monochromatic k -term a.p. Hence, $r_k(w(k; l) - 1) = l$.

Since any l -colouring of $[1, w(k; l)]$ has a monochromatic k -term a.p., by Corollary 2, $r_k(w(k; l)) = l + 1$. By first part of the corollary, $r_k(w(k; l + 1) - 1) = l + 1$. Again, by Corollary 2, $r_k(n) = l + 1$ for $w(k; l) \leq n \leq w(k; l + 1) - 1$. $\square$

For $l \geq 1$, a family of sets $\{A_1, A_2, \dots, A_l\}$ is called a partition of $[1, n]$ if

- (a) $A_i \cap A_j = \emptyset$ for $i \neq j$;
- (b) $\bigcup_{i=1}^l A_i = [1, n]$.

Here, we allow $A_j = \emptyset$ for some j . If each A_j avoids any k -term a.p., the partition $\{A_1, A_2, \dots, A_l\}$ is called a free k -term a.p. partition.

Let $r_k(n) = l$. Then, there exists a l -colouring of $[1, n]$ that avoids any monochromatic k -term a.p. For $1 \leq i \leq l$, let

$$A_i = \{a \in [1, n] : a \text{ is coloured with colour } i\}.$$

Note that $\{A_1, A_2, \dots, A_l\}$ is a free k -term a.p. partition of $[1, n]$. So, $r_k(n)$ can be considered as the minimum size of a free k -term a.p. partition of $[1, n]$.

A subset $A \subseteq \mathbb{N}$ is said to have a pure arithmetic progression of length k (or pure k -term a.p.), if there exists a $d \in \mathbb{N}$ such that

$$d, 2d, 3d, \dots, kd \in A.$$

Given a subset $A \subseteq \mathbb{N}$ and $c \in \mathbb{N}$, we write

$$c - A = \{c - a : a \in A\} \text{ and } c + A = \{c + a : a \in A\}.$$

Lemma 2. Let $\{A_1, A_2, \dots, A_l\}$ be a partition of $[1, n]$ such that each A_i does not contain any k -term a.p. Suppose there is a A_{i_0} that does not contain any pure $(k - 1)$ -term a.p. Let $B_{i_0} = ((n + 1) - A_{i_0}) \cup \{n + 1\}$ and $B_j = (n + 1) - A_j$ for all $j \neq i_0$. Then, $\{B_1, B_2, \dots, B_l\}$ is a free k -term a.p. partition of $[1, n + 1]$.

Proof. Note that $(n+1) - [1, n] = [1, n]$. So, it is not hard to see that $\{B_1, B_2, \dots, B_l\}$ is a partition of $[1, n+1]$. Now, if B_j contains a k -term a.p. for some $j \neq i_0$, say $\{a + sd\}_{0 \leq s \leq k-1}$, then A_j contains the k -term a.p. $\{((n+1) - a - (k-1)d) + ld\}_{0 \leq l \leq k-1}$, a contradiction. So, we may assume that B_j does not contain any k -term a.p. for all $j \neq i_0$.

Suppose B_{i_0} contains a k -term a.p. for some $j \neq i_0$, say $\{a' + ld'\}_{0 \leq l \leq k-1}$. If $a' + (k-1)d' \neq n+1$, then A_{i_0} contains the k -term a.p. $\{((n+1) - a' - (k-1)d') + sd'\}_{0 \leq s \leq k-1}$, a contradiction. So, we may assume that $a' + (k-1)d' = n+1$. This implies that A_{i_0} contains the pure $(k-1)$ -term a.p. $\{sd'\}_{1 \leq s \leq k-1}$, a contradiction. $\square$

Theorem 3. Suppose $r_k(n) = l$. Then, $r_k(n+1) = l$ if and only if there exists a free k -term a.p. partition $\{A_1, A_2, \dots, A_l\}$ of $[1, n]$ such that A_{i_0} does not contain any pure $(k-1)$ -term a.p. for some $i_0 \in [1, l]$.

Proof. If there exists a free k -term a.p. partition $\{A_1, A_2, \dots, A_l\}$ of $[1, n]$ and A_{i_0} does not contain any pure $(k-1)$ -term a.p., then by Lemma 2, $\{B_1, B_2, \dots, B_l\}$ is a free k -term a.p. partition of $[1, n+1]$ where $B_{i_0} = ((n+1) - A_{i_0}) \cup \{n+1\}$ and $B_j = (n+1) - A_j$ for all $j \neq i_0$. It follows from Corollary 2 that $r_k(n+1) = l$.

Suppose $r_k(n+1) = l$. Then, there exists a free k -term a.p. partition $\{C_1, C_2, \dots, C_l\}$ of $[1, n+1]$. By relabelling if necessary, we may assume that $(n+1) \in C_1$. Let $D_1 = (n+1) - (C_1 \setminus \{n+1\})$ and $D_j = (n+1) - C_j$ for $2 \leq j \leq l$. Note that $\{D_1, D_2, \dots, D_l\}$ is a free k -term a.p. partition of $[1, n]$. If D_1 contains a pure $(k-1)$ -term a.p., say $\{sd\}_{1 \leq s \leq k-1}$, then $\{(n+1) - (k-1)d + sd\}_{0 \leq s \leq k-1}$ is a k -term a.p. in C_1 , a contradiction. Hence, D_1 does not contain any pure $(k-1)$ -term a.p. $\square$

Corollary 4. Suppose $r_k(n) = l$ and $r_k(n+1) = l+1$. Then, $r_k(n+j) = l+1$ for all $1 \leq j \leq 2k-2$.

Proof. Let $\{A_1, A_2, \dots, A_l\}$ be a free k -term a.p. partition of $[1, n]$. Since $r_k(n+1) = l+1$, by Theorem 3, each A_i contains a pure $(k-1)$ -term a.p. Let

$$\begin{aligned} B_i &= (k-1) + A_i \text{ for all } 1 \leq i \leq l; \\ B_{l+1} &= \{1, 2, \dots, k-1, n+k, n+k+1, \dots, n+2k-2\}. \end{aligned}$$

Note that $\{B_1, B_2, \dots, B_{l+1}\}$ is a free k -term a.p. partition of $[1, n+2k-2]$. Therefore, $r_k(n+j) \leq l+1$ for all $1 \leq j \leq 2k-2$. The equality follows from Corollary 2. $\square$

Corollary 5.

$$w(k; l+1) \geq k + 2l(k-1).$$

Proof. By Corollary 3, $r_k(w(k; l) - 1) = l$ and $r_k(n) = l+1$ for $w(k; l) \leq n \leq w(k; l+1) - 1$. It follows from Corollary 4 that

$$w(k; l+1) - 1 \geq (w(k; l) - 1) + 2k - 2,$$

which is equivalent to $w(k; l+1) \geq w(k; l) + 2k - 2$. Since $w(k; 1) = k$, we have

$$\begin{aligned} w(k; l+1) &\geq w(k; l) + 2(k-1) \\ &\geq w(k; l-1) + 4(k-1) \\ &\geq w(k; l-2) + 6(k-1) \\ &\vdots \\ &\geq w(k; 1) + 2l(k-1) = k + 2l(k-1). \end{aligned}$$

$\square$

Let $a, b \in \mathbb{N}$ with $a < b$. We shall write $[a, b] = \{a, a + 1, \dots, b\}$ and $[a, a] = \{a\}$.

Theorem 4. $r_k(n) = 2$ for all $k \leq n \leq 2(k-1)^2$.

Proof. For $1 \leq i \leq k-1$ and $0 \leq j \leq k-2$, let

$$A(i, j) = [1, i] \cup \bigcup_{s=1}^j [i + 2s(k-1) - (k-2), i + 2s(k-1)]$$

$$B(i, j) = [i + 2j(k-1) + 1, 2i + 2j(k-1)] \cup \bigcup_{s=1}^j [i + (2s-1)(k-1) - (k-2), i + (2s-1)(k-1)].$$

In particular, when $j = 0$

$$A(i, 0) = [1, i]$$

$$B(i, 0) = [i + 1, 2i].$$

Note that $\{A(i, j), B(i, j)\}$ is a partition of $[1, 2i + 2j(k-1)]$. Now,

$$(2i + 2j(k-1) + 1) - [i + 2j(k-1) + 1, 2i + 2j(k-1)] = [1, i]$$

and

$$(2i + 2j(k-1) + 1) - [i + (2s-1)(k-1) - (k-2), i + (2s-1)(k-1)]$$

$$= [i + 2(j-s+1)(k-1) - (k-2), i + 2(j-s+1)(k-1)].$$

So, $(2i + 2j(k-1) + 1) - B(i, j) = A(i, j)$.

Clearly, $\{A(i, 0), B(i, 0)\}$ is a free k -term a.p. partition for $1 \leq i \leq k-1$. Suppose $\{A(i_0, j_0), B(i_0, j_0)\}$ is a free k -term a.p. partition for some $1 \leq i_0 \leq k-2$ and $0 \leq j_0 \leq k-2$. We claim that $\{A(i_0 + 1, j_0), B(i_0 + 1, j_0)\}$ is also a free k -term a.p. partition.

Since $i_0 \leq k-2$, $A(i_0, j_0)$ does not contain $\{m\}_{1 \leq m \leq k-1}$. Suppose $A(i_0, j_0)$ contains a pure $(k-1)$ -term a.p. $\{md\}_{1 \leq m \leq k-1}$ where $d \geq 2$. Note that

$$d(k-1) \in [i_0 + 2s'(k-1) - (k-2), i_0 + 2s'(k-1)],$$

for some $1 \leq s' \leq j_0$. If $2 \leq d \leq k-1$, there exists a $m' \in [1, k-2]$ such that

$$m'd \in [i_0 + (2s'-1)(k-1) - (k-2), i_0 + (2s'-1)(k-1)] \subseteq B(i_0, j_0),$$

a contradiction. So, we may assume that $d \geq k$. If both $m_1 d$ and $(m_1 + 1)d$ are in

$$[i + 2s(k-1) - (k-2), i + 2s(k-1)],$$

for some $1 \leq s \leq j_0$, then $d \leq k-2$, a contradiction. So, we may assume that for each s , $[i + 2s(k-1) - (k-2), i + 2s(k-1)]$ contains at most one element of the form md where $1 \leq m \leq k-1$. This implies that $d \in [1, i_0]$ and $d \leq i_0 \leq k-2$, a contradiction. Hence, $A(i_0, j_0)$ does not contain any pure $(k-1)$ -term a.p. By Lemma 2,

$$\{((2i_0 + 2j_0(k-1) + 1) - A(i_0, j_0)) \cup \{2i_0 + 2j_0(k-1) + 1\}, (2i_0 + 2j_0(k-1) + 1) - B(i_0, j_0)\}$$

$$= \{B(i_0, j_0) \cup \{2i_0 + 2j_0(k-1) + 1\}, A(i_0, j_0)\},$$

is a free k -term a.p. partition. Note that

$$((2i_0 + 2j_0(k-1) + 2) - A(i_0, j_0)) \cup \{2i_0 + 2j_0(k-1) + 2\} = B(i_0 + 1, j_0),$$

and

$$(2i_0 + 2j_0(k-1) + 2) - (B(i_0, j_0) \cup \{2i_0 + 2j_0(k-1) + 1\}) = A(i_0 + 1, j_0).$$

By Lemma 2, $\{A(i_0 + 1, j_0), B(i_0 + 1, j_0)\}$ is a free k -term a.p. partition.

Suppose $\{A(k-1, j_0), B(k-1, j_0)\}$ is a free k -term a.p. partition for some $0 \leq j_0 \leq k-3$. We claim that $\{A(1, j_0 + 1), B(1, j_0 + 1)\}$ is also a free k -term a.p. partition. Note that

$$\begin{aligned} A(k-1, j_0) &= [1, k-1] \cup \bigcup_{s=1}^{j_0} [1 + 2s(k-1), (2s+1)(k-1)] \\ B(k-1, j_0) &= [k + 2j_0(k-1), 2(j_0+1)(k-1)] \cup \\ &\quad \bigcup_{s=1}^{j_0} [1 + (2s-1)(k-1), 2s(k-1)]. \end{aligned}$$

Clearly, $B(k-1, j_0)$ does not contain $\{m\}_{1 \leq m \leq k-1}$. Suppose $B(k-1, j_0)$ contains a pure $(k-1)$ -term a.p., say $\{md\}_{1 \leq m \leq k-1}$ where $d \geq 2$. If $2 \leq d \leq k-1$, then $d \in [1, k-1] \subseteq A(k-1, j_0)$, a contradiction. So, we may assume that $d \geq k$. If both m_1d and $(m_1+1)d$ are in

$$[1 + (2s-1)(k-1), 2s(k-1)],$$

for some $1 \leq s \leq j_0$, then $d \leq k-2$, a contradiction. So, we may assume that for each s , $[1 + (2s-1)(k-1), 2s(k-1)]$ contains at most one element of the form md where $1 \leq m \leq k-1$. Since $j_0 \leq k-3$, we must have both $(k-2)d$ and $(k-1)d$ in $[k + 2j_0(k-1), 2(j_0+1)(k-1)]$. This implies that $d \leq k-2$, a contradiction. Hence, $B(k-1, j_0)$ does not contain any pure $(k-1)$ -term a.p. Note that

$$\begin{aligned} ((2(j_0+1)(k-1) + 1) - B(k-1, j_0)) \cup \{2(j_0+1)(k-1) + 1\} \\ = A(k-1, j_0) \cup \{2(j_0+1)(k-1) + 1\}, \end{aligned}$$

and

$$(2(j_0+1)(k-1) + 1) - A(k-1, j_0) = B(k-1, j_0).$$

By Lemma 2, $\{A(k-1, j_0) \cup \{2(j_0+1)(k-1) + 1\}, B(k-1, j_0)\}$ is a free k -term a.p. partition. Note that

$$(2(j_0+1)(k-1) + 2) - B(k-1, j_0) \cup \{2(j_0+1)(k-1) + 2\} = B(1, j_0 + 1),$$

and

$$(2i_0 + 2j_0(k-1) + 2) - (A(k-1, j_0) \cup \{2(j_0+1)(k-1) + 1\}) = A(1, j_0 + 1).$$

By Lemma 2, $\{A(i_0 + 1, j_0), B(i_0 + 1, j_0)\}$ is a free k -term a.p. partition. By double induction, $\{A(i, j), B(i, j)\}$ is a free k -term a.p. partition of $[1, 2i + 2j(k-1)]$. In particular, $\{A(k-1, k-2), B(k-1, k-2)\}$ is a free k -term a.p. partition of $[1, 2(k-1)^2]$. Hence, $r_k(n) = 2$ for all $k \leq n \leq 2(k-1)^2$. $\square$

Note that Theorem 4 is best possible because when $k = 3$, $r_3(n) = 2$ for all $3 \leq n \leq 8$ and $r_3(9) = 3$. Note also that for $k \geq 4$, $2(k-1)^2 \geq k^2$, therefore $r_k(k^2) = 2$.

The following corollary follows from Corollary 3.

Corollary 6. $w(k; 2) \geq 2(k-1)^2 + 1$.

Lemma 3. Let $p \geq 3$ be a prime and $A \subseteq \mathbb{N}$. For each $i \in [1, p]$, let

$$A_i = \{b \in A : b \equiv i \pmod{p}\}.$$

Suppose $A_{i_0} = \emptyset$ for some i_0 . If A contains a p -term a.p., $\{a + sd\}_{0 \leq s \leq p-1}$, then $\gcd(p, d) = p$ and the p -term a.p. is contained entirely in A_{a_0} for some $a_0 \in [1, p]$ with $a_0 \equiv a \pmod{p}$.

Proof. Suppose $\gcd(p, d) = 1$. Then,

$$\{a, a + d, \dots, a + (p-1)d\} \equiv \{1, 2, \dots, p\} \pmod{p}.$$

This implies that $A_i \neq \emptyset$ for all i , a contradiction. Hence, $\gcd(p, d) = p$ and the p -term a.p. is contained entirely in A_{a_0} . $\square$

For each real number x , let $\lceil x \rceil$ be the smallest integer such that $\lceil x \rceil - 1 < x \leq \lceil x \rceil$.

Lemma 4. Let $p \geq 3$ be a prime. If $p \leq k$, then

$$r_k(np) \leq r_k(n) + \left\lceil \frac{r_k(n)}{p-1} \right\rceil$$

Proof. For each $j \in [1, p]$, let

$$A_j = \{a \in [1, np] : a \equiv j \pmod{p}\}.$$

and

$$f_j : [1, n] \rightarrow A_j$$

be defined by $f_j(u) = j + (u-1)p$ for all $u \in [1, n]$. Note that f_j is a bijection.

Let $r_k(n) = l$ and $\{B_1, B_2, \dots, B_l\}$ be a free k -term a.p. partition of $[1, n]$. Then,

$$\{f_j(B_1), f_j(B_2), \dots, f_j(B_l)\}$$

is a free k -term a.p. partition of A_j . Let

$$A_{i,j} = f_j(B_i),$$

for all $1 \leq i \leq l$ and $1 \leq j \leq p$. Then, $A_j = \bigcup_{i=1}^l A_{i,j}$.

Let $l = q(p-1) + e$ where $0 \leq e < p-1$. Let

$$C_1 = \{1, 2, \dots, p-1\}$$

$$C_2 = \{1 + (p-1), 2 + (p-1), \dots, p-1 + (p-1)\}$$

$$\vdots$$

$$C_q = \{1 + (q-1)(p-1), 2 + (q-1)(p-1), \dots, p-1 + (q-1)(p-1)\}$$

$$C_{q+1} = \{1 + q(p-1), 2 + q(p-1), \dots, e + q(p-1)\}.$$

Then, $\{C_1, C_2, \dots, C_{q+1}\}$ is a partition of $[1, l]$ if $e \geq 1$ and $\{C_1, C_2, \dots, C_q\}$ is a partition of $[1, l]$ if $e = 0$. Let $q_0 = \left\lceil \frac{l}{p-1} \right\rceil$. Then, $q_0 = q+1$ if $e \geq 1$ and $q_0 = q$ if $e = 0$.

For each $b \in \mathbb{N}$, let $\bar{b} \in [1, p]$ be such that $\bar{b} \equiv b \pmod{p}$. Now, for each $t \in [1, q_0]$, let

$$D_t = \bigcup_{b \in C_t} A_{b, \bar{b}}$$

Since $|C_t| = p-1$, there is a i_0 such that $D_t(i_0) = \{d \in D_t : d \equiv i_0 \pmod{p}\} = \emptyset$. If D_t contains a k -term a.p., say $\{a + sd\}_{0 \leq s \leq k-1}$, then it contains the p -term a.p.

$\{a + sd\}_{0 \leq s \leq p-1}$. By Lemma 3, $\gcd(p, d) = p$. This implies that $A_{b_1, \bar{b}_1}$ contains the k -term a.p. $\{a + sd\}_{0 \leq s \leq k-1}$ for some $b_1 \in C_t$. Since $A_{b_1, \bar{b}_1} = f_{\bar{b}_1}(B_{b_1})$, B_{b_1} contains the k -term a.p. $\{a' + sd'\}_{0 \leq s \leq k-1}$ where $a' = \frac{a - \bar{b}_1}{p} + 1$ and $d' = \frac{d}{p}$. This is not possible as $\{B_1, B_2, \dots, B_l\}$ is a free k -term a.p. partition of $[1, n]$. Thus, D_t does not contain any k -term a.p.

For each $i \in [1, l]$, let

$$E_i = \left(\bigcup_{j=1}^p A_{i,j} \right) \setminus A_{i,\bar{i}}$$

Clearly, $E_i(\bar{i}) = \{d \in E_i : d \equiv \bar{i} \pmod{p}\} = \emptyset$. If E_i contains a k -term a.p., say $\{a + sd\}_{0 \leq s \leq k-1}$, then it contains the p -term a.p. $\{a + sd\}_{0 \leq s \leq p-1}$. By Lemma 3, $\gcd(p, d) = p$. This implies that A_{i,j_1} contains the k -term a.p. $\{a + sd\}_{0 \leq s \leq k-1}$ for some $j_1 \in [1, p] \setminus \{\bar{i}\}$. Since $A_{i,j_1} = f_{j_1}(B_i)$, B_i contains the k -term a.p. $\{a' + sd'\}_{0 \leq s \leq k-1}$ where $a' = \frac{a - j_1}{p} + 1$ and $d' = \frac{d}{p}$. This is not possible as $\{B_1, B_2, \dots, B_l\}$ is a free k -term a.p. partition of $[1, n]$. Thus, E_i does not contain any k -term a.p.

Now,

$$\{E_1, E_2, \dots, E_l, D_1, D_2, \dots, D_{q_0}\}$$

is a free k -term a.p. partition of $[1, pn]$. Hence, $r_k(np) \leq l + q_0$ and the lemma follows. $\square$

Theorem 5. Let $p \geq 3$ be a prime. Then, $r_p(p) = 2$,

$$r_p(p^2) = \begin{cases} 2, & \text{for } p \geq 5; \\ 3, & \text{for } p = 3, \end{cases}$$

and for $m \geq 3$,

$$r_p(p^m) < \begin{cases} (p+1) \left(\frac{p}{p-1} \right)^{m-2} - p + 1, & \text{for } p \geq 5; \\ 5 \left(\frac{3}{2} \right)^{m-2} - 2, & \text{for } p = 3. \end{cases}$$

Proof. Clearly, $r_p(p) = 2$. By Lemma 4,

$$\begin{aligned} r_p(p^m) &< r_p(p^{m-1}) + \frac{r_p(p^{m-1})}{p-1} + 1 \\ &= \left(\frac{p}{p-1} \right) r_p(p^{m-1}) + 1 \\ &< \left(\frac{p}{p-1} \right) \left(\left(\frac{p}{p-1} \right) r_p(p^{m-2}) + 1 \right) + 1 \\ &= \left(\frac{p}{p-1} \right)^2 r_p(p^{m-2}) + \left(\frac{p}{p-1} \right) + 1 \\ &\vdots \\ &< \left(\frac{p}{p-1} \right)^{m-2} r_p(p^2) + \left(\frac{p}{p-1} \right)^{m-3} + \dots + \left(\frac{p}{p-1} \right) + 1. \end{aligned}$$

Suppose $p \geq 5$. Then, $2(p-1)^2 \geq p^2$. By Theorem 4, $r_p(p^2) = 2$. Thus,

$$\begin{aligned} r_p(p^m) &< 2\left(\frac{p}{p-1}\right)^{m-2} + \left(\frac{p}{p-1}\right)^{m-3} + \cdots + \left(\frac{p}{p-1}\right) + 1 \\ &= 2\left(\frac{p}{p-1}\right)^{m-2} + \frac{\left(\frac{p}{p-1}\right)^{m-2} - 1}{\left(\frac{p}{p-1}\right) - 1} \\ &= 2\left(\frac{p}{p-1}\right)^{m-2} + \left(\left(\frac{p}{p-1}\right)^{m-2} - 1\right)(p-1) \\ &= (p+1)\left(\frac{p}{p-1}\right)^{m-2} - (p-1). \end{aligned}$$

Suppose $p = 3$. Then, $r_3(3^2) = 3$. So,

$$\begin{aligned} r_3(3^m) &< \left(\frac{3}{2}\right)^{m-2} r_3(3^2) + \left(\frac{3}{2}\right)^{m-3} + \cdots + \left(\frac{3}{2}\right) + 1 \\ &= 3\left(\frac{3}{2}\right)^{m-2} + \left(\frac{3}{2}\right)^{m-3} + \cdots + \left(\frac{3}{2}\right) + 1 \\ &= 3\left(\frac{3}{2}\right)^{m-2} + \left(\left(\frac{3}{2}\right)^{m-2} - 1\right)2 \\ &= 5\left(\frac{3}{2}\right)^{m-2} - 2. \end{aligned}$$

□

Corollary 7. For a fixed $m \geq 2$, there exists an integer $n_0 = n_0(m)$ such that for all prime $p \geq n_0$,

$$r_p(p^m) \leq m.$$

Proof. Since $r_p(p^m) = 2$ for $m = 2$, we may assume that $m \geq 3$. Now, the corollary follows Theorem 5 and by noting that $\lim_{p \rightarrow \infty} \left((p+1)\left(\frac{p}{p-1}\right)^{m-2} - p + 1\right) = m$. □

The following corollary follows from Corollaries 3 and 7.

Corollary 8. For a fixed $m \geq 2$, there exists an integer $n_0 = n_0(m)$ such that for all prime $p \geq n_0$,

$$w(p; m) \geq p^m + 1.$$

3. Conclusions

In this paper, we gave the essential conditions for $r_k(n) = r_k(n+1)$ and gave an upper bound for $r_p(p^m)$ for any prime $p \geq 3$ and integer $m \geq 2$. Suppose $p \geq 3$ be a prime. We hope that exact values of $r_p(p^m)$ can be obtained for some small values of $m \geq 3$ in future. Here, we end this paper with the following conjecture. Note that the conjecture is true for $m = 2$.

Conjecture 6. For a fixed $m \geq 2$, there exists an integer $n_0 = n_0(m)$ such that for all prime $p \geq n_0$,

$$r_p(p^m) = m.$$

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