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Schur Complement-Based Infinity Norm Bounds for the Inverse of GDSDD Matrices

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Abstract: Based on the Schur complement, some upper bounds for the infinity norm of the inverse of generalized doubly strictly diagonally dominant matrices are obtained. In addition, it is shown that the new bound improves the previous bounds. Numerical examples are given to illustrate our results. By using the infinity norm bound, a lower bound for the smallest singular value is given.

Keywords: GDSDD matrices; Schur complements; Infinity norm bounds; singular value

1. Introduction

Throughout the paper, let n be an integer number, $N := \{1, 2, \dots, n\}$ be the set of all indices, $C^{n \times n}$ (resp. $R^{n \times n}$) denote the set of all $n \times n$ complex (resp. real) matrices, $I \in R^{n \times n}$ be an identity matrix, $|A| = [|a_{ij}|] \in R^{n \times n}$, $r_i(A) := \sum_{j=1, j \neq i}^n |a_{ij}|$ denote deleted i th row sum, and $J := \{i \in N : |a_{ii}| > r_i(A)\}$.

Definition 1 ([1]). Let $A = [a_{ij}] \in C^{n \times n}$ ($n \geq 2$), then A is called:

(i) A row diagonally dominant (DD) if for all $i \in N$,

$$|a_{ii}| \geq r_i(A); \quad (1)$$

(ii) A strictly diagonally dominant (SDD) if all the strict inequalities in (1) hold;

(iii) A doubly strictly diagonally dominant (DSDD) matrix, if:

$$|a_{ii}||a_{jj}| > r_i(A)r_j(A), \quad i, j \in N, i \neq j. \quad (2)$$

The class of generalized doubly strictly diagonally dominant (GDSDD) matrices was presented by Gao and Wang in [2].

Definition 2 ([2]). A matrix A is called a GDSDD matrix if $J \neq \emptyset$ and there exist proper subsets N_1, N_2 of N such that $N_1 \cap N_2 = \emptyset$, $N_1 \cup N_2 = N$ and:

$$(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) > \beta_i \alpha_j \quad (3)$$

for any $i \in N_1$ and $j \in N_2$, where with $s = i$ or j ,

$$\alpha_s = \sum_{\substack{t \in N_1 \\ t \neq s}} |a_{st}|, \quad \beta_s = \sum_{\substack{t \in N_2 \\ t \neq s}} |a_{st}|.$$

Here the α_s and β_s may be interpreted as the sums of the absolute values of the nondiagonal elements in row s that fall in the columns N_1 and N_2 , respectively. When $n = 2$, a GDSDD matrix is nothing but a DSDD matrix. A matrix A satisfying (3) may not

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be generalized doubly diagonally dominant for another pair subsets N_1 and N_2 . Assuming that the matrix order is $n \geq 2$, we adopt the notation: $GDSDD_n^{N_1, N_2}$ for generalized doubly strictly diagonally dominant.

Definition 3 ([3]). A matrix A is called an H -matrix if its comparison matrix $\mu(A) = [\mu_{ij}]$ defined by:

$$\mu_{ij} = \begin{cases} |a_{ij}| & \text{if } i = j, \\ -|a_{ij}| & \text{if } i \neq j, \end{cases}$$

is an M -matrix, i.e., $(\mu(A))^{-1} \geq 0$.

It is shown in [3] that if A is an H -matrix, then:

$$(\mu(A))^{-1} \geq |A^{-1}|. \quad (4)$$

In addition, it was shown that SDD matrices, $DSDD$ matrices, and $GDSDD$ matrices are subclass of H -matrices [4].

Upper bounds for the infinity norm of the inverse of nonsingular matrices can be used to the convergence analysis of matrix splitting and matrix multi-splitting iterative methods for solving the large sparse of linear equations. Moreover, the upper bounds can also be applied to the smallest singular value of matrices [5,6].

One traditional method for finding upper bounds for the infinity norm of the inverse of nonsingular matrices is to use the definition and properties of a given matrix class; see [7–10] for details. The first work may due to Varah [10], who in 1975 presented the following upper bound for the infinity norm of the inverse for SDD matrices.

Theorem 1 ([10]). If $A = [a_{ij}] \in C^{n \times n}$ is SDD , then:

$$\|A^{-1}\|_{\infty} \leq \max_{i \in N} \frac{1}{|a_{ii}| - r_i(A)}.$$

However, if the diagonal dominance of A (i.e. $|a_{ii}| - r_i(A)$) is weak, Varah's bound may yield a large value. In 2020, based on the Schur complement, Li [11] obtained two upper bounds for the infinity norm of inverse of SDD matrices.

Using the definition and properties of $DSDD$ matrices, Liu, Zhang and Liu in [9] obtained an upper bound of $\|A^{-1}\|_{\infty}$ for a $DSDD$ matrix A as follows.

Theorem 2 ([7]). If $A = [a_{ij}] \in C^{n \times n}$ is $DSDD$, then:

$$\|A^{-1}\|_{\infty} \leq \max_{\substack{i, j \in N \\ i \neq j}} \frac{|a_{jj}| + r_i(A)}{|a_{ii}||a_{jj}| - r_i(A)r_j(A)}.$$

In 2020, based on the Schur complement, Sang [12] obtained two upper bounds for the infinity norm of $DSDD$ matrices.

And Moraa in [13] give an upper bound for the infinity norm of $GDSDD$ matrices as follows.

Theorem 3. If $A = [a_{ij}] \in C^{n \times n}$ is $GDSDD_n^{N_1, N_2}$, then:

$$\|A^{-1}\|_{\infty} \leq \max \left\{ \max_{i \in N_1, j \in N_2} \frac{|a_{jj}| - \beta_j + \beta_i}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \alpha_j \beta_i}, \max_{i \in N_1, j \in N_2} \frac{|a_{ii}| - \alpha_i + \alpha_j}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \alpha_j \beta_i} \right\}. \quad (5)$$

In this paper, based on the Schur complement, we present some upper bounds for the infinity norm of the inverse of *GDSDD* matrices, and numerical examples are given to show the effectiveness of the obtained results. In addition, applying these new bounds, a lower bound for the smallest singular value of *GDSDD* matrices is obtained.

2. Schur Complement-Based Infinity Bounds for the Inverse of *GDSDD* Matrices

Firstly, we recall the Schur complement of matrices.

Let α be a proper subset of N and denote by $|\alpha|$ the cardinality of α and by $\alpha' = N - \alpha$ the complement to α in N . For nonempty index sets $\alpha, \beta \subseteq N$, we write $A(\alpha, \beta)$ to mean the submatrix of $A \in C^{n \times n}$ lying in the rows indexed by α and the columns indexed by β . $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$. Throughout this paper, supposing $\alpha = \{i_1, i_2, \dots, i_l\} \subset N$, the complement $\alpha' = N - \alpha = \{j_1, j_2, \dots, j_m\}$ and the elements of α and α' are both conventionally arranged in increasing order. Furthermore, if $A(\alpha)$ is nonsingular, we define the Schur complement of A with respect to $A(\alpha)$, which is denoted by A/α or simply A/α , to be:

$$A/\alpha = A(\alpha') - A(\alpha', \alpha)A(\alpha)^{-1}A(\alpha, \alpha').$$

Liu, Huang, and Zhang in [14] proved that the Schur complements of *GDSDD* matrices are *GDSDD* matrices. Many similar or related results had been obtained; see [9,15,16] for details.

For a given nonempty proper subset α , there is always a permutation matrix P such that:

$$P^T A P = \begin{bmatrix} A(\alpha) & A(\alpha, \alpha') \\ A(\alpha', \alpha) & A(\alpha') \end{bmatrix}.$$

It is well known that the inverse of a permutation matrix is also a permutation matrix and the infinity norm for the inverse of a permutation matrix equals to 1 ($\|P\|_\infty = 1$). Hence, if A is nonsingular, then:

$$\|A^{-1}\|_\infty = \|P(P^T A P)^{-1}P^T\|_\infty \leq \|((P^T A P)^{-1})\|_\infty.$$

So we next only consider the upper bound for $\|((P^T A P)^{-1})\|_\infty$. For the sake of simplicity, we consider that:

$$P^T A P = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad (6)$$

where $A_{11} = A(\alpha)$ and $A_{22} = A(\alpha')$.

Lemma 1 ([17]). Let $A \in C^{n \times n}$ with the form (6), and I_{11} (resp. I_{22}) be the identity matrix of order l (resp. m). If A_{11} is nonsingular, then:

$$S(P^T A P)Q = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix},$$

where

$$S = \begin{bmatrix} I_{11} & 0 \\ -A_{21}A_{11}^{-1} & I_{22} \end{bmatrix} \text{ and } Q = \begin{bmatrix} I_{11} & -A_{11}^{-1}A_{12} \\ 0 & I_{22} \end{bmatrix}.$$

From Lemma 1, we obtain:

$$A = (P^T)^{-1}S^{-1}S(P^T A P)Q Q^{-1}P^{-1},$$

and hence:

$$A^{-1} = P Q (S(P^T A P) Q)^{-1} S P^T,$$

which implies that:

$$\|A^{-1}\|_{\infty} = \|PQ(SP^TAPQ)^{-1}SP^T\|_{\infty} \leq \|Q\|_{\infty} \|(SP^TAPQ)^{-1}\|_{\infty} \|S\|_{\infty}. \quad (7)$$

This implies that if upper bounds for $\|S\|_{\infty}$, $\|Q\|_{\infty}$, $\|A_{11}^{-1}\|_{\infty}$, and $\|(A/\alpha)^{-1}\|_{\infty}$ are given, then the product of these bounds could be regarded as an upper bound for $\|A^{-1}\|_{\infty}$ by (7). It is not difficult to compute that:

$$\begin{aligned} \|S\|_{\infty} &= 1 + \|A_{21}A_{11}^{-1}\|_{\infty}, \\ \|Q\|_{\infty} &= 1 + \|A_{11}^{-1}A_{12}\|_{\infty}, \end{aligned}$$

and

$$\|(SP^TAPQ)^{-1}\|_{\infty} = \max\left\{\|A_{11}^{-1}\|_{\infty}, \|(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}\|_{\infty}\right\}.$$

In 2020, Li [11] gave an upper bound for $\|S\|_{\infty}$ as follows:

Lemma 2 ([11]). Let $A = [a_{ij}] \in C^{n \times n}$ be nonsingular with $a_{ii} \neq 0$ for $i \in N$, and $\emptyset \neq \alpha \subset N$. If A_{11} is nonsingular, and:

$$1 > \max_{i \in \alpha} \frac{\max_{j \in \alpha, j \neq i} |a_{ji}|}{|a_{ii}|} \cdot (l - 1), \quad (8)$$

then

$$\|S\|_{\infty} \leq \gamma(\alpha) = 1 + l \cdot \max_{i \in \alpha} \frac{\max_{j \in \alpha'} |a_{ji}|}{|a_{ii}|} \cdot \left(1 - \max_{i \in \alpha} \frac{\max_{j \in \alpha, j \neq i} |a_{ji}|}{|a_{ii}|} \cdot (l - 1)\right)^{-1}. \quad (9)$$

Next, we only consider upper bounds for $\|Q\|_{\infty}$, $\|A_{11}^{-1}\|_{\infty}$ and $\|(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}\|_{\infty}$ when A is $GDSDD_n^{N_1, N_2}$.

Lemma 3 ([14]). Let A be $GDSDD_n^{N_1, N_2}$. Then $A(N_1)$ and $A(N_2)$ are SDD.

Lemma 4 ([14]). Let A be $GDSDD_n^{N_1, N_2}$. Then $N_1 \subseteq J$ or $N_2 \subseteq J$.

In this paper, we assume $N_1 \subseteq J$.

Lemma 5 ([14]). Let A be $GDSDD_n^{N_1, N_2}$ and $\alpha \subset N$. Then $A(\alpha)$ is $GDSDD_{|\alpha|}^{N_1 \cap \alpha, N_2 \cap \alpha}$.

Lemma 6 ([14]). Let A be $GDSDD_n^{N_1, N_2}$. Then for any proper subset α of N , A/α is $GDSDD_{|N-\alpha|}^{N_1-\alpha, N_2-\alpha}$.

Lemma 7 ([18]). Let $A = [a_{ij}] \in C^{n \times n}$ be SDD, and $B = [b_{ij}] \in C^{n \times m}$. Then:

$$\|A^{-1}B\|_{\infty} \leq \max_{i \in N} \frac{R_i(B)}{|a_{ii}| - r_i(A)}, \quad (10)$$

where $R_i(B) = \sum_{k=1}^m |b_{ik}|$.

Lemma 8. Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ be $\text{GDSDD}_n^{N_1, N_2}$, and $B = [b_{ij}] \in \mathbb{C}^{n \times m}$. Then:

$$\|A^{-1}B\|_{\infty} \leq \max \left\{ \max_{\substack{i \in N_1 \\ j \in N_2}} \frac{(|a_{jj}| - \beta_j)R_i(B) + \beta_i R_j(B)}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \beta_i \alpha_j}, \max_{\substack{i \in N_1 \\ j \in N_2}} \frac{(|a_{ii}| - \alpha_i)R_j(B) + \alpha_j R_i(B)}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \beta_i \alpha_j} \right\},$$

$$\text{where } R_i(B) = \sum_{k=1}^m |b_{ik}|.$$

Proof. Since there exists an m -dimensional vector $x = (x_1, x_2, \dots, x_m)^T$ such that $\|x\|_{\infty} = 1$ and:

$$\|A^{-1}B\|_{\infty} = \|A^{-1}Bx\|_{\infty} = \|y\|_{\infty} = \max_{i \in N} |y_i|, \quad (11)$$

where $y = A^{-1}Bx = (y_1, y_2, \dots, y_n)^T$, we have $Ay = Bx$, and let $|y_p| = \max_{i \in N_1} |y_i|$, $|y_q| = \max_{j \in N_2} |y_j|$, which implies that:

$$a_{ii}y_i + \sum_{\substack{j \in N \\ j \neq i}} a_{ij}y_j = \sum_{k=1}^m b_{ik}x_k, \quad i \in N.$$

Then,

$$\begin{aligned} \sum_{k=1}^m b_{pk}x_k &= a_{pp}y_p + \sum_{\substack{j \in N \\ j \neq p}} a_{pj}y_j = a_{pp}y_p + \sum_{\substack{j \in N_1 \\ j \neq p}} a_{pj}y_j + \sum_{j \in N_2} a_{pj}y_j, \\ \sum_{k=1}^m b_{qk}x_k &= a_{qq}y_q + \sum_{\substack{j \in N \\ j \neq q}} a_{qj}y_j = a_{qq}y_q + \sum_{j \in N_1} a_{qj}y_j + \sum_{j \in N_2, j \neq q} a_{qj}y_j. \end{aligned}$$

We can obtain:

$$\begin{aligned} \sum_{k=1}^m |b_{pk}| \max_{1 \leq k \leq m} |x_k| &\geq \sum_{k=1}^m |b_{pk}| |x_k| \geq \left| \sum_{k=1}^m b_{pk}x_k \right| \\ &= |a_{pp}y_p + \sum_{\substack{j \in N \\ j \neq p}} a_{pj}y_j| \\ &= |a_{pp}y_p + \sum_{\substack{j \in N_1 \\ j \neq p}} a_{pj}y_j + \sum_{j \in N_2} a_{pj}y_j| \\ &\geq |a_{pp}y_p| - \left| \sum_{\substack{j \in N_1 \\ j \neq p}} a_{pj}y_j \right| - \left| \sum_{j \in N_2} a_{pj}y_j \right| \\ &\geq |a_{pp}| |y_p| - \sum_{\substack{j \in N_1 \\ j \neq p}} |a_{pj}| |y_j| - \sum_{j \in N_2} |a_{pj}| |y_j| \\ &\geq |a_{pp}| |y_p| - \sum_{\substack{j \in N_1 \\ j \neq p}} |a_{pj}| |y_p| - \sum_{j \in N_2} |a_{pj}| |y_q| \\ &= |a_{pp}| |y_p| - \alpha_p |y_p| - \beta_p |y_q| \end{aligned}$$

and

$$\begin{aligned} \sum_{k=1}^m |b_{qk}| \max_{1 \leq k \leq m} |x_k| &\geq \sum_{k=1}^m |b_{qk}| |x_k| \geq \left| \sum_{k=1}^m b_{qk}x_k \right| \\ &= |a_{qq}y_q + \sum_{\substack{j \in N \\ j \neq q}} a_{qj}y_j| \end{aligned}$$

$$\begin{aligned}
&= |a_{qq}y_q + \sum_{j \in N_1} a_{qj}y_j + \sum_{\substack{j \in N_2 \\ j \neq q}} a_{qj}y_j| \\
&\geq |a_{qq}y_q| - |\sum_{j \in N_1} a_{qj}y_j| - |\sum_{\substack{j \in N_2 \\ j \neq q}} a_{qj}y_j| \\
&\geq |a_{qq}||y_q| - \sum_{j \in N_1} |a_{qj}||y_j| - \sum_{\substack{j \in N_2 \\ j \neq q}} |a_{qj}||y_j| \\
&\geq |a_{qq}||y_q| - \sum_{j \in N_1} |a_{qj}||y_p| - \sum_{\substack{j \in N_2 \\ j \neq q}} |a_{qj}||y_q| \\
&= |a_{qq}||y_p| - \alpha_q|y_p| - \beta_q|y_q|.
\end{aligned}$$

That is to say,

$$\sum_{k=1}^m |b_{pk}| \max_{1 \leq k \leq m} |x_k| \geq |a_{pp}||y_p| - \alpha_p|y_p| - \beta_p|y_q| \quad (12)$$

and

$$\sum_{k=1}^m |b_{qk}| \max_{1 \leq k \leq m} |x_k| \geq |a_{qq}||y_p| - \alpha_q|y_p| - \beta_q|y_q|. \quad (13)$$

The proof is divided into two cases.

Case 1: $|y_p| \geq |y_q|$. Then, $(12) \times (|a_{qq}| - \beta_q) + (13) \times \beta_p$ yields

$$(|a_{qq}| - \beta_q) \sum_{k=1}^m |b_{pk}| \max_{1 \leq k \leq m} |x_k| + \beta_p \sum_{k=1}^m |b_{qk}| \max_{1 \leq k \leq m} |x_k| \geq |y_p|[(|a_{pp}| - \alpha_p)(|a_{qq}| - \beta_q) - \beta_p\alpha_q],$$

which leads to:

$$\begin{aligned}
\|A^{-1}B\|_{\infty} &= \max_{i \in N} |y_i| = |y_p| \\
&\leq \frac{(|a_{qq}| - \beta_q) \sum_{k=1}^m |b_{pk}| \max_{1 \leq k \leq m} |x_k| + \beta_p \sum_{k=1}^m |b_{qk}| \max_{1 \leq k \leq m} |x_k|}{(|a_{pp}| - \alpha_p)(|a_{qq}| - \beta_q) - \alpha_q\beta_p} \\
&= \frac{(|a_{qq}| - \beta_q) \sum_{k=1}^m |b_{pk}| + \beta_p \sum_{k=1}^m |b_{qk}|}{(|a_{pp}| - \alpha_p)(|a_{qq}| - \beta_q) - \alpha_q\beta_p} \\
&\leq \max_{i \in N_1, j \in N_2} \frac{(|a_{jj}| - \beta_j) \sum_{k=1}^m |b_{ik}| + \beta_i \sum_{k=1}^m |b_{jk}|}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \alpha_j\beta_i} \\
&= \max_{i \in N_1, j \in N_2} \frac{(|a_{jj}| - \beta_j)R_i(B) + \beta_i R_j(B)}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \alpha_j\beta_i}.
\end{aligned} \quad (14)$$

Case 2: $|y_p| < |y_q|$. Then, $(12) \times \alpha_q + (13) \times (|a_{pp}| - \alpha_p)$ yields:

$$\alpha_q \sum_{k=1}^m |b_{pk}| \max_{1 \leq k \leq m} |x_k| + (|a_{pp}| - \alpha_p) \sum_{k=1}^m |b_{qk}| \max_{1 \leq k \leq m} |x_k| \geq |y_q|[(|a_{pp}| - \alpha_p)(|a_{qq}| - \beta_q) - \beta_p\alpha_q],$$

which leads to:

$$\begin{aligned}
\|A^{-1}B\|_{\infty} &= \max_{i \in N} |y_i| = |y_q| \\
&\leq \frac{\alpha_q \sum_{k=1}^m |b_{pk}| \max_{1 \leq k \leq m} |x_k| + (|a_{pp}| - \alpha_p) \sum_{k=1}^m |b_{qk}| \max_{1 \leq k \leq m} |x_k|}{(|a_{pp}| - \alpha_p)(|a_{qq}| - \beta_q) - \alpha_q \beta_p} \\
&= \frac{\alpha_q \sum_{k=1}^m |b_{pk}| + (|a_{pp}| - \alpha_p) \sum_{k=1}^m |b_{qk}|}{(|a_{pp}| - \alpha_p)(|a_{qq}| - \beta_q) - \alpha_q \beta_p} \\
&\leq \max_{i \in N_1, j \in N_2} \frac{\alpha_j \sum_{k=1}^m |b_{ik}| + (|a_{ii}| - \alpha_i) \sum_{k=1}^m |b_{jk}|}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \alpha_j \beta_i} \\
&= \max_{i \in N_1, j \in N_2} \frac{(|a_{ii}| - \alpha_i)R_j(B) + \alpha_j R_i(B)}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \alpha_j \beta_i}.
\end{aligned} \tag{15}$$

The conclusion follows from equality (11) and inequalities (14) and (15). $\square$

Corollary 1. Let $A = [a_{ij}] \in C^{n \times n}$ be $GDSDD_n^{N_1, N_2}$, and $\emptyset \neq \alpha \subset N$. Then:

$$\|Q\|_{\infty} = 1 + \|A_{11}^{-1}A_{12}\|_{\infty} \leq 1 + \varphi(\alpha), \tag{16}$$

where

$$\varphi(\alpha) = \max \left\{ \max_{\substack{i \in N_1 \cap \alpha \\ j \in N_2 \cap \alpha}} \frac{(|a_{jj}| - \beta_j)R_i(A_{12}) + \beta_i R_j(A_{12})}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \beta_i \alpha_j}, \max_{\substack{i \in N_1 \cap \alpha \\ j \in N_2 \cap \alpha}} \frac{(|a_{ii}| - \alpha_i)R_j(A_{12}) + \alpha_j R_i(A_{12})}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \beta_i \alpha_j} \right\}.$$

Specially, if $\alpha \subset N_1$ or $\alpha \subset N_2$, then A_{11} is SDD, and

$$\|Q\|_{\infty} = 1 + \|A_{11}^{-1}A_{12}\|_{\infty} \leq 1 + \max_{i \in \alpha} \frac{R_i(A_{12})}{|a_{ii}| - r_i(A_{11})}. \tag{17}$$

For the sake of convenience, denote:

$$[l] = \{1, 2, \dots, l\},$$

$$[m] = \{1, 2, \dots, m\},$$

$$v_{jk} := (a_{j_k i_1}, \dots, a_{j_k i_l}),$$

$$w_{jk} := (a_{i_1 j_k}, \dots, a_{i_l j_k})^T,$$

$$|v_{jk}| := (|a_{j_k i_1}|, \dots, |a_{j_k i_l}|),$$

$$|w_{jk}| := (|a_{i_1 j_k}|, \dots, |a_{i_l j_k}|)^T,$$

for any $k = 1, 2, \dots, m$. And thus:

$$\sum_{k \in [m]} |w_{jk}| = \left(\sum_{k \in [m]} |a_{i_1 j_k}|, \dots, \sum_{k \in [m]} |a_{i_l j_k}| \right)^T = \sum_{j_k \in \alpha'} |w_{j_k}|. \tag{18}$$

Lemma 9. Let $x = \{\mu[A(\alpha)]\}^{-1} \sum_{k \in [m]} |w_{j_k}| = (x_1, \dots, x_l)^T$, $x_g = \max_{k \in [l]} x_k$. If $i_g \in \alpha$ is a strictly diagonally dominant row, then:

$$0 \leq x_k < 1, k \in [l]. \tag{19}$$

Proof.

$$x = \{\mu[A(\alpha)]\}^{-1} \sum_{k \in [m]} |w_{jk}| = (x_1, \dots, x_l)^T, \text{ i.e., } \mu[A(\alpha)]x = \sum_{k \in [m]} |w_{jk}|. \quad (20)$$

For any $\alpha \subset N$, A_{11} is GDSDD, so is $\mu(A(\alpha))$, and consequently, $\mu(A(\alpha))$ is an H -matrix, and $\{\mu[A(\alpha)]\}^{-1} \geq 0$ by (4). Hence, x is nonnegative. Let $x_g = \max_{k \in [l]} x_k$. By i_g is a strictly diagonally dominant row, then:

$$|a_{i_g i_g}| > \sum_{v \in [l], v \neq g} |a_{i_g i_v}| + \sum_{k \in [m]} |a_{i_g j_k}|. \quad (21)$$

By the g th equality of (20) and inequality (21), we have:

$$\sum_{k \in [m]} |a_{i_g j_k}| = |a_{i_g i_g}|x_g - \sum_{v \in [l], v \neq g} |a_{i_g i_v}|x_v \geq x_g \left(|a_{i_g i_g}| - \sum_{v \in [l], v \neq g} |a_{i_g i_v}| \right),$$

and hence:

$$x_g \leq \frac{\sum_{k \in [m]} |a_{i_g j_k}|}{|a_{i_g i_g}| - \sum_{v \in [l], v \neq g} |a_{i_g i_v}|} < 1,$$

which implies that:

$$0 \leq x_k < 1, k \in [l].$$

The proof is completed. $\square$

Theorem 4. Let $A = [a_{ij}]$ be $\text{GDSDD}_n^{N_1, N_2}$ with $a_{ii} \neq 0, i \in N$, and $\emptyset \neq \alpha \subset N$. Assume that (8) holds, if $\alpha \subset N_1$, then:

$$\|A^{-1}\|_{\infty} \leq \gamma(\alpha) \cdot \left(1 + \max_{i \in \alpha} \frac{R_i(A_{12})}{|a_{ii}| - r_i(A_{11})}\right) \cdot \theta(\alpha),$$

where

$$\theta(\alpha) = \max \left\{ \max_{i \in \alpha} \frac{1}{|a_{ii}| - r_i(A_{11})}, \eta(\alpha) \right\},$$

$$\eta(\alpha) = \max \left\{ \max_{\substack{i, j \in \alpha' \\ i \in N_1 - \alpha, j \in N_2}} \frac{|a_{jj}| - \beta_j(A) + \beta_i(A) + \sum_{k \in \alpha} |a_{jk}| + \sum_{k \in \alpha} |a_{ik}|}{y_{i,j} - z_{i,j}}, \right. \\ \left. \max_{\substack{i, j \in \alpha' \\ i \in N_1 - \alpha, j \in N_2}} \frac{|a_{ii}| + \alpha_j(A) - \sum_{k \in N_1 - \alpha} |a_{ik}| + \sum_{k \in \alpha} |a_{ik}|}{y_{i,j} - z_{i,j}} \right\},$$

$$y_{i,j} = (|a_{ii}| - \sum_{k \in N_1 - \alpha} |a_{ik}| - |v_i| \{\mu[A(\alpha)]\}^{-1} \sum_{k \in N_1 - \alpha} |w_k|)(|a_{jj}| - \beta_j(A) - |v_j| \{\mu[A(\alpha)]\}^{-1} \sum_{k \in N_2} |w_k|),$$

$$z_{i,j} = (\sum_{k \in N_1 - \alpha} |a_{jk}| + |v_j| \{\mu[A(\alpha)]\}^{-1} \sum_{k \in N_1 - \alpha} |w_k|)(\beta_i(A) + |v_i| \{\mu[A(\alpha)]\}^{-1} \sum_{k \in N_2} |w_k|).$$

Proof. Denote the (s, t) -entry of A/α by (a'_{st}) . Then for any $s, t \in [m]$,

$$a'_{st} = a_{jsjt} - v_{js} [A(\alpha)]^{-1} w_{jt}, \quad (22)$$

and by inequality (4), we have:

$$\begin{aligned} |a'_{st}| &= |a_{jsjt} - v_{js} [A(\alpha)]^{-1} w_{jt}| \\ &\leq |a_{jsjt}| + |v_{js}| |[A(\alpha)]|^{-1} |w_{jt}| \end{aligned} \quad (23)$$

$$\begin{aligned} |a'_{st}| &\leq |a_{jsjt}| + |v_{js}| \{\mu[A(\alpha)]\}^{-1} |w_{jt}|, \\ |a'_{st}| &= |a_{jsjt} - v_{js} [A(\alpha)]^{-1} w_{jt}| \\ &\geq |a_{jsjt}| - |v_{js}| |[A(\alpha)]|^{-1} |w_{jt}| \\ &\geq |a_{jsjt}| - |v_{js}| \{\mu[A(\alpha)]\}^{-1} |w_{jt}|. \end{aligned} \quad (24)$$

Since A is $GDSDD_n^{N_1, N_2}$, $\alpha \subset N_1$, by Lemmas 5 and 6, we have A_{11} is SDD , and A/α is $GDSDD_{|N-\alpha|}^{N_1-\alpha, N_2-\alpha}$. Applying the Varah's bound to A_{11} , we get:

$$\|A_{11}^{-1}\|_{\infty} \leq \max_{i \in \alpha} \frac{1}{|a_{ii}| - r_i(A_{11})}. \quad (25)$$

Applying Theorem 3 to A/α yields:

$$\|(A/\alpha)^{-1}\|_{\infty} \leq \max \left\{ \begin{aligned} &\max_{\substack{s, t \in [m] \\ j_s \in N_1 - \alpha, j_t \in N_2}} \frac{|a'_{tt}| - \beta_t(A/\alpha) + \beta_s(A/\alpha)}{(|a'_{ss}| - \alpha_s(A/\alpha))(|a'_{tt}| - \beta_t(A/\alpha)) - \alpha_t(A/\alpha)\beta_s(A/\alpha)}, \\ &\max_{\substack{s, t \in [m] \\ j_s \in N_1 - \alpha, j_t \in N_2}} \frac{|a'_{ss}| - \alpha_s(A/\alpha) + \alpha_t(A/\alpha)}{(|a'_{ss}| - \alpha_s(A/\alpha))(|a'_{tt}| - \beta_t(A/\alpha)) - \alpha_t(A/\alpha)\beta_s(A/\alpha)} \end{aligned} \right\}. \quad (26)$$

By Theorem 2 in [14], we have:

$$\begin{aligned} &(|a'_{ss}| - \alpha_s(A/\alpha))(|a'_{tt}| - \beta_t(A/\alpha)) - \alpha_t(A/\alpha)\beta_s(A/\alpha) \\ &\geq (|a_{jsjs}| - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |a_{jsjk}| - |v_{js}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1 - \alpha} |w_{jk}|)(|a_{jtjt}| - \sum_{\substack{j_k \in N_2 \\ k \neq t}} |a_{jtjk}| \\ &- |v_{jt}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2} |w_{jk}|) - (\sum_{j_k \in N_1 - \alpha} |a_{jtjk}| + |v_{jt}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1 - \alpha} |w_{jk}|)(\sum_{j_k \in N_2} |a_{jsjk}| \\ &+ |v_{js}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2} |w_{jk}|) \\ &> 0. \end{aligned} \quad (27)$$

Now, the upper bounds of $|a'_{tt}| - \beta_t(A/\alpha) + \beta_s(A/\alpha)$ and $|a'_{ss}| - \alpha_s(A/\alpha) + \alpha_t(A/\alpha)$ are considered. Since $\alpha \subset N_1 \subseteq J$, it satisfies Lemma 9, which implies that (19) holds.

By (19), (20), (23) and (24), we have:

$$\begin{aligned} &|a'_{tt}| - \beta_t(A/\alpha) + \beta_s(A/\alpha) \\ &= |a'_{tt}| - \sum_{\substack{j_k \in N_2 \\ k \neq t}} |a'_{tk}| + \sum_{j_k \in N_2} |a'_{sk}| \\ &\leq |a_{jtjt}| + |v_{jt}| \{\mu[A(\alpha)]\}^{-1} |w_{jt}| - \sum_{\substack{j_k \in N_2 \\ k \neq t}} (|a_{jtjk}| - |v_{jt}| \{\mu[A(\alpha)]\}^{-1} |w_{jk}|) \\ &+ \sum_{j_k \in N_2} (|a_{jsjk}| + |v_{js}| \{\mu[A(\alpha)]\}^{-1} |w_{jk}|) \quad (\text{by inequalities (23) and (24)}) \\ &= |a_{jtjt}| + |v_{jt}| \{\mu[A(\alpha)]\}^{-1} |w_{jt}| - \sum_{\substack{j_k \in N_2 \\ k \neq t}} |a_{jtjk}| + |v_{jt}| \{\mu[A(\alpha)]\}^{-1} \sum_{\substack{j_k \in N_2 \\ k \neq t}} |w_{jk}| \end{aligned}$$

$$\begin{aligned}
& + \sum_{j_k \in N_2} |a_{jsj_k}| + |v_{j_s}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2} |w_{j_k}| \\
& = |a_{j_tj_t}| - \sum_{\substack{j_k \in N_2 \\ k \neq t}} |a_{j_tj_k}| + \sum_{j_k \in N_2} |a_{jsj_k}| + |v_{j_t}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2} |w_{j_k}| \\
& + |v_{j_s}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2} |w_{j_k}| \\
& = |a_{j_tj_t}| - \sum_{\substack{j_k \in N_2 \\ k \neq t}} |a_{j_tj_k}| + \sum_{j_k \in N_2} |a_{jsj_k}| + (|v_{j_t}| + |v_{j_s}|) \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2} |w_{j_k}| \\
& \leq |a_{j_tj_t}| - \sum_{\substack{j_k \in N_2 \\ k \neq t}} |a_{j_tj_k}| + \sum_{j_k \in N_2} |a_{jsj_k}| + (|v_{j_t}| + |v_{j_s}|) \{\mu[A(\alpha)]\}^{-1} \sum_{k \in [m]} |w_{j_k}| \text{ (by (18) and } N_2 \subset \alpha') \\
& = |a_{j_tj_t}| - \sum_{\substack{j_k \in N_2 \\ k \neq t}} |a_{j_tj_k}| + \sum_{j_k \in N_2} |a_{jsj_k}| + \sum_{k \in [l]} (|a_{j_ti_k}| + |a_{jsi_k}|) x_k \\
& \leq |a_{j_tj_t}| - \sum_{\substack{j_k \in N_2 \\ k \neq t}} |a_{j_tj_k}| + \sum_{j_k \in N_2} |a_{jsj_k}| + \sum_{k \in [l]} (|a_{j_ti_k}| + |a_{jsi_k}|) \text{ (by inequality (19))} \\
& = |a_{j_tj_t}| - \beta_{j_t}(A) + \beta_{j_s}(A) + \sum_{k \in [l]} |a_{j_ti_k}| + \sum_{k \in [l]} |a_{jsi_k}|.
\end{aligned} \tag{28}$$

Similarly,

$$\begin{aligned}
& |a'_{ss}| - \alpha_s(A/\alpha) + \alpha_t(A/\alpha) \\
& = |a'_{ss}| - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |a'_{sk}| + \sum_{j_k \in N_1 - \alpha} |a'_{tk}| \\
& \leq |a_{jsj_s}| + |v_{j_s}| \{\mu[A(\alpha)]\}^{-1} |w_{j_s}| - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} (|a_{jsj_k}| + |v_{j_s}| \{\mu[A(\alpha)]\}^{-1} |w_{j_k}|) \\
& + \sum_{j_k \in N_1 - \alpha} (|a_{j_tj_k}| + |v_{j_t}| \{\mu[A(\alpha)]\}^{-1} |w_{j_k}|) \text{ (by inequalities (23) and (24))} \\
& = |a_{jsj_s}| + |v_{j_s}| \{\mu[A(\alpha)]\}^{-1} |w_{j_s}| - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |a_{jsj_k}| + |v_{j_s}| \{\mu[A(\alpha)]\}^{-1} \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |w_{j_k}| \\
& + \sum_{j_k \in N_1 - \alpha} |a_{j_tj_k}| + |v_{j_t}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1 - \alpha} |w_{j_k}| \\
& = |a_{jsj_s}| - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |a_{jsj_k}| + \sum_{j_k \in N_1 - \alpha} |a_{j_tj_k}| + |v_{j_s}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1 - \alpha} |w_{j_k}| \\
& + |v_{j_t}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1 - \alpha} |w_{j_k}| \\
& = |a_{jsj_s}| - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |a_{jsj_k}| + \sum_{j_k \in N_1 - \alpha} |a_{j_tj_k}| + (|v_{j_s}| + |v_{j_t}|) \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1 - \alpha} |w_{j_k}| \\
& \leq |a_{jsj_s}| - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |a_{jsj_k}| + \sum_{j_k \in N_1 - \alpha} |a_{j_tj_k}| + (|v_{j_s}| + |v_{j_t}|) \{\mu[A(\alpha)]\}^{-1} \sum_{k \in [m]} |w_{j_k}| \text{ (by (18) and } N_1 - \alpha \subset \alpha') \\
& = |a_{jsj_s}| - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |a_{jsj_k}| + \sum_{j_k \in N_1 - \alpha} |a_{j_tj_k}| + \sum_{k \in [l]} (|a_{jsi_k}| + |a_{j_ti_k}|) x_k \\
& \leq |a_{jsj_s}| - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |a_{jsj_k}| + \sum_{j_k \in N_1 - \alpha} |a_{j_tj_k}| + \sum_{k \in [l]} (|a_{jsi_k}| + |a_{j_ti_k}|) \text{ (by inequality (19))} \\
& = |a_{jsj_s}| + \alpha_{j_t}(A) - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |a_{jsj_k}| + \sum_{k \in [l]} |a_{jsi_k}|.
\end{aligned} \tag{29}$$

Furthermore, by (26)–(29), we have:

$$\begin{aligned} \|(A/\alpha)^{-1}\|_{\infty} &\leq \max \left\{ \max_{\substack{s,t \in [m] \\ j_s \in N_1 - \alpha \\ j_t \in N_2 - \alpha}} \frac{|a_{j_t j_t}| - \beta_{j_t}(A) + \beta_{j_s}(A) + \sum_{k \in [l]} |a_{j_t i_k}| + \sum_{k \in [l]} |a_{j_s i_k}|}{y_{j_s, j_t} - z_{j_s, j_t}}, \right. \\ &\quad \left. \max_{\substack{s,t \in [m] \\ j_s \in N_1 - \alpha \\ j_t \in N_2 - \alpha}} \frac{|a_{j_s j_s}| + \alpha_{j_t}(A) - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |a_{j_s j_k}| + \sum_{k \in [l]} |a_{j_s i_k}|}{y_{j_s, j_t} - z_{j_s, j_t}} \right\} \\ &= \max \left\{ \max_{\substack{i,j \in \alpha' \\ i \in N_1 - \alpha, j \in N_2}} \frac{|a_{jj}| - \beta_j(A) + \beta_i(A) + \sum_{k \in \alpha} |a_{jk}| + \sum_{k \in \alpha} |a_{ik}|}{y_{i,j} - z_{i,j}}, \right. \\ &\quad \left. \max_{\substack{i,j \in \alpha' \\ i \in N_1 - \alpha, j \in N_2}} \frac{|a_{ii}| + \alpha_j(A) - \sum_{\substack{k \in N_1 - \alpha \\ k \neq i}} |a_{ik}| + \sum_{k \in \alpha} |a_{ik}|}{y_{i,j} - z_{i,j}} \right\}, \end{aligned} \quad (30)$$

where

$$\begin{aligned} y_{j_s, j_t} &= (|a_{j_s j_s}| - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |a_{j_s j_k}| - |v_{j_s}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1 - \alpha} |w_{j_k}|)(|a_{j_t j_t}| \\ &\quad - \sum_{\substack{j_k \in N_2 \\ k \neq t}} |a_{j_t j_k}| - |v_{j_t}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2} |w_{j_k}|), \\ z_{j_s, j_t} &= (\sum_{j_k \in N_1 - \alpha} |a_{j_t j_k}| + |v_{j_t}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1 - \alpha} |w_{j_k}|)(\sum_{j_k \in N_2} |a_{j_s j_k}| \\ &\quad + |v_{j_s}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2} |w_{j_k}|). \end{aligned}$$

By Corollary 1, we have:

$$\|Q\|_{\infty} \leq 1 + \max_{i \in \alpha} \frac{R_i(A_{12})}{|a_{ii}| - r_i(A_{11})}. \quad (31)$$

Furthermore, by (7), (9), (25), (30), and (31), the conclusion follows. $\square$

Theorem 5. Let $A = [a_{ij}]$ be $GDSDD_n^{N_1, N_2}$ with $a_{ii} \neq 0$, $i \in N$, and $\emptyset \neq \alpha \subset N$. Assume that (8) and (19) hold, if $\alpha \subset N_2$, then:

$$\|A^{-1}\|_{\infty} \leq \gamma(\alpha) \cdot (1 + \max_{i \in \alpha} \frac{R_i(A_{12})}{|a_{ii}| - r_i(A_{11})}) \cdot \delta(\alpha),$$

where

$$\delta(\alpha) = \max \left\{ \max_{i \in \alpha} \frac{1}{|a_{ii}| - r_i(A_{11})}, \xi(\alpha) \right\},$$

$$\zeta(\alpha) = \max \left\{ \max_{\substack{i,j \in \alpha' \\ i \in N_1, j \in N_2 - \alpha}} \frac{|a_{jj}| + \beta_i(A) - \sum_{k \in N_2 - \alpha, k \neq j} |a_{jk}| + \sum_{k \in \alpha} |a_{jk}|}{f_{i,j} - g_{i,j}}, \right. \\ \left. \max_{\substack{i,j \in \alpha' \\ i \in N_1, j \in N_2 - \alpha}} \frac{|a_{ii}| - \alpha_i(A) + \alpha_j(A) + \sum_{k \in \alpha} |a_{ik}| + \sum_{k \in \alpha} |a_{jk}|}{f_{i,j} - g_{i,j}} \right\},$$

$$f_{i,j} = (|a_{ii}| - \alpha_i(A) - |v_i| \{\mu[A(\alpha)]\}^{-1} \sum_{k \in N_1} |w_k|)(|a_{jj}| - \sum_{\substack{k \in N_2 - \alpha \\ k \neq j}} |a_{jk}| - |v_j| \{\mu[A(\alpha)]\}^{-1} \sum_{k \in N_2 - \alpha} |w_k|),$$

$$g_{i,j} = (\beta_j(A) + |v_j| \{\mu[A(\alpha)]\}^{-1} \sum_{k \in N_1} |w_k|)(\sum_{k \in N_2 - \alpha} |a_{ik}| + |v_i| \{\mu[A(\alpha)]\}^{-1} \sum_{k \in N_2 - \alpha} |w_k|).$$

Proof. Since A is $GDSDD_n^{N_1, N_2}$, $\alpha \subset N_2$, by Lemmas 5 and 6, we have A_{11} is SDD , and A/α is $GDSDD_{|N-\alpha|}^{N_1, N_2-\alpha}$. Applying the Varah's bound to A_{11} , we get:

$$\|A_{11}^{-1}\|_{\infty} \leq \max_{i \in \alpha} \frac{1}{|a_{ii}| - r_i(A_{11})}. \quad (32)$$

Applying Theorem 3 to A/α yields:

$$\|(A/\alpha)^{-1}\|_{\infty} \leq \max \left\{ \max_{\substack{s,t \in [m], \\ j_s \in N_1 - \alpha, j_t \in N_2}} \frac{|a'_{tt}| - \beta_t(A/\alpha) + \beta_s(A/\alpha)}{(|a'_{ss}| - \alpha_s(A/\alpha))(|a'_{tt}| - \beta_t(A/\alpha)) - \alpha_t(A/\alpha)\beta_s(A/\alpha)}, \right. \\ \left. \max_{\substack{s,t \in [m], \\ j_s \in N_1 - \alpha, j_t \in N_2}} \frac{|a'_{ss}| - \alpha_s(A/\alpha) + \alpha_t(A/\alpha)}{(|a'_{ss}| - \alpha_s(A/\alpha))(|a'_{tt}| - \beta_t(A/\alpha)) - \alpha_t(A/\alpha)\beta_s(A/\alpha)} \right\}. \quad (33)$$

By Theorem 2 in [14], we have:

$$\begin{aligned} & (|a'_{ss}| - \alpha_s(A/\alpha))(|a'_{tt}| - \beta_t(A/\alpha)) - \alpha_t(A/\alpha)\beta_s(A/\alpha) \\ & \geq (|a'_{jsjs}| - \sum_{\substack{j_k \in N_1 \\ k \neq s}} |a_{jsjk}| - |v_{js}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1} |w_{jk}|)(|a_{j_tj_t}| - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a_{j_tj_k}| \\ & - |v_{j_t}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2 - \alpha} |w_{jk}|) - (\sum_{j_k \in N_1} |a_{j_tj_k}| + |v_{j_t}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1} |w_{jk}|)(\sum_{j_k \in N_2 - \alpha} |a_{jsj_k}| \\ & + |v_{js}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2 - \alpha} |w_{jk}|) \\ & > 0. \end{aligned} \quad (34)$$

Now, the upper bounds of $|a'_{tt}| - \beta_t(A/\alpha) + \beta_s(A/\alpha)$ and $|a'_{ss}| - \alpha_s(A/\alpha) + \alpha_t(A/\alpha)$ are considered.

By (19), (20), (23), and (24), we have:

$$\begin{aligned} & |a'_{tt}| - \beta_t(A/\alpha) + \beta_s(A/\alpha) \\ & = |a'_{tt}| - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a'_{tk}| + \sum_{j_k \in N_2 - \alpha} |a'_{sk}| \\ & \leq |a_{j_tj_t}| + |v_{j_t}| \{\mu[A(\alpha)]\}^{-1} |w_{j_t}| - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} (|a_{j_tj_k}| - |v_{j_t}| \{\mu[A(\alpha)]\}^{-1} |w_{j_k}|) \end{aligned}$$

$$\begin{aligned}
& + \sum_{j_k \in N_2 - \alpha} (|a_{jsj_k}| + |v_{js}| \{\mu[A(\alpha)]\}^{-1} |w_{j_k}|) \text{ (by inequalities (23) and (24))} \\
& = |a_{jtj_t}| + |v_{j_t}| \{\mu[A(\alpha)]\}^{-1} |w_{j_t}| - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a_{j_tj_k}| + |v_{j_t}| \{\mu[A(\alpha)]\}^{-1} \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |w_{j_k}| \\
& + \sum_{j_k \in N_2 - \alpha} |a_{jsj_k}| + |v_{js}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2 - \alpha} |w_{j_k}| \\
& = |a_{jtj_t}| - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a_{j_tj_k}| + \sum_{j_k \in N_2 - \alpha} |a_{jsj_k}| + |v_{j_t}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2 - \alpha} |w_{j_k}| \\
& + |v_{js}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2 - \alpha} |w_{j_k}| \\
& = |a_{jtj_t}| - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a_{j_tj_k}| + \sum_{j_k \in N_2 - \alpha} |a_{jsj_k}| + (|v_{j_t}| + |v_{js}|) \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2 - \alpha} |w_{j_k}| \quad (35) \\
& \leq |a_{jtj_t}| - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a_{j_tj_k}| + \sum_{j_k \in N_2 - \alpha} |a_{jsj_k}| + (|v_{j_t}| + |v_{js}|) \{\mu[A(\alpha)]\}^{-1} \sum_{k \in [m]} |w_{j_k}| \text{ (by (18) and } N_2 - \alpha \subset \alpha') \\
& = |a_{jtj_t}| - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a_{j_tj_k}| + \sum_{j_k \in N_2 - \alpha} |a_{jsj_k}| + \sum_{k \in [l]} (|a_{j_ti_k}| + |a_{jsi_k}|) x_k \\
& \leq |a_{jtj_t}| - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a_{j_tj_k}| + \sum_{j_k \in N_2 - \alpha} |a_{jsj_k}| + \sum_{k \in [l]} (|a_{j_ti_k}| + |a_{jsi_k}|) \text{ (by inequality (19))} \\
& = |a_{jtj_t}| + \beta_{js}(A) - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a_{j_tj_k}| + \sum_{k \in [l]} |a_{j_ti_k}|.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& |a'_{ss}| - \alpha_s(A/\alpha) + \alpha_t(A/\alpha) \\
& = |a'_{ss}| - \sum_{\substack{j_k \in N_1 \\ k \neq s}} |a'_{sk}| + \sum_{j_k \in N_1} |a'_{tk}| \\
& \leq |a_{jsj_s}| + |v_{js}| \{\mu[A(\alpha)]\}^{-1} |w_{j_s}| - \sum_{\substack{j_k \in N_1 \\ k \neq s}} (|a_{jsj_k}| - |v_{js}| \{\mu[A(\alpha)]\}^{-1} |w_{j_k}|) \\
& + \sum_{j_k \in N_1} (|a_{jtj_k}| + |v_{j_t}| \{\mu[A(\alpha)]\}^{-1} |w_{j_k}|) \text{ (by inequalities (23) and (24))} \\
& = |a_{jsj_s}| + |v_{js}| \{\mu[A(\alpha)]\}^{-1} |w_{j_s}| - \sum_{\substack{j_k \in N_1 \\ k \neq s}} |a_{jsj_k}| + |v_{js}| \{\mu[A(\alpha)]\}^{-1} \sum_{\substack{j_k \in N_1 \\ k \neq s}} |w_{j_k}| \\
& + \sum_{j_k \in N_1} |a_{jtj_k}| + |v_{j_t}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1} |w_{j_k}| \\
& = |a_{jsj_s}| - \sum_{\substack{j_k \in N_1 \\ k \neq s}} |a_{jsj_k}| + \sum_{j_k \in N_1} |a_{jtj_k}| + |v_{js}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1} |w_{j_k}| \\
& + |v_{j_t}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1} |w_{j_k}| \\
& = |a_{jsj_s}| - \sum_{\substack{j_k \in N_1 \\ k \neq s}} |a_{jsj_k}| + \sum_{j_k \in N_1} |a_{jtj_k}| + (|v_{js}| + |v_{j_t}|) \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1} |w_{j_k}| \\
& \leq |a_{jsj_s}| - \sum_{\substack{j_k \in N_1 \\ k \neq s}} |a_{jsj_k}| + \sum_{j_k \in N_1} |a_{jtj_k}| + (|v_{js}| + |v_{j_t}|) \{\mu[A(\alpha)]\}^{-1} \sum_{k \in [m]} |w_{j_k}| \text{ (by (18) and } N_1 \subset \alpha')
\end{aligned}$$

$$\begin{aligned}
&= |a_{jsjs}| - \sum_{\substack{j_k \in N_1 \\ k \neq s}} |a_{jsjk}| + \sum_{j_k \in N_1} |a_{jtjk}| + \sum_{k \in [l]} (|a_{jsik}| + |a_{jti_k}|) x_k \\
&\leq |a_{jsjs}| - \sum_{\substack{j_k \in N_1 \\ k \neq s}} |a_{jsjk}| + \sum_{j_k \in N_1} |a_{jtjk}| + \sum_{k \in [l]} (|a_{jsik}| + |a_{jti_k}|) \text{ (by inequalities (19))} \\
&= |a_{jsjs}| - \alpha_{js}(A) + \alpha_{jt}(A) + \sum_{k \in [l]} |a_{jsik}| + \sum_{k \in [l]} |a_{jti_k}|.
\end{aligned} \tag{36}$$

Furthermore, by (33)–(36), we have:

$$\begin{aligned}
\|(A/\alpha)^{-1}\|_{\infty} &\leq \max \left\{ \max_{\substack{s,t \in [m] \\ j_s \in N_1, j_t \in N_2 - \alpha}} \frac{|a_{jtjt}| + \beta_{js}(A) - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a_{jtjk}| + \sum_{k \in [l]} |a_{jti_k}|}{f_{js,jt} - g_{js,jt}}, \right. \\
&\quad \left. \max_{\substack{s,t \in [m] \\ j_s \in N_1, j_t \in N_2 - \alpha}} \frac{|a_{jsjs}| - \alpha_{js}(A) + \alpha_{jt}(A) + \sum_{k \in [l]} |a_{jsik}| + \sum_{k \in [l]} |a_{jti_k}|}{f_{js,jt} - g_{js,jt}} \right\} \\
&= \max \left\{ \max_{\substack{i,j \in \alpha' \\ i \in N_1, j \in N_2 - \alpha}} \frac{|a_{jj}| + \beta_i(A) - \sum_{\substack{k \in N_2 - \alpha \\ k \neq j}} |a_{jk}| + \sum_{k \in \alpha} |a_{jk}|}{f_{i,j} - g_{i,j}}, \right. \\
&\quad \left. \max_{\substack{i,j \in \alpha' \\ i \in N_1, j \in N_2 - \alpha}} \frac{|a_{ii}| - \alpha_i(A) + \alpha_j(A) + \sum_{k \in \alpha} |a_{ik}| + \sum_{k \in \alpha} |a_{jk}|}{f_{i,j} - g_{i,j}} \right\},
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
f_{js,jt} &= (|a_{jsjs}| - \sum_{\substack{j_k \in N_1 \\ k \neq s}} |a_{jsjk}| - |v_{js}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1} |w_{jk}|) (|a_{jtjt}| \\
&\quad - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a_{jtjk}| - |v_{jt}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2 - \alpha} |w_{jk}|), \\
g_{js,jt} &= (\sum_{j_k \in N_1} |a_{jtjk}| + |v_{jt}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1} |w_{jk}|) (\sum_{j_k \in N_2 - \alpha} |a_{jsjk}| \\
&\quad + |v_{js}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2 - \alpha} |w_{jk}|).
\end{aligned}$$

By Corollary 1, we have:

$$\|Q\|_{\infty} \leq 1 + \max_{i \in \alpha} \frac{R_i(A_{12})}{|a_{ii}| - r_i(A_{11})}. \tag{38}$$

Furthermore, by (7), (9), (32), (37), and (38), the conclusion follows. $\square$

Theorem 6. Let $A = [a_{ij}]$ be $GDSDD_n^{N_1, N_2}$ with $a_{ii} \neq 0, i \in N$, and $\emptyset \neq \alpha \subset N$. Assume that (8) and (19) hold, if $\alpha \not\subseteq N_1, \alpha \not\subseteq N_2, N_1 \not\subseteq \alpha$, and $N_2 \not\subseteq \alpha$, then:

$$\|A^{-1}\|_{\infty} \leq \gamma(\alpha) \cdot (1 + \varphi(\alpha)) \cdot \zeta(\alpha),$$

where

$$\varphi(\alpha) = \max \left\{ \max_{\substack{i \in N_1 \cap \alpha \\ j \in N_2 \cap \alpha}} \frac{(|a_{jj}| - \beta_j)R_i(A_{12}) + \beta_i R_j(A_{12})}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \beta_i \alpha_j}, \max_{\substack{i \in N_1 \cap \alpha \\ j \in N_2 \cap \alpha}} \frac{(|a_{ii}| - \alpha_i)R_j(A_{12}) + \alpha_j R_i(A_{12})}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \beta_i \alpha_j} \right\},$$

$$\zeta(\alpha) = \max\{\omega(\alpha), \tau(\alpha)\},$$

$$\omega(\alpha) = \max \left\{ \max_{\substack{i \in N_1 \cap \alpha \\ j \in N_2 \cap \alpha}} \frac{|a_{jj}| - \beta_j + \beta_i}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \beta_i \alpha_j}, \max_{\substack{i \in N_1 \cap \alpha \\ j \in N_2 \cap \alpha}} \frac{|a_{ii}| - \alpha_i + \alpha_j}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \beta_i \alpha_j} \right\},$$

$$\tau(\alpha) = \max \left\{ \max_{\substack{i, j \in \alpha' \\ i \in N_1 - \alpha, j \in N_2 - \alpha}} \frac{|a_{jj}| + \beta_i(A) - \sum_{k \in N_2 - \alpha} |a_{jk}| + \sum_{k \in N_1 \cap \alpha} |a_{ik}| + \sum_{k \in \alpha} |a_{jk}|}{p_{i,j} - q_{i,j}}, \right. \\ \left. \max_{\substack{i, j \in \alpha' \\ i \in N_1 - \alpha, j \in N_2 - \alpha}} \frac{|a_{ii}| + \alpha_j(A) - \sum_{k \in N_2 - \alpha} |a_{ik}| + \sum_{k \in N_2 \cap \alpha} |a_{jk}| + \sum_{k \in \alpha} |a_{ik}|}{p_{i,j} - q_{i,j}} \right\},$$

$$p_{i,j} = (|a_{ii}| - \sum_{\substack{k \in N_1 - \alpha \\ k \neq s}} |a_{ik}| - |v_i| \{\mu[A(\alpha)]\})^{-1} \sum_{k \in N_1 - \alpha} |w_k| (|a_{jj}| - \sum_{\substack{k \in N_2 - \alpha \\ k \neq t}} |a_{jk}| - |v_j| \{\mu[A(\alpha)]\})^{-1} \sum_{k \in N_2 - \alpha} |w_k|),$$

$$q_{i,j} = (\sum_{k \in N_1 - \alpha} |a_{jk}| + |v_j| \{\mu[A(\alpha)]\})^{-1} \sum_{k \in N_1 - \alpha} |w_k| (\sum_{k \in N_2 - \alpha} |a_{ik}| + |v_i| \{\mu[A(\alpha)]\})^{-1} \sum_{k \in N_2 - \alpha} |w_k|).$$

Proof. Since A is $GDSDD_n^{N_1, N_2}$, $\alpha \not\subseteq N_1$, $\alpha \not\subseteq N_2$, $N_1 \not\subseteq \alpha$ and $N_2 \not\subseteq \alpha$, by Lemmas 5 and 6, we have A_{11} is $GDSDD_{|\alpha|}^{N_1 \cap \alpha, N_2 \cap \alpha}$, and A/α is $GDSDD_{|N - \alpha|}^{N_1 - \alpha, N_2 - \alpha}$. Applying Theorem 3 to A_{11} , we get:

$$\|A_{11}^{-1}\|_{\infty} \leq \max \left\{ \max_{\substack{i \in N_1 \cap \alpha \\ j \in N_2 \cap \alpha}} \frac{|a_{jj}| - \beta_j + \beta_i}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \beta_i \alpha_j}, \max_{\substack{i \in N_1 \cap \alpha \\ j \in N_2 \cap \alpha}} \frac{|a_{ii}| - \alpha_i + \alpha_j}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \beta_i \alpha_j} \right\}. \quad (39)$$

Applying Theorem 3 to A/α yields:

$$\|(A/\alpha)^{-1}\|_{\infty} \leq \max \left\{ \max_{\substack{s, t \in [m] \\ j_s \in N_1 - \alpha, j_t \in N_2 - \alpha}} \frac{|a'_{tt}| - \beta_t(A/\alpha) + \beta_s(A/\alpha)}{(|a'_{ss}| - \alpha_s(A/\alpha))(|a'_{tt}| - \beta_t(A/\alpha)) - \alpha_t(A/\alpha)\beta_s(A/\alpha)}, \right. \\ \left. \max_{\substack{s, t \in [m] \\ j_s \in N_1 - \alpha, j_t \in N_2 - \alpha}} \frac{|a'_{ss}| - \alpha_s(A/\alpha) + \alpha_t(A/\alpha)}{(|a'_{ss}| - \alpha_s(A/\alpha))(|a'_{tt}| - \beta_t(A/\alpha)) - \alpha_t(A/\alpha)\beta_s(A/\alpha)} \right\}. \quad (40)$$

By Theorem 2 in [14], we have:

$$\begin{aligned}
& (|a'_{ss}| - \alpha_s(A/\alpha))(|a'_{tt}| - \beta_t(A/\alpha)) - \alpha_t(A/\alpha)\beta_s(A/\alpha) \\
& \geq (|a_{js}| - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |a_{jsj_k}| - |v_{js}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1 - \alpha} |w_{j_k}|)(|a_{jt}| - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a_{jtj_k}| \\
& - |v_{jt}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2 - \alpha} |w_{j_k}|) - (\sum_{j_k \in N_1 - \alpha} |a_{jtj_k}| + |v_{jt}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1 - \alpha} |w_{j_k}|)(\sum_{j_k \in N_2 - \alpha} |a_{jsj_k}| \\
& + |v_{js}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2 - \alpha} |w_{j_k}|) \\
& > 0.
\end{aligned} \tag{41}$$

Now, the upper bounds of $|a'_{tt}| - \beta_t(A/\alpha) + \beta_s(A/\alpha)$ and $|a'_{ss}| - \alpha_s(A/\alpha) + \alpha_t(A/\alpha)$ are considered.

By (19), (20), (23), and (24), we have:

$$\begin{aligned}
& |a'_{tt}| - \beta_t(A/\alpha) + \beta_s(A/\alpha) \\
& = |a'_{tt}| - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a'_{tk}| + \sum_{j_k \in N_2 - \alpha} |a'_{sk}| \\
& \leq |a_{jt}| + |v_{jt}| \{\mu[A(\alpha)]\}^{-1} |w_{jt}| - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} (|a_{jtj_k}| - |v_{jt}| \{\mu[A(\alpha)]\}^{-1} |w_{j_k}|) \\
& + \sum_{j_k \in N_2 - \alpha} (|a_{jsj_k}| + |v_{js}| \{\mu[A(\alpha)]\}^{-1} |w_{j_k}|) \text{ (by inequalities (23) and (24))} \\
& = |a_{jt}| + |v_{jt}| \{\mu[A(\alpha)]\}^{-1} |w_{jt}| - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a_{jtj_k}| + |v_{jt}| \{\mu[A(\alpha)]\}^{-1} \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |w_{j_k}| \\
& + \sum_{j_k \in N_2 - \alpha} |a_{jsj_k}| + |v_{js}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2 - \alpha} |w_{j_k}| \\
& = |a_{jt}| - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a_{jtj_k}| + \sum_{j_k \in N_2 - \alpha} |a_{jsj_k}| + |v_{jt}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2 - \alpha} |w_{j_k}| \\
& + |v_{js}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2 - \alpha} |w_{j_k}| \\
& = |a_{jt}| - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a_{jtj_k}| + \sum_{j_k \in N_2 - \alpha} |a_{jsj_k}| + (|v_{jt}| + |v_{js}|) \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2 - \alpha} |w_{j_k}| \\
& \leq |a_{jt}| - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a_{jtj_k}| + \sum_{j_k \in N_2 - \alpha} |a_{jsj_k}| + (|v_{jt}| + |v_{js}|) \{\mu[A(\alpha)]\}^{-1} \sum_{k \in [m]} |w_{j_k}| \text{ (by (18) and } N_2 - \alpha \subset \alpha') \\
& = |a_{jt}| - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a_{jtj_k}| + \sum_{j_k \in N_2 - \alpha} |a_{jsj_k}| + \sum_{k \in [l]} (|a_{jti_k}| + |a_{jsi_k}|) x_k \\
& \leq |a_{jt}| - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a_{jtj_k}| + \sum_{j_k \in N_2 - \alpha} |a_{jsj_k}| + \sum_{k \in [l]} (|a_{jti_k}| + |a_{jsi_k}|) \text{ (by inequality (19))} \\
& = |a_{jt}| - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a_{jtj_k}| + \sum_{j_k \in N_2 - \alpha} |a_{jsj_k}| + \sum_{k \in [l]} |a_{jti_k}| + \sum_{k \in [l]} |a_{jsi_k}|.
\end{aligned} \tag{42}$$

Similarly,

$$\begin{aligned}
& |a'_{ss}| - \alpha_s(A/\alpha) + \alpha_t(A/\alpha) \\
&= |a'_{ss}| - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |a'_{sk}| + \sum_{j_k \in N_1 - \alpha} |a'_{tk}| \\
&\leq |a_{jsjs}| + |v_{js}| \{\mu[A(\alpha)]\}^{-1} |w_{js}| - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} (|a_{jsjk}| - |v_{js}| \{\mu[A(\alpha)]\}^{-1} |w_{jk}|) \\
&+ \sum_{j_k \in N_1 - \alpha} (|a_{jijk}| + |v_{jt}| \{\mu[A(\alpha)]\}^{-1} |w_{jk}|) \text{ (by inequalities (23) and (24))} \\
&= |a_{jsjs}| + |v_{js}| \{\mu[A(\alpha)]\}^{-1} |w_{js}| - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |a_{jsjk}| + |v_{js}| \{\mu[A(\alpha)]\}^{-1} \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |w_{jk}| \\
&+ \sum_{j_k \in N_1 - \alpha} |a_{jijk}| + |v_{jt}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1 - \alpha} |w_{jk}| \\
&= |a_{jsjs}| - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |a_{jsjk}| + \sum_{j_k \in N_1 - \alpha} |a_{jijk}| + |v_{js}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1 - \alpha} |w_{jk}| \\
&+ |v_{jt}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1 - \alpha} |w_{jk}| \\
&= |a_{jsjs}| - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |a_{jsjk}| + \sum_{j_k \in N_1 - \alpha} |a_{jijk}| + (|v_{js}| + |v_{jt}|) \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1 - \alpha} |w_{jk}| \\
&\leq |a_{jsjs}| - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |a_{jsjk}| + \sum_{j_k \in N_1 - \alpha} |a_{jijk}| + (|v_{js}| + |v_{jt}|) \{\mu[A(\alpha)]\}^{-1} \sum_{k \in [m]} |w_{jk}| \text{ (by (18) and } N_1 - \alpha \subset \alpha') \\
&= |a_{jsjs}| - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |a_{jsjk}| + \sum_{j_k \in N_1 - \alpha} |a_{jijk}| + \sum_{k \in [l]} (|a_{jsik}| + |a_{jti_k}|) x_k \\
&\leq |a_{jsjs}| - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |a_{jsjk}| + \sum_{j_k \in N_1 - \alpha} |a_{jijk}| + \sum_{k \in [l]} (|a_{jsik}| + |a_{jti_k}|) \text{ (by inequality (19))} \\
&= |a_{jsjs}| - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |a_{jsjk}| + \sum_{j_k \in N_1 - \alpha} |a_{jijk}| + \sum_{k \in [l]} |a_{jsik}| + \sum_{k \in [l]} |a_{jti_k}|.
\end{aligned} \tag{43}$$

Furthermore, by (40)–(43), we have:

$$\begin{aligned}
\|(A/\alpha)^{-1}\|_{\infty} &\leq \max \left\{ \max_{\substack{s, t \in [m] \\ j_s \in N_1 - \alpha \\ j_t \in N_2 - \alpha}} \frac{|a_{jti_t}| - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a_{jtk}| + \sum_{j_k \in N_2 - \alpha} |a_{jsjk}| + \sum_{k \in [l]} |a_{jti_k}| + \sum_{k \in [l]} |a_{jsi_k}|}{p_{jsjt} - q_{jsjt}}, \right. \\
&\quad \left. \max_{\substack{s, t \in [m] \\ j_s \in N_1 - \alpha \\ j_t \in N_2 - \alpha}} \frac{|a_{jsjs}| - \sum_{j_k \in N_1 - \alpha, k \neq s} |a_{jsjk}| + \sum_{j_k \in N_1 - \alpha} |a_{jijk}| + \sum_{k \in [l]} |a_{jsi_k}| + \sum_{k \in [l]} |a_{jti_k}|}{p_{jsjt} - q_{jsjt}} \right\} \\
&= \max \left\{ \max_{\substack{i, j \in \alpha' \\ i \in N_1 - \alpha \\ j \in N_2 - \alpha}} \frac{|a_{jj}| + \beta_i(A) - \sum_{\substack{k \in N_2 - \alpha \\ k \neq j}} |a_{jk}| + \sum_{k \in N_1 \cap \alpha} |a_{ik}| + \sum_{k \in \alpha} |a_{jk}|}{p_{ij} - q_{ij}}, \right.
\end{aligned}$$

$$\max_{\substack{i,j \in \alpha' \\ i \in N_1 - \alpha \\ j \in N_2 - \alpha}} \left\{ \frac{|a_{ii}| - \sum_{k \in N_2 - \alpha} |a_{ik}| + \alpha_j(A) + \sum_{k \in N_2 \cap \alpha} |a_{jk}| + \sum_{k \in \alpha} |a_{ik}|}{p_{i,j} - q_{i,j}} \right\}, \quad (44)$$

where

$$\begin{aligned} p_{js,jt} &= (|a_{jsjs}| - \sum_{\substack{j_k \in N_1 - \alpha \\ k \neq s}} |a_{jsj_k}| - |v_{js}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1 - \alpha} |w_{j_k}|)(|a_{jtjt}| \\ &\quad - \sum_{\substack{j_k \in N_2 - \alpha \\ k \neq t}} |a_{jtj_k}| - |v_{jt}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2 - \alpha} |w_{j_k}|), \\ q_{js,jt} &= (\sum_{j_k \in N_1 - \alpha} |a_{jtj_k}| + |v_{jt}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_1 - \alpha} |w_{j_k}|)(\sum_{j_k \in N_2 - \alpha} |a_{jsj_k}| \\ &\quad + |v_{js}| \{\mu[A(\alpha)]\}^{-1} \sum_{j_k \in N_2 - \alpha} |w_{j_k}|). \end{aligned}$$

By Corollary 1, we have:

$$\|Q\|_{\infty} \leq \max \left\{ \max_{\substack{i \in N_1 \cap \alpha \\ j \in N_2 \cap \alpha}} \frac{(|a_{jj}| - \beta_j)R_i(A_{12}) + \beta_i R_j(A_{12})}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \beta_i \alpha_j}, \max_{\substack{i \in N_1 \cap \alpha \\ j \in N_2 \cap \alpha}} \frac{(|a_{ii}| - \alpha_i)R_j(A_{12}) + \alpha_j R_i(A_{12})}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \beta_i \alpha_j} \right\}. \quad (45)$$

Furthermore, by (7), (9), (39), (44), and (45), the conclusion follows. $\square$

Theorem 7. Let $A = [a_{ij}]$ be $GDSDD_n^{N_1, N_2}$ with $a_{ii} \neq 0, i \in N$, and $\emptyset \neq \alpha \subset N$. Assume that (8) holds, if $N_1 \subset \alpha$ or $N_2 \subset \alpha$, then:

$$\|A^{-1}\|_{\infty} \leq \gamma(\alpha) \cdot (1 + \varphi(\alpha)) \cdot \max\{\phi(\alpha), \max_{\substack{j \in \alpha' \\ k \in \alpha' \\ k \neq j}} \frac{1}{|a_{jj}| - \sum_{k \in \alpha' \\ k \neq j} |a_{jk}| - w_j(A)}\},$$

where

$$\phi(\alpha) = \max \left\{ \max_{\substack{i \in N_1 \cap \alpha \\ j \in N_2 \cap \alpha}} \frac{|a_{jj}| - \beta_j + \beta_i}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \beta_i \alpha_j}, \max_{\substack{i \in N_1 \cap \alpha \\ j \in N_2 \cap \alpha}} \frac{|a_{ii}| - \alpha_i + \alpha_j}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \beta_i \alpha_j} \right\},$$

$$w_j(A) = \max_{i \in \alpha} \frac{\sum_{k \in \alpha'} |a_{ik}|}{|a_{ii}| - \sum_{\substack{k \in \alpha \\ k \neq i}} |a_{ik}|} \sum_{k \in \alpha} |a_{jk}|.$$

Proof. Since A is $GDSDD_n^{N_1, N_2}$, $N_1 \subset \alpha$ or $N_2 \subset \alpha$, by Lemmas 5 and 6, we have A_{11} is $GDSDD_{|\alpha|}^{N_1 \cap \alpha, N_2 \cap \alpha}$ and A/α is SDD , consequently, nonsingular. Let:

$$A/\alpha = A_{22} - A_{21}A_{11}^{-1}A_{12} = (a'_{st}).$$

Then by Theorem 1 in [14], we have:

$$|a'_{tt}| - r_t(A/\alpha) \geq |a_{jtjt}| - \sum_{\substack{k \in [m] \\ k \neq t}} |a_{jtj_k}| - w_{jt}(A).$$

Applying the Varah's bound to A/α , we get:

$$\|(A/\alpha)^{-1}\|_{\infty} \leq \max_{j \in \alpha'} \frac{1}{|a_{jj}| - \sum_{\substack{k \in \alpha' \\ k \neq j}} |a_{jk}| - w_j(A)}, \quad (46)$$

where

$$w_j(A) = \max_{i \in \alpha} \frac{\sum_{k \in \alpha'} |a_{ik}|}{|a_{ii}| - \sum_{\substack{k \in \alpha \\ k \neq i}} |a_{ik}|} \sum_{k \in \alpha} |a_{jk}|.$$

By Theorem 3,

$$\|A_{11}^{-1}\|_{\infty} \leq \max \left\{ \max_{\substack{i \in N_1 \cap \alpha \\ j \in N_2 \cap \alpha}} \frac{|a_{jj}| - \beta_j + \beta_i}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \beta_i \alpha_j}, \max_{\substack{i \in N_1 \cap \alpha \\ j \in N_2 \cap \alpha}} \frac{|a_{ii}| - \alpha_i + \alpha_j}{(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) - \beta_i \alpha_j} \right\}. \quad (47)$$

From (7), (9), (46), (47), and Corollary 1, the conclusion follows. $\square$

Theorem 8. Let $A = [a_{ij}]$ be $GDSDD_n^{N_1, N_2}$ with $a_{ii} \neq 0$, $i \in N$, and $\emptyset \neq \alpha \subset N$. Assume that (8) holds, if $N_1 = \alpha$ or $N_2 = \alpha$, then:

$$\|A^{-1}\|_{\infty} \leq \gamma(\alpha) \cdot \left(1 + \max_{i \in \alpha} \frac{R_i(A_{12})}{|a_{ii}| - r_i(A_{11})} \right) \cdot \max \left\{ \max_{i \in \alpha} \frac{1}{|a_{ii}| - r_i(A_{11})}, \max_{j \in \alpha'} \frac{1}{|a_{jj}| - \sum_{\substack{k \in \alpha' \\ k \neq j}} |a_{jk}| - w_j(A)} \right\},$$

where

$$w_j(A) = \max_{i \in \alpha} \frac{\sum_{k \in \alpha'} |a_{ik}|}{|a_{ii}| - \sum_{\substack{k \in \alpha \\ k \neq i}} |a_{ik}|} \sum_{k \in \alpha} |a_{jk}|.$$

Proof. Since A is $GDSDD_n^{N_1, N_2}$, $N_1 = \alpha$ or $N_2 = \alpha$, by Lemmas 5 and 6, we have A_{11} and A/α is SDD , consequently, nonsingular. Applying the Varah's bound to A_{11} , we get:

$$\|A_{11}^{-1}\|_{\infty} \leq \max_{i \in \alpha} \frac{1}{|a_{ii}| - r_i(A_{11})}. \quad (48)$$

From Corollary 1, we can obtain:

$$\|Q\|_{\infty} \leq 1 + \max_{i \in \alpha} \frac{R_i(A_{12})}{|a_{ii}| - r_i(A_{11})}. \quad (49)$$

Let,

$$A/\alpha = A_{22} - A_{21}A_{11}^{-1}A_{12} = (a'_{st}).$$

Then by Theorem 1 in [14], we have:

$$|a'_{tt}| - r_t(A/\alpha) \geq |a_{tj_t}| - \sum_{\substack{k \in [m] \\ k \neq t}} |a_{tj_k}| - w_{j_t}(A).$$

Applying the Varah's bound to A/α , we get:

$$\|(A/\alpha)^{-1}\|_{\infty} \leq \max_{j \in \alpha'} \frac{1}{|a_{jj}| - \sum_{\substack{k \in \alpha' \\ k \neq j}} |a_{jk}| - w_j(A)}, \quad (50)$$

where

$$w_j(A) = \max_{i \in \alpha} \frac{\sum_{\substack{k \in \alpha' \\ k \neq i}} |a_{ik}|}{|a_{ii}| - \sum_{\substack{k \in \alpha \\ k \neq i}} |a_{ik}|} \sum_{k \in \alpha} |a_{jk}|.$$

From (7), (9), (48)–(50), the conclusion follows. $\square$

Theorem 9. Let $A = [a_{ij}]$ be $\text{GDSDD}_n^{N_1, N_2}$, and $n \geq 3$. Then:

$$\|A^{-1}\|_{\infty} \leq \Delta(A) = \min_{i \in N} \Delta_i(A),$$

where

$$\Delta_i(A) = \left(1 + \frac{\max_{j \in N, j \neq i} |a_{ji}|}{|a_{ii}|}\right) \left(1 + \frac{\max_{j \in N, j \neq i} |a_{ij}|}{|a_{ii}|}\right) \tilde{\Delta}_i(A),$$

$$\tilde{\Delta}_i(A) = \max \left\{ \frac{1}{|a_{ii}|}, \Delta'(A) \right\},$$

$$\Delta'(A) = \max \left\{ \begin{array}{l} \max_{\substack{j \in N_1 - \{i\} \\ k \in N_2 - \{i\}}} \frac{|c_{kk}| - \sum_{\substack{p \in N_2 \\ p \neq k, i}} |c_{kp}| + \sum_{p \in N_2, p \neq i} |c_{jp}|}{(|c_{jj}| - \sum_{\substack{p \in N_1 \\ p \neq j, i}} |c_{jp}|)(|c_{kk}| - \sum_{\substack{p \in N_2 \\ p \neq k, i}} |c_{kp}|) - \sum_{p \in N_1, p \neq i} |c_{kp}| \sum_{p \in N_2, p \neq i} |c_{jp}|}, \\ \max_{\substack{j \in N_1 - \{i\} \\ k \in N_2 - \{i\}}} \frac{|c_{jj}| - \sum_{\substack{p \in N_1 \\ p \neq j, i}} |c_{jp}| + \sum_{p \in N_1, p \neq i} |c_{kp}|}{(|c_{jj}| - \sum_{\substack{p \in N_1 \\ p \neq j, i}} |c_{jp}|)(|c_{kk}| - \sum_{\substack{p \in N_2 \\ p \neq k, i}} |c_{kp}|) - \sum_{p \in N_1, p \neq i} |c_{kp}| \sum_{p \in N_2, p \neq i} |c_{jp}|} \end{array} \right\},$$

and

$$c_{jk} = a_{jk} - \frac{a_{ji}a_{ik}}{a_{ii}}, \quad j, k \in N - \{i\}.$$

Proof. Since A is GDSDD , then for any $\emptyset \neq \alpha \subset N$, by Lemmas 5 and 6, we have A_{11} is $\text{GDSDD}_{|\alpha|}^{N_1 \cap \alpha, N_2 \cap \alpha}$, A/α is $\text{GDSDD}_{|N-\alpha|}^{N_1 - \alpha, N_2 - \alpha}$, and consequently, they are nonsingular. Taking $\alpha = \{i\}$, then $A_{11} = a_{ii}$, $\alpha' = N - \{i\}$ and:

$$[A(\alpha)]^{-1} = \frac{1}{a_{ii}}. \quad (51)$$

We have:

$$\|S\|_{\infty} = 1 + \frac{\max_{s \in [m]} |a_{jsi}|}{|a_{ii}|} = 1 + \frac{\max_{j \in N, j \neq i} |a_{ji}|}{|a_{ii}|}, \quad (52)$$

and

$$\|Q\|_{\infty} = 1 + \frac{\max_{s \in [m]} |a_{ij_s}|}{|a_{ii}|} = 1 + \frac{\max_{j \in N, j \neq i} |a_{ij}|}{|a_{ii}|}. \quad (53)$$

Let $A/\alpha = (a'_{st})$. By (22), we have:

$$|a'_{st}| = |a_{j_s j_t} - \frac{a_{j_s i} a_{ij_t}}{a_{ii}}| = |c_{j_s j_t}|, \quad s, t \in [m], \quad j_s, j_t \in N - \{i\}.$$

By calculation, we obtain for $j_s \in N_1 - \{i\}$, $j_t \in N_2 - \{i\}$,

$$\begin{aligned} \alpha_s(A/\alpha) &= \sum_{\substack{j_p \in N_1 - \{i\} \\ p \neq s}} \left| a_{j_s j_p} - \frac{a_{j_s i} a_{ij_p}}{a_{ii}} \right| = \sum_{\substack{j_p \in N_1 \\ j_p \neq j_s, i}} |c_{j_s j_p}|, \\ \beta_s(A/\alpha) &= \sum_{j_p \in N_2 - \{i\}} \left| a_{j_s j_p} - \frac{a_{j_s i} a_{ij_p}}{a_{ii}} \right| = \sum_{\substack{j_p \in N_2 \\ j_p \neq i}} |c_{j_s j_p}|, \\ \alpha_t(A/\alpha) &= \sum_{\substack{j_p \in N_1 - \{i\} \\ j_p \neq i}} \left| a_{j_t j_p} - \frac{a_{j_t i} a_{ij_p}}{a_{ii}} \right| = \sum_{\substack{j_p \in N_1 \\ j_p \neq i}} |c_{j_t j_p}|, \end{aligned}$$

and

$$\beta_t(A/\alpha) = \sum_{\substack{j_p \in N_2 - \{i\} \\ p \neq t}} \left| a_{j_t j_p} - \frac{a_{j_t i} a_{ij_p}}{a_{ii}} \right| = \sum_{\substack{j_p \in N_2 \\ j_p \neq j_t, i}} |c_{j_t j_p}|.$$

By (40), we have:

$$\|(A/\alpha)^{-1}\|_{\infty} \leq \max \left\{ \begin{aligned} & \max_{\substack{j \in N_1 - \{i\} \\ k \in N_2 - \{i\}}} \frac{|c_{kk}| - \sum_{\substack{p \in N_2 \\ p \neq k, i}} |c_{kp}| + \sum_{p \in N_2, p \neq i} |c_{jp}|}{(c_{jj} - \sum_{\substack{p \in N_1 \\ p \neq j, i}} |c_{jp}|)(|c_{kk}| - \sum_{\substack{p \in N_2 \\ p \neq k, i}} |c_{kp}|) - \sum_{p \in N_1, p \neq i} |c_{kp}| \sum_{p \in N_2, p \neq i} |c_{jp}|}, \\ & \max_{\substack{j \in N_1 - \{i\} \\ k \in N_2 - \{i\}}} \frac{|c_{jj}| - \sum_{\substack{p \in N_1 \\ p \neq j, i}} |c_{jp}| + \sum_{p \in N_1, p \neq i} |c_{kp}|}{(c_{jj} - \sum_{\substack{p \in N_1 \\ p \neq j, i}} |c_{jp}|)(|c_{kk}| - \sum_{\substack{p \in N_2 \\ p \neq k, i}} |c_{kp}|) - \sum_{p \in N_1, p \neq i} |c_{kp}| \sum_{p \in N_2, p \neq i} |c_{jp}|} \end{aligned} \right\}. \quad (54)$$

By (7), (51)–(54), we have $\|A^{-1}\|_{\infty} \leq \Delta_i(A)$. By the arbitrariness of i , the conclusion follows. $\square$

In the following, we prove that the bound in Theorem 9 generally improves the bound obtained by Theorem 3 in [11] for *SDD* matrices and Theorem 9 in [12] for *DSDD* matrices.

Theorem 10. Let $A = [a_{ij}] \in C^{n \times n}$ be *SDD*. Then,

$$\|A^{-1}\|_{\infty} \leq \Delta(A) \leq B_{Schur}(A) = \min_{i \in N} B_{Schur}(i),$$

where $\Delta(A)$ is defined in Theorem 9 and:

$$B_{Schur}(i) = 2 \left(1 + \frac{\max_{j \in N, j \neq i} |a_{ji}|}{|a_{ii}|} \right) \max \left\{ \frac{1}{|a_{ii}|}, \max_{j \in N, j \neq i} \frac{1}{|a_{jj}| - r_j(A) + \frac{|a_{ii}| - r_i(A)}{|a_{ii}|} |a_{ji}|} \right\}.$$

Proof. Since A is SDD , A is $GDSDD$ with $N_1 = N, N_2 = \emptyset$. From Theorem 9, we can obtain:

$$\|A^{-1}\|_{\infty} \leq \Delta(A) = \min_{i \in N} \Delta_i(A),$$

where

$$\Delta_i(A) = \left(1 + \frac{\max_{j \in N, j \neq i} |a_{ji}|}{|a_{ii}|} \right) \left(1 + \frac{\max_{j \in N, j \neq i} |a_{ij}|}{|a_{ii}|} \right) \tilde{\Delta}_i(A),$$

$$\tilde{\Delta}_i(A) = \max \left\{ \frac{1}{|a_{ii}|}, \Delta'(A) \right\},$$

$$\Delta'(A) = \max_{j \in N, j \neq i} \frac{1}{|c_{jj}| - \sum_{p \in N, p \neq j, i} |c_{jp}|},$$

and

$$c_{jk} = a_{jk} - \frac{a_{ji}a_{ik}}{a_{ii}}, \quad j, k \in N - \{i\}.$$

From Theorem 3 of [11],

$$\|A^{-1}\|_{\infty} \leq B_{Schur}(A) = \min_{i \in N} B_{Schur}(i),$$

where

$$B_{Schur}(i) = 2 \left(1 + \frac{\max_{j \in N, j \neq i} |a_{ji}|}{|a_{ii}|} \right) \max \left\{ \frac{1}{|a_{ii}|}, \max_{j \in N, j \neq i} \frac{1}{|a_{jj}| - r_j(A) + \frac{|a_{ii}| - r_i(A)}{|a_{ii}|} |a_{ji}|} \right\}.$$

By calculation,

$$\begin{aligned} |c_{jj}| - \sum_{p \in N, p \neq j, i} |c_{jp}| &= |a_{jj} - \frac{a_{ji}a_{ij}}{a_{ii}}| - \sum_{p \in N, p \neq j, i} |a_{jp} - \frac{a_{ji}a_{ip}}{a_{ii}}| \\ &\geq |a_{jj}| - \frac{|a_{ji}||a_{ij}|}{|a_{ii}|} - \sum_{p \in N, p \neq j, i} (|a_{jp}| + \frac{|a_{ji}||a_{ip}|}{|a_{ii}|}) \\ &= |a_{jj}| - \frac{|a_{ji}||a_{ij}|}{|a_{ii}|} - (\sum_{p \in N, p \neq j, i} |a_{jp}| + \sum_{p \in N, p \neq j, i} \frac{|a_{ji}||a_{ip}|}{|a_{ii}|}) \\ &= |a_{jj}| - \sum_{p \in N, p \neq j, i} |a_{jp}| - \frac{|a_{ji}||a_{ij}|}{|a_{ii}|} - \sum_{p \in N, p \neq j, i} \frac{|a_{ji}||a_{ip}|}{|a_{ii}|} \\ &= |a_{jj}| - \sum_{p \in N, p \neq j, i} |a_{jp}| - |a_{ji}| + |a_{ji}|(1 - \frac{|a_{ij}|}{|a_{ii}|} - \sum_{p \in N, p \neq j, i} \frac{|a_{ip}|}{|a_{ii}|}) \\ &= |a_{jj}| - r_j(A) + |a_{ji}|(1 - \frac{r_i(A)}{|a_{ii}|}) \\ &= |a_{jj}| - r_j(A) + \frac{|a_{ii}| - r_i(A)}{|a_{ii}|} |a_{ji}|. \end{aligned}$$

Then,

$$\max_{j \in N, j \neq i} \frac{1}{|c_{jj}| - \sum_{p \in N, p \neq j, i} |c_{jp}|} \leq \max_{j \in N, j \neq i} \frac{1}{|a_{jj}| - r_j(A) + \frac{|a_{ii}| - r_i(A)}{|a_{ii}|} |a_{ji}|}.$$

Since A is SDD ,

$$|a_{ii}| > r_i(A) = \sum_{j \in N, j \neq i} |a_{ij}|, \quad i \in N.$$

For all $i \in N$, it is easy to get:

$$\frac{\sum_{j \in N, j \neq i} |a_{ij}|}{|a_{ii}|} < 1,$$

$$\left(1 + \frac{\max_{j \in N, j \neq i} |a_{ij}|}{|a_{ii}|}\right) < 2.$$

Therefore,

$$\Delta_i(A) < B_{Schur}(i), \quad i \in N.$$

Furthermore,

$$\Delta(A) < B_{Schur}(A).$$

The proof is completed. $\square$

Theorem 10 shows that the bound in Theorem 9 generally improves the bound obtained by Theorem 3 in [11] for SDD matrices.

Theorem 11. Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ be $DSDD$. Then:

$$\|A^{-1}\|_{\infty} \leq \Delta(A) \leq \Pi(A) = \min_{i \in N} \pi_i(A),$$

where $\Delta(A)$ is defined in Theorem 9,

$$\pi_i(A) = \left(1 + \frac{\max_{j \in N, j \neq i} |a_{ji}|}{|a_{ii}|}\right) \left(1 + \frac{\max_{j \in N, j \neq i} |a_{ij}|}{|a_{ii}|}\right) \tilde{\pi}_i(A),$$

$$\tilde{\pi}_i(A) = \max \left\{ \frac{1}{|a_{ii}|}, \max_{\substack{j, k \in N - \{i\} \\ j \neq k}} \frac{|c_{kk}| + \sum_{p \in N, p \neq i, j} |c_{jp}|}{|c_{jj}| |c_{kk}| - \sum_{p \in N, p \neq i, j} |c_{jp}| \sum_{p \in N, p \neq i, k} |c_{kp}|} \right\},$$

and

$$c_{jk} = a_{jk} - \frac{a_{ji}a_{ik}}{a_{ii}}, \quad j, k \in N - \{i\}.$$

Proof. Since A is $DSDD$, there is at most one i_0 such that $|a_{i_0 i_0}| \leq r_{i_0}(A)$. If there does not exist an i_0 such that $|a_{i_0 i_0}| \leq r_{i_0}(A)$, then A is SDD , the proof is the same as Theorem 10. If there exists only one i_0 such that $|a_{i_0 i_0}| \leq r_{i_0}(A)$, then A is $GDSDD$ with $N_1 = N - \{i_0\}$, $N_2 = \{i_0\}$. From Theorem 9, we can obtain:

$$\|A^{-1}\|_{\infty} \leq \Delta(A) = \min_{i \in N} \Delta_i(A),$$

where

$$\Delta_i(A) = \left(1 + \frac{\max_{j \in N, j \neq i} |a_{ji}|}{|a_{ii}|}\right) \left(1 + \frac{\max_{j \in N, j \neq i} |a_{ij}|}{|a_{ii}|}\right) \tilde{\Delta}_i(A),$$

$$\tilde{\Delta}_i(A) = \max \left\{ \frac{1}{|a_{ii}|}, \Delta'(A) \right\},$$

$$\Delta'(A) = \max \left\{ \max_{\substack{j \in N - \{i_0\} - \{i\} \\ k \in \{i_0\} - \{i\}}} \frac{|c_{kk}| + \sum_{p \in \{i_0\}, p \neq i} |c_{jp}|}{(|c_{jj}| - \sum_{p \in N - \{i_0\}, p \neq j, i} |c_{jp}|) |c_{kk}| - \sum_{p \in N - \{i_0\}, p \neq i} |c_{kp}| \sum_{p \in \{i_0\}, p \neq i} |c_{jp}|}, \right. \\ \left. \max_{\substack{j \in N - \{i_0\} - \{i\} \\ k \in \{i_0\} - \{i\}}} \frac{|c_{jj}| - \sum_{p \in N - \{i_0\}, p \neq j, i} |c_{jp}| + \sum_{p \in N - \{i_0\}, p \neq i} |c_{kp}|}{(|c_{jj}| - \sum_{p \in N - \{i_0\}, p \neq j, i} |c_{jp}|) |c_{kk}| - \sum_{p \in N - \{i_0\}, p \neq i} |c_{kp}| \sum_{p \in \{i_0\}, p \neq i} |c_{jp}|} \right\}.$$

From Theorem 9 of [12],

$$\|A^{-1}\|_{\infty} \leq \Pi(A) = \min_{i \in N} \pi_i(A),$$

where

$$\pi_i(A) = \left(1 + \frac{\max_{j \in N, j \neq i} |a_{ji}|}{|a_{ii}|}\right) \left(1 + \frac{\max_{j \in N, j \neq i} |a_{ij}|}{|a_{ii}|}\right) \tilde{\pi}_i(A),$$

$$\tilde{\pi}_i(A) = \max \left\{ \frac{1}{|a_{ii}|}, \max_{\substack{j, k \in N - \{i\} \\ j \neq k}} \frac{|c_{kk}| + \sum_{p \in N, p \neq i, j} |c_{jp}|}{|c_{jj}| |c_{kk}| - \sum_{p \in N, p \neq i, j} |c_{jp}| \sum_{p \in N, p \neq i, k} |c_{kp}|} \right\}.$$

For $i \neq i_0$,

$$\Delta'(A) = \max \left\{ \max_{\substack{j \in N - \{i_0\} - \{i\} \\ k = i_0}} \frac{|c_{i_0 i_0}| + |c_{j i_0}|}{|c_{jj}| |c_{i_0 i_0}| - |c_{i_0 i_0}| \sum_{p \in N - \{i_0\}, p \neq j, i} |c_{jp}| - |c_{j i_0}| \sum_{p \in N - \{i_0\}, p \neq i} |c_{i_0 p}|}, \right. \\ \left. \max_{\substack{j \in N - \{i_0\} - \{i\} \\ k = i_0}} \frac{|c_{jj}| - \sum_{p \in N - \{i_0\}, p \neq j, i} |c_{jp}| + \sum_{p \in N - \{i_0\}, p \neq i} |c_{i_0 p}|}{|c_{jj}| |c_{i_0 i_0}| - |c_{i_0 i_0}| \sum_{p \in N - \{i_0\}, p \neq j, i} |c_{jp}| - |c_{j i_0}| \sum_{p \in N - \{i_0\}, p \neq i} |c_{i_0 p}|} \right\}.$$

Obviously,

$$\begin{aligned} & \max_{\substack{j, k \in N - \{i\} \\ j \neq k}} \frac{|c_{kk}| + \sum_{p \in N, p \neq i, j} |c_{jp}|}{|c_{jj}| |c_{kk}| - \sum_{p \in N, p \neq i, j} |c_{jp}| \sum_{p \in N, p \neq i, k} |c_{kp}|} \\ & \geq \max_{\substack{j \in N - \{i_0\} - \{i\} \\ k = i_0}} \frac{|c_{i_0 i_0}| + \sum_{p \in N, p \neq i, j} |c_{jp}|}{|c_{jj}| |c_{i_0 i_0}| - \sum_{p \in N, p \neq i, j} |c_{jp}| \sum_{p \in N, p \neq i, i_0} |c_{i_0 p}|} \end{aligned}$$

$$\begin{aligned}
& \max_{\substack{j,k \in N-\{i\} \\ j \neq k}} \frac{|c_{kk}| + \sum_{p \in N, p \neq i, j} |c_{jp}|}{|c_{jj}| |c_{kk}| - \sum_{p \in N, p \neq i, j} |c_{jp}| \sum_{p \in N, p \neq i, k} |c_{kp}|} \\
& \geq \max_{\substack{j=i_0 \\ k \in N-\{i_0\}-\{i\}}} \frac{|c_{kk}| + \sum_{p \in N, p \neq i, i_0} |c_{i_0 p}|}{|c_{i_0 i_0}| |c_{kk}| - \sum_{p \in N, p \neq i, i_0} |c_{i_0 p}| \sum_{p \in N, p \neq i, k} |c_{kp}|}.
\end{aligned}$$

We can obtain:

$$\begin{aligned}
|c_{i_0 i_0}| + |c_{j i_0}| & \leq |c_{i_0 i_0}| + \sum_{p \in N, p \neq i, j} |c_{jp}|, \\
|c_{jj}| |c_{i_0 i_0}| - \sum_{p \in N, p \neq i, j} |c_{jp}| \sum_{p \in N, p \neq i, i_0} |c_{i_0 p}| & = |c_{jj}| |c_{i_0 i_0}| - (|c_{j i_0}| + \sum_{p \in N, p \neq i, j, i_0} |c_{jp}|) \sum_{p \in N, p \neq i, i_0} |c_{i_0 p}| \\
& = |c_{jj}| |c_{i_0 i_0}| - \sum_{p \in N, p \neq i, j, i_0} |c_{jp}| \sum_{p \in N, p \neq i, i_0} |c_{i_0 p}| - |c_{j i_0}| \sum_{p \in N, p \neq i, i_0} |c_{i_0 p}| \\
& \leq |c_{jj}| |c_{i_0 i_0}| - |c_{i_0 i_0}| \sum_{p \in N, p \neq i, j, i_0} |c_{jp}| - |c_{j i_0}| \sum_{p \in N, p \neq i, i_0} |c_{i_0 p}| \\
& = |c_{jj}| |c_{i_0 i_0}| - |c_{i_0 i_0}| \sum_{p \in N-\{i_0\}, p \neq i, j} |c_{jp}| - |c_{j i_0}| \sum_{p \in N-\{i_0\}, p \neq i} |c_{i_0 p}|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \max_{\substack{j \in N-\{i_0\}-\{i\} \\ k=i_0}} \frac{|c_{i_0 i_0}| + |c_{j i_0}|}{|c_{jj}| |c_{i_0 i_0}| - |c_{i_0 i_0}| \sum_{p \in N-\{i_0\}, p \neq j, i} |c_{jp}| - |c_{j i_0}| \sum_{p \in N-\{i_0\}, p \neq i} |c_{i_0 p}|} \\
& \leq \max_{\substack{j \in N-\{i_0\}-\{i\} \\ k=i_0}} \frac{|c_{i_0 i_0}| + \sum_{p \in N, p \neq i, j} |c_{jp}|}{|c_{jj}| |c_{i_0 i_0}| - \sum_{p \in N, p \neq i, j} |c_{jp}| \sum_{p \in N, p \neq i, i_0} |c_{i_0 p}|}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \max_{\substack{j \in N-\{i_0\}-\{i\} \\ k=i_0}} \frac{|c_{jj}| - \sum_{p \in N-\{i_0\}, p \neq j, i} |c_{jp}| + \sum_{p \in N-\{i_0\}, p \neq i} |c_{i_0 p}|}{(|c_{jj}| - \sum_{p \in N-\{i_0\}, p \neq j, i} |c_{jp}|) |c_{i_0 i_0}| - |c_{j i_0}| \sum_{p \in N-\{i_0\}, p \neq i} |c_{i_0 p}|} \\
& \leq \max_{\substack{j=i_0 \\ k \in N-\{i_0\}-\{i\}}} \frac{|c_{kk}| + \sum_{p \in N, p \neq i, i_0} |c_{i_0 p}|}{|c_{i_0 i_0}| |c_{kk}| - \sum_{p \in N, p \neq i, i_0} |c_{i_0 p}| \sum_{p \in N, p \neq i, k} |c_{kp}|}.
\end{aligned}$$

Then,

$$\begin{aligned}
& \max \left\{ \max_{\substack{j \in N-\{i_0\}-\{i\} \\ k=i_0}} \frac{|c_{i_0 i_0}| + |c_{j i_0}|}{(|c_{jj}| - \sum_{p \in N-\{i_0\}, p \neq j, i} |c_{jp}|) |c_{i_0 i_0}| - \sum_{p \in N-\{i_0\}, p \neq i} |c_{i_0 p}| |c_{j i_0}|}, \right. \\
& \left. \max_{\substack{j \in N-\{i_0\}-\{i\} \\ k=i_0}} \frac{|c_{jj}| - \sum_{p \in N-\{i_0\}, p \neq j, i} |c_{jp}| + \sum_{p \in N-\{i_0\}, p \neq i} |c_{i_0 p}|}{(|c_{jj}| - \sum_{p \in N-\{i_0\}, p \neq j, i} |c_{jp}|) |c_{i_0 i_0}| - \sum_{p \in N-\{i_0\}, p \neq i} |c_{i_0 p}| |c_{j i_0}|} \right\} \\
& \leq \max \left\{ \max_{\substack{j \in N-\{i_0\}-\{i\} \\ k=i_0}} \frac{|c_{i_0 i_0}| + \sum_{p \in N, p \neq i, j} |c_{jp}|}{(|c_{jj}| |c_{i_0 i_0}| - \sum_{p \in N, p \neq i, j} |c_{jp}| \sum_{p \in N, p \neq i, i_0} |c_{i_0 p}|)}, \right.
\end{aligned}$$

$$\left. \max_{\substack{j=i_0 \\ k \in N - \{i_0\} - \{i\}}} \frac{|c_{kk}| + \sum_{p \in N, p \neq i, i_0} |c_{i_0 p}|}{|c_{i_0 i_0}| |c_{kk}| - \sum_{p \in N, p \neq i, i_0} |c_{i_0 p}| \sum_{p \in N, p \neq i, k} |c_{kp}|} \right\}$$

$$\leq \max_{\substack{j, k \in N - \{i\} \\ j \neq k}} \frac{|c_{kk}| + \sum_{p \in N, p \neq i, j} |c_{jp}|}{|c_{jj}| |c_{kk}| - \sum_{p \in N, p \neq i, j} |c_{jp}| \sum_{p \in N, p \neq i, k} |c_{kp}|}.$$

Therefore,

$$\tilde{\Delta}_i(A) \leq \tilde{\pi}_i(A).$$

For $i = i_0$,

$$\tilde{\Delta}_{i_0}(A) = \frac{1}{|a_{i_0 i_0}|},$$

$$\tilde{\pi}_{i_0}(A) = \max \left\{ \frac{1}{|a_{i_0 i_0}|}, \max_{\substack{j, k \in N - \{i_0\} \\ j \neq k}} \frac{|c_{kk}| + \sum_{p \in N, p \neq i_0, j} |c_{jp}|}{|c_{jj}| |c_{kk}| - \sum_{p \in N, p \neq i_0, j} |c_{jp}| \sum_{p \in N, p \neq i_0, k} |c_{kp}|} \right\},$$

$$\tilde{\Delta}_{i_0}(A) \leq \tilde{\pi}_{i_0}(A).$$

Furthermore,

$$\tilde{\Delta}_i(A) \leq \tilde{\pi}_i(A), \quad i \in N,$$

$$\Delta_i(A) \leq \Pi_i(A), \quad i \in N,$$

$$\Delta(A) \leq \Pi(A).$$

The proof is completed. $\square$

Theorem 11 shows that the bound in Theorem 9 generally improves the bound obtained by Theorem 9 in [12] for DSDD matrices.

We illustrate our results by the following two examples.

Example 1. Consider the bound for $\|A^{-1}\|_\infty$ of a SDD matrix A , where:

$$A = \begin{bmatrix} 14 & 4 & 4 & 5 \\ 1 & 19 & 4 & 5 \\ 2 & 3 & 13 & 5 \\ 4 & 5 & 4 & 14 \end{bmatrix}.$$

By Theorem 1, we have:

$$\|A^{-1}\|_\infty \leq 1.$$

By Theorem 2, we have:

$$\|A^{-1}\|_\infty \leq 1.$$

By Theorem 3, we have:

$$\|A^{-1}\|_{\infty} \leq 1.$$

By Theorem 3 in [11], we have:

$$\|A^{-1}\|_{\infty} \leq 0.8727.$$

By Theorem 9 in [12], we have:

$$\|A^{-1}\|_{\infty} \leq 0.2200.$$

By Theorem 9, we have:

$$\|A^{-1}\|_{\infty} \leq 0.2141.$$

In fact, $\|A^{-1}\|_{\infty} = 0.1464$. This example shows that our bound is better than those in Theorems 1–3 and Theorem 3 in [11], Theorem 9.

Example 2. Consider the bound for $\|A^{-1}\|_{\infty}$ of a GDSDD matrix A for $N_1 = \{1, 2\}$, $N_2 = \{3, 4\}$,

$$A = \begin{bmatrix} 12 & 5 & 1 & 1 \\ 4 & 13 & 3 & 1 \\ 2 & 0 & 3 & 2 \\ 3 & 2 & 1 & 6 \end{bmatrix},$$

$$A^{-1} = \begin{bmatrix} 0.0971 & -0.0354 & 0.0073 & -0.0127 \\ -0.0177 & 0.0812 & -0.0808 & 0.0163 \\ -0.0408 & 0.0336 & 0.3521 & -0.1162 \\ -0.0358 & -0.0150 & -0.0354 & 0.1869 \end{bmatrix}.$$

Note that:

$$\|A^{-1}\|_{\infty} = 0.5426.$$

We know that A is neither a SDD matrix nor a DSDD matrix, and the bound of $\|A^{-1}\|_{\infty}$ can not be obtained by Theorems 1 and 2, Theorem 3 in [11], or Theorem 9 in [12], but it can be estimated by Theorems 3 and 9 in this paper.

By Theorem 3, we have:

$$\|A^{-1}\|_{\infty} \leq 11.$$

The bound for $\|A^{-1}\|_{\infty}$ can be estimated by Theorem 9 as follows:

$$\|A^{-1}\|_{\infty} \leq 1.0986.$$

3. Applications to the Smallest Singular Value for GDSDD Matrices

The singular values of an $n \times m$ complex matrix A are the eigenvalues of $\sqrt{AA^*}$ and are denoted:

$$\sigma_1(A) \geq \sigma_2(A) \geq \dots \geq \sigma_n(A) \geq 0.$$

The smallest singular value plays a special role in expressing properties of matrices. It indicates not only whether A is nonsingular, but also how far from the singular matrices

A is. In addition, the spectral condition number $\sigma_1(A)/\sigma_n(A)$ is important in studying numerical calculations involving A [6]. Hence, lower bounds for the smallest singular value of matrices are of interest.

For SDD matrices case, Varah in [10] obtained a lower bound for $\sigma_n(A)$ which is listed as follows.

Theorem 12 ([10]). *If a matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ and its transpose A^T are all SDD , then:*

$$\sigma_n(A) \geq \sqrt{\min_{i \in N} \{|a_{ii}| - r_i(A)\} \cdot \min_{i \in N} \{|a_{ii}| - r_i(A^T)\}}.$$

Besides the lower bound in Theorem 12, there are other existing lower bounds for $\sigma_n(A)$, see [5,6,11] and references therein. Based on Theorem 9, we can get a new lower bound for $\sigma_n(A)$.

Theorem 13. *If a matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ and its transpose A^T are all $GDSDD$, then:*

$$\sigma_n(A) \geq \sqrt{\frac{1}{\Delta(A) \cdot \Delta(A^T)}},$$

where $\Delta(A)$ is defined as in Theorem 9.

Proof. Since A^T is $GDSDD$, then:

$$\|A^{-1}\|_1 = \|(A^{-1})^T\|_\infty = \|(A^T)^{-1}\|_\infty \leq \Delta(A^T).$$

By the well-known inequality (see Theorem 2 in [5]):

$$\|A^{-1}\|_2^2 \leq \|A^{-1}\|_1 \|A^{-1}\|_\infty,$$

we have:

$$\|A^{-1}\|_2^2 \leq \Delta(A^T) \cdot \Delta(A),$$

and hence:

$$\|A^{-1}\|_2^{-1} \geq \sqrt{\frac{1}{\Delta(A) \cdot \Delta(A^T)}}.$$

By $\|A^{-1}\|_2^{-1} = \sigma_n(A)$, the conclusion follows. $\square$

4. Conclusions

Based on the fact that the Schur complement of a $GDSDD$ matrix is also a $GDSDD$ matrix, we in Theorems 4–8 presented an upper bound of $\|A^{-1}\|_\infty$ for a $GDSDD$ matrix A for any nonempty proper subset α of N . And then, for the case $|\alpha| = 1$, we in Theorem 9 gave an upper bound for $\|A^{-1}\|_\infty$. Moreover, we show that the bound in Theorem 9 generally improves the bound obtained by Theorem 3 in [11] for SDD matrices and Theorem 9 in [12] for $DSDD$ matrices.

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