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On a Periodic Boundary Value Problem for Fractional Quasilinear Differential Equations with a Self-Adjoint Positive Operator in Hilbert Spaces

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Abstract: In this paper we study the existence of a mild solution of a periodic boundary value problem for fractional quasilinear differential equations in a Hilbert spaces. We assume that a linear part in equations is a self-adjoint positive operator with dense domain in Hilbert space and a nonlinear part is a map obeying Carathéodory type conditions. We find the mild solution of this problem in the form of a series in a Hilbert space. In the space of continuous functions, we construct the corresponding resolving operator, and for it, by using Schauder theorem, we prove the existence of a fixed point. At the end of the paper, we give an example for a boundary value problem for a diffusion type equation.

Keywords: fractional differential equation; Caputo fractional derivative; self-adjoint positive operator; fixed point; Mittag–Leffler function; Green function; completely continuous map

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1. Introduction

Fractional calculus and the theory of fractional differential equations have gained significant popularity and importance over the past three decades, mainly due to demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering (see monographs [1,2], papers [3–9]). This branch of mathematics really provides many useful tools for various problems related to special functions of mathematical physics, as well as their extensions and generalizations for one or more variables (see e.g., paper [10]). In addition, the field of application of fractional analysis in modern science includes research in the theory of fluid flows, rheology, dynamic processes in self-similar and porous structures, diffusion transfer, electrical networks, the theory of control of dynamic systems, the theory of viscoelasticity, electrochemistry of optics, and much more (see monograph [11], paper [12]).

At the same time, many problems of mathematical physics, the theory of elasticity, hydrodynamics and others lead to the study of boundary value problems for partial differential equations with appropriate boundary conditions. If the differential equation is linear with respect to the desired function and the boundary conditions are linear and homogeneous, then a unified approach based on the introduction and study of the properties of the so-called operator of the boundary value problem can be applied to such problems (see monographs [13,14], papers [15–20]). The resulting operators possess the properties of linearity, but they turn out to be unbounded in the chosen Hilbert space, and therefore, they are specified not on the entire space but only on some dense set. Usually these operators turn out to be a self-adjoint operator possessing the property of positive definiteness or semi-boundedness (see monograph [21]).

For example, in paper [22], the authors study a class of a fractional differential equations in Hilbert spaces the following type

$${}^C D_0^q u(t) + Au(t) = 0, \quad t \in [0, T],$$

where ${}^C D_0^q$ is the Caputo derivative of order $q \in (1, 2)$, A is linear self-adjoint positive operator. The authors introduce notions of weak and strong solutions for these equation and present conditions under which there exist solutions.

In the present paper, for a fractional order $q \in (1, 2)$ semilinear differential equation of the following form:

$${}^C D_0^q x(t) = Ax(t) + v(t, x(t)), \quad t \in [0, T], \quad (1)$$

in a separable Hilbert space H , with a scalar product $\langle \cdot, \cdot \rangle$, we prove the existence of a mild solution satisfying the following periodic boundary value conditions

$$x(0) = x(T), \quad x'(0) = x'(T), \quad (2)$$

where $A : D(A) \subset H \rightarrow H$ is linear self-adjoint positive operator such that $\overline{D(A)} = H$ and for every $x \in D(A)$:

$$\langle Ax, x \rangle \geq a \|x\|^2, \quad (3)$$

where $a > 0$, and $v : [0, T] \times H \rightarrow H$ is a nonlinear map. Notice, for the quasilinear case with a fractional derivative of order $q \in (1, 2)$, this kind of problems were not study until now.

2. Preliminaries

To solve the problem, we will need the following information from fractional calculus. (see, e.g., monographs [1,2]).

Definition 1. The fractional integral of an order $q > 0$ of a function $g : [0, T] \rightarrow E$ is the function $I_0^q g$ of the following form:

$$I_0^q g(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s) ds,$$

where Γ is the Euler gamma-function

$$\Gamma(q) = \int_0^\infty x^{q-1} e^{-x} dx.$$

Definition 2. The Caputo fractional derivative of an order $q \geq 0$ of a function $g \in C^n([0, T]; E)$ is the function ${}^C D_0^q g$ of the following form:

$${}^C D_0^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n = [q] + 1.$$

Definition 3 (Cf. [23]). A function of the form

$$E_{q,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(qn + \beta)}, \quad q > 0, \beta \in \mathbb{C}, z \in \mathbb{C}$$

is called the Mittag-Leffler function.

Denote $E_{q,1}$ by E_q . The Mittag-Leffler function has a great importance in the theory of fractional differential equations. Let us consider the Cauchy type problem for a differential equation of a fractional order $1 < q < 2$:

$${}^C D_0^q x(t) = \lambda x(t) + g(t), \quad t \in [0, T], \quad (4)$$

$$x(0) = c_1, \quad x'(0) = c_2, \quad (5)$$

here $\lambda \in \mathbb{R}$, and $g : [0, T] \rightarrow \mathbb{R}$ is a function for which there exists $I_0^q f$. The solution of problem (4) and (5) is a function $x \in C([0, T], \mathbb{R})$, satisfying conditions (5) such that the fractional derivative ${}^C D_0^q x \in C([0, T], \mathbb{R})$ and satisfying Equation (4). The unique solution of problem (4) and (5) (see [1]) is the following function

$$x(t) = c_1 E_q(\lambda t^q) + c_2 t E_{q,2}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) g(s) ds. \quad (6)$$

In the paper [19], by using the Green function method, the authors study for a semilinear fractional differential equation of an order $q \in (1, 2)$ in a separable Banach space E of the form

$${}^C D_0^q x(t) = \lambda x(t) + g(t), \quad t \in [0, T],$$

the existence of a mild solution satisfying the boundary value conditions

$$x(0) = x(T), \quad x'(0) = x'(T);$$

here, $\lambda > 0$ and the function $g \in L^\infty([0, T]; E)$.

The last problem (see [19]) has the unique solution

$$x(t) = \int_0^T G(t, s, \lambda) g(s) ds,$$

where the Green function $G(t, s, \lambda)$ has the following form

$$G(t, s, \lambda) = \begin{cases} \frac{(1-E_q(\lambda T^q))(T-s)^{q-1} E_{q,q}(\lambda(T-s)^q) + t E_{q,2}(\lambda T^q)(T-s)^{q-2} E_{q,q-1}(\lambda(T-s)^q)}{(1-E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q) E_{q,2}(\lambda T^q)} E_q(\lambda t^q) + \\ \frac{(1-E_q(\lambda T^q))(T-s)^{q-2} E_{q,q-1}(\lambda(T-s)^q) + T^{-1} E_{q,0}(\lambda T^q)(T-s)^{q-1} E_{q,q}(\lambda(T-s)^q)}{(1-E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q) E_{q,2}(\lambda T^q)} t E_{q,2}(\lambda t^q) + \\ (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q), & 0 \leq s \leq t < T, \\ \frac{(1-E_q(\lambda T^q))(T-s)^{q-1} E_{q,q}(\lambda(T-s)^q) + t E_{q,2}(\lambda T^q)(T-s)^{q-2} E_{q,q-1}(\lambda(T-s)^q)}{(1-E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q) E_{q,2}(\lambda T^q)} E_q(\lambda t^q) + \\ \frac{(1-E_q(\lambda T^q))(T-s)^{q-2} E_{q,q-1}(\lambda(T-s)^q) + T^{-1} E_{q,0}(\lambda T^q)(T-s)^{q-1} E_{q,q}(\lambda(T-s)^q)}{(1-E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q) E_{q,2}(\lambda T^q)} t E_{q,2}(\lambda t^q), & 0 \leq t < s < T, \end{cases} \quad (7)$$

where the following condition

$$(1 - E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q) E_{q,2}(\lambda T^q) \neq 0, \quad (8)$$

is supposed for $\lambda > 0$. Below, it is assumed that the Green function G satisfies the condition (G) the Green function G does not change its sign on $[0, T] \times [0, T] \times (0, +\infty)$;

In paper [19], it was shown that if condition (G) is satisfied, then

$$\int_0^T |G(t, s, \lambda)| ds \equiv \frac{1}{\lambda}. \quad (9)$$

In the following, we will assume that $\lambda > 0$.

We will need the following statements.

Lemma 1 (Cf. [24], Theorem 5.6.2). *The set of Lebesgue points of every function $\xi \in L^\infty([0, T]; E)$ is the set of full measure for $[0, T]$.*

The next lemma is a direct consequence of Theorem 5.8.4 and Corollary 3.9.5 in [24].

Lemma 2. For every function $\xi \in L^\infty([0, T]; E)$, there is a sequence of functions $\{\xi_n\} \subset C([0, T]; E)$, such that $\xi_n(t) \rightarrow \xi(t)$ at all Lebesgue points of the function ξ on $[0, T]$; moreover $\|\xi_n\|_{C([0, T]; E)} \leq \|\xi\|_{L^\infty([0, T]; E)}$.

For example the following sequence, constructed on the basis of the Steklov projector, satisfies the conditions of Lemma 2

$$\xi_n(t) = \frac{1}{2n} \int_{t-\frac{1}{n}}^{t+\frac{1}{n}} \hat{\xi}(s) ds, \text{ for } t \in [0, T];$$

$$\hat{\xi}(t) = \begin{cases} \xi(t), & \text{if } t \in [0, T]; \\ 0, & \text{if } t \notin [0, T]. \end{cases}$$

Notice, that the function ξ is defined on $[0, T]$, and the auxiliary function $\hat{\xi}$, used for the construction of the functions ξ_n , is defined on the entire real axis $\mathbb{R}$.

Lemma 3. Let $\xi \in L^\infty([0, T]; E)$ and a constant $C > 0$ such that $\|\xi\|_{L^\infty([0, T]; E)} \leq C$, then a function

$$y(t) = \int_0^T G(t, s, \lambda) \xi(s) ds, \quad t \in [0, T],$$

is continuous; moreover

$$\|y\|_{C([0, T]; E)} \leq \frac{C}{\lambda}. \quad (10)$$

Proof. Formula (7) of Green's function implies for all $t \in [0, T]$ that $G(t, \cdot, \lambda)$ is not continuous only at a point $s = T$, and at this point, it has a summable singularity; therefore if $\xi \in C([0, T]; E)$, then a function

$$y(t) = \int_0^T G(t, s, \lambda) \xi(s) ds$$

is continuous.

Let $\xi \in L^\infty([0, T]; E)$, then by using of Lemma 2 and Lemma 1, there exists a sequence of functions $\{\xi_n\} \subset C([0, T]; E)$, such that $\xi_n(t) \rightarrow \xi(t)$ for a.e. $t \in [0, T]$. At the same time, the functions

$$y_n(t) = \int_0^T G(t, s, \lambda) \xi_n(s) ds$$

are continuous. We know $\|\xi_n\|_{C([0, T]; E)} \leq \|\xi\|_{L^\infty([0, T]; E)}$; then, by Lebesgue's theorem, we have

$$\lim_{n \rightarrow \infty} y_n(t) = \lim_{n \rightarrow \infty} \int_0^T G(t, s, \lambda) \xi_n(s) ds = \int_0^T G(t, s, \lambda) \xi(s) ds = y(t).$$

Now, we show that the sequence $\{y_n\}$ really converges uniformly to the function y . Let $\varepsilon > 0$ be an arbitrary number, then for $0 < d < T$, we have

$$\begin{aligned} \|y_n(t) - y(t)\|_E &= \left\| \int_0^T G(t, s, \lambda) \xi_n(s) ds - \int_0^T G(t, s, \lambda) \xi(s) ds \right\|_E \leq \\ &\int_0^T |G(t, s, \lambda)| \|\xi_n(s) - \xi(s)\|_E ds = \\ &\int_0^{T-d} |G(t, s, \lambda)| \|\xi_n(s) - \xi(s)\|_E ds + \int_{T-d}^T |G(t, s, \lambda)| \|\xi_n(s) - \xi(s)\|_E ds = Z_1 + Z_2, \end{aligned}$$

where

$$Z_1 = \int_0^{T-d} |G(t, s, \lambda)| \|\xi_n(s) - \xi(s)\|_E ds,$$

$$Z_2 = \int_{T-d}^T |G(t, s, \lambda)| \|\xi_n(s) - \xi(s)\|_E ds.$$

Notice, that it follows from the inequality $\|\xi\|_{L^\infty([0,T];H)} \leq C$, for each $n : \|\xi_n\|_{C([0,T];H)} \leq C$. Since the Green function $G(t, s, \lambda)$ is not continuous only at $s = T$, and at this point, it has a summable singularity, we can take $0 < \delta_1 < \frac{\varepsilon}{6C}$ and for it to find $d > 0$ such that

$$\int_{T-d}^T |G(t, s, \lambda)| ds < \delta_1,$$

for all $t \in [0, T]$. Then, for Z_2 , we have

$$Z_2 = \int_{T-d}^T |G(t, s, \lambda)| \|\xi_n(s) - \xi(s)\|_E ds \leq 2C \int_{T-d}^T |G(t, s, \lambda)| ds \leq 2C\delta_1 < \frac{\varepsilon}{3},$$

uniformly respectively to $t \in [0, T]$.

The Green function G is continuous by $(t, s) \in [0, T] \times [0, T - d]$; therefore, there exists a constant $M > 0$ such that Green's function is uniformly bounded by a constant M with respect to the variables $(t, s) \in [0, T] \times [0, T - d]$. By the Egorov theorem, there exists a set $\Delta_{\delta_2} \subset [0, T - d]$ with measure $\mu(\Delta_{\delta_2}) < \delta_2 < \frac{\varepsilon}{6CM}$, such that the sequence $\{\xi_n\}$ converges uniformly to the function ξ on the set $[0, T - d] \setminus \Delta_{\delta_2}$. We represent

$$Z_1 = \int_0^{T-d} |G(t, s, \lambda)| \|\xi_n(s) - \xi(s)\|_E ds = \int_{[0, T-d] \setminus \Delta_{\delta_2}} |G(t, s, \lambda)| \|\xi_n(s) - \xi(s)\|_E ds + \int_{[0, T-d] \cap \Delta_{\delta_2}} |G(t, s, \lambda)| \|\xi_n(s) - \xi(s)\|_E ds = I_1 + I_2,$$

where

$$I_1 = \int_{[0, T-d] \setminus \Delta_{\delta_2}} |G(t, s, \lambda)| \|\xi_n(s) - \xi(s)\|_E ds,$$

$$I_2 = \int_{[0, T-d] \cap \Delta_{\delta_2}} |G(t, s, \lambda)| \|\xi_n(s) - \xi(s)\|_E ds.$$

Then for I_2 , we have

$$I_2 = \int_{[0, T-d] \cap \Delta_{\delta_2}} |G(t, s, \lambda)| \|\xi_n(s) - \xi(s)\|_E ds \leq 2CM\delta_2 < \frac{\varepsilon}{3}$$

uniformly respectively to $t \in [0, T]$. It remains for us to consider the estimate for I_1 . For $0 < \delta_3 < \frac{\varepsilon}{3MT}$, there exists N such that $\|\xi_n(s) - \xi(s)\|_E < \delta_3$ for each $n \geq N$, and $s \in [0, T - d]$. Then, for I_1 , we have

$$I_1 = \int_{[0, T-d] \setminus \Delta_{\delta_2}} |G(t, s, \lambda)| \|\xi_n(s) - \xi(s)\|_E ds \leq MT\delta_3 < \frac{\varepsilon}{3}.$$

Therefore, for each $\varepsilon > 0$, we may choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, such that

$$\|y_n(t) - y(t)\|_E < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

uniformly respectively to $t \in [0, T]$.

Therefore, the function y is continuous as the limit of a uniformly convergent sequence of continuous functions.

Now, we will show the validity of estimate (10). By using Expression (9), we have

$$\|y\|_{C([0,T];E)} = \max_{t \in [0,T]} \left\| \int_0^T G(t,s,\lambda) \xi(s) ds \right\|_E \leq C \int_0^T |G(t,s,\lambda)| ds = \frac{C}{\lambda}.$$

□

3. Results

We assume that there exists a sequence $\{\lambda_n\}$ of distinct simple eigenvalues of the operator A , where $\lambda_n > 0$, and $\lambda_n \rightarrow \infty$. Then, for the corresponding eigenvectors e_n , such that $Ae_n = \lambda_n e_n$, $\{e_n\}$ is an orthonormal basis in H , and for each $x \in D(A)$, we have

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n.$$

We will assume that the operator A satisfies a condition

(A) a series $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty$, and that the nonlinearity $v : [0, T] \times H \rightarrow H$ obeys the following conditions:

(v1) for each $x \in H$ the function $v(\cdot, x) : [0, T] \rightarrow H$ is measurable;

(v2) for a.e. $t \in [0, T]$ the function $v(t, \cdot) : H \rightarrow H$ is continuous;

(v3) for each $r > 0$ there exists a real function $\omega_r \in L_+^\infty[0, T]$ such that, for each $x \in H$ with $\|x\|_H \leq r$, we have

$$\|v(t, x)\|_H \leq \omega_r(t), \text{ for a.e. } t \in [0, T].$$

From the condition (A), it follows that the inverse operator A^{-1} is bounded. From the conditions (v1)–(v3), it follows that for a function $x \in C([0, T]; H)$ the function $v(\cdot, x(\cdot))$ belongs to $L^\infty([0, T]; H)$.

We will find the mild solutions of problem (1) and (2) in the form

$$x(t) = \sum_{n=1}^{\infty} x_n(t) e_n, \quad (11)$$

where $x_n(t) = \langle x(t), e_n \rangle$ are yet unknown functions. Let $f(t) = v(t, x(t))$, $t \in [0, T]$, and $f(t) = \sum_{n=1}^{\infty} f_n(t) e_n$, where $f_n(t) = \langle f(t), e_n \rangle$, then for a.e. $t \in [0, T]$ and every n , we have $|f_n(t)| \leq \|\omega_r\|_{L^\infty([0,T];H)}$, where $r = \|x\|_{C([0,T];H)}$.

By means of the scalar product in H , we multiply Equation (1) by e_n , and we have

$${}^C D_0^q x(t), e_n \rangle = \langle Ax(t), e_n \rangle + \langle f(t), e_n \rangle,$$

and take into account that the operator A is self-adjoint, so we obtain

$${}^C D_0^q x_n(t) = \lambda_n x_n(t) + f_n(t). \quad (12)$$

The unique solution of problem (12), (2) is the function (see Preliminaries)

$$x_n(t) = \int_0^T G(t,s,\lambda_n) f_n(s) ds. \quad (13)$$

where for all n , Green's function has the form as in (7).

From condition (G), for each n , we have

$$|x_n(t)| = \left| \int_0^T G(t,s,\lambda_n) f_n(s) ds \right| \leq \int_0^T |G(t,s,\lambda_n)| |f_n(s)| ds \leq \frac{\|\omega_r\|_{L^\infty([0,T];H)}}{\lambda_n},$$

and therefore,

$$\sum_{n=1}^{\infty} |x_n(t)|^2 \leq \sum_{n=1}^{\infty} \frac{\|\omega_r\|_{L^\infty([0,T];H)}^2}{\lambda_n^2},$$

and by using the condition (A), a series $\sum_{n=1}^{\infty} |x_n(t)|^2$ is converges.

Then by (11) the mild solution of problem (1) and (2) will be have the following form

$$x(t) = \sum_{n=1}^{\infty} \int_0^T G(t, s, \lambda_n) f_n(s) ds \cdot e_n.$$

Consider the operator F defined in the following way:

$$Fx(t) = \sum_{n=1}^{\infty} \int_0^T G(t, s, \lambda_n) f_n(s) ds \cdot e_n,$$

where $f_n(s) = \langle f(s), e_n \rangle$ and $f(s) = v(s, x(s))$. Notice, that by using Lemma 3, we have $F : C([0, T]; H) \rightarrow C([0, T]; H)$. If a function $x \in C([0, T]; H)$ is a mild solution of periodic boundary value problem (1) and (2), then it is a fixed point of operator F . Therefore, in the following, we will show the existence of fixed points of the operator F .

Theorem 1. Let conditions (A), (G), (v1)–(v3) and (8) be satisfied, then the operator F is completely continuous.

Proof. Let $\Omega \subset C([0, T]; H)$ be a bounded set and $r_\Omega = \sup_{x \in \Omega} \|x\|_{C([0,T];H)}$. For fixed $t \in [0, T]$, we denote $F(\Omega)(t) = \{x(t) | x \in F(\Omega)\}$. Let us first prove the validity of the statement for a fixed value of n . Consider the operator

$$F_n x(t) = \int_0^T G(t, s, \lambda_n) f_n(s) ds \cdot e_n,$$

where $f_n(s) = \langle f(s), e_n \rangle$ and $f(s) = v(s, x(s))$.

Obviously, the operator transforms a bounded set into a bounded one; therefore, $F_n(\Omega)(t)$ is relatively compact for all n and $t \in [0, T]$.

Now, we must demonstrate that the set of functions

$$M = \left\{ F_n(x)(t) = \int_0^T G(t, s, \lambda_n) f_n(s) ds \cdot e_n : f_n(s) = \langle f(s), e_n \rangle, f(s) = v(s, x(s)), x \in \Omega \right\}$$

is equicontinuous. It is sufficient to show that

$$M' = \left\{ \int_0^t (t-s)^{q-1} E_{q,q}(\lambda_n(t-s)^q) f_n(s) ds \cdot e_n : f_n(s) = \langle f(s), e_n \rangle, f(s) = v(s, x(s)), x \in \Omega \right\}.$$

is equicontinuous, since, by virtue of the properties of the Mittag-Leffler function, the other terms in the formula of the Green function (7) are equicontinuous.

Let us fix $\epsilon > 0$. If we take $t_1, t_2 \in [0, T]$ such that $0 < t_1 < t_2 \leq T$ then for arbitrary $x \in \Omega$ we have

$$\left\| \int_0^{t_2} (t_2-s)^{q-1} E_{q,q}(\lambda_n(t_2-s)^q) f_n(s) e_n ds - \int_0^{t_1} (t_1-s)^{q-1} E_{q,q}(\lambda_n(t_1-s)^q) f_n(s) e_n ds \right\|_H \leq$$

$$\left\| \int_{t_1}^{t_2} (t_2-s)^{q-1} E_{q,q}(\lambda_n(t_2-s)^q) f_n(s) e_n ds \right\|_H +$$

$$\left\| \int_0^{t_1} \left((t_2-s)^{q-1} E_{q,q}(\lambda_n(t_2-s)^q) - (t_1-s)^{q-1} E_{q,q}(\lambda_n(t_1-s)^q) \right) f_n(s) e_n ds \right\|_H = Z_1 + Z_2,$$

where

$$Z_1 = \left\| \int_{t_1}^{t_2} (t_2 - s)^{q-1} E_{q,q}(\lambda_n(t_2 - s)^q) f_n(s) e_n ds \right\|_H,$$

$$Z_2 = \left\| \int_0^{t_1} \left((t_2 - s)^{q-1} E_{q,q}(\lambda_n(t_2 - s)^q) - (t_1 - s)^{q-1} E_{q,q}(\lambda_n(t_1 - s)^q) \right) f_n(s) e_n ds \right\|_H.$$

By using condition (v3), we can take $\delta_1 > 0$ such that the condition $|t_2 - t_1| < \delta_1$ implies the following estimate:

$$Z_1 \leq \|\omega_{r_\Omega}\|_{L^\infty([0,T];H)} E_{q,q}(\lambda_n T^q) \frac{(t_2 - t_1)^q}{q} < \frac{\epsilon}{2}.$$

Consider the function $p : [0, T] \rightarrow \mathbb{R}, p(\tau) = \tau^{q-1} E_{q,q}(\lambda_n \tau^q)$. The function p is continuous on the segment $[0, T]$; hence, by using the Cantor theorem, it is uniformly continuous on the segment $[0, T]$, i.e., for each $\gamma > 0$, there exists $\delta_2 > 0$, such that $|\tau_2 - \tau_1| < \delta_2, \tau_1, \tau_2 \in [0, T]$, implies

$$\left| \tau_2^{q-1} E_{q,q}(\lambda_n \tau_2^q) - \tau_1^{q-1} E_{q,q}(\lambda_n \tau_1^q) \right| < \gamma < \frac{\epsilon}{2 \|\omega_{r_\Omega}\|_{L^\infty([0,T];H)} T}.$$

Now, taking $\tau = t - s$, we get

$$Z_2 \leq \|\omega_{r_\Omega}\|_{L^\infty([0,T];H)} \gamma t_1 < \frac{\epsilon}{2}.$$

Thus, for each n , the operator F_n is completely continuous. For any finite m , the operator

$$\sum_{n=1}^m \int_0^T G(t, s, \lambda_n) f_n(s) ds \cdot e_n,$$

where $f_n(s) = \langle f(s), e_n \rangle$ and $f(s) = v(s, x(s))$, is completely continuous also.

Now, the complete continuity of F follows from the next relations:

$$\begin{aligned} \lim_{m \rightarrow \infty} \left\| Fx(t) - \sum_{n=1}^m \int_0^T G(t, s, \lambda_n) f_n(s) ds \cdot e_n \right\|_H &= \lim_{m \rightarrow \infty} \left\| \sum_{n=m+1}^{\infty} \int_0^T G(t, s, \lambda_n) f_n(s) ds \cdot e_n \right\|_H \\ &\leq \lim_{m \rightarrow \infty} \left(\sum_{n=m+1}^{\infty} \frac{1}{\lambda_n^2} \|\omega_{r_\Omega}\|_{L^\infty([0,T];H)}^2 \right)^{1/2} = \lim_{m \rightarrow \infty} \left(\sum_{n=m+1}^{\infty} \frac{1}{\lambda_n^2} \right)^{1/2} \|\omega_{r_\Omega}\|_{L^\infty([0,T];H)} = 0. \end{aligned}$$

□

Now, we prove the main result of the paper.

Theorem 2. Let conditions (v1), (v2), (A), (G), (8) be fulfilled. Additionally, let us assume that condition (v3) has the following form:

(v3') there exists a real function $\alpha \in L_+^\infty([0, T])$ such that

$$\|v(t, \xi)\|_H \leq \alpha(t)(1 + \|\xi\|_H).$$

If

$$k = \left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \right)^{1/2} \|\alpha\|_{L^\infty([0,T];H)} < 1, \quad (14)$$

then problem (1) and (2) has a mild solution.

Proof. Taking an arbitrary function $x \in C([0, T]; H)$, we will have for $t \in [0, T]$ the following estimate:

$$\begin{aligned} \|Fx(t)\|_H &\leq \left\| \sum_{n=1}^{\infty} \int_0^T G(t, s, \lambda_n) f_n(s) ds \cdot e_n \right\|_H \leq \\ &\left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \|\alpha\|_{L^\infty([0, T]; H)}^2 (1 + \|x\|_{C([0, T]; H)})^2 \right)^{1/2} = \\ &\left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \right)^{1/2} \|\alpha\|_{L^\infty([0, T]; H)} (1 + \|x\|_{C([0, T]; H)}) = k(1 + \|x\|_{C([0, T]; H)}). \end{aligned}$$

So, if we will take

$$R \geq \frac{k}{1-k},$$

then the inequality $\|x\|_{C([0, T]; H)} \leq R$ implies $\|Fx\|_{C([0, T]; H)} \leq R$. Therefore, the operator F mapping the closed ball $B_R(0) \subset C([0, T]; H)$ into itself. Now, since the operator F is completely continuous, by the Schauder theorem, it has a fixed point, which is a mild solution of periodic boundary value problem (1) and (2). $\square$

4. Example

Let us consider the following boundary value problem for a system governed by a partial fractional differential equation of a diffusion type:

$$\partial_t^q u(t, s) = -\frac{\partial^2 u}{\partial s^2}(t, s) + \mu u(t, s) + \psi(t, s, u(t, s)), \quad (15)$$

$$u(0, s) = u(T, s), \quad \partial_t u(0, s) = \partial_t u(T, s), \quad (16)$$

$$u(t, 0) = u(t, \pi) = 0, \quad (17)$$

here, $t \in [0, T]$, $s \in [0, \pi]$, $\mu > 0$, ∂_t^q is the Caputo partial derivative in t of order $1 < q < 2$, $\psi: [0, T] \times [0, \pi] \times \mathbb{R} \rightarrow \mathbb{R}$.

Considering $u(t, s)$ as $x(t)(s)$, where $x: [0, T] \rightarrow L^2[0, \pi]$, we will reduce the above problem to abstract problem (1) and (2) in the space $H := L^2[0, \pi]$. In so doing, the operator A is defined by the formula

$$Az = -\frac{d^2 z}{ds^2} + \mu z,$$

and

$$D(A) = \left\{ z \in W_2^2[0, \pi] : z(0) = z(\pi) = 0 \right\}.$$

where $W_2^2[0, \pi]$ is a Sobolev space.

It is well known that the operator A is self-adjoint and positive, but for the convenience of the reader, we give a proof of these properties. Indeed

$$\begin{aligned} \langle Az, y \rangle &= - \int_0^\pi y(s) \frac{d^2 z(s)}{ds^2} ds + \mu \int_0^\pi y(s) z(s) ds = \\ &\int_0^\pi \frac{dy(s)}{ds} \frac{dz(s)}{ds} ds - y(s) \frac{dz(s)}{ds} \Big|_0^\pi + \mu \int_0^\pi y(s) z(s) ds = \\ &\int_0^\pi \frac{dy(s)}{ds} \frac{dz(s)}{ds} ds + \mu \int_0^\pi y(s) z(s) ds = \langle z, Ay \rangle, \text{ for } z, y \in D(A), \end{aligned}$$

then

$$\begin{aligned} \langle Az, z \rangle &= \int_0^\pi \frac{dz(s)}{ds} \frac{dz(s)}{ds} ds + \mu \int_0^\pi z^2(s) ds = \\ &\int_0^\pi \left(\frac{dz(s)}{ds} \right)^2 ds + \mu \int_0^\pi z^2(s) ds \geq 0, \end{aligned}$$

and $\langle Az, z \rangle = 0$ if and only if $z = 0$.

For a function $z \in D(A)$ we have

$$z(s) = \int_0^s 1 \cdot \frac{dz(\tau)}{d\tau} d\tau,$$

and therefore,

$$z^2(s) \leq \int_0^s 1^2 d\tau \cdot \int_0^s \left(\frac{dz(\tau)}{d\tau} \right)^2 d\tau = s \int_0^s \left(\frac{dz(\tau)}{d\tau} \right)^2 d\tau \leq s \int_0^\pi \left(\frac{dz(\tau)}{d\tau} \right)^2 d\tau,$$

and

$$\int_0^\pi z^2(s) ds \leq \int_0^\pi s ds \int_0^\pi \left(\frac{dz(\tau)}{d\tau} \right)^2 d\tau = \frac{\pi^2}{2} \int_0^\pi \left(\frac{dz(s)}{ds} \right)^2 ds,$$

and hence,

$$\begin{aligned} \langle Az, z \rangle &= \int_0^\pi \left(\frac{dz(s)}{ds} \right)^2 ds + \mu \int_0^\pi z^2(s) ds \geq \\ &\frac{2}{\pi^2} \int_0^\pi z^2(s) ds + \mu \int_0^\pi z^2(s) ds = \left(\frac{2}{\pi^2} + \mu \right) \|z\|_{L^2[0,\pi]}^2. \end{aligned}$$

Notice, in this case, the distinct simple eigenvalues of the operator A are $\lambda_n = 4\pi^2 n^2 + \mu$, where $n = 0, 1, 2, \dots$

We will suppose that the function ψ generates the superposition function $v: [0, T] \times H \rightarrow H$ defined as

$$v(t, x)(s) = \psi(t, s, x(s)).$$

In order to conclude that this function is well defined, it is sufficient to suppose that the function ψ is continuous, $\psi(t, \cdot, \xi) \in H$ for all $(t, \xi) \in [0, T] \times \mathbb{R}$ and ψ has a sublinear growth in the third variable:

$$|\psi(t, s, \xi)| \leq a + b|\xi|, \quad \forall (t, s) \in [0, T] \times [0, \pi],$$

where a and b are some nonnegative constants, then the superposition operator v has sublinear growth also.

Now, we can substitute problem (15)–(17) with problem (1) and (2) in the space $H = L^2[0, \pi]$:

$$\begin{aligned} {}^C D_0^q x(t) &= Ax(t) + v(t, x(t)), \\ x(0) &= x(T), \quad x'(0) = x'(T). \end{aligned}$$

Therefore, all conditions of Theorem 2 are fulfilled, and we conclude that problem (15)–(17) has a unique mild solution.

5. Conclusions

We studied the periodic boundary value problem for fractional quasilinear differential equations in a Hilbert space. The existence of a mild solution to this problem is obtained under the assumption that the linear part of the equation is a self-adjoint positive operator with dense domain in Hilbert space, whereas the nonlinear part is a map obeying Carathéodory type conditions. We were looking for the mild solution to this problem in the form of a series in a Hilbert space. In the space of continuous functions, we constructed the corresponding resolving operator, and for it, by using the Schauder theorem, we proved the existence of a fixed point. An example concerning the existence of a mild solution for a system governed by a partial fractional-order differential equation is presented.

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