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On the Weak Solutions of a Delay Composite Functional Integral Equation of Volterra-Stieltjes Type in Reflexive Banach Space

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Abstract: Differential and integral equations in reflexive Banach spaces have gained great attention and have been investigated in many studies and monographs. Inspired by those, we study the existence of the solution to a delay functional integral equation of Volterra-Stieltjes type and its corresponding delay-functional integro-differential equation in reflexive Banach space E . Sufficient conditions for the uniqueness of the solutions are given. The continuous dependence of the solutions on the delay function, the initial data, and some other parameters are proved.

Keywords: weakly continuous; Volterra-Stieltjes integral; reflexive Banach space; continuous dependence

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1. Introduction

The existence of weak solutions for ordinary differential equations in Banach spaces has been investigated in many papers, for example, in Cichoń [1,2], Cramer et al. [3], Knight [4], Kubiacyk and Szufła [5], and [6–11] and the references therein for fractional-order differential equations in Banach spaces, and [12–14] for quadratic integral equations in reflexive Banach algebra.

The integral equations of the Volterra-Stieltjes type have been considered by many authors; for recent publications of these integral equation and the properties of the Volterra-Stieltjes integral equations see, for example, [15–19] and references therein.

Let E be a reflexive Banach space with norm $\|\cdot\|_E$ and E^* be the dual of E . Let $C(I, E)$, $I = [0, T]$ be the class of strongly continuous functions $x : I \rightarrow E$ with norm $\|x\|_C = \sup_{t \in I} \|x(t)\|_E$.

Consider the delay second-order nonlinear functional integral equation of Volterra-Stieltjes type

$$x(t) = f_1\left(t, \int_0^{m(t)} f_2(t, s, x(s)) d_s g(t, s)\right), \quad t \in I \quad (1)$$

and its corresponding initial value problem

$$\frac{dx(t)}{dt} = f_1\left(t, \int_0^{m(t)} f_2(t, s, x(s)) d_s g(t, s)\right), \quad a.e., t \in I, \quad (2)$$

$$x(0) = x_0 \quad (3)$$

where f_1, f_2 are weakly continuous on the bounded interval I with value in the reflexive Banach space E , $g : I \times I \rightarrow R$ is nondecreasing in the second argument, m is a continuous and decreasing function on I , and the symbol d_s indicates integration with respect to s .

D. N. Sidorov [20] studied the the majorant integral equation

$$x(t) = \int_0^t m(s) \gamma(x(s)) ds \quad (4)$$

and the corresponding Cauchy problem for the differential equation

$$\frac{d}{dt} = m(s)\gamma(x(s)), \quad x(0) = 0. \quad (5)$$

He obtained sufficient conditions for the existence of a Kantorovich principal solution of the nonlinear Volterra integral equation of the second kind (4) on the half-line $[0, \infty)$ and on a finite interval I . He suggested a method for computing the boundary of an interval outside which the solution can blow up.

Here, we prove the existence of local solutions $x \in C(I, E)$ of the integral Equation (1) and the initial value problem (2) and (3). The sufficient conditions for the uniqueness of the solutions will be given. The continuous dependence of the solutions on the initial data x_0 and on the functions m , f_2 and g will be proved. An example will be given to illustrate our results.

The following lemma and theorems will be needed in our proof (see [21,22]).

Theorem 1. Let E be a normed space with $x_0 \neq 0$. Then there exists a $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|_E$.

Lemma 1. A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology. Consequently, in reflexive Banach space, the subset is weakly relatively compact if and only if it is bounded in the norm topology.

Theorem 2. (O'Regan fixed point theorem) Let E be a Banach space, and let Q be a nonempty, bounded, closed and convex subset of $C(I, E)$ and let $F : Q \rightarrow Q$ be weakly sequentially continuous and assume that $\{FQ(t)\}$ is relatively weakly compact in E for each $t \in I$. Then, F has a fixed point in the set Q (see [22]).

2. Existence of at Least One Solution

2.1. Functional Integral Equation

Consider now the delay functional integral Equation (1) under the following assumptions:

- (i) $m : I \rightarrow I$, $m(t) \leq t$ is continuous and increasing.
- (ii) $f_1 : I \times E \rightarrow E$ is weakly continuous and weakly satisfies the Lipschitz condition

$$|\varphi(f_1(t, x) - f_1(t, y))| \leq k_1 |\varphi(x - y)|, \quad k_1 > 0, \quad \forall (t, x), (t, y) \in I \times E, \quad \varphi \in E^*.$$

- (iii) $f_2 : I \times I \times E \rightarrow E$ is weakly continuous and there exist two positive constants k_2 and b , such that $\|f_2(t, s, x)\|_E \leq b + k_2 \|x\|_E$.
- (iv) The function $g : I \times R \rightarrow R$ is continuous with

$$\mu = \max\{\sup |g(t, m(t))| + \sup |g(t, 0)|\}, \quad \text{on } I.$$

- (v) For all $t_1, t_2 \in I$ such that $t_1 < t_2$, the function $s \rightarrow g(t_2, s) - g(t_1, s)$ is nondecreasing on I .
- (vi) $g(0, s) = 0$ for any $s \in I$.
- (vii) $k_1 k_2 \mu < 1$

Remark 1. From the assumption (ii) set $y = 0$, we get

$$\begin{aligned} |\varphi(f_1(t, x(t)))| - |\varphi(f_1(t, 0))| &\leq |\varphi(f_1(t, x(t)) - f_1(t, 0))| \leq k_1 |\varphi(x)|, \\ |\varphi(f_1(t, x(t)))| &\leq k_1 |\varphi(x)| + |\varphi(f_1(t, 0))|. \end{aligned}$$

and

$$|\varphi(f_1(t, x(s)))| \leq k_1 |\varphi(x)| + f^*$$

where $f^* = \sup_{t \in I} |\varphi(f_1(t, 0))| \geq 0$.

Definition 1. By a weak solution of (1), we mean a function $x \in C(I, E)$ that satisfies the delay composite functional integral Equation (1). This is equivalent to finding a solution $x \in C(I, E)$ of functional integral equation

$$\varphi(x(t)) = \varphi\left(f_1\left(t, \int_0^{m(t)} f_2(t, s, x(s)) d_s g(t, s)\right)\right), \quad t \in I, \quad \forall \varphi \in E^*.$$

Now, we have the following theorem.

Theorem 3. Let the assumptions (i)–(vii) be satisfied; then, the functional Equation (1) has at least one weak solution $x \in C(I, E)$.

Proof. Define the operator A by

$$Ax(t) = f_1\left(t, \int_0^{m(t)} f_2(t, s, x(s)) d_s g(t, s)\right), \quad t \in I$$

and the set Q_r by

$$Q_r = \{x \in C(I, E) : \|x\|_E \leq r\}, \quad r = \frac{f^* + k_1 b \mu}{1 - k_1 k_2 \mu}.$$

Now, let $x \in Q_r$, then

$$\begin{aligned} \|Ax(t)\|_E = |\varphi(Ax(t))| &= \left| \varphi\left(f_1\left(t, \int_0^{m(t)} f_2(t, s, x(s)) d_s g(t, s)\right)\right) \right| \\ &\leq f^* + k_1 \left| \varphi\left(\int_0^{m(t)} f_2(t, s, x(s)) d_s g(t, s)\right) \right| \\ &\leq f^* + k_1 \varphi\left(\int_0^{m(t)} |f_2(t, s, x(s))| d_s \left(\bigvee_{z=0}^s g(t, z)\right)\right) \\ &\leq f^* + k_1 \int_0^{m(t)} \left(b + k_2 |\varphi(x)|\right) d_s \left(\bigvee_{z=0}^s g(t, z)\right) \\ &\leq f^* + k_1 \left(b + k_2 \|x\|\right) \int_0^{m(t)} d_s g(t, s) \\ &\leq f^* + k_1 (b + k_2 r) (g(t, m(t)) - g(t, 0)) \\ &\leq f^* + k_1 (b + k_2 r) \mu = r, \end{aligned}$$

then

$$\|Ax\|_C = \sup_{t \in I} |\varphi(Ax(t))| \leq r.$$

This proves that the class of functions $\{Ax\}$ is uniformly bounded on Q_r .

Now we show that $A : Q_r \rightarrow Q_r$. Let $x \in Q_r$ and define

$$\theta(\delta) = \sup_{x \in Q_r} \{|\varphi(f_2(t_2, s, x(s)) - f_2(t_1, s, x(s)))| : t_1, t_2 \in I, t_1 < t_2, |t_2 - t_1| < \delta, \|x\|_E \leq r\},$$

then, from the uniform continuity of the functions $f_2 : I \times I \times Q_r \rightarrow R$, $f_1 : I \times Q_r \rightarrow R$, and assumptions (ii) and (iii), we deduce that $\theta(\delta) \rightarrow 0$, as $\delta \rightarrow 0$ (δ is independent of $x \in Q_r$); similarly, we have

$$\theta_1(\delta) = \sup_{x \in Q_r} \{|\varphi(f_1(t_2, x(s)) - f_1(t_1, x(s)))| : t_1, t_2 \in I, t_1 < t_2, |t_2 - t_1| < \delta, \|x\|_E \leq r\},$$

and $\theta_1(\delta) \rightarrow 0$, as $\delta \rightarrow 0$ (δ is independent of $x \in Q_r$).

Then we have

$$\begin{aligned}
|\varphi(Ax(t_2) - Ax(t_1))| &= \left| \varphi(f_1(t_2, \int_0^{m(t_2)} f_2(t_2, s, x(s)) d_s g(t_2, s)) \right. \\
&\quad - \varphi(f_1(t_2, \int_0^{m(t_1)} f_2(t_1, s, x(s)) d_s g(t_1, s))) \\
&\quad + \varphi(f_1(t_2, \int_0^{m(t_1)} f_2(t_1, s, x(s)) d_s g(t_1, s))) \\
&\quad \left. - \varphi(f_1(t_1, \int_0^{m(t_1)} f_2(t_1, s, x(s)) d_s g(t_1, s))) \right| \\
&\leq k_1 \left| \varphi \left(\int_0^{m(t_2)} f_2(t_2, s, x(s)) d_s g(t_2, s) - \int_0^{m(t_1)} f_2(t_1, s, x(s)) d_s g(t_1, s) \right) \right| \\
&\quad + \theta_1(\delta) \\
&\leq k_1 \left| \varphi \left(\int_0^{m(t_2)} f_2(t_2, s, x(s)) d_s g(t_2, s) - \int_0^{m(t_2)} f_2(t_1, s, x(s)) d_s g(t_2, s) \right. \right. \\
&\quad \left. \left. + \int_0^{m(t_2)} f_2(t_1, s, x(s)) d_s g(t_2, s) - \int_0^{m(t_1)} f_2(t_1, s, x(s)) d_s g(t_1, s) \right) \right| + \theta_1(\delta) \\
&\leq k_1 \left| \varphi \left(\int_0^{m(t_2)} [f_2(t_2, s, x(s)) - f_2(t_1, s, x(s))] d_s g(t_2, s) \right. \right. \\
&\quad \left. \left. + \int_0^{m(t_1)} f_2(t_1, s, x(s)) d_s g(t_2, s) + \int_{m(t_1)}^{m(t_2)} f_2(t_1, s, x(s)) d_s g(t_2, s) \right. \right. \\
&\quad \left. \left. - \int_0^{m(t_1)} f_2(t_1, s, x(s)) d_s g(t_1, s) \right) \right| + \theta_1(\delta) \\
&\leq k_1 \left| \int_0^{m(t_2)} \varphi([f_2(t_2, s, x(s)) - f_2(t_1, s, x(s))] d_s g(t_2, s) \right. \\
&\quad \left. + \int_0^{m(t_1)} \varphi(f_2(t_1, s, x(s))) d_s [g(t_2, s) - g(t_1, s)] \right. \\
&\quad \left. + \int_{m(t_1)}^{m(t_2)} \varphi(f_2(t_1, s, x(s))) d_s g(t_2, s) \right| + \theta_1(\delta) \\
&\leq k_1 \left(\int_0^{m(t_2)} |\varphi(f_2(t_2, s, x(s)) - f_2(t_1, s, x(s)))| d_s \left(\bigvee_{z=0}^s g(t_2, z) \right) \right. \\
&\quad \left. + \int_0^{m(t_1)} |\varphi(f_2(t_1, s, x(s)))| d_s \left(\bigvee_{z=0}^s [g(t_2, z) - g(t_1, z)] \right) \right. \\
&\quad \left. + \int_{m(t_1)}^{m(t_2)} |\varphi(f_2(t_1, s, x(s)))| d_s \left(\bigvee_{z=0}^s g(t_2, z) \right) \right) + \theta_1(\delta) \\
&\leq k_1 \theta(\delta) \int_{m(t_1)}^{m(t_2)} d_s \left(\bigvee_{z=0}^s g(t_2, z) \right) \\
&\quad + k_1(b + k_2 r) \int_0^{m(t_1)} d_s \left(\bigvee_{z=0}^s [g(t_2, z) - g(t_1, z)] \right) \\
&\quad + k_1(b + k_2 r) \int_{m(t_1)}^{m(t_2)} d_s \left(\bigvee_{z=0}^s g(t_2, z) \right) + \theta_1(\delta) \\
&\leq k_1 \theta(\delta) [g(t_2, m(t_2)) - g(t_2, m(t_1))] \\
&\quad + k_1(b + k_2 r) [g(t_2, m(t_1)) - g(t_1, m(t_1)) - g(t_2, 0) + g(t_1, 0)] \\
&\quad + k_1(b + k_2 r) [g(t_2, m(t_2)) - g(t_2, m(t_1))] + \theta_1(\delta).
\end{aligned}$$

Then

$$\begin{aligned}\|Ax(t_2) - Ax(t_1)\|_E &= \sup_{t_1, t_2 \in I} |\varphi(Ax(t_2) - Ax(t_1))| \\ &\leq k_1 \theta(\delta) [g(t_2, m(t_2)) - g(t_2, m(t_1))] \\ &+ k_1(b + k_2 r) [g(t_2, m(t_1)) - g(t_1, m(t_1)) - g(t_2, 0) + g(t_1, 0)] \\ &+ k_1(b + k_2 r) [g(t_2, m(t_2)) - g(t_2, m(t_1))] + \theta_1(\delta).\end{aligned}$$

Hence, we deduce that $A : Q_r \rightarrow Q_r$.

Note that Q_r is nonempty closed, bounded, convex, and strongly equi-continuous subset of $C(I, E)$. Then according to Lemma 1, $\{AQ\}$ is relatively weakly compact.

Next, to prove that the operator A is weakly sequentially continuous, we take $\{x_n\} \subset Q_r$, and $\varphi(x_n) \rightarrow \varphi(x)$, in $Q_r \subseteq E$; then

$$\varphi(Ax_n(t)) = \varphi(f_1(t, \int_0^{m(t)} f_2(t, s, x_n(s)) d_s g(t, s)))$$

and

$$\lim_{n \rightarrow \infty} \varphi(Ax_n(t)) = \lim_{n \rightarrow \infty} \varphi(f_1(t, \int_0^{m(t)} f_2(t, s, x_n(s)) d_s g(t, s))).$$

Applying Lebesgue dominated convergence Theorem (see [23]), then from assumption (iii) we have

$$\begin{aligned}\lim_{n \rightarrow \infty} \varphi(Ax_n(t)) &= \varphi(f_1(t, \lim_{n \rightarrow \infty} \int_0^{m(t)} f_2(t, s, x_n(s)) d_s g(t, s))) \\ &= \varphi(f_1(t, \int_0^{m(t)} f_2(t, s, \lim_{n \rightarrow \infty} x_n(s)) d_s g(t, s))) \\ &= \varphi(f_1(t, \int_0^{m(t)} f_2(t, s, x(s)) d_s g(t, s))) = \varphi(Ax(t)).\end{aligned}$$

Then $\varphi(Ax_n(t)) \rightarrow \varphi(Ax(t))$, which means that the operator A is weakly sequentially continuous.

Since all conditions of Theorem 2 are satisfied, the operator A has at least one fixed point $x \in Q_r$, and the integral Equation (1) has at least one weak solution $x \in C(I, E)$. This completes the proof. $\square$

2.2. Initial Value Problem

In order to study the existence of at least one solution of the initial value problem (2) and (3), we relax assumption (ii).

(viii) $f_1 : I \times R \rightarrow R$ is weakly-weakly continuous and there exists $k_1 > 0$ and a weakly continuous function $a(t)$ where $a : I \rightarrow I$, such that

$$|\varphi(f_1(t, y))| \leq a(t) + k_1 |\varphi(y)|.$$

For the initial value problem (2) and (3), we have the following lemma.

Lemma 2. *If the solution of the initial value problem (2) and (3) exists, then it can be represented by the integral equation*

$$x(t) = x_0 + \int_0^t f_1(s, \int_0^{m(s)} f_2(s, \theta, x(\theta)) d_\theta g(s, \theta)) ds, \quad t \in I, \quad (6)$$

Proof. Integrating (2), we obtain

$$\begin{aligned}\int_0^t \frac{dx}{ds} &= \int_0^t f_1(s, \int_0^{m(s)} f_2(s, \theta, x(\theta)) d_\theta g(s, \theta)) ds, \\ x(t) - x(0) &= \int_0^t f_1(s, \int_0^{m(s)} f_2(s, \theta, x(\theta)) d_\theta g(s, \theta)) ds\end{aligned}$$

However, $x(0) = x_0$; then

$$x(t) = x_0 + \int_0^t f_1(s, \int_0^{m(s)} f_2(s, \theta, x(\theta)) d_\theta g(s, \theta)) ds.$$

Now, differentiating (6), we get

$$\frac{dx}{dt} = \frac{d}{dt}x_0 + \frac{d}{dt} \int_0^t f_1(s, \int_0^{m(s)} f_2(s, \theta, x(\theta)) d_\theta g(s, \theta)) ds, \quad t \in I$$

$$\frac{dx}{dt} = f_1(t, \int_0^{m(t)} f_2(t, s, x(s)) d_s g(t, s)), \quad a.e., t \in (0, T],$$

Now, at $t = 0$ in (6) we obtain

$$x(0) = x_0 + \int_0^0 f_1(t, \int_0^{m(t)} f_2(s, \theta, x(\theta)) d_\theta g(s, \theta)) ds.$$

Hence

$$x(0) = x_0.$$

Hence, (2), (3), and (6) are equivalent. $\square$

Now, we have the following existence theorem.

Theorem 4. Let the assumptions (i), (iii)–(vi), and (viii) be satisfied. If $k_2 k_1 \mu T < 1$, then the initial value problem (2) and (3) has at least one solution $x \in C(I, E)$.

Proof. Define the set Q by

$$Q = \{x \in C(I, E) : \|x\|_E \leq r_1\}, \quad r_1 = \frac{|x_0| + a + k_1 b \mu T}{1 - k_2 k_1 \mu T}.$$

where $a = \int_0^t a(s) ds$. It is clear that Q is a nonempty, bounded, closed, and convex set.

Define the operator F by

$$Fx(t) = x_0 + \int_0^t f_1(s, \int_0^{m(s)} f_2(s, \theta, x(\theta)) d_\theta g(s, \theta)) ds.$$

Let $x \in Q$, then

$$\begin{aligned}
\|Fx(t)\|_E = \varphi|Fx(t)| &= |\varphi(x_0 + \int_0^t f_1(s, \int_0^{m(s)} f_2(s, \theta, x(\theta)) d_\theta g(s, \theta)) ds)| \\
&\leq \|x_0\|_E + \int_0^t |\varphi(f_1(s, \int_0^{m(s)} f_2(s, \theta, x(\theta)) d_\theta g(s, \theta)))| ds \\
&\leq \|x_0\|_E + \int_0^t (\varphi(a(s)) + k_1 \int_0^{m(s)} |\varphi(f_2(s, \theta, x(\theta)))| d_\theta g(s, \theta)) ds \\
&\leq \|x_0\|_E + a + k_1 \int_0^t \int_0^{m(s)} |\varphi(f_2(s, \theta, x(\theta)))| d_\theta \bigvee_{z=0}^\theta g(s, z) ds \\
&\leq \|x_0\|_E + a + k_1 \int_0^t \int_0^{m(s)} (b + k_2 |\varphi(x)|) d_\theta \bigvee_{z=0}^\theta g(s, z) ds \\
&\leq \|x_0\|_E + a + k_1 (b + k_2 \|x\|) (g(s, m(s)) - g(s, 0)) T \\
&\leq \|x_0\|_E + a + k_1 (b + k_2 r_1) \mu T = r_1.
\end{aligned}$$

Then

$$\|Fx\|_C = \sup_{t \in I} \|Fx(t)\|_E \leq r_1.$$

This proves that the class of functions $\{Fx\}$ is uniformly bounded on Q .

Let $x \in Q$, $\delta > 0$ be given and $t_1, t_2 \in I$, $t_1 < t_2$, such that $|t_2 - t_1| < \delta$, then

$$\begin{aligned}
|\varphi(Fx(t_2) - Fx(t_1))| &= |\varphi(\int_0^{t_2} f_1(s, \int_0^{m(s)} f_2(s, \theta, x(\theta)) d_\theta g(s, \theta)) ds \\
&\quad - \int_0^{t_1} f_1(s, \int_0^{m(s)} f_2(s, \theta, x(\theta)) d_\theta g(s, \theta)) ds)| \\
&\leq |\varphi(\int_0^{t_1} f_1(s, \int_0^{m(s)} f_2(s, \theta, x(\theta)) d_\theta g(s, \theta)) ds \\
&\quad + \int_{t_1}^{t_2} f_1(s, \int_0^{m(s)} f_2(s, \theta, x(\theta)) d_\theta g(s, \theta)) ds \\
&\quad - \int_0^{t_1} f_1(s, \int_0^{m(s)} f_2(s, \theta, x(\theta)) d_\theta g(s, \theta)) ds)| \\
&\leq \int_{t_1}^{t_2} |\varphi(f_1(s, \int_0^{m(s)} f_2(s, \theta, x(\theta)) d_\theta g(s, \theta))| d_\theta g(s, \theta) ds \\
&\leq \int_{t_1}^{t_2} (a(s) + k_1 |\varphi(f_2(s, \theta, x(\theta)))| d_\theta g(s, \theta)) ds \\
&\leq \int_{t_1}^{t_2} (a(s) + k_1 \int_0^{m(s)} |\varphi(f_2(s, \theta, x(\theta)))| d_\theta \bigvee_{z=0}^\theta g(s, z)) ds \\
&\leq \int_{t_1}^{t_2} (a(s) + k_1 \int_0^{m(s)} (b + k_2 |\varphi(x)|) d_\theta g(s, \theta)) ds \\
&\leq [a + k_1 (b + k_2 \|x\|) (g(s, m(s)) - g(s, 0))] |t_2 - t_1| \\
&\leq [a + k_1 (b + k_2 r_1) \mu] |t_2 - t_1|.
\end{aligned}$$

Then

$$\|Fx(t_2) - Fx(t_1)\|_C = \sup_{t \in I} |\varphi(Fx(t_2) - Fx(t_1))| \leq [a + k_1 (b + k_2 r_1) \mu] |t_2 - t_1|.$$

The above inequality means that the operator F maps Q into itself.

Note that Q is nonempty closed, bounded, convex, and strongly equi-continuous subset of $C(I, E)$. Then according to Lemma 1, $\{FQ\}$ is relatively weakly compact.

To prove that the operator F is weakly sequentially continuous, let $\{x_n\} \subset Q$, and $\varphi(x_n) \rightarrow \varphi(x)$, in $Q \subseteq E$ then

$$\begin{aligned}\varphi(Fx_n(t)) &= \varphi(x_0 + \int_0^t f_1(s, \int_0^{m(s)} f_2(s, \theta, x_n(\theta)) d_\theta g(s, \theta)) ds) \\ \lim_{n \rightarrow \infty} \varphi(Fx_n(t)) &= \lim_{n \rightarrow \infty} \varphi(x_0 + \int_0^t f_1(s, \int_0^{m(s)} f_2(s, \theta, x(\theta)) d_\theta g(s, \theta)) ds)\end{aligned}$$

Applying Lebesgue dominated convergence Theorem (see[23]), then

$$\begin{aligned}\varphi(Fx_n(t)) &= \varphi(x_0 + \int_0^t \lim_{n \rightarrow \infty} f_1(s, \int_0^{m(s)} f_2(s, \theta, x_n(\theta)) d_\theta g(s, \theta)) ds) \\ &= \varphi(x_0 + \int_0^t f_1(s, \int_0^{m(s)} f_2(s, \theta, \lim_{n \rightarrow \infty} x_n(\theta)) d_\theta g(s, \theta)) ds) \\ &= \varphi(x_0 + \int_0^t f_1(s, \int_0^{m(s)} f_2(s, \theta, x_0(\theta)) d_\theta g(s, \theta)) ds) = \varphi(Fx_0(t)).\end{aligned}$$

Then F is weakly sequentially continuous and by Theorem 2, F has a fixed point $x \in Q$ and (6) has a weak solution. Consequently, (2) and (3) have at least one weak solution $x \in C(I, E)$. $\square$

3. Uniqueness of the Solution

Here, we study a sufficient condition for the uniqueness of the solution $x \in C[I, E]$ of the delay functional integral Equation (1). Consider the following assumptions:

(iii)* $f_2 : I \times I \times E \rightarrow E$ is weakly continuous and satisfies the weakly Lipschitz condition,

$$|\varphi(f_2(t, s, x) - f_2(t, s, y))| \leq k_2 |\varphi(x - y)|.$$

From the assumption (iii)*, we have

$$\begin{aligned}|\varphi(f_2(t, s, x(s)))| - |\varphi(f_2(t, s, 0))| &\leq |\varphi(f_2(t, s, x(s)) - f_2(t, s, 0))| \leq k_2 |\varphi(x)| \\ |\varphi(f_2(t, s, x(s)))| &\leq k_2 |\varphi(x)| + |\varphi(f_2(t, s, 0))|\end{aligned}$$

then

$$|\varphi(f_2(t, s, x(s)))| \leq k_2 |\varphi(x)| + b,$$

where $b = \sup_t |\varphi(f_2(t, s, 0))|$.

Theorem 5. Let the assumptions (i) and (ii), (iii)*-(iv)-(vii) be satisfied; then the solution $x \in C(I, E)$ of the integral Equation (1) is unique.

Proof. Let x_1, x_2 be two weakly solutions of the integral Equation (1) in the reflexive Banach space E : we have

$$\begin{aligned}
|\varphi(x_1(t) - x_2(t))| &= |\varphi(f_1(t, \int_0^{m(t)} f_2(t, s, x_1(s)) d_s g(t, s)) \\
&\quad - f_1(t, \int_0^{m(t)} f_2(t, s, x_2(s)) d_s g(t, s)))| \\
&\leq k_1 |\varphi(\int_0^{m(t)} f_2(t, s, x_1(s)) d_s g(t, s) - \int_0^{m(t)} f_2(t, s, x_2(s)) d_s g(t, s))| \\
&\leq k_1 \int_0^{m(t)} |\varphi(f_2(t, s, x_1(s)) - f_2(t, s, x_2(s)))| d_s (\bigvee_{z=0}^s g(t, z)) \\
&\leq k_1 k_2 \int_0^{m(t)} |\varphi(x_1(s) - x_2(s))| d_s (\bigvee_{z=0}^s g(t, z)) \\
&\leq k_1 k_2 \|x_1 - x_2\|_C \int_0^{m(t)} d_s g(t, s),
\end{aligned}$$

therefore

$$\begin{aligned}
\|x_1 - x_2\|_C &\leq k_1 k_2 \|x_1 - x_2\|_C [g(t, m(t)) - g(t, 0)] \\
&\leq k_1 k_2 \mu \|x_1 - x_2\|_C
\end{aligned}$$

and

$$(1 - k_1 k_2 \mu) \|x_1 - x_2\|_C \leq 0.$$

However, $k_1 k_2 \mu < 1$; then

$$\|x_1 - x_2\|_C = 0.$$

Thus, $x_1 = x_2$ and the solution of the functional integral Equation (1) is unique. $\square$

To study the uniqueness of the solution of the initial value problem (2) and (3), we replace the assumption (viii) by

(viii)* $f_1 : I \times R \rightarrow R$ is weakly continuous and satisfying weakly Lipschitz condition

$$|\varphi(f_1(t, x) - f_1(t, y))| \leq k_1 |\varphi(x - y)|, \quad k_1 > 0, \quad \forall (t, x), (t, y) \in I \times E$$

Theorem 6. Let the assumptions (i), (iii)–(vi), (viii)* be satisfied. If $k_1 k_2 \mu T < 1$, then the weak solution $x \in C(I, E)$ of the initial value problem (2) and (3) is unique.

Proof. Let $x_1 = x_2$ be two weak solutions of the functional integral Equation (6), we get

$$\begin{aligned}
\|x_1(t) - x_2(t)\|_E = |\varphi(x_1(t) - x_2(t))| &= |\varphi(x_0 + \int_0^t f_1(s, \int_0^{m(s)} f_2(s, \theta, x_1(\theta)) d_\theta g(s, \theta)) ds \\
&\quad - (x_0 + \int_0^t f_1(s, \int_0^{m(s)} f_2(s, \theta, x_2(\theta)) d_\theta g(s, \theta)) ds))| \\
&\leq k_1 \int_0^t |\varphi(\int_0^{m(s)} f_2(s, \theta, x_1(\theta)) d_\theta g(s, \theta) \\
&\quad - \int_0^{m(s)} f_2(s, \theta, x_2(\theta)) d_\theta g(s, \theta))| ds \\
&\leq k_1 k_2 \int_0^t \int_0^{m(s)} |\varphi(x_1(\theta) - x_2(\theta))| d_\theta (\bigvee_{z=0}^\theta g(s, z)) ds \\
&\leq k_1 k_2 \|x_1 - x_2\|_C \int_0^t \int_0^{m(s)} d_\theta g(s, \theta) ds \\
&\leq k_1 k_2 \|x_1 - x_2\|_C \int_0^t [g(s, m(s)) - g(s, 0)] ds,
\end{aligned}$$

thus

$$\begin{aligned}\|x_1 - x_2\|_C &\leq k_1 k_2 \mu \|x_1 - x_2\|_C \int_0^t ds \\ &\leq k_1 k_2 \mu T \|x_1 - x_2\|_C \\ \|x_1 - x_2\|_C [1 - k_1 k_2 \mu T] &\leq 0.\end{aligned}$$

However, $k_1 k_2 \mu T < 1$, then

$$\|x_1 - x_2\|_C = 0.$$

This means that

$$x_1 - x_2 = 0 \Rightarrow x_1 = x_2.$$

Thus, the solution of the integral Equation (6) is unique; consequently, the solution of the initial value problem (2) and (3) is unique. $\square$

4. Continuous Dependence

Definition 2. The solutions of the functional integral Equation (1) and the initial value problem (2) and (3) depend continuously on the delay function m if $\forall \epsilon > 0, \exists \delta > 0$, such that

$$|m(t) - m^*(t)| \leq \delta \Rightarrow \|x - x^*\|_C \leq \epsilon.$$

Theorem 7. Let the assumptions of Theorem 5 be satisfied; then the solution of the functional integral Equation (1) depends continuously on the delay function m .

Proof. Let $\delta > 0$ be given such that $|m(t) - m^*(t)| \leq \delta, \forall t \in I$; we obtain

$$\begin{aligned}|\varphi(x(t) - x^*(t))| &\leq \varphi(f_1(t, \int_0^{m(t)} f_2(t, s, x(s)) d_s g(t, s)) - f_1(t, \int_0^{m^*(t)} f_2(t, s, x^*(s)) d_s g(t, s))) \\ &\leq k_1 |\int_0^{m(t)} \varphi(f_2(t, s, x(s)) d_s g(t, s)) - \int_0^{m^*(t)} \varphi(f_2(t, s, x^*(s)) d_s g(t, s))| \\ &\leq k_1 |\int_0^{m(t)} \varphi(f_2(t, s, x(s)) d_s g(t, s)) - \int_0^{m(t)} \varphi(f_2(t, s, x^*(s)) d_s g(t, s)) \\ &\quad + \int_0^{m(t)} \varphi(f_2(t, s, x^*(s)) d_s g(t, s)) - \int_0^{m^*(t)} \varphi(f_2(t, s, x^*(s)) d_s g(t, s))| \\ &\leq k_1 \int_0^{m(t)} |\varphi(f_2(t, s, x(s)) - f_2(t, s, x^*(s)))| d_s (\bigvee_{z=0}^s g(t, z)) \\ &\quad + \int_{m^*(t)}^{m(t)} |\varphi(f_2(t, s, x^*(s)))| d_s (\bigvee_{z=0}^s g(t, z)) \\ &\leq k_1 [k_2 \int_0^{m(t)} |\varphi(x(s) - x^*(s))| d_s g(t, s) + \int_{m^*(t)}^{m(t)} (b + k_2 |\varphi(x)|) d_s g(t, s)] \\ &\leq k_1 [k_2 \|x - x^*\|_C [g(t, m^*(t)) - g(t, 0)] + (b + k_2 r) \int_{m^*(t)}^{m(t)} d_s g(t, s)],\end{aligned}$$

and hence

$$\|x - x^*\|_C \leq k_1 k_2 \mu \|x - x^*\|_C + k_1 (b + k_2 r) |g(t, m(t)) - g(t, m^*(t))|.$$

However, from the continuity of g , we have

$$|m(t) - m^*(t)| \leq \delta \Rightarrow |g(t, m(t)) - g(t, m^*(t))| < \epsilon_1,$$

then

$$\begin{aligned}\|x - x^*\|_C &\leq k_1 k_2 \|x - x^*\|_C \mu + k_1(b + k_2 r) \epsilon_1 \\ \|x - x^*\|_C (1 - k_1 k_2 \mu) &\leq k_1(b + k_2 r) \epsilon_1 \\ \|x - x^*\|_C &\leq \frac{k_1(b + k_2 r) \epsilon_1}{1 - k_1 k_2 \mu} = \epsilon.\end{aligned}$$

This completes the proof. $\square$

Theorem 8. Let the assumptions of Theorem 6 be satisfied; then the solution of the initial value problem (2) and (3) depends continuously on the delay function m .

Proof. Let $\delta > 0$ be given such that $|m(t) - m^*(t)| \leq \delta, \forall t \geq 0$; then

$$\begin{aligned}\|x(t) - x^*(t)\|_E &\leq \varphi\left(\int_0^t f_1(s, \int_0^{m(s)} f_2(s, \theta, x(\theta)) d_\theta g(s, \theta)) ds\right) \\ &\quad - \int_0^t f_1(s, \int_0^{m^*(s)} f_2(s, \theta, x^*(\theta)) d_\theta g(s, \theta)) ds \\ &\leq k_1 \left| \int_0^t \int_0^{m(s)} \varphi(f_2(s, \theta, x(\theta))) d_\theta g(s, \theta) - \int_0^{m^*(t)} \varphi(f_2(s, \theta, x^*(\theta))) d_\theta g(s, \theta) \right| ds \\ &\leq k_1 \int_0^t \left| \int_0^{m(s)} \varphi(f_2(s, \theta, x(\theta))) d_\theta g(s, \theta) - \int_0^{m^*(s)} \varphi(f_2(s, \theta, x^*(\theta))) d_\theta g(s, \theta) \right| ds \\ &\quad + \int_0^{m^*(s)} \varphi(f_2(s, \theta, x^*(\theta))) d_\theta g(s, \theta) - \int_0^{m^*(s)} \varphi(f_2(s, \theta, x^*(\theta))) d_\theta g(s, \theta) | ds \\ &\leq k_1 \int_0^t \left(\int_0^{m(s)} |\varphi(f_2(s, \theta, x(\theta))) - \varphi(f_2(s, \theta, x^*(\theta)))| \left(\bigvee_{z=0}^\theta g(s, z) \right) ds \right. \\ &\quad \left. + \int_{m^*(s)}^{m(s)} |\varphi(f_2(s, \theta, x^*(\theta)))| \left(\bigvee_{z=0}^\theta g(s, z) \right) ds \right) \\ &\leq k_1 (k_2 \int_0^t \int_0^{m(s)} |\varphi(x(\theta)) - \varphi(x^*(\theta))| \left(\bigvee_{z=0}^\theta g(s, z) \right) ds + \int_{m^*(s)}^{m(s)} (b + k_2 |\varphi(x)|) \left(\bigvee_{z=0}^\theta g(s, z) \right) ds) \\ &\leq k_1 (k_2 \|x - x^*\|_C \int_0^t \int_0^{m^*(s)} d_\theta g(s, \theta) + \int_{m^*(s)}^{m(s)} (b + k_2 |\varphi(x)|) d_\theta g(s, \theta)) ds \\ &\leq k_1 k_2 \|x - x^*\|_C \int_0^t ([g(s, m(\theta)) - g(s, 0)] + k_1(b + k_2 r_1) \int_{m^*(s)}^{m(s)} d_\theta g(s, \theta)) ds. \\ &\leq k_3 k_2 \mu T \|x - x^*\|_C + k_1 T (b + k_2 r_1) [g(s, m(\theta)) - g(s, m^*(\theta))].\end{aligned}$$

However, from the continuity of g , we have

$$|m(s) - m^*(s)| \leq \delta \Rightarrow |g(s, m(\theta)) - g(s, m^*(\theta))| < \epsilon_1$$

then

$$\begin{aligned}\|x - x^*\|_C &\leq k_1 k_2 \mu T \|x - x^*\|_C + k_1 T (b + k_1 r_2) \epsilon_1 \\ \|x - x^*\|_C (1 - k_1 k_2 \mu T) &\leq k_1 T (b + k_2 r_1) \epsilon_1 \\ \|x - x^*\|_C &\leq \frac{k_1 T (b + k_2 r_1) \epsilon_1}{1 - k_1 k_2 \mu T} = \epsilon.\end{aligned}$$

This completes the proof. $\square$

Definition 3. The solutions of the functional integral Equation (1) and the initial value problem (2) and (3) depend continuously on the function $f_2(t, s, x(s))$ if $\forall \epsilon > 0, \exists \delta > 0$, such that

$$|\varphi(f_2(t, s, x(s))) - \varphi(f_2^*(t, s, x(s)))| \leq \delta \Rightarrow \|x - x^*\| \leq \epsilon.$$

Now we will study the continuous dependence on the function $f_2(t, s, x(s))$.

Theorem 9. *Let the assumptions of Theorem 5 be satisfied; then the solution of the functional integral Equation (1) depends continuously on the function f_2 .*

Proof. Let

$$|\varphi(f_2(t, s, x(s)) - f_2^*(t, s, x(s)))| \leq \delta,$$

then we get

$$\begin{aligned} \|x(t) - x^*(t)\|_E &= \varphi(f_1(t, \int_0^{m(t)} f_2(t, s, x(s)) d_s g(t, s)) - f_1(t, \int_0^{m(t)} f_2^*(t, s, x^*(s)) d_s g(t, s))) \\ &\leq k_1 |\int_0^{m(t)} \varphi(f_2(t, s, x(s))) d_s g(t, s) - \int_0^{m(t)} \varphi(f_2^*(t, s, x^*(s))) d_s g(t, s)| \\ &\leq k_1 |\int_0^{m(t)} \varphi(f_2(t, s, x(s))) d_s g(t, s) - \int_0^{m(t)} \varphi(f_2^*(t, s, x(s))) d_s g(t, s) \\ &\quad + \int_0^{m(t)} \varphi(f_2^*(t, s, x(s))) d_s g(t, s) - \int_0^{m(t)} \varphi(f_2^*(t, s, x^*(s))) d_s g(t, s)| \\ &\leq k_1 (\int_0^{m(t)} |\varphi(f_2(t, s, x(s)) - f_2^*(t, s, x(s)))| d_s (\bigvee_{z=0}^s g(t, z)) \\ &\quad + \int_0^{m(t)} |\varphi(f_2^*(t, s, x(s)) - f_2^*(t, s, x^*(s)))| d_s (\bigvee_{z=0}^s g(t, z))) \\ &\leq k_1 [\delta \int_0^{m(t)} d_s g(t, s) + k_2 \int_0^{m(t)} |\varphi(x(s) - x^*(s))| d_s g(t, s)] \\ &\leq k_1 \delta [g(t, m(t)) - g(t, 0)] + k_1 k_2 \|x - x^*\|_C [g(t, m(t)) - g(t, 0)] \\ &\leq k_1 \delta \mu + k_1 k_2 \mu \|x - x^*\|_C, \end{aligned}$$

then

$$\begin{aligned} \|x - x^*\|_C [1 - k_1 k_2 \mu] &\leq k_1 \delta \mu \\ \|x - x^*\|_C &\leq \frac{k_1 \delta \mu}{1 - k_1 k_2 \mu} = \epsilon. \end{aligned}$$

This completes the proof. $\square$

Theorem 10. *Let the assumptions of Theorem 6 be satisfied; then the solution of initial value problem (2) and (3) depends continuously on the function f_2 .*

Proof. Let

$$|\varphi(f_2(t, s, x(s)) - f_2^*(t, s, x(s)))| \leq \delta,$$

then we get

$$\begin{aligned}
 \|x(t) - x^*(t)\|_E &= \varphi\left(\int_0^t f_1(s, \int_0^{m(s)} f_2(s, \theta, x(\theta)) d_\theta g(s, \theta)) ds\right. \\
 &\quad \left.- \int_0^t f_1(s, \int_0^{m(s)} f_2^*(s, \theta, x^*(\theta)) d_\theta g(s, \theta)) ds\right) \\
 &\leq k_1 \left| \int_0^t \left[\int_0^{m(s)} \varphi(f_2(s, \theta, x(\theta))) d_\theta g(s, \theta) - \int_0^{m(s)} \varphi(f_2^*(s, \theta, x^*(\theta))) d_\theta g(s, \theta) \right] ds \right| \\
 &\leq k_1 \left| \int_0^t \left[\int_0^{m(s)} \varphi(f_2(s, \theta, x(\theta))) d_\theta g(s, \theta) - \int_0^{m(s)} \varphi(f_2^*(s, \theta, x(\theta))) d_\theta g(s, \theta) \right] \right. \\
 &\quad \left. + \int_0^{m(s)} \varphi(f_2^*(s, \theta, x(\theta))) d_\theta g(s, \theta) - \int_0^{m(s)} \varphi(f_2^*(s, \theta, x^*(\theta))) d_\theta g(s, \theta) \right] ds \right| \\
 &\leq k_1 \int_0^t \left| \int_0^{m(s)} |\varphi(f_2(s, \theta, x(\theta)) - f_2^*(s, \theta, x(\theta)))| d_\theta \left(\bigvee_{z=0}^\theta g(s, z) \right) \right. \\
 &\quad \left. + \int_0^{m(s)} |\varphi(f_2^*(s, \theta, x(\theta)) - f_2^*(s, \theta, x^*(\theta)))| d_\theta \left(\bigvee_{z=0}^\theta g(s, z) \right) \right] ds \\
 &\leq k_1 \int_0^t [\delta \int_0^{m(s)} d_\theta g(s, \theta) + k_2 \int_0^{m(s)} |\varphi(x(\theta) - x^*(\theta))| d_\theta g(s, \theta)] ds \\
 &\leq k_1 \delta T [g(s, m(s)) - g(s, 0)] + k_1 k_2 T \|x - x^*\|_C [g(t, m(s)) - g(s, 0)] \\
 &\leq k_1 \delta \mu T + k_1 k_2 \mu T \|x - x^*\|_C,
 \end{aligned}$$

then

$$\begin{aligned}
 \|x - x^*\|_C [1 - k_1 k_2 \mu T] &\leq k_1 \delta \mu T \\
 \|x - x^*\|_C &\leq \frac{k_1 \delta \mu T}{1 - k_1 k_2 \mu T} = \epsilon.
 \end{aligned}$$

This completes the proof. $\square$

Definition 4. The solution of the initial value problem (2) and (3) depends continuously on the initial data x_0 if $\forall \epsilon > 0, \exists \delta > 0$, such that

$$|x_0 - x_0^*| \leq \delta \Rightarrow \|x - x^*\|_C \leq \epsilon.$$

Theorem 11. Let the assumptions of Theorem 6 be satisfied; then, the solution of the initial value problem (2) and (3) depends continuously on the initial data x_0 .

Proof. Let

$$|x_0 - x_0^*| \leq \delta.$$

$$\begin{aligned}
 \|x(t) - x^*(t)\|_E &= \varphi\left(x_0 + \int_0^t f_1(s, \int_0^{m(s)} f_2(s, \theta, x(\theta)) d_\theta g(s, \theta)) ds\right. \\
 &\quad \left.- x_0^* + \int_0^t f_1(s, \int_0^{m(s)} f_2(s, \theta, x(\theta)) d_\theta g(s, \theta)) ds\right) \\
 &\leq |x_0 - x_0^*| + k_1 \left| \int_0^t \left[\int_0^{m(s)} \varphi(f_2(s, \theta, x(\theta))) d_\theta g(s, \theta) \right. \right. \\
 &\quad \left. \left. - \int_0^{m(s)} \varphi(f_2(s, \theta, x^*(\theta))) d_\theta g(s, \theta) \right] ds \right| \\
 &\leq \delta + k_1 k_2 \int_0^t \left[\int_0^{m(s)} |\varphi(x(\theta) - x^*(\theta))| d_\theta g(s, \theta) \right] ds \\
 &\leq \delta + k_1 k_2 \|x - x^*\|_C \int_0^t \left[\int_0^{m(s)} d_\theta g(s, \theta) \right] ds \\
 &\leq \delta + k_1 k_2 \|x - x^*\|_C \int_0^t [g(s, m(s)) - g(s, 0)] ds \\
 &\leq \delta + k_1 k_2 T \mu \|x - x^*\|_C,
 \end{aligned}$$

then

$$\begin{aligned}\|x - x^*\|_C(1 - k_1 k_2 T \mu) &\leq \delta \\ \|x - x^*\|_C &\leq \frac{\delta}{1 - k_1 k_2 T \mu} = \epsilon.\end{aligned}$$

This completes the proof. $\square$

Definition 5. The solutions of the functional integral Equation (1) and the initial value problem (2) and (3) depend continuously on the function $g(t, s)$ if $\forall \epsilon > 0, \exists \delta > 0$, such that

$$|g(t, s) - g^*(t, s)| \leq \delta \Rightarrow \|x - x^*\| \leq \epsilon.$$

Theorem 12. Let the assumptions of Theorem 5 be satisfied; then the solution of the functional integral Equation (1) depends continuously on the function $g(t, s)$.

Proof. Let $\delta > 0$ be given such that $|g(t, s) - g^*(t, s)| \leq \delta, \forall t \geq 0$, then

$$\begin{aligned}\|x(t) - x^*(t)\|_E &\leq \varphi(f_1(t, \int_0^{m(t)} f_2(t, s, x(s)) d_s g(t, s))) - \varphi(f_1(t, \int_0^{m(t)} f_2(t, s, x^*(s)) d_s g^*(t, s))) \\ &\leq k_1 \left| \int_0^{m(t)} \varphi(f_2(t, s, x(s)) d_s g(t, s)) - \int_0^{m(t)} \varphi(f_2(t, s, x^*(s)) d_s g^*(t, s)) \right| \\ &\leq k_1 \left| \int_0^{m(t)} \varphi(f_2(t, s, x(s)) d_s g(t, s)) - \int_0^{m(t)} \varphi(f_2(t, s, x^*(s)) d_s g(t, s)) \right| \\ &\quad + \left| \int_0^{m(t)} \varphi(f_2(t, s, x^*(s)) d_s g(t, s)) - \int_0^{m(t)} \varphi(f_2(t, s, x^*(s)) d_s g^*(t, s)) \right| \\ &\leq k_1 \left| \int_0^{m(t)} \varphi(f_2(t, s, x(s)) - (f_2(t, s, x^*(s))) d_s g(t, s)) \right| \\ &\quad + \left| \int_0^{m(t)} \varphi(f_2(t, s, x^*(s)) d_s (g(t, s) - g^*(t, s))) \right| \\ &\leq k_1 k_2 \left| \int_0^{m(t)} |\varphi(x(t) - x^*(t))| d_s g(t, s) + \int_0^{m(t)} (b + k_2 |\varphi(x)|) d_s (g(t, s) - g^*(t, s)) \right| \\ &\leq k_1 k_2 \mu \|x - x^*\|_C + (b + k_2 \|x\|)((g(t, m(t))) - g^*(t, m(t))) - (g(t, 0)) - g^*(t, 0)) \\ &\leq k_1 k_2 \mu \|x - x^*\|_C + (b + k_2 r) 2\delta,\end{aligned}$$

then

$$\begin{aligned}\|x - x^*\|_C[1 - k_1 k_2 \mu] &\leq (b + k_2 r) 2\delta \\ \|x - x^*\|_C &\leq \frac{(b + k_2 r) 2\delta}{1 - k_1 k_2 \mu} = \epsilon.\end{aligned}$$

This completes the proof. $\square$

Theorem 13. Let the assumptions of Theorem 6 be satisfied then the solution of initial value problem (2) and (3) depends continuously on the function $g(t, s)$.

Proof. Let

$$|g(t, s) - g^*(t, s)| \leq \delta,$$

then we get

$$\begin{aligned}
\|x(t) - x^*(t)\|_E &= \varphi\left(\int_0^t f_1(s, \int_0^{m(s)} f_2(s, \theta, x(\theta)) d_\theta g(t, s)) ds\right. \\
&\quad \left.- \int_0^t f_1(s, \int_0^{m(s)} f_2(s, \theta, x^*(\theta)) d_\theta g^*(s, \theta) ds)\right) \\
&\leq k_1 \left| \int_0^t \left[\int_0^{m(s)} \varphi(f_2(s, \theta, x(\theta))) d_\theta g(s, \theta) - \int_0^{m(s)} \varphi(f_2(s, \theta, x^*(\theta))) d_\theta g^*(s, \theta) \right] ds \right| \\
&\leq k_1 \left| \int_0^t \left[\int_0^{m(s)} \varphi(f_2(s, \theta, x(\theta))) d_\theta g(s, \theta) - \int_0^{m(s)} \varphi(f_2(s, \theta, x^*(\theta))) d_\theta g(s, \theta) \right. \right. \\
&\quad \left. \left. + \int_0^{m(s)} \varphi(f_2(s, \theta, x^*(\theta))) d_\theta g(s, \theta) - \int_0^{m(s)} \varphi(f_2(s, \theta, x^*(\theta))) d_\theta g^*(s, \theta) \right] ds \right| \\
&\leq k_1 \left| \int_0^t \left[\int_0^{m(s)} \varphi(f_2(s, \theta, x(\theta)) - f_2(s, \theta, x^*(\theta))) d_\theta g(s, \theta) \right. \right. \\
&\quad \left. \left. + \int_0^{m(s)} \varphi(f_2(s, \theta, x^*(\theta))) d_\theta (g(s, \theta) - g^*(s, \theta)) \right] ds \right| \\
&\leq k_1 \left| \int_0^t \left[k_2 \int_0^{m(s)} |\varphi(x(\theta) - x^*(\theta))| d_\theta g(s, \theta) \right. \right. \\
&\quad \left. \left. + \int_0^{m(s)} (b + k_2 |\varphi(x^*(\theta))|) d_\theta (g(s, \theta) - g^*(s, \theta)) \right] ds \right| \\
&\leq k_1 \int_0^t [k_2 \mu \|x - x^*\|_C \\
&\quad + (b + k_2 r)[(g(s, m(s)) - g^*(s, m(s))) - (g(s, 0) - g^*(s, 0))] ds] \\
&\leq k_1 k_2 \mu T \|x - x^*\|_C + k_1 (b + k_2 r) 2\delta,
\end{aligned}$$

$$\begin{aligned}
\|x - x^*\|_C [1 - k_1 k_2 \mu T] &\leq k_1 (b + k_2 r) 2\delta T \\
\|x - x^*\|_C &\leq \frac{k_1 (b + k_2 r) 2\delta T}{1 - k_1 k_2 \mu T} = \epsilon.
\end{aligned}$$

This completes the proof. $\square$

5. Examples

Example 1. Consider the delay functional integral equation

$$x(t) = t + \frac{1}{3} \int_0^t (t^2 + ts + tx^2(s)) ds, \quad t \in I \quad (7)$$

Obviously, this equation is a special case of (1). Indeed, if we put

$$f_1(t, x) = t + \frac{1}{3}x$$

$$f_2(t, s, x) = t + s + x^2(s)$$

and $g(t, s) = t(t + s - 1) \Rightarrow d_s g(t, s) = t ds, m(t) = t$.

It is clear that the functions f_1, f_2, g satisfy the assumptions of Theorem 5; then, the functional integral Equation (7) has one weak solution $x \in C(I, E)$.

Example 2. Consider the initial value problem

$$\frac{dx(t)}{dt} = t + \frac{t^2}{2} + \int_0^{m(t)} \frac{t}{t+s} (t + s + x^2(s)) ds, \quad a.e., t \in I, \quad (8)$$

$$x(0) = x_0 \quad (9)$$

where $f_1(t, x) = t + \frac{t^2}{2} + x$, $f_2(t, s, x) = t + s + x^2(s)$ and

$$g(t, s) = \begin{cases} t \ln \frac{t+s}{t}, & \text{for } t \in (0, T], s \in I, \\ 0, & \text{for } t = 0, s \in I, \end{cases}$$

It is clear that this problem is a special case of the initial value problems (2) and (3), Where $d_s g(t, s) = \frac{t}{t+s} ds$, f_1, f_2, g satisfy the assumptions (iii)–(vi) and (viii) in Theorem 6; then the initial value problem (8) and (9) has a one weak solution $x \in C(I, E)$.

6. Conclusions

The theory of differential equations in abstract Banach spaces has been established by some authors from different viewpoints, for example, [24–27]. Here we have proved the existence of solutions $x \in C(I, E)$ for the delay functional integral Equation (1) and its corresponding initial value problem (2) and (3) which have been studied in a reflexive Banach space E . The continuous solutions of (1)–(3) on the delay function m and the function f_2 and the continuous solution of (2) and (3) on the initial data x_0 have been proved. As an application, two examples are given.

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