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Consensus Control of Leaderless and Leader-Following Coupled PDE-ODEs Modeled Multi-Agent Systems

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Abstract: This paper discusses consensus control of nonlinear coupled parabolic PDE-ODE-based multi-agent systems (PDE-ODEMASs). First, a consensus controller of leaderless PDE-ODEMASs is designed. Based on a Lyapunov-based approach, coupling strengths are obtained for leaderless PDE-ODEMASs to achieve leaderless consensus. Furthermore, a consensus controller in the leader-following PDE-ODEMAS is designed and the corresponding coupling strengths are obtained to ensure the leader-following consensus. Two examples show the effectiveness of the proposed methods.

Keywords: consensus; PDE-ODEs; MASs; leader-following; coupling strengths

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1. Introduction

Consensus in multi-agent systems (MASs) is to achieve a common group objective when agents have different initial states [1–4]. It has received great attention in the past decade as a result from its wide applications in flocking of mobile robots [5], opinion dynamics in social networks [6], formation of unmanned vehicles [7–9], microgrid energy management [10], traffic flow [11], etc.

In a pioneering contribution, many important control methods were proposed for consensus of MASs, focusing on models based on ordinary differential equations [12–20]. Actually, there are many practical cases in nature and discipline fields with spatio-temporal characteristics, modeled by coupled partial differential equations (PDEs) [21–25]. Applied to overhead cranes [26], hormonal therapy [27], traffic flow [28], etc., another class of spatio-temporal models is based on coupled partial differential equations—ordinary differential equations (PDE-ODEs) [29–31]. Therefore, it is important to research consensus control of PDE-based coupled MASs (PDEMASs) or coupled PDE-ODE-based MASs (PDE-ODEMASs).

More recently, there have been many important results related to PDEMASs. Ref. [32] studied a distributed adaptive controller of uncertain leader-following parabolic PDEMASs; ref. [33] studied consensus control for parabolic and second-order hyperbolic PDEMASs; ref. [34] studied distributed P-type iterative learning for PDEMASs with time delay; refs. [35,36] studied iterative learning control for PDEMASs without and with time delay; ref. [37] studied boundary control of 3-D PDEMASs with arbitrarily large boundary input delay; refs. [38,39] studied consensus and input constraint consensus of nonlinear PDE-MASs using boundary control. However, consensus control for PDE-ODEMASs has not been addressed yet, which is a new challenge.

Motivated by the above, this paper studies consensus control of nonlinear coupled parabolic PDE-ODEMASs with Neumann boundary conditions. First, dealing with the leaderless case, a consensus controller of leaderless PDE-ODEMASs is designed. The leaderless consensus error system is obtained and one Lyapunov functional candidate is given. Using Wirtinger's inequality and matrix properties, coupling strengths are obtained for leaderless PDE-ODEMASs to achieve cluster consensus. Furthermore, dealing with the leader-following case, a consensus controller of leader-following PDE-ODEMASs is designed. The leader-following consensus error system is obtained and another Lyapunov functional candidate is given. The corresponding coupling strengths are obtained to ensure leader-following consensus.

The remainder of this paper is organized as follows. The problem formulation is given in Section 2. Section 3 presents a consensus control design of the leaderless PDE-ODEMAS and Section 4 gives that of the leader-following PDE-ODEMAS. An example to illustrate the effectiveness of the proposed method is presented in Sections 5 and 6 offers some concluding remarks.

Notations: $\lambda_{\max}(\cdot), \lambda_2(\cdot)$ stand for the maximum eigenvalue and smallest nonzero eigenvalue of $\cdot$, respectively. $\otimes$ is a Kronecker product of matrices. The identity matrix of n order is denoted by I_n. $||\cdot||$ denotes the Euclidean norm for vectors in $\boldsymbol{R}^n$ or the induced 2-norm for matrices in $\boldsymbol{R}^{m\times n}$.

2. Problem Formulation

Consider a nonlinear PDE-ODEMAS as

$$
\begin{aligned}
\dot{x}_i(t) &= f(x_i(t)) + \int_0^1 w(y_i(\xi,t))d\xi + u_i(t), \\
\frac{\partial y_i(\xi,t)}{\partial t} &= \alpha \frac{\partial^2 y_i(\xi,t)}{\partial \xi^2} + p(y_i(\xi,t)) \\
&\quad + q(x_i(t)) + U_i(\xi,t),
\end{aligned} \tag{1}
$$

such that

$$
\begin{aligned}
&\left.\frac{\partial y_i(\xi,t)}{\partial \xi}\right|_{\xi=0} = 0, \left.\frac{\partial y_i(\xi,t)}{\partial \xi}\right|_{\xi=1} = 0, \\
&x_i(0) = x_i^0, y_i(\xi,0) = y_i^0(\xi),
\end{aligned} \tag{2}
$$

where $(\xi,t) \in [0,1]\times[0,\infty)$, respectively, mean the spatial variable and time variable; $x_i(t), y_i(\xi,t) \in \mathbb{R}^n$ are the states; $u_i(t), U_i(\xi,t) \in \mathbb{R}^n$ are the control inputs; $x_i^0, y_i^0(\xi)$ are bounded and $y_i^0(\xi)$ is continuous; α is a positive scalar; $i \in \{1,2,\cdots,N\}$; and $f(\cdot), w(\cdot), p(\cdot), q(\cdot) \in \mathbb{R}^n$ are sufficiently smooth nonlinear functions.

Define consensus error $e_i(t) \triangleq x_i(t) - \frac{1}{N}\sum_{j=1}^N x_j(t)$ and $\varepsilon_i(\xi,t) \triangleq y_i(\xi,t) - \frac{1}{N}\sum_{j=1}^N y_j(\xi,t)$.

Definition 1. *For the leaderless PDE-ODEMAS (1), (2) with any initial conditions, if*

$$
\lim_{t\to\infty} e_i(t) \to 0, \lim_{t\to\infty} \varepsilon_i(\xi,t) \to 0, \tag{3}
$$

for any $i \in \{1,2,\cdots,N\}$, then the leaderless PDE-ODEMAS (1), (2) achieves consensus.

Lemma 1 ([40]). *Let κ be a differentiable function with $\kappa(0) = 0$ and $\kappa(1) = 0$, then*

$$
\int_0^1 \kappa^T(s)\kappa(s)ds \le \pi^{-2}\int_0^1 \dot{\kappa}^T(s)\dot{\kappa}(s)ds. \tag{4}
$$

Lemma 2 ([41]). *For an undirected connected graph with Laplacian matrix L, and $x \in \mathbb{R}^n$ such that $1_N^T x = 0$, then*

$$
\lambda_2(L)x^T x \le x^T L x. \tag{5}
$$

If Laplacian matrix $L \in \mathbb{R}^{N\times N}$ is symmetric, then $0 = \lambda_1(\cdot) < \lambda_2(\cdot) \leq \cdots \leq \lambda_N(\cdot)$. The smallest nonzero eigenvalue of $\lambda_2(\cdot)$ is known as the algebraic connectivity of graphs [41].

Assumption 1. *Assume $f(\cdot), p(\cdot), q(\cdot), w(\cdot)$ satisfy the Lipschitz condition, i.e., for any ν_1 and $\nu_2 \in \mathbb{R}^n$, there exist scalars $\gamma_1, \gamma_2, \gamma_3, \gamma_4 > 0$ such that*

$$\begin{aligned} |f(\nu_1) - f(\nu_2)| &\leq \gamma_1 |\nu_1 - \nu_2|, \\ |p(\nu_1) - p(\nu_2)| &\leq \gamma_2 |\nu_1 - \nu_2|, \\ |q(\nu_1) - q(\nu_2)| &\leq \gamma_3 |\nu_1 - \nu_2|, \\ |w(\nu_1) - w(\nu_2)| &\leq \gamma_4 |\nu_1 - \nu_2|. \end{aligned} \tag{6}$$

3. Consensus Control of the Leaderless PDE-ODEMAS

To achieve consensus of the leaderless PDE-ODEMAS (1), the consensus controller is designed as:

$$\begin{aligned} u_i(t) &= d\sum_{j=1}^{N} a_{ij}(x_j(t) - x_i(t)), \\ U_i(\xi, t) &= k\sum_{j=1}^{N} b_{ij}(y_j(\xi, t) - y_i(\xi, t)), \end{aligned} \tag{7}$$

where d and k are the coupling strengths to be determined, $i \in \{1, 2, \cdots, N\}$. Assume that the topological structure $A = (a_{ij})_{N\times N}$ is defined as: $a_{ij} = a_{ji} > 0 (i \neq j)$ if the agent i connects to j, otherwise $a_{ij} = 0 (i \neq j)$; $a_{ii} = 0$. The topological structure $B = (b_{ij})_{N\times N}$ is defined the same as A.

The consensus error system can be obtained from (1), (2), and (7) that

$$\begin{aligned} \dot{e}_i(t) &= f(x_i(t)) - \frac{1}{N}\sum_{j=1}^{N} f(x_j(t)) + \int_0^1 w(y_i(\xi, t))d\xi - \frac{1}{N}\sum_{j=1}^{N}\int_0^1 w(y_j(\xi, t))d\xi \\ &\quad + d\sum_{j=1}^{N} a_{ij}(x_j(t) - x_i(t)), \\ \frac{\partial \varepsilon_i(\xi, t)}{\partial t} &= \alpha\frac{\partial^2 \varepsilon_i(\xi, t)}{\partial \xi^2} + p(y_i(\xi, t)) - \frac{1}{N}\sum_{j=1}^{N} p(y_j(\xi, t)) + q(x_i(t)) \\ &\quad - \frac{1}{N}\sum_{j=1}^{N} q(x_j(t)) + k\sum_{j=1}^{N} b_{ij}(y_j(\xi, t) - y_i(\xi, t)), \end{aligned} \tag{8}$$

such that

$$\begin{aligned} &\frac{\partial \varepsilon_i(\xi, t)}{\partial \xi}\bigg|_{\xi=0} = 0, \frac{\partial \varepsilon_i(\xi, t)}{\partial \xi}\bigg|_{\xi=1} = 0, \\ &e_i(0) = e_i^0(\xi), \varepsilon_i(\xi, 0) = \varepsilon_i^0(\xi), \end{aligned} \tag{9}$$

where $e_i^0 \triangleq x_i^0 - \frac{1}{N}\sum_{j=1}^{N} x_j^0$ and $\varepsilon_i^0(\xi) \triangleq y_i^0(\xi) - \frac{1}{N}\sum_{j=1}^{N} y_j^0(\xi)$.

Theorem 1. *Under Assumption 1, assume the graphs A and B are connected. Using the controller (7), the leaderless PDE-ODEMAS (1), (2) achieves consensus if*

$$\begin{aligned} d &> \frac{\gamma_1 + \frac{1}{2}\gamma_3^2 + \frac{1}{2}}{\lambda_2(L_a)}, \\ k &> \max\{\frac{\gamma_2 + \frac{1}{2}\gamma_4^2 + \frac{1}{2} - \alpha\pi^2}{\lambda_2(L_b)}, 0\}. \end{aligned} \tag{10}$$

Proof. Consider the following Lyapunov function as

$$V_1(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)e_i(t) + \frac{1}{2}\sum_{i=1}^{N}\int_0^1 \varepsilon_i^T(\xi,t)\varepsilon_i(\xi,t)d\xi. \tag{11}$$

We have

$$\begin{aligned}
\dot{V}_1(t) =& \sum_{i=1}^{N} e_i^T(t)\dot{e}_i(t) + \sum_{i=1}^{N}\int_0^1 \varepsilon_i^T(\xi,t)\frac{\partial \varepsilon_i(\xi,t)}{\partial t}d\xi \\
=& \sum_{i=1}^{N} e_i^T(t)[f(x_i(t)) - \frac{1}{N}\sum_{j=1}^{N} f(x_j(t))] \\
&+ \sum_{i=1}^{N} e_i^T(t)[\int_0^1 w(y_i(\xi,t))d\xi - \frac{1}{N}\sum_{j=1}^{N}\int_0^1 w(y_j(\xi,t))d\xi] \\
&+ \sum_{i=1}^{N} e_i^T(t)d\sum_{j=1}^{N} a_{ij}(e_j(t) - e_i(t)) + \sum_{i=1}^{N}\int_0^1 \varepsilon_i^T(\xi,t)\Theta\frac{\partial^2 \varepsilon_i(\xi,t)}{\partial \xi^2}d\xi \\
&+ \sum_{i=1}^{N}\int_0^1 \varepsilon_i^T(\xi,t)(p(y_i(\xi,t)) - \frac{1}{N}\sum_{j=1}^{N} p(y_j(\xi,t)))d\xi \\
&+ \sum_{i=1}^{N}\int_0^1 \varepsilon_i^T(\xi,t)(q(x_i(t)) - \frac{1}{N}\sum_{j=1}^{N} q(x_j(t)))d\xi \\
&+ \sum_{i=1}^{N}\int_0^1 \varepsilon_i^T(\xi,t)k\sum_{j=1}^{N} b_{ij}(\varepsilon_j(\xi,t) - \varepsilon_i(\xi,t))d\xi.
\end{aligned} \tag{12}$$

According to the matrix property,

$$\begin{aligned}
&\sum_{i=1}^{N} e_i^T(t)d\sum_{j=1}^{N} a_{ij}(e_j(t) - e_i(t)) \\
=& - de^T(t)(L_a \otimes I_n)e(t) \\
\leq& - d\lambda_2(L_a)e^T(t)e(t),
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
&\sum_{i=1}^{N}\int_0^1 \varepsilon_i^T(\xi,t)k\sum_{j=1}^{N} b_{ij}(\varepsilon_i(\xi,t) - \varepsilon_j(\xi,t))d\xi \\
=& - k\int_0^1 \varepsilon^T(\xi,t)(L_b \otimes I_n)\varepsilon(\xi,t)d\xi \\
\leq& - k\lambda_2(L_b)\int_0^1 \varepsilon^T(\xi,t)\varepsilon(\xi,t)d\xi,
\end{aligned} \tag{14}$$

where $\lambda_2(\cdot)$ denotes the smallest nonzero eigenvalue of $\cdot$, $L_{a,ij} = -a_{ij}$ when $i \neq j$, $L_{a,ii} = \sum_{j=1}^{N} a_{ij}$, $L_{b,ij} = -b_{ij}$ when $i \neq j$, $L_{b,ii} = \sum_{j=1}^{N} b_{ij}$. Therefore, L_a, L_b are Laplacian matrices. Using Lemma 1, for $\alpha > 0$,

$$
\begin{aligned}
&\int_0^1 \sum_{i=1}^N \varepsilon_i^T(\xi,t)\alpha\varepsilon_{i,\xi\xi}(\xi,t)d\xi \\
=&-\alpha\int_0^1 \varepsilon_\xi^T(\xi,t)\varepsilon_\xi(\xi,t)d\xi \\
\leq&-\alpha\pi^2\int_0^1 \varepsilon^T(\xi,t)\varepsilon(\xi,t)d\xi.
\end{aligned} \tag{15}
$$

Using Assumption 1, owing to $\sum\limits_{i=1}^N e_i^T(t)(f(\frac{1}{N}\sum_{j=1}^N x_j(t)) - \frac{1}{N}\sum_{j=1}^N f(x_j(t))) = 0$ and $\sum\limits_{i=1}^N \varepsilon_i^T(\xi,t)(f(\frac{1}{N}\sum_{j=1}^N p(y_i(\xi,t))) - \frac{1}{N}\sum_{j=1}^N p(y_i(\xi,t)) = 0$, we have

$$
\sum_{i=1}^N e_i^T(t)(f(x_i(t)) - \frac{1}{N}\sum_{j=1}^N f(x_j(t))) \leq \gamma_1\sum_{i=1}^N e_i^2(t), \tag{16}
$$

and

$$
\sum_{i=1}^N \int_0^1 \varepsilon_i^T(\xi,t)(p(y_i(\xi,t)) - \frac{1}{N}\sum_{j=1}^N p(y_j(\xi,t)))d\xi \leq \gamma_2\int_0^1 \varepsilon_i^2(\xi,t)d\xi. \tag{17}
$$

In the same way,

$$
\begin{aligned}
&\sum_{i=1}^N \int_0^1 \varepsilon_i^T(\xi,t)(q(x_i(t)) - \frac{1}{N}\sum_{j=1}^N q(x_j(t)))d\xi \\
=&\sum_{i=1}^N \int_0^1 \varepsilon_i^T(\xi,t)(q(x_i(t)) - q(\frac{1}{N}\sum_{j=1}^N x_j(t)))d\xi \\
\leq&\frac{1}{2}\sum_{i=1}^N \int_0^1 \varepsilon_i^T(\xi,t)\varepsilon_i(\xi,t)d\xi + \frac{1}{2}\sum_{i=1}^N \int_0^1 (q(x_i(t)) - q(\frac{1}{N}\sum_{j=1}^N x_j(t)))^2 d\xi \\
\leq&\frac{1}{2}\sum_{i=1}^N \int_0^1 \varepsilon_i^T(\xi,t)\varepsilon_i(\xi,t)d\xi + \frac{1}{2}\gamma_3^2\sum_{i=1}^N e_i^2(t),
\end{aligned} \tag{18}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^N e_i^T(t)\int_0^1 (w(y_i(\xi,t)) - \frac{1}{N}\sum_{j=1}^N w(y_j(\xi,t)))d\xi \\
=&\sum_{i=1}^N e_i^T(t)\int_0^1 (w(y_i(\xi,t)) - w(\frac{1}{N}\sum_{j=1}^N y_j(\xi,t)))d\xi \\
\leq&\frac{1}{2}\sum_{i=1}^N [e_i^T(t)e_i(t)d\xi + \int_0^1 (w(y_i(\xi,t)) - w(\frac{1}{N}\sum_{j=1}^N y_j(\xi,t)))^2 d\xi] \\
\leq&\frac{1}{2}\sum_{i=1}^N e_i^T(t)e_i(t) + \frac{1}{2}\sum_{i=1}^N \gamma_4^2\int_0^1 \varepsilon_i^2(\xi,t)d\xi.
\end{aligned} \tag{19}
$$

Substituting (13)–(19) into (12),

$$
\begin{aligned}
\dot{V}_1(t) \leq&-\rho_1 e^T(t)e(t) - \rho_2\int_0^1 \varepsilon^2(\xi,t)d\xi \\
\leq&-2\rho V(t),
\end{aligned} \tag{20}
$$

where $e \triangleq [e_1^T, e_2^T, \cdots, e_N^T]^T$, $\varepsilon \triangleq [\varepsilon_1^T, \varepsilon_2^T, \cdots, \varepsilon_N^T]^T$, $\rho_1 \triangleq -\gamma_1 - \frac{1}{2}\gamma_3^2 - \frac{1}{2} + d\lambda_2(L_a)$, $\rho_2 \triangleq -\gamma_2 - \frac{1}{2}\gamma_4^2 - \frac{1}{2} + \alpha\pi^2 + k\lambda_2(L_b)$, and $\rho \triangleq \min\{\rho_1, \rho_2\}$.

Taking d and k as (10) yields,

$$\rho_1 > 0, \rho_2 > 0. \tag{21}$$

It follows from (20) and (21) that $V_1(t) \leq V_1(0)\exp\{-2\rho t\}$, which implies $e_i(t) \to 0$ and $\varepsilon_i(\xi, t) \to 0$ as $t \to \infty$ This completes the proof. □

4. Consensus Control of the Leader-Following PDE-ODEMAS

The leader agent is supposed to be

$$\begin{aligned}
\dot{x}_0(t) &= f(x_0(t)) + \int_0^1 w(y_0(\xi, t))d\xi, \\
\frac{\partial y_0(\xi, t)}{\partial t} &= \alpha\frac{\partial^2 y_0(\xi, t)}{\partial \xi^2} + p(y_0(\xi, t)) + q(x_0(t)),
\end{aligned} \tag{22}$$

such that

$$\begin{aligned}
&\left.\frac{\partial y_0(\xi, t)}{\partial \xi}\right|_{\xi=0} = 0, \left.\frac{\partial y_0(\xi, t)}{\partial \xi}\right|_{\xi=1} = 0, \\
&x_0(0) = x_0^0, y_0(\xi, 0) = y_0^0(\xi),
\end{aligned} \tag{23}$$

where $x_0^0, y_0^0(\xi)$ are bounded and $y_0^0(\xi)$ is continuous.

The leader-following consensus controller is designed as:

$$\begin{aligned}
u_i(t) &= d[\sum_{j=1}^{N} a_{ij}(x_j(t) - x_i(t)) + \delta_i(x_0(t) - x_i(t))], \\
U_i(\xi, t) &= k[\sum_{j=1}^{N} b_{ij}(y_j(\xi, t) - y_i(\xi, t)) + \rho_i(y_0(\xi, t) - y_i(\xi, t))],
\end{aligned} \tag{24}$$

where $\delta_i > 0$ if x_i can obtain the information of x_0; otherwise, $\delta_i = 0$; and $\rho_i > 0$ if y_i can obtain the information of y_0; otherwise, $\rho_i = 0$.

Let $\tilde{e}_i(t) = x_i(t) - x_0(t)$ and $\tilde{\varepsilon}_i(\xi, t) = y_i(\xi, t) - y_0(\xi, t)$. The leader-following consensus error system is obtained as

$$\begin{aligned}
\dot{\tilde{e}}_i(t) &= f(x_i(t)) - f(x_0(t)) + \int_0^1 w(y_i(\xi, t))d\xi \\
&\quad - \int_0^1 w(y_0(\xi, t))d\xi - d\sum_{j=1}^{N} g_{ij}\tilde{e}_j(t), \\
\frac{\partial \tilde{\varepsilon}_i(\xi, t)}{\partial t} &= \alpha\frac{\partial^2 \tilde{\varepsilon}_i(\xi, t)}{\partial \xi^2} + p(y_i(\xi, t)) - p(y_0(\xi, t)) \\
&\quad + q(x_i(t)) - q(x_0(t)) \\
&\quad - k\sum_{j=1}^{N} h_{ij}\tilde{\varepsilon}_j(\xi, t),
\end{aligned} \tag{25}$$

such that

$$\begin{aligned}
&\left.\frac{\partial \tilde{\varepsilon}_i(\xi, t)}{\partial \xi}\right|_{\xi=0} = 0, \left.\frac{\partial \tilde{\varepsilon}_i(\xi, t)}{\partial \xi}\right|_{\xi=1} = 0, \\
&\tilde{e}_i(0) = \tilde{e}_i^0(\xi), \tilde{\varepsilon}_i(\xi, 0) = \tilde{\varepsilon}_i^0(\xi),
\end{aligned} \tag{26}$$

where $G = [g_{ij}] = L_A + diag\{\delta_i\}, H = [h_{ij}] = L_B + diag\{\rho_i\}$, $\tilde{e}_i^0 \triangleq x_i^0 - x_0^0$ and $\tilde{\varepsilon}_i^0(\xi) \triangleq y_i^0(\xi) - y_0^0(\xi)$.

Definition 2. *For the leader-following PDE-ODEMAS (22), (23) with any initial conditions, if*

$$\lim_{t\to\infty} \tilde{e}_i(t) \to 0, \lim_{t\to\infty} ||\tilde{\varepsilon}_i(\xi,t)|| \to 0, \tag{27}$$

for any $i \in \{1, 2, \cdots, N\}$, then the leader-following PDE-ODEMAS (22), (23) achieves consensus.

Theorem 2. *Under Assumption 1, assume the graphs A and B are connected. Using the controller (26), the leader-following PDE-ODEMAS (1) achieves consensus if*

$$\begin{aligned} d &> \frac{\gamma_1 + \frac{1}{2}\gamma_3^2 + \frac{1}{2}}{\lambda_{\min}(G)}, \\ k &> \max\{\frac{\gamma_2 + \frac{1}{2}\gamma_4^2 + \frac{1}{2} - \alpha\pi^2}{\lambda_{\min}(H)}, 0\}. \end{aligned} \tag{28}$$

Proof. Consider the Lyapunov functional candidate as

$$V_2(t) = \frac{1}{2}\sum_{i=1}^N \tilde{e}_i^T(t)\tilde{e}_i(t) + \frac{1}{2}\sum_{i=1}^N \int_0^1 \tilde{\varepsilon}_i^T(\xi,t)\tilde{\varepsilon}_i(\xi,t)d\xi. \tag{29}$$

One has

$$\begin{aligned} \dot{V}_2(t) =& \sum_{i=1}^N \tilde{e}_i^T(t)\dot{\tilde{e}}_i(t) + \sum_{i=1}^N \int_0^1 \tilde{\varepsilon}_i^T(\xi,t)\frac{\partial \tilde{\varepsilon}_i(\xi,t)}{\partial t}d\xi \\ =& \sum_{i=1}^N \tilde{e}_i^T(t)(f(x_i(t)) - f(x_0(t))) \\ &+ \sum_{i=1}^N \tilde{e}_i^T(t)[\int_0^1 w(y_i(\xi,t))d\xi - \int_0^1 w(y_0(\xi,t))d\xi] \\ &- \sum_{i=1}^N \tilde{e}_i^T(t)d\sum_{j=1}^N g_{ij}\tilde{e}_j(t) \\ &+ \sum_{i=1}^N \int_0^1 \tilde{\varepsilon}_i^T(\xi,t)\alpha\frac{\partial^2 \tilde{\varepsilon}_i(\xi,t)}{\partial \xi^2}d\xi \\ &+ \sum_{i=1}^N \int_0^1 \tilde{\varepsilon}_i^T(\xi,t)(p(y_i(\xi,t)) - p(y_0(\xi,t)))d\xi \\ &+ \sum_{i=1}^N \int_0^1 \tilde{\varepsilon}_i^T(\xi,t)(q(x_i(t)) - q(x_0(t)))d\xi \\ &- \sum_{i=1}^N \int_0^1 \tilde{\varepsilon}_i^T(\xi,t)k\sum_{j=1}^N h_{ij}\tilde{\varepsilon}_j(\xi,t)d\xi. \end{aligned} \tag{30}$$

Since G and H are symmetric positive definite matrices,

$$\begin{aligned} &- \sum_{i=1}^N \tilde{e}_i^T(t)d\sum_{j=1}^N g_{ij}(\tilde{e}_j(t)) \\ =& - d\tilde{e}^T(t)(G\otimes I_n)\tilde{e}(t) \\ \le& - d\lambda_{\min}(G)\tilde{e}^T(t)\tilde{e}(t), \end{aligned} \tag{31}$$

and

$$\begin{aligned}&\sum_{i=1}^{N}\int_0^1 \tilde{\varepsilon}_i^T(\xi,t)k\sum_{j=1}^{N}h_{ij}\tilde{\varepsilon}_j(\xi,t)d\xi\\ =&-k\int_0^1 \tilde{\varepsilon}^T(\xi,t)(H\otimes I_n)\tilde{\varepsilon}(\xi,t)d\xi\\ \leq&-k\lambda_{\min}(H)\int_0^1 \tilde{\varepsilon}^T(\xi,t)\tilde{\varepsilon}(\xi,t)d\xi,\end{aligned} \tag{32}$$

where $\tilde{e} \triangleq [\tilde{e}_1^T,\tilde{e}_2^T,\cdots,\tilde{e}_N^T]^T$, $\tilde{\varepsilon} \triangleq [\tilde{\varepsilon}_1^T,\tilde{\varepsilon}_2^T,\cdots,\tilde{\varepsilon}_N^T]^T$, $\lambda_{\min}(\cdot)$ denotes the smallest nonzero eigenvalue and G, H are symmetric positive definite matrices.

Considering (13)–(19), and substituting (31)–(32) into (30),

$$\begin{aligned}\dot{V}_2(t)\leq&(\gamma_1+\frac{1}{2}\gamma_3^2+\frac{1}{2}-d\lambda_{\min}(G))\tilde{e}^T(t)\tilde{e}(t)+(\gamma_2+\frac{1}{2}\gamma_4^2\\ &+\frac{1}{2}-\alpha\pi^2-k\lambda_{\min}(H))\int_0^1\tilde{\varepsilon}^T(\xi,t)\tilde{\varepsilon}(\xi,t)d\xi.\end{aligned} \tag{33}$$

In a similar way to the analysis in Theorem 1, the proof can be completed. □

Remark 1. *Many papers have investigated stabilization control methods for PDE-ODE systems [29–31,42], while this paper investigates consensus control for PDE-ODE-based MASs, considering control based on coupling.*

Remark 2. *Many significant results were obtained for consensus control modeled by PDEMASs [32–39]. Different from PDEMASs, this paper investigates consensus control methods for PDE-ODEMASs, as well as considering leaderless and leader-following models.*

5. Numerical Simulation

Example 1. *Consider the leaderless PDE-ODEMAS (1) and (2) with coefficients as*

$$\begin{aligned}&\alpha=0.8, f(\cdot)=w(\cdot)=p(\cdot)=q(\cdot)=\tanh(\cdot),\\ &a_{ij}=b_{ij}=1,\ \text{and}\ i\neq j,\ \text{for}\ i,j=1,2,3,4,\\ &n=2,\end{aligned} \tag{34}$$

and with random initial conditions.

It is obvious that $f(\cdot), p(\cdot), q(\cdot)$, and $w(\cdot)$ satisfy the Lipschitz condition with $\gamma_1=\gamma_2=\gamma_3=\gamma_4=1$.

With Theorem 1, according to (10), $d>0.50$ and $k>0$ are obtained. Therefore, we take $d=0.51$ and $k=0.01$. It can be seen in Figures 1 and 2 that the leaderless PDE-ODEMAS achieves consensus with control gains $d=0.51$ and $k=0.01$.

From another point of view, $d=0.49$ and $k=0$ do not satisfy (10). It can be seen in Figures 3 and 4 that the leaderless PDE-ODEMAS cannot achieve consensus with control gains $d=0.49$ and $k=0$.

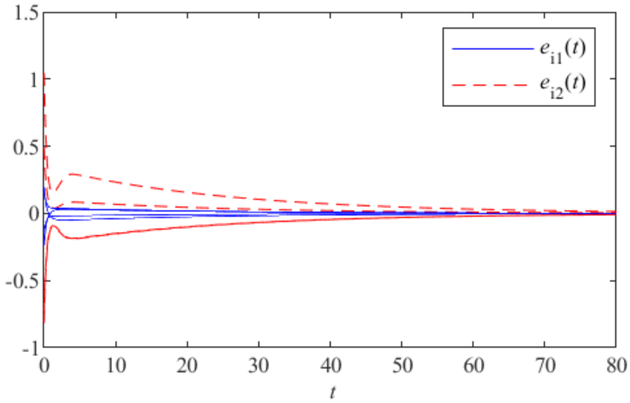

Figure 1. $e_i(t)$ withthe control gains $d = 0.51$ and $k = 0.01$ in Example 1.

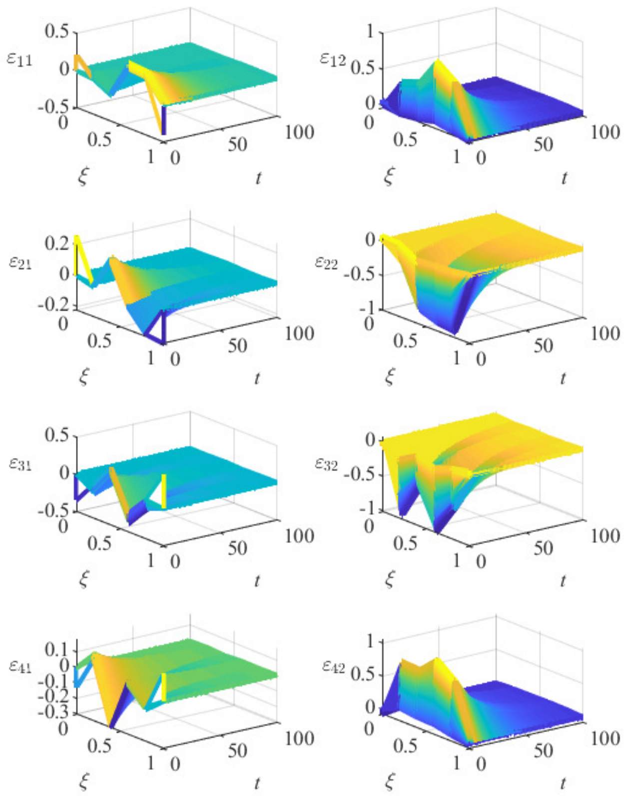

Figure 2. $\varepsilon_i(\xi, t)$ with the control gains $d = 0.51$ and $k = 0.01$ in Example 1.

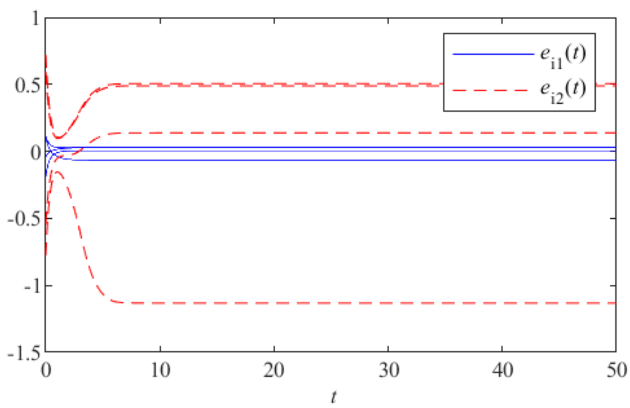

Figure 3. $e_i(t)$ with the control gains $d = 0.49$ and $k = 0$ in Example 1.

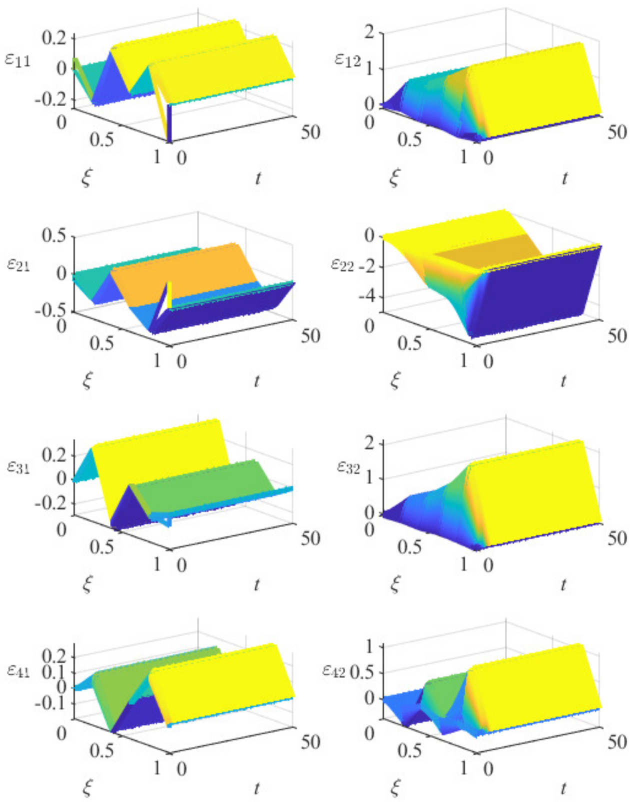

Figure 4. $\varepsilon_i(\xi, t)$ with the control gains $d = 0.49$ and $k = 0$ in Example 1.

Example 2. *Consider a nonlinear leader-following PDE-ODEMAS composed of 1 leader agent (22) and (23) and 4 following agents (1) and (2) with coefficients the same as Example 1. In the same way, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 1$ are obtained. Choose $\delta_i = \rho_i = 1$. With Theorem 2, according to (28), $d > 2.0$ and $k > 0$ are obtained. Therefore, we take $d = 2.1$ and $k = 0.1$. It can be seen in Figures 5 and 6 that the leader-following PDE-ODEMAS achieves consensus.*

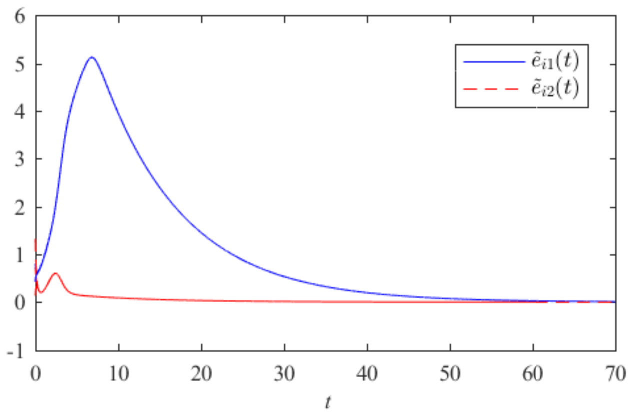

Figure 5. $\tilde{e}_i(t)$ with the control gains $d = 2.1$ and $k = 0.1$ in Example 2.

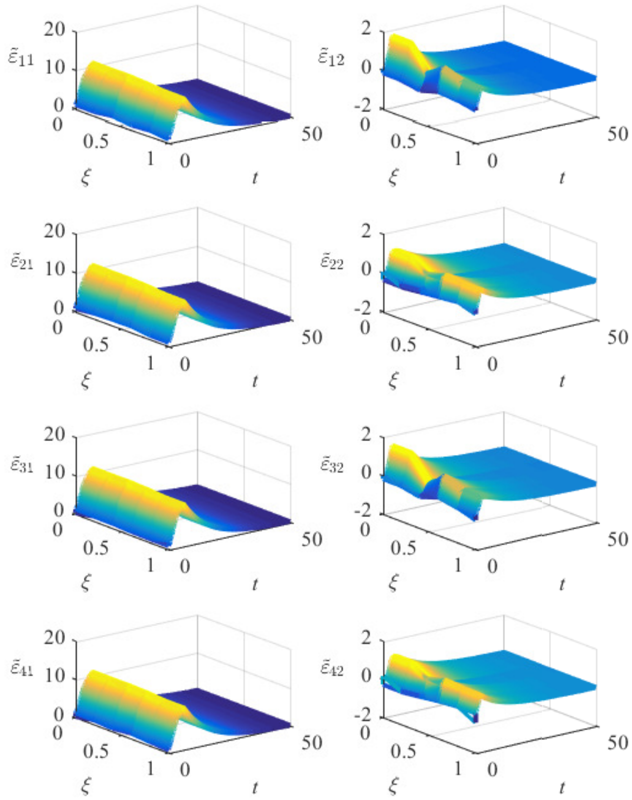

Figure 6. $\tilde{\varepsilon}_i(\xi, t)$ with the control gains $d = 2.1$ and $k = 0.1$ in Example 2.

From another point of view, $d = 1.9$ *and* $k = 0$ *do not satisfy (28). It can be seen in Figures 7 and 8 that the leader-following PDE-ODEMAS cannot achieve consensus with control gains* $d = 0.49$ *and* $k = 0$.

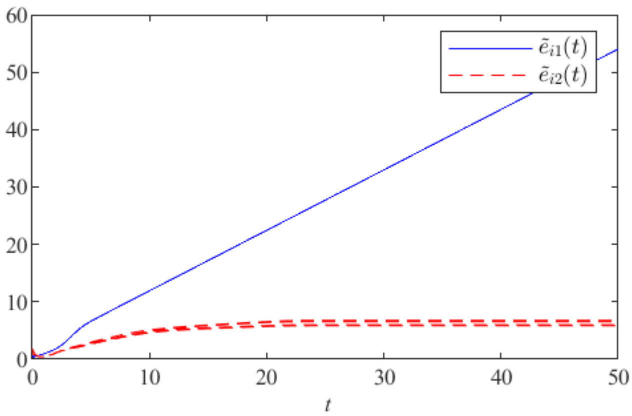

Figure 7. $\tilde{e}_i(t)$ with the control gains $d = 1.9$ and $k = 0$ in Example 2.

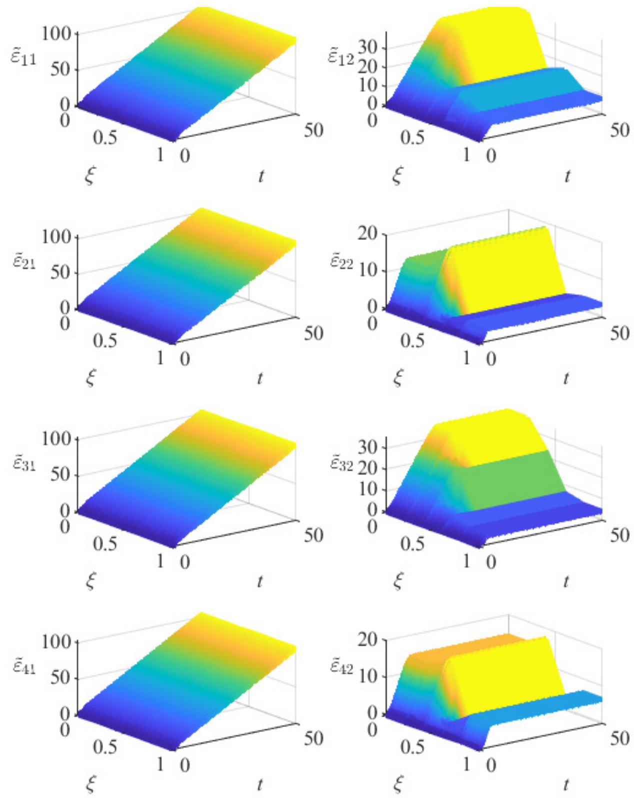

Figure 8. $\tilde{\varepsilon}_i(\xi, t)$ with control gains $d = 1.9$ and $k = 0$ in Example 2.

6. Conclusions

This paper has studied consensus control of the PDE-ODEMASs. First, a consensus controller of the leaderless PDE-ODEMASs was designed. We have shown that the cluster consensus behavior can be reached for the given coupling strengths for the leaderless PDE-ODEMASs. Then, a consensus controller in the leader-following PDE-ODEMASs was designed. Leader-following consensus behavior can be arrived at for the given coupling strengths for the leader-following PDE-ODEMASs. In numerical simulations, it shows the obtained gains according to the proposed methods can ensure consensus of both leaderless and leader-following PDE-ODEMASs. On the contrary, the control with gains a little bit less than those according to the proposed methods cannot achieve consensus. There are often a great number of agents in the real world and, in future, pinning consensus, only controlling a few agents of the PDE-ODEMASs, will be studied, as well as time delays.

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