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A Generalized Approach of the Gilpin–Ayala Model with Fractional Derivatives under Numerical Simulation

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Abstract: In this article, we study the existence and uniqueness of multiple positive periodic solutions for a Gilpin–Ayala predator-prey model under consideration by applying asymptotically periodic functions. The result of this paper is completely new. By using Comparison Theorem and some technical analysis, we showed that the classical nonlinear fractional model is bounded. The Banach contraction mapping principle was used to prove that the model has a unique positive asymptotical periodic solution. We provide an example and numerical simulation to inspect the correctness and availability of our essential outcomes.

Keywords: asymptotically; periodic functions; Gilpin–Ayala prey-predator

MSC: 34D05; 34N05

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1. Introduction of the Model

In 1973, Ayala et al. conducted tests on natural product fly flow to test the legitimacy of ten models of competition [1]. The Gilpin–Ayala biological system is one of the foremost vital organic numerical models. One of the model's best ways of bookkeeping for exploration is given by [1]

$$\begin{cases} U'(\mathfrak{s}) = \zeta_1(\mathfrak{s})U(\mathfrak{s})\left[1-\left(\dfrac{U(\mathfrak{s})}{\omega_1(\mathfrak{s})}\right)^{\nu_1} - c_1(\mathfrak{s})\dfrac{V(\mathfrak{s})}{\omega_2(\mathfrak{s})}\right], \\ V'(\mathfrak{s}) = \zeta_2(\mathfrak{s})U(\mathfrak{s})\left[1-\left(\dfrac{V(\mathfrak{s})}{\omega_2(\mathfrak{s})}\right)^{\nu_2} - c_2(\mathfrak{s})\dfrac{U(\mathfrak{s})}{\omega_1(\mathfrak{s})}\right]. \end{cases} \quad (1)$$

As we all know, many competitive systems, including ecosystems, economic systems and some social systems, can be described by the Lotka–Volterra model. When $\nu_1 = \nu_2 = 1$, system (1) changes into the Lotka–Volterra competitive model. In the past decade, many generalizations and modifications to system (1) have been proposed and studied [2–10].

More so, many authors have taken into account several important factors in the Lotka–Volterra predator-prey model in order to get a more realistic model. In [11], the authors introduced a complex model presented by

$$\begin{cases} U_i'(\mathfrak{s}) = U_i(\mathfrak{s})\left[b_i(\mathfrak{s}) - \sum_{j=1}^{n} a_{ij}(\mathfrak{s})U_i^{\alpha_{ij}}(\mathfrak{s})\right], & i = 1,\ldots,n \\ U_i'(\mathfrak{s}) = U_i(\mathfrak{s})\left[b_i(\mathfrak{s}) + \sum_{j=1}^{n} a_{ij}(\mathfrak{s})U_i^{\alpha_{ij}}(\mathfrak{s}) - \sum_{j=m+1}^{n} a_{ij}(\mathfrak{s})U_i^{\alpha_{ij}}(\mathfrak{s})\right], & i = m+1,\ldots,n \end{cases} \tag{2}$$

where U_i is the size of the i-th prey population, and for $i = m+1,\ldots,n$, U_i is the size of the i-th predator population and $\alpha_{ij} > 0$, i, $j = 1,\ldots,n$, are the parameters that modify the classical Lotka–Volterra model and they represent a nonlinear measure of interspecific interference. Liao et al. investigated the two-species Gilpin–Ayala competition predator-prey system using the harvesting terms as follows:

$$\begin{cases} U'(\mathfrak{s}) = \zeta_1(\mathfrak{s})U(\mathfrak{s})\left[1 - \left(\dfrac{U(\mathfrak{s})}{\omega_1(\mathfrak{s})}\right)^{\nu_1} - c_1(\mathfrak{s})\dfrac{V(\mathfrak{s})}{\omega_2(\mathfrak{s})}\right] - \check{\zeta}_1, \\ V'(\mathfrak{s}) = \zeta_2(\mathfrak{s})U(\mathfrak{s})\left[1 - \left(\dfrac{V(\mathfrak{s})}{\omega_2(\mathfrak{s})}\right)^{\nu_2} - c_2(\mathfrak{s})\dfrac{U(\mathfrak{s})}{\omega_1(\mathfrak{s})}\right] - \check{\zeta}_2, \end{cases} \tag{3}$$

where $\zeta_i(\mathfrak{s}) > 0$, $\omega_i(\mathfrak{s}) > 0$, $\check{\zeta}_i > 0$, the functions $c_i \in C([0,\infty),(0,\infty))$ are λ-periodic functions, ν_i are positive constants for $i = 1,2$ and U, and V represents the number of individuals in the prey and predator population [2]. On the other hand, in model (3), the interaction between populations is assumed to be instantaneous, whereas in reality, this interaction always has a delay time due, for example, to the time of maturation or the gestation time of the population, for this, several authors have observed that it is more natural to assume that the growth rate also depends on the past, which can result from a variety of causes, such as the hatching period, the slowness of food replacement, or the profit of the stock of food, which takes us to a functional differential equation with delay or distributed delay [12–15]. Amdouni et al. considered the following Gilpin–Ayala competitive system with delays, distributed delay, feedback control, and the effect of a toxic substance, which is given by the following model

$$\begin{cases} U_i'(\mathfrak{s}) = U(\mathfrak{s})\Bigg[\zeta_i(\mathfrak{s}) - \sum_{j=1}^{m} a_{ij}(\mathfrak{s})\zeta_j^{\nu_{ij}}(\mathfrak{s}) - \sum_{j=1}^{m} b_{ij}(\mathfrak{s})\zeta_j^{\sigma_{ij}}(\mathfrak{s} - \tau_{ij}(\mathfrak{s})) - h_i(\mathfrak{s})d_i(\mathfrak{s}) \\ \qquad - \sum_{\substack{j=1 \\ j\neq i}}^{m} c_{ij}(\mathfrak{s})\zeta_j^{\nu_{ii}}(\mathfrak{s})\zeta_j^{\nu_{ij}}(\mathfrak{s}) - \sum_{j=1}^{m}\int_{-q_{ij}}^{0} g_{ij}(\mathfrak{s},\eta)\zeta_j^{\omega_{ii}}(\mathfrak{s}+\eta)\,\mathrm{d}\eta\Bigg], \\ d_i'(\mathfrak{s}) = l_i(\mathfrak{s}) - f_i(\mathfrak{s})d_i(\mathfrak{s}) + k_i(\mathfrak{s})\zeta_i^{\nu_{ii}}(\mathfrak{s}), \end{cases} \tag{4}$$

where ν_{ij}, σ_{ij}, and ω_{ij} are positive constants, $g_{ij}(\mathfrak{s},\eta)$ are nonnegative, pseudo almost periodic functions with respect to $\mathfrak{s}$ uniformly in $\eta \in [-q_{ij},0]$, and d_i, ζ_i, h_i, l_i, k_i, f_i are all nonnegative pseudo almost periodic functions defined in $\mathbb{R}$. More so, in recent years, many authors have used fractional theory for modeling many phenomena, such as physics, biology, ecology, etc. [16–20]. The authors of [20] reviewed the basic ideas of fractional differential equations and their applications to nonlinear biochemical reaction models and applied the idea to a nonlinear model of enzyme inhibitor reactions with a suggested method that provides a good step forward in understanding the model dynamics in complex enzymatic reactions. Nikan et al., in [21], focused on an efficient meshless numerical method for seeking accurate solutions to the nonlinear time-fractional fourth-order diffusion problem:

$$\frac{\partial U(\mathfrak{s},\tau)}{\partial \tau} - \frac{\partial^{\beta}\Delta U(\mathfrak{s},\tau)}{\partial \tau^{\beta}} - \delta U(\mathfrak{s},\tau) + \Delta^2 U(\mathfrak{s},\tau) = g(\mathfrak{s},\tau) + G(U), \tag{5}$$

for $\mathfrak{s} \in \Omega \subset \mathbb{R}^2$ and $0 < \tau \leq \top$, under initial and boundary conditions $U(\mathfrak{s}, 0) = \hbar(\mathfrak{s})$ for $\mathfrak{s} \in \overline{\Omega}$ and $U(\mathfrak{s}, \tau) = \Delta U(\mathfrak{s}, \tau) = 0$, $\mathfrak{s} \in \partial\Omega$ for $\tau > 0$, where $0 < \beta < 1$, $\mathfrak{s}(t_1, t_2)$ stands for the space variable, $\partial\Omega$ is the closed curve bounding the region, $\overline{\Omega} = \Omega \cup \partial\Omega$ represents the space domain, $g(\mathfrak{s}, \tau)$ is the forcing term with sufficient smoothness, and $\hbar(\mathfrak{s})$ is a given continuous function. The symbols Δ and Δ^2 denote the Laplacian and double Laplacian operators corresponding to the space directions, respectively. An improved asymptotic expansion approximation was constructed, and the asymptotic expansion was approximated numerically using the Runge–Kutta methods and hybrid finite difference methods in [22].

In addition, the fractional calculus yields an excellent description of the interactions and changes in ecosystems. Furthermore, the fractional derivative is not a neighborhood of the initial state but of the past state. For this reason, this theory allows us to describe a real object more than any other theory. Motivated by the above, in this paper, we consider the fractional prey-predator Gilpin–Ayala model given by

$$\begin{cases} {}^{C}\mathcal{D}_{\mathfrak{s}}^{\gamma} U(\mathfrak{s}) = U(\mathfrak{s})\left[\zeta_1(\mathfrak{s}) - c_{11}(\mathfrak{s})U^{\alpha_1}(\mathfrak{s}) - c_{12}(\mathfrak{s})\int_{-q}^{0} \grave{\zeta}_1(\eta)V(\mathfrak{s}+\eta)\,\mathrm{d}\eta - \varphi_1(\mathfrak{s})\right], \\ {}^{C}\mathcal{D}_{\mathfrak{s}}^{\gamma} V(\mathfrak{s}) = V(\mathfrak{s})\left[\zeta_2(\mathfrak{s}) - c_{22}(\mathfrak{s})V^{\alpha_2}(\mathfrak{s}) + c_{12}(\mathfrak{s})\int_{-q}^{0} \grave{\zeta}_2(\eta)U(\mathfrak{s}+\eta)\,\mathrm{d}\eta - \varphi_2(\mathfrak{s})\right], \end{cases} \tag{6}$$

for $\mathfrak{s} \geq 0$. The initial conditions associated with system (6) are of the form:

$$U(\mathfrak{s}) = \phi_1(\mathfrak{s}), \quad V(\mathfrak{s}) = \phi_2(\mathfrak{s}), \quad \mathfrak{s} \in \Lambda =: [-q, 0], \tag{7}$$

where ${}^{C}\mathcal{D}_{\mathfrak{s}}^{\gamma}$ denotes the Caputo fractional derivative of order $0 < \gamma < 1$, $\phi_i \in C_b(\Lambda)$; that is,

$$C_b(\Lambda) = \left\{\phi \in C(\Lambda) \,:\, \phi \text{ is bounded}\right\},$$

$\phi_i(\mathfrak{s}) \geq 0$ and $\alpha_i > 0$ for $\mathfrak{s} \in \Lambda$, $\phi_i(0) > 0$, $\sup_{\mathfrak{s}\in\Lambda} \phi_i(\mathfrak{s}) < \infty$, for $i = 1, 2$, and ζ_1, ζ_2, c_{11}, c_{22}, c_{12}, φ_1, φ_2 are all nonnegative S-asymptotically λ-periodic functions with the declaration in Table 1.

Table 1. The declaration of the symbols in system (6).

Symbols	Declaration
$\mathfrak{s}$	Time variable
$U(\mathfrak{s})$	Prey population density
$V(\mathfrak{s})$	Predator population density
$\zeta_1(\mathfrak{s}), \zeta_2(\mathfrak{s})$	Natural growth rates
$c_{11}(\mathfrak{s}), c_{22}(\mathfrak{s})$	Intraspecific competition rates
$c_{12}(\mathfrak{s})$	Predation rates
$\grave{\zeta}_1(\mathfrak{s}), \grave{\zeta}_2(\mathfrak{s})$	Kernel functions with innite distributed delay
$\varphi_1(\mathfrak{s}), \varphi_2(\mathfrak{s})$	Manual control functions

The solution of System (6) with the initial values is equivalent to the following Volterra integral equation

$$\begin{aligned} U(\mathfrak{s}) &= \phi_1(0) + \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} \mathfrak{F}_1(\eta, U(\eta), V(\eta))\,\mathrm{d}\eta, \\ V(\mathfrak{s}) &= \phi_2(0) + \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} \mathfrak{F}_2(\eta, U(\eta), V(\eta))\,\mathrm{d}\eta, \end{aligned}$$

where

$$\begin{aligned}\mathfrak{F}_1(\mathfrak{s}, U(\mathfrak{s}), V(\mathfrak{s})) &= U(\mathfrak{s})\Big[\zeta_1(\mathfrak{s}) - c_{11}(\mathfrak{s})U^{\alpha_1}(\mathfrak{s}) \\ &\qquad - c_{12}(\mathfrak{s})\int_{-q}^{0} \grave{\zeta}_1(\eta)V(\mathfrak{s}+\eta)\mathrm{d}\eta - \varphi_1(\mathfrak{s})\Big], \\ \mathfrak{F}_2(\mathfrak{s}, U(\mathfrak{s}), V(\mathfrak{s})) &= V(\mathfrak{s})\Big[\zeta_2(\mathfrak{s}) - c_{22}(\mathfrak{s})V^{\alpha_2}(\mathfrak{s}) \\ &\qquad + c_{12}(\mathfrak{s})\int_{-q}^{0} \grave{\zeta}_2(\eta)U(\mathfrak{s}+\eta)\mathrm{d}\eta - \varphi_2(\mathfrak{s})\Big]. \end{aligned} \tag{8}$$

Our model is more complicated and accurate since the theory of fractional calculus has received extensive attention. The importance of fractional calculus in our paper is to describe the interactions and changes in ecosystems.

The rest of the paper can be sketched out as: In Section 2, we mainly introduce the basic concepts, important and necessary propositions results of fractional calculus, and S-asymptotically λ-periodic with related assumptions. In Section 3.1, the positivity and boundedness solution of system (6) is obtained. Model (6) is studied with S-asymptotically λ-periodic functions, coefficient, distributed delay, and control terms, which extends the characterization of the ecological model. In addition, by Banach's fixed point theorem, the existence of an S-asymptotically λ-periodic fractional Gilpin–Ayala predator-prey model with distributed delay and a control term is obtained in Section 3.2. Further, we shall show that the unique solution is globally asymptotically stable in Section 3.3. Numerical examples and simulations are provided in Section 3.4. Finally, Section 4 provides a conclusion.

2. Preliminaries

We consider the space of all continuous and bounded functions $\phi : [\mathfrak{s}_\circ, \infty) \to \mathbb{R}^n$, which is denoted by $C_b([\mathfrak{s}_\circ, \infty), \mathbb{R}^n)$ with the norm uniform as $\|\phi\| = \sup_{\mathfrak{s}\geq\mathfrak{s}_\circ} |\phi(\mathfrak{s})|$. The space of r-order continuous and differentiable functions are presented by $C^r([\mathfrak{s}_\circ, \infty), \mathbb{R}^n)$. The fractional integral of order $\gamma > 0$ of a given function ϕ is defined by [23]

$$\mathcal{I}^{\gamma}_{\mathfrak{s}_\circ}\phi(\mathfrak{s}) = \int_{\mathfrak{s}_\circ}^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)}\,\phi(\eta)\,\mathrm{d}\eta, \quad \mathfrak{s} \geq \mathfrak{s}_\circ.$$

Using the definition, the fractional Riemann–Liouville derivative of order $0 < \gamma < 1$ of ϕ is defined as [23]

$${}^{RL}\mathcal{D}^{\gamma}_{\mathfrak{s}_\circ}\phi(\mathfrak{s}) = \mathcal{D}^1\left(\mathcal{I}^{1-\gamma}_{\mathfrak{s}_\circ}\phi\right)(\mathfrak{s}) = \frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\int_{\mathfrak{s}_\circ}^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{-\gamma}}{\Gamma(1-\gamma)}\,\phi(\eta)\,\mathrm{d}\eta.$$

The Caputo derivative with order $\gamma > 0$ of function $\phi(\mathfrak{s}) \in C^{m+1}([\mathfrak{s}_\circ, \infty))$ is defined as [23]

$${}^{C}\mathcal{D}^{\gamma}_{\mathfrak{s}_\circ}\phi(\mathfrak{s}) = \int_{\mathfrak{s}_\circ}^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{m-\alpha-1}}{\Gamma(m-\gamma)}\,\phi^{(m)}(\eta)\,\mathrm{d}\eta, \quad \mathfrak{s} \geq \mathfrak{s}_\circ,$$

where $m-1 < \gamma < m$, m is a positive integer number, and $\Gamma(\cdot)$ is the Euler's gamma function. The one-parameter and two-parameter Mittag–Leffler functions are defined as

$$E_{\gamma}(\mathfrak{z}) = \sum_{k=0}^{+\infty} \frac{\mathfrak{z}^k}{\Gamma(\gamma k+1)}, \quad E_{\gamma,\acute{\gamma}}(\mathfrak{z}) = \sum_{k=0}^{+\infty} \frac{\mathfrak{z}^k}{\Gamma(\gamma k+\acute{\gamma})},$$

where the real part $Re(\gamma)$ of complex number γ is $Re(\gamma) > 0$, $\mathfrak{z}$ and $\acute{\gamma}$ are also both complex numbers.

Definition 1 ([12]). *A function $\phi \in C_b(\mathbb{R}^+)$ is called S-asymptotically λ-periodic if there exists $\lambda > 0$ such that*

$$\lim_{\mathfrak{s} \to \infty} \Big| \phi(\mathfrak{s} + \lambda) - \phi(\mathfrak{s}) \Big| = 0.$$

In this case, we say that λ is an asymptotic period of ϕ.

We denote by $SAP_\lambda(\mathbb{R})$ the space of all S-asymptotically λ-periodic functions endowed with the following norm

$$\|\phi\|_\lambda = \sup_{\mathfrak{s} \geq 0} e^{-\mathfrak{s}} |\phi(\mathfrak{s})|, \quad \forall \phi \in SAP_\lambda(\mathbb{R}),$$

where $SAP_\lambda(\mathbb{R})$ is a Banach space [6].

Lemma 1 ([5]). *Let $\phi(\mathfrak{s}) \in C^1([0, \infty))$ and $0 < \gamma \leq 1$ then the following inequality holds true almost everywhere*

$${}^C\mathcal{D}_{\mathfrak{s}}^{\gamma} |\phi(\mathfrak{s})| \leq \mathrm{sgn}(\phi(\mathfrak{s})) {}^C\mathcal{D}_{\mathfrak{s}}^{\gamma} \phi(\mathfrak{s}).$$

Corollary 1 ([24]). *Let $\phi \in C([0, r])$, ${}^C\mathcal{D}_{\mathfrak{s}}^{\gamma} \phi(\mathfrak{s}) \in C([0, r])$ and $0 < \gamma < 1$. If*

$${}^C\mathcal{D}_{\mathfrak{s}}^{\gamma} \phi(\mathfrak{s}) \geq 0 \ \Big({}^C\mathcal{D}_{\mathfrak{s}}^{\gamma} \phi(\mathfrak{s}) \leq 0 \Big), \quad \forall \mathfrak{s} \in (0, r),$$

then $\phi(\mathfrak{s})$ is a non-decreasing function (non-increasing function).

We consider the following assumptions

(A1) The kernel satisfies

$$\grave{\zeta}_i(\mathfrak{s}) \leq e^{\mu_i \mathfrak{s}}, \qquad \forall \mathfrak{s} \in \Lambda, \ \mu_i > 0, \ i = 1, 2; \tag{9}$$

(A2)

$$\underline{\zeta}_1 \left[\bar{c}_{11} M^{\alpha_1} + \bar{c}_{12} M \frac{1 - e^{-q\mu_1}}{\mu_1} + \overline{\varphi}_1 \right]^{-1} < 1; \tag{10}$$

(A3)

$$\frac{\underline{\zeta}_2 + \bar{c}_{12} M q \grave{\underline{\zeta}}_2}{\bar{c}_{22} M^{\alpha_2} + \overline{\varphi}_2} < 1; \tag{11}$$

(A4)

$$\begin{aligned} &\left| \underline{\zeta}_1 - \underline{c}_{11}(1 + \alpha_1) M^{\alpha_1} - \underline{\varphi}_1 - \underline{c}_{12} M \left[\frac{1 - e^{-q\mu_1}}{\mu_1} + \frac{1 - e^{-(\mu_1 - 1)q}}{\mu_1 - 1} \right] \right| < 1, \\ &\left| \underline{\zeta}_2 - \underline{c}_{22}(1 + \alpha_2) M^{\alpha_2} - \underline{\varphi}_2 + \bar{c}_{12} M \left[\frac{1 - e^{-q\mu_2}}{\mu_2} + \frac{1 - e^{-(\mu_2 - 1)q}}{\mu_2 - 1} \right] \right| < 1, \end{aligned} \tag{12}$$

where $\underline{\zeta}_1 = \inf_{\mathfrak{s} \in \Lambda} \zeta_1$, $\bar{c}_{11} = \sup_{\mathfrak{s} \in \Lambda} c_{11}$, $\bar{c}_{12} = \sup_{\mathfrak{s} \in \Lambda} c_{12}$, $\underline{\zeta}_2 = \inf_{\mathfrak{s} \in \Lambda} \zeta_2$, $\bar{c}_{22} = \sup_{\mathfrak{s} \in \Lambda} c_{22}$, $\underline{\zeta}_1 = \inf_{\mathfrak{s} \in \Lambda} \zeta_1$, $\underline{c}_{11} = \inf_{\mathfrak{s} \in \Lambda} c_{11}$, $\underline{c}_{22} = \inf_{\mathfrak{s} \in \Lambda} c_{22}$, $\underline{c}_{12} = \inf_{\mathfrak{s} \in \Lambda} c_{12}$, $\overline{\varphi}_i = \sup_{\mathfrak{s} \in \Lambda} \varphi_i$, $\underline{\varphi}_i = \inf_{\mathfrak{s} \in \Lambda} \varphi_i$ for $i = 1, 2$ and $M = \max\{M_1, M_2\}$,

$$M_1 > \left(\frac{\bar{\zeta}_1}{\underline{c}_{11}} \right)^{1/\alpha_1}, \quad M_2 > \left(\frac{\bar{\zeta}_2}{\underline{c}_{22}} + M_1 \frac{\bar{c}_{12}(1 - e^{-q\mu_2})}{\mu_2 \underline{c}_{22}} \right)^{1/\alpha_2}. \tag{13}$$

3. Main Results

3.1. Positivity and Boundedness of the Solution

First, we state the following lemma.

Lemma 2. *System (6) with the initial conditions of (7) has a positive solution.*

Proof. Let $\mathcal{X}(\mathfrak{s}) = (U(\mathfrak{s}),\ V(\mathfrak{s}))$ be a solution of system (6). First, we show that $U(\mathfrak{s}) \geq 0,\ \forall\, \mathfrak{s} \geq 0$. Suppose that it is false, so we can find $\mathfrak{s}_1 > 0$ such that

$$\begin{cases} U(\mathfrak{s}) > 0, & \mathfrak{s} \in [0, \mathfrak{s}_1), \\ U(\mathfrak{s}) < 0, & \mathfrak{s} > \mathfrak{s}_1. \end{cases}$$

Under the first equation of system (6), we get

$${}^{C}\mathcal{D}_{\mathfrak{s}}^{\gamma} U(\mathfrak{s})\Big|_{\mathfrak{s}=\mathfrak{s}_1} = 0.$$

By Corollary 1, we obtain $U(\mathfrak{s}_1^+) = 0$, which contradicts the fact that $U(\mathfrak{s}_1^+) < 0$. Therefore, $U(\mathfrak{s}) \geq 0$ for all $\mathfrak{s} \geq 0$. Secondly, by the same way, we can obtain that $V(\mathfrak{s}) \geq 0$ for all $\mathfrak{s} \geq 0$. This completes the proof. □

Lemma 3. *Under* (A1)-(A3) *there exists* $\check{T} > 0$ *such that*

$$m < V(\mathfrak{s}),\ U(\mathfrak{s}) < M, \quad \forall\, \mathfrak{s} \geq \check{T}.$$

Proof. Let $\mathcal{X}(\mathfrak{s}) = (U(\mathfrak{s}), V(\mathfrak{s}))$ be a solution of system (6). From the first equation of system (6) we have

$${}^{C}\mathcal{D}_{\mathfrak{s}}^{\gamma} U(\mathfrak{s}) \leq U(\mathfrak{s})(\zeta_1(\mathfrak{s}) - c_{11}(\mathfrak{s}) U^{\alpha_1}(\mathfrak{s})).$$

Suppose that there exists $\check{T}_1 > 0$ such that for $\mathfrak{s} > \check{T}_1$,

$$U(\mathfrak{s}) < M_1, \tag{14}$$

or

$$U(\mathfrak{s}) > M_1. \tag{15}$$

If Inequality (14) holds, then $U(\mathfrak{s}) < M_1 < M$ for $\mathfrak{s} > \check{T}_1$ and if inequality (15) holds, then for $\mathfrak{s} \geq \check{T}_1$, we get

$$\begin{aligned} {}^{C}\mathcal{D}_{\mathfrak{s}}^{\gamma} U(\mathfrak{s}) \leq U(\mathfrak{s})(\zeta_1(\mathfrak{s}) - c_{11}(\mathfrak{s}) U^{\alpha_1}(\mathfrak{s})) \ &\leq\ -\underline{c}_{11} U(\mathfrak{s}) \left(U^{\alpha_1}(\mathfrak{s}) - \frac{\overline{\zeta}_1}{\underline{c}_{11}} \right) \\ &\leq\ -\underline{c}_{11} U(\mathfrak{s}) \left(M_1^{\alpha_1} - \frac{\overline{\zeta}_1}{\underline{c}_{11}} \right). \end{aligned}$$

Therefore, the comparison theorem (see [3]) gives

$$U(\mathfrak{s}) \leq U(0)\, E_{\gamma} \left(-\underline{c}_{11} \left(M_1^{\alpha_1} - \frac{\overline{\zeta}_1}{\underline{c}_{11}} \right) \mathfrak{s}^{\gamma} \right).$$

For $\mathfrak{s} \to \infty$, we obtain $U(\mathfrak{s}) \to 0$, which contradicts the fact that $U(\mathfrak{s}) > M_1$. Now, we turn our attention to $V(\mathfrak{s})$. Suppose that there exists $\check{T}_2 > \check{T}_1$ such that for $\mathfrak{s} > \check{T}_2$, we have

$$V(\mathfrak{s}) < M_2, \tag{16}$$

or

$$V(\mathfrak{s}) > M_2. \tag{17}$$

Then, $V(\mathfrak{s}) < M_2$ for $\mathfrak{s} > \check{T}_2 > \check{T}_1$ whenever inequality (16) holds, and then $V(\mathfrak{s}) > M_2$ for $\mathfrak{s} \geq \check{T}_2$ whenever inequality (17) holds. By the second equation of system (6) we have

$$
\begin{aligned}
{}^{C}\mathcal{D}_{\mathfrak{s}}^{\gamma} V(\mathfrak{s}) &\leq V(\mathfrak{s})\left(\zeta_2(\mathfrak{s}) - c_{22}(\mathfrak{s})V^{\alpha_2}(\mathfrak{s}) + c_{12}(\mathfrak{s})\int_{-q}^{0} \grave{\zeta}_2(\eta)U(\mathfrak{s}+\eta)\,\mathrm{d}\eta\right) \\
&\leq -\underline{c}_{22}V(\mathfrak{s})\left(V^{\alpha_2} - \frac{\overline{\zeta}_2}{\underline{c}_{22}} - \frac{\overline{c}_{12}}{\underline{c}_{22}}\int_{-q}^{0} e^{\mu_2\eta}\, U(\mathfrak{s}+\eta)\,\mathrm{d}\eta\right) \\
&\leq -\underline{c}_{22}V(\mathfrak{s})\left(M_2^{\alpha_2} - \frac{\overline{\zeta}_2}{\underline{c}_{22}} - M_1\frac{\overline{c}_{12}(1-e^{-q\mu_2})}{\mu_2\underline{c}_{22}}\right).
\end{aligned}
$$

The comparison theorem leads

$$
V(\mathfrak{s}) \leq V(0)\, E_{\gamma}\left(-\underline{c}_{22}\left(M_2^{\alpha_2} - \frac{\overline{\zeta}_1}{\underline{c}_{22}} - M_1\frac{\overline{c}_{12}(1-e^{-q\mu_2})}{\mu_2\underline{c}_{22}}\right)\mathfrak{s}^{\gamma}\right).
$$

Similarly, we obtain that $V(\mathfrak{s}) \to 0$ as $\mathfrak{s} \to \infty$, which contradicts inequality (17). Let

$$
M = \max\left\{M_1, M_2\right\},
$$

then $0 < U(\mathfrak{s}), V(\mathfrak{s}) \leq M$ for $\mathfrak{s} \geq \check{T}_2$. Next, we have to show that $U(\mathfrak{s}) \geq m_1$. The first equation of system (6) gives

$$
{}^{C}\mathcal{D}_{\mathfrak{s}}^{\gamma} U(\mathfrak{s}) \geq U(\mathfrak{s})\left(\zeta_1(\mathfrak{s}) - \left(c_{11}(\mathfrak{s})M^{\alpha_1} + c_{12}(\mathfrak{s})M\int_{-q}^{0} \grave{\zeta}_1(\eta)\,\mathrm{d}\eta + \varphi_1(\mathfrak{s})\right)\right).
$$

Let

$$
\begin{aligned}
{}^{C}\mathcal{D}_{\mathfrak{s}}^{\gamma}\mathfrak{H}(\mathfrak{s}) &= \mathfrak{H}(\mathfrak{s})\left(\underline{\zeta}_1 - \left(\overline{c}_{11}M^{\alpha_1} + \overline{c}_{12}M\frac{1-e^{-q\mu_1}}{\mu_1} + \overline{\varphi}_1\right)\right), \quad \mathfrak{s} \geq 0, \\
\mathfrak{H}(\mathfrak{s}) &= \varphi(\mathfrak{s}), \quad \mathfrak{s} \in \Lambda.
\end{aligned}
$$

By the fractional comparison principle (see [4]), we get $U(\mathfrak{s}) \geq \mathfrak{H}(\mathfrak{s})$. Now let us prove that $\mathfrak{H}(\mathfrak{s}) > m_1$. Suppose that there exists $\check{T}_3 > \check{T}_2$ such that for $\mathfrak{s} > \check{T}_3$, we have

$$
\mathfrak{H}(\mathfrak{s}) < m_1, \tag{18}
$$

or

$$
\mathfrak{H}(\mathfrak{s}) > m_1. \tag{19}
$$

If inequality (19) holds, then $\mathfrak{H}(\mathfrak{s}) > m_2$ for $\mathfrak{s} \geq \check{T}_3$. If (18) holds, then for $\mathfrak{s} \geq \check{T}_3$,

$$
{}^{C}\mathcal{D}_{\mathfrak{s}}^{\gamma}\mathfrak{H}(\mathfrak{s}) = \mathfrak{H}(\mathfrak{s})\left(\overline{c}_{11}M^{\alpha_1} + \overline{c}_{12}M\frac{1-e^{-q\mu_1}}{\mu_1} + \overline{\varphi}_1\right)\left(\frac{\underline{\zeta}_1}{\overline{c}_{11}M^{\beta_1} + \overline{c}_{12}M\frac{1-e^{-q\mu_1}}{\mu_1} + \overline{\varphi}_1} - 1\right).
$$

Therefore,

$$
\begin{aligned}
\mathfrak{H}(\mathfrak{s}) &= \varphi(0)\, E\left(\left(\overline{c}_{11}M^{\alpha_1} + \overline{c}_{12}M\frac{1-e^{-q\mu_1}}{\mu_1} + \overline{\varphi}_1\right)\right. \\
&\qquad \left.\times\left(\frac{\underline{\zeta}_1}{\overline{c}_{11}M^{\alpha_1} + \overline{c}_{12}M\frac{1-e^{-q\mu_1}}{\mu_1} + \overline{\varphi}_1} - 1\right)\mathfrak{s}^{\gamma}\right), \quad \mathfrak{s} \geq 0.
\end{aligned}
$$

By (A2), $\mathfrak{H}(\mathfrak{s}) \to \infty$ as $\mathfrak{s} \to \infty$, which contradicts (18). Consequently, there exists $\check{T}_4 > 0$ such that for $\mathfrak{s} \geq \check{T}_4$,

$$
0 < m_1 < U(\mathfrak{s}) < M.
$$

Now, we have to prove that $V(\mathfrak{s}) > m_2$ for $\mathfrak{s} \geq \check{T}_4$. Suppose that there exists $\check{T}_5 > \check{T}_4$ such that for $\mathfrak{s} > \check{T}_5$, we have

$$V(\mathfrak{s}) < m_2, \tag{20}$$

or

$$V(\mathfrak{s}) > m_2. \tag{21}$$

If (21) holds, then $V(\mathfrak{s}) > m_2$ for $\mathfrak{s} \geq \check{T}_5$. If (20) holds, then for $\mathfrak{s} \geq \check{T}_5$,

$$^C\mathcal{D}_{\mathfrak{s}}^{\gamma} V(\mathfrak{s}) \geq V(\mathfrak{s})(\bar{c}_{22}M^{\alpha_2} + \overline{\varphi}_2)\left(\frac{\underline{\zeta}_2 + \bar{c}_{12}Mq\check{\underline{\zeta}}_2(\eta)}{\bar{c}_{22}M^{\alpha_2} + \overline{\varphi}_2} - 1\right).$$

Therefore,

$$V(\mathfrak{s}) \geq \varphi_2(0)\, E\left([\bar{c}_{22}M^{\alpha_2} + \overline{\varphi}_2]\left(\frac{\underline{\zeta}_2 + \bar{c}_{12}Mq_{\cdot 2}(\eta)}{\bar{c}_{22}M^{\alpha_2} + \overline{\varphi}_2} - 1\right)\mathfrak{s}^{\gamma}\right),$$

for $\mathfrak{s} \geq 0$. For $\mathfrak{s} \to \infty$, $V(\mathfrak{s}) \to \infty$, which contradicts the fact that $V(\mathfrak{s}) < m_2$. Therefore, there exists $\check{T} > \check{T}_5$ such that $m < V(\mathfrak{s}), U(\mathfrak{s}) < M$ where $m = \min\{m_1,\ m_2\}$. □

3.2. Existence and Uniqueness of the Solution

Lemma 4. *Let $\mathfrak{H} \in C_b([0,\infty),\mathbb{R})$, $0 < \gamma < 1$. Then*

$$\mathfrak{s} \to J(\mathfrak{s}) := \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{H}(\eta)\,\mathrm{d}\eta,$$

is bounded.

Proof. To prove the boundedness of J, we have to prove the boundedness of the intervals

$$[0,\rho],\ [\rho,2\rho],\ \ldots,\ [n\rho,(n+1)\rho], \quad n \in \mathbb{N}.$$

Let us prove the boundedness of J in $[0,\rho]$. For $\mathfrak{s} \in [0,\rho]$, we have

$$|J(\mathfrak{s})| = \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)}\,|\mathfrak{H}(\eta)|\,\mathrm{d}\eta \leq \rho^{\gamma}\,\frac{\|\mathfrak{H}\|}{\Gamma(\gamma)}.$$

Therefore,

$$\sup_{\mathfrak{s}\in[0,\rho]} |J(\mathfrak{s})| \leq \rho^{\gamma}\frac{\|\mathfrak{H}\|}{\Gamma(\gamma)}.$$

Hence, J is bounded on $[0,\rho]$. Similarly, using the same method in $[\rho,2\rho],\ldots,[n\rho,(n+1)\rho]$, $n \in \mathbb{N}$, we prove that J is bounded. □

Theorem 1. *Under* (A1)-(A4)*. System* (6) *has a unique asymptotically λ-periodic solution.*

Proof. Let $\Xi : SAP_{\lambda}(\mathbb{R}) \to C_b([0,\infty),\mathbb{R})$ be the operator defined by

$$(\Xi X)(\mathfrak{s}) = \begin{pmatrix}(\Xi U)(\mathfrak{s})\\ (\Xi V)(\mathfrak{s})\end{pmatrix} = \begin{pmatrix}\phi_1(0) + \displaystyle\int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)}\,\mathfrak{F}_1(\eta,U(\eta),V(\eta))\,\mathrm{d}\eta \\ \phi_2(0) + \displaystyle\int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{F}_2(\eta,U(\eta),V(\eta))\,\mathrm{d}\eta\end{pmatrix},$$

where $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are defined by (8). Next, we prove the above theorem in two steps.

Step 1: Now, we show that $(\Xi\mathcal{X})(.) \in SAP_\lambda(\mathbb{R})$.
Let $\mathcal{X} = (U, V) \in SAP_\lambda(\mathbb{R})$,

$$\begin{aligned}
&\big(\Xi\mathcal{X}\big)(\mathfrak{s}+\lambda) - \big(\Xi\mathcal{X}\big)(\mathfrak{s})\\
= & \begin{pmatrix} (\Xi U)(\mathfrak{s}+\lambda) - (\Xi U)(\mathfrak{s}) \\ (\Xi V)(\mathfrak{s}+\lambda) - (\Xi V)(\mathfrak{s}) \end{pmatrix}\\
= & \begin{pmatrix} \displaystyle\int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} \mathfrak{F}_1(\eta+\lambda, U(\eta+\lambda), V(\eta+\lambda)) - \mathfrak{F}_1(\eta, U(\eta), V(\eta))\, \mathrm{d}\eta \\ \displaystyle\int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-s)^{\gamma-1}}{\Gamma(\gamma)} \mathfrak{F}_2(\eta+\lambda, U(\eta+\lambda), V(\eta+\lambda)) - \mathfrak{F}_2(\eta, U(\eta), V(\eta))\, \mathrm{d}\eta \end{pmatrix}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathfrak{F}_1\big(\eta+\lambda, U(\eta+\lambda), V(\eta+\lambda)\big) - \mathfrak{F}_1(\eta, U(\eta), V(\eta))\\
= \;& U(\eta+\lambda)\Big[\zeta_1(\eta+\lambda) - c_{11}(\eta+\lambda)U^{\alpha_1}(\eta+\lambda)\\
& -c_{12}(\eta+\lambda)\int_{-q}^0 \check{\zeta}_1(u)V(\eta+\lambda+u)\,\mathrm{d}u - \varphi_1(\eta+\lambda)\Big]\\
& -U(\eta)\Big[\zeta_1(\eta) - c_{11}(\eta)U^{\alpha_1}(\eta) - c_{12}(\eta)\int_{-q}^0 \check{\zeta}_1(u)V(\eta+u)\,\mathrm{d}u - \varphi_1(\eta)\Big]\\
= \;& \zeta_1(\eta+\lambda)[U(\eta+\lambda) - U(\eta)] + U(\eta)[\zeta_1(\eta+\lambda) - \zeta_1(\eta)]\\
& +U(\eta)[\varphi_1(\eta) - \varphi_1(\eta+\lambda)] + \varphi_1(\eta+\lambda)[U(\eta) - U(\eta+\lambda)]\\
& +U^{1+\alpha_1}(\eta+\lambda)[c_{11}(\eta+\lambda) - c_{11}(\eta)] + c_{11}(\eta)\Big[U^{1+\alpha_1}(\eta+\lambda) - U^{1+\alpha_1}(\eta)\Big]\\
& +\int_{-q}^0 \check{\zeta}_1(u)\Big[V(\eta+u)c_{12}(\eta)[U(\eta) - U(\eta+\lambda)]\\
& +U(\eta+\lambda)c_{12}(\eta+\lambda)\big[V(\eta+u) - V(\eta+\lambda+u)\big]\\
& +V(\eta+u)U(\eta+\lambda)\big[c_{12}(\eta) - c_{12}(\eta+\lambda)\big]\Big]\,\mathrm{d}u,
\end{aligned}$$

and

$$\begin{aligned}
&\mathfrak{F}_2(\eta+\lambda, U(\eta+\lambda), V(\eta+\lambda)) - \mathfrak{F}_2(\eta, U(\eta), V(\eta))\\
= \;& V(\eta+\lambda)\Big[\zeta_2(\eta+\lambda) - c_{22}(\eta+\lambda)V^{\alpha_2}(\eta+\lambda)\\
& +c_{12}(\eta+\lambda)\int_{-q}^0 \check{\zeta}_2(u)U(\eta+u)\,\mathrm{d}u - \varphi_2(\eta)\Big]\\
& -V(\eta)\Big[\zeta_2(\eta) - c_{22}(\eta)V^{\alpha_2}(\eta) + c_{12}(\eta)\int_{-q}^0 \check{\zeta}_2(u)U(\eta+u)\,\mathrm{d}u - \varphi_2(\eta)\Big]\\
= \;& \check{\zeta}_2(\eta+\lambda)[V(\eta+\lambda) - V(\eta)] + V(\eta)[\check{\zeta}_2(\eta+\lambda) - \check{\zeta}_2(\eta)]\\
& +V(\eta)[\varphi_2(\eta) - \varphi_2(\eta+\lambda)] + \varphi_2(\eta+\lambda)[V(\eta) - V(\eta+\lambda)]\\
& +V^{1+\alpha_2}(\eta+\lambda)[c_{22}(\eta+\lambda) - c_{22}(\eta)]\\
& +c_{22}(\eta)[V^{1+\alpha_2}(\eta+\lambda) - V^{1+\alpha_2}(\eta)]\\
& +\int_{-q}^0 \check{\zeta}_2(u)\Big[U(\eta+\lambda+u)c_{12}(\eta+\lambda)[-V(\eta) + V(\eta+\lambda)]\\
& +V(\eta)c_{12}(\eta)[-U(\eta+u) + U(\eta+\lambda+u)]\\
& +U(\eta+\lambda+u)V(\eta)[-c_{12}(\eta) + c_{12}(\eta+\lambda)]\Big]\,\mathrm{d}u.
\end{aligned}$$

Since, $\Psi(\mathfrak{s}) = |p_1^{\mathfrak{s}} - p_2^{\mathfrak{s}}|$ are monotonically increasing functions, p_1, $p_2 > 0$ and $\mathfrak{s} \geq 1$. Then

$$\begin{aligned} |U^{1+\alpha_1}(\eta+\lambda) - U^{1+\alpha_1}(\eta)| &\leq (1+\alpha_1)|U(\eta+\lambda) - U(\eta)|M^{\alpha_1} \\ |V^{1+\alpha_2}(\eta+\lambda) - V^{1+\alpha_2}(s)| &\leq (1+\alpha_2)|V(\eta+\lambda) - V(\eta)|M^{\alpha_2}. \end{aligned}$$

Since ζ_1, ζ_2, φ_1, φ_2, c_{11}, c_{12}, c_{22}, U, V are asymptotically λ-periodic functions, then for each $\varepsilon > 0$, there exists $\mathfrak{s}_\varepsilon > 0$ such that for $\mathfrak{s} \geq \mathfrak{s}_\varepsilon$ we obtain

$$\begin{array}{cc} |\zeta_1(\mathfrak{s}+\lambda) - \zeta_1(\mathfrak{s})| < \varepsilon, & |\zeta_2(\mathfrak{s}+\lambda) - \zeta_2(\mathfrak{s})| < \varepsilon, \\ |\varphi_1(\mathfrak{s}+\lambda) - \varphi_1(\mathfrak{s})| < \varepsilon, & |\varphi_2(\mathfrak{s}+\lambda) - \varphi_2(\mathfrak{s})| < \varepsilon, \\ |U(\mathfrak{s}+\lambda) - U(\mathfrak{s})| < \varepsilon, & |V(\mathfrak{s}+\lambda) - V(\mathfrak{s})| < \varepsilon, \end{array}$$

and $|c_{11}(\mathfrak{s}+\lambda) - c_{11}(\mathfrak{s})| < \varepsilon$, $|c_{22}(\mathfrak{s}+\lambda) - c_{22}(\mathfrak{s})| < \varepsilon$, $|c_{12}(\mathfrak{s}+\lambda) - c_{12}(\mathfrak{s})| < \varepsilon$, which leads

$$\begin{aligned} & \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} \left|\mathfrak{F}_1(\eta+\lambda, U(\eta+\lambda), V(\eta+\lambda)) - \mathfrak{F}_1(\eta, U(\eta), V(\eta))\right| \mathrm{d}\eta \\ \leq\; & \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} \bigg(|\zeta_1(\eta+\lambda)|\,|U(\eta+\lambda) - U(\eta)| + |U(\eta)|\,|\zeta_1(\eta+\lambda) - \zeta_1(\eta)| \\ & +|U(\eta)|\,|\varphi(\eta) - \varphi_1(\eta+\lambda) + \varphi(\eta+\lambda)[U(\eta) - U(\eta+\lambda)] \\ & +|U^{1+\alpha_1}(\eta+\lambda)|\,|c_{11}(\eta+\lambda) - c_{11}(\eta)| + |c_{11}(\eta)|\,|U^{1+\alpha_1}(\eta+\lambda) - U^{1+\alpha_1}(\eta)| \\ & + \int_{-q}^0 \check{\zeta}_1(u)\Big[|V(\eta+u)|\,|c_{12}(\eta)|\,|U(\eta) - U(\eta+\lambda)| \\ & +|U(\eta+\lambda)|\,|c_{12}(\eta+\lambda)|\,|V(\eta+u) - V(\eta+\lambda+u)| \\ & +|V(\eta+u)|\,|U(\eta+\lambda)|\,|c_{12}(\eta) - c_{12}(\eta+\lambda)|\Big] \mathrm{d}u\bigg) \mathrm{d}\eta \\ \leq\; & \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} \bigg(|\zeta_1(\eta+\lambda)|\varepsilon + 2\varepsilon|U(\eta)| + \varepsilon\varphi(\eta+\lambda) \\ & +|U^{1+\alpha_1}(\eta+\lambda)|\varepsilon + |c_{11}(\eta)|(1+\alpha_1)M^{\alpha_1}\varepsilon + \int_{-q}^0 \check{\zeta}_1(u)\Big[\varepsilon|V(\eta+u)||c_{12}(\eta)| \\ & +|U(\eta+\lambda)|\,|c_{12}(\eta+\lambda)|\varepsilon + \varepsilon|V(\eta+u)|\,|U(\eta+\lambda)|\Big] \mathrm{d}u\bigg) \mathrm{d}\eta \\ \leq\; & \varepsilon\int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} \Big(|\zeta_1(\eta+\lambda)| + 2|U(\eta)| + |\varphi(\eta+\lambda)| \\ & +|U^{1+\alpha_1}(\eta+\lambda)| + |c_{11}(\eta)|(1+\alpha_1)M^{\alpha_1} \\ & +\frac{1-e^{-q\mu_1}}{\mu_1}\big[\|V\|\,|c_{12}(\eta)| + \|U\|\,|c_{12}(\eta+\lambda)| + M|U(\eta+\lambda)|\big]\Big) \mathrm{d}\eta. \end{aligned}$$

Similarly,

$$\begin{aligned} & \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} \left|\mathfrak{F}_2(\eta+\lambda, U(\eta+\lambda), V(\eta+\lambda)) - \mathfrak{F}_2(\eta, U(\eta), V(\eta))\right| \mathrm{d}\eta \\ \leq\; & \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} \bigg(|\zeta_2(\eta+\lambda)|\,|V(\eta+\lambda) - V(\eta)| + |V(\eta)|\,|\zeta_2(\eta+\lambda) - \zeta_2(\eta)| \\ & +|V(\eta)|\,|\varphi(\eta) - \varphi_2(\eta+\lambda)| + |\varphi_2(\eta+\lambda)|\,|V(\eta) - V(\eta+\lambda)| \\ & +|V^{1+\alpha_2}(\eta+\lambda)|\,|c_{22}(\eta+\lambda) - c_{22}(\eta)| + |c_{22}(\eta)|\,|V^{1+\alpha_2}(\eta+\lambda) - V^{1+\alpha_2}(\eta)| \\ & + \int_{-q}^0 \check{\zeta}_2(u)\Big[|U(\eta+\lambda+u)|\,|c_{12}(\eta+\lambda)|\,|-V(\eta) + V(\eta+\lambda)| \\ & +|V(\eta)|\,|c_{12}(\eta)|\,|-U(\eta+u) + U(\eta+\lambda+u)| \\ & +|U(\eta+\lambda+u)|\,|V(\eta)|\,|-c_{12}(\eta) + c_{12}(\eta+\lambda)|\Big] \mathrm{d}u\bigg) \mathrm{d}\eta \end{aligned}$$

$$
\begin{aligned}
\leq\ & \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} \bigg(|\zeta_2(\eta+\lambda)|\varepsilon + |V(\eta)|\varepsilon + |V(\eta)|\varepsilon + |\varphi_2(\eta+\lambda)|\varepsilon + |V^{1+\alpha_2}(\eta+\lambda)|\varepsilon \\
& + c_{22}(\eta)(1+\alpha_2)M^{\alpha_2}\varepsilon + \int_{-q}^{0} e^{u\mu_2} \Big[|U(\eta+\lambda+u)|\, |c_{12}(\eta+\lambda)|\varepsilon + |V(\eta)|\, |c_{12}(\eta)|\varepsilon \\
& + |U(\eta+\lambda+u)|\, |V(\eta)|\varepsilon \Big] \mathrm{d}u \bigg) \mathrm{d}\eta \\
\leq\ & \varepsilon \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\alpha)} \Big[|\zeta_2(\eta+\lambda)| + 2|V(\eta)| + |\varphi_2(\eta+\lambda)| + |V^{1+\alpha_2}(\eta+\lambda)| \\
& + c_{22}(\eta)(1+\alpha_2)M^{\alpha_2} + \frac{1-e^{-q\mu_2}}{\mu_2} \Big[\|U\|\, |c_{12}(\eta+\lambda)| + \|V\|\, |c_{12}(\eta)| + \|U\|\, |V(\eta)| \Big] \Big] \mathrm{d}\eta.
\end{aligned}
$$

By Lemma 4, there exists $\varsigma > 0$, such that

$$
\begin{aligned}
& \sup_{\mathfrak{s}\geq 0} \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} |\zeta_i(\eta+\lambda)| \,\mathrm{d}\eta < \varsigma, \\
& \sup_{\mathfrak{s}\geq 0} \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} |\varphi_i(\eta+\lambda)| \,\mathrm{d}\eta < \varsigma, \qquad i = 1,2 \\
& \sup_{\mathfrak{s}\geq 0} \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} |c_{12}(\eta+\lambda)| \,\mathrm{d}\eta < \varsigma, \\
& \sup_{\mathfrak{s}\geq 0} \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} |U(\eta+\lambda)| \,\mathrm{d}\eta < \varsigma, \\
& \sup_{\mathfrak{s}\geq 0} \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} |V(\eta+\lambda)| \,\mathrm{d}\eta < \varsigma, \\
& \sup_{\mathfrak{s}\geq 0} \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} |U(\eta)| \,\mathrm{d}\eta < \varsigma, \\
& \sup_{\mathfrak{s}\geq 0} \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} |V(\eta)| \,\mathrm{d}\eta < \varsigma, \\
& \sup_{\mathfrak{s}\geq 0} \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} |c_{ii}(\eta+\lambda)| \,\mathrm{d}\eta < \varsigma, \qquad i = 1,2.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
& \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} \left| \mathfrak{F}_1(\eta+\lambda, U(\eta+\lambda), V(\eta+\lambda)) - \mathfrak{F}_1(\eta, U(\eta), V(\eta)) \right| \mathrm{d}\eta \\
\leq\ & \varepsilon\varsigma \left(5 + (1+\alpha_1)M^{\alpha_1} + 3M \frac{1-e^{-q\mu_1}}{\mu_1} \right)
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} \left| \mathfrak{F}_2(\eta+\lambda, U(\eta+\lambda), V(\eta+\lambda)) - \mathfrak{F}_2(\eta, U(\eta), V(\eta)) \right| \mathrm{d}\eta \\
\leq\ & \varepsilon\varsigma \left(5 + (1+\alpha_2)M^{\alpha_2} + 3M \frac{1-e^{-q\mu_2}}{\mu_2} \right).
\end{aligned}
$$

From the above estimates, we obtain

$$\begin{aligned} |(\Xi U)(\mathfrak{s}+\lambda)-(\Xi U)(\mathfrak{s})| &\leq \varepsilon\varsigma\left(5+(1+\alpha_1)M^{\alpha_1}+3M\frac{1-e^{-q\mu_1}}{\mu_1}\right) \\ |(\Xi V)(\mathfrak{s}+\lambda)-(\Xi V)(\mathfrak{s})| &\leq \varepsilon\varsigma\left(5+(1+\alpha_2)M^{\alpha_2}+3M\frac{1-e^{-q\mu_2}}{\mu_2}\right). \end{aligned}$$

For $\mathfrak{s} \geq \mathfrak{s}_\varepsilon$, we have

$$\lim_{\mathfrak{s}\to\infty}|(\Xi U)(\mathfrak{s}+\lambda)-(\Xi U)(\mathfrak{s})|=0 \quad \text{and} \quad \lim_{\mathfrak{s}\to\infty}|(\Xi V)(\mathfrak{s}+\lambda)-(\Xi V)(\mathfrak{s})|=0.$$

In addition,

$$\begin{aligned} |(\Xi U)(\mathfrak{s})| &\leq |\phi_1(0)|+\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)}|U(\eta)|\,|\zeta_1(\eta)-c_{11}(\eta)U^{\alpha_1}(\eta) \\ &\quad -c_{12}(\eta)\int_{-q}^0\check{\zeta}_1(u)V(\eta+u)\,\mathrm{d}u-\varphi_1(\eta)|\,\mathrm{d}\eta \\ &\leq |\varphi_1(0)|+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\gamma)}\,|U(\eta)|\,|\zeta_1(\eta)|\,\mathrm{d}\eta \\ &\leq |\varphi_1(0)|+\frac{\|U\|}{\Gamma(\gamma)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\eta)^{\gamma-1}|\zeta_1(\eta)|\,\mathrm{d}\eta<\infty, \end{aligned}$$

which results that $(\Xi U)(SAP_\lambda(\mathbb{R}))\subset SAP_\lambda(\mathbb{R})$.

Step 2: Let $\mathcal{X}=(U,V)$, $\mathfrak{Z}=(A,B)\in SAP_\lambda(\mathbb{R})$, $\|\mathcal{X}\|_\lambda=\max\{|U|_\lambda,\,|V|_\lambda\}$,

$$\begin{aligned} &e^{-\mathfrak{s}}\Big[(\Xi\mathcal{X})(\mathfrak{s})-(\Xi\mathfrak{Z})(\mathfrak{s})\Big] \\ =&\begin{pmatrix} e^{-\mathfrak{s}}[(\Xi U)(\mathfrak{s})-(\Xi A)(\mathfrak{s})] \\ e^{-\mathfrak{s}}[(\Xi V)(\mathfrak{s})-(\Xi B)(\mathfrak{s})] \end{pmatrix} \\ =&\begin{pmatrix} e^{-\mathfrak{s}}\displaystyle\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)}\,[\mathfrak{F}_1(\eta,U(\eta),V(\eta))-\mathfrak{F}_1(\eta,A(\eta),B(\eta))]\,\mathrm{d}\eta \\ e^{-\mathfrak{s}}\displaystyle\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)}[\mathfrak{F}_2(\eta,U(\eta),V(\eta))-\mathfrak{F}_2(\eta,A(\eta),B(\eta))]\,\mathrm{d}\eta, \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} &e^{-\mathfrak{s}}\Big(\mathfrak{F}_1(\eta,U(\eta),V(\eta))-\mathfrak{F}_1(\eta,A(\eta),B(\eta))\Big) \\ =\;& e^{-\mathfrak{s}}\Big(\zeta_1(\eta)[U(\eta)-A(\eta)]-\varphi_1(\eta)[U(\eta)-A(\eta)]-c_{11}(\eta)[U^{1+\alpha_1}(\eta)-A^{1+\alpha_1}(\eta)] \\ &-\int_{-q}^0\check{\zeta}_1(u)(c_{12}(\eta)[(U(\eta)-A(\eta))V(\eta+u)+A(\eta)[V(\eta+u)-B(\eta+u)]])\,\mathrm{d}u\Big) \\ =\;& e^{-(\mathfrak{s}-\eta)}\Big([\zeta_1(\eta)-c_{11}(\eta)(1+\alpha_1)M^{\alpha_1}-\varphi_1(\eta) \\ &-c_{12}(\eta)\int_{-q}^0\check{\zeta}_1(u)V(\eta+u)]e^{-\eta}[U(\eta)-A(\eta)]\,\mathrm{d}u \\ &-\int_{-q}^0\check{\zeta}_1(u)c_{12}(\eta)A(\eta)e^{-(\eta+u)}e^{-u}[V(\eta+u)-B(\eta+u)]\,\mathrm{d}u\Big), \end{aligned}$$

which gives,

$$
\begin{aligned}
&\int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)}\, e^{-\mathfrak{s}} \Big| \mathfrak{F}_1(\eta, U(\eta), V(\eta)) - \mathfrak{F}_1(\eta, A(\eta), B(\eta)) \Big| \,\mathrm{d}\eta \\
\leq\ & \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)}\, e^{-(\mathfrak{s}-\eta)} \Big| \zeta_1(\eta) - \varphi_1(\eta) - c_{11}(\eta)(1+\alpha_1)M^{\alpha_1} \\
& - c_{12}(\eta) \int_{-q}^{0} \grave{\zeta}_1(u) V(\eta+u) - \int_{-q}^{0} \grave{\zeta}_1(u) c_{12}(\eta) A(\eta) e^{-u} \Big| \, \|\mathcal{X} - \mathfrak{Z}\|_\lambda \\
\leq\ & \|\mathcal{X} - \mathfrak{Z}\|_\lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)}\, e^{-(\mathfrak{s}-\eta)} |\overline{\zeta_1} - \underline{c}_{11}(1+\alpha_1)M^{\alpha_1} - \underline{\varphi}_1 \\
& - \bar{c}_{12} M \Big[\int_{-q}^{0} e^{u\mu_1} \,\mathrm{d}u + \int_{-q}^{0} e^{u\mu_1} e^{-u} \,\mathrm{d}u \Big] \Big| \,\mathrm{d}\eta \\
\leq\ & \|\mathcal{X} - \mathfrak{Z}\|_\lambda \frac{1}{\Gamma(\gamma)} \int_0^{\mathfrak{s}} (\mathfrak{s}-\eta)^{\gamma-1} e^{-(\mathfrak{s}-\eta)} |\overline{\zeta_1} - \underline{c}_{11}(1+\alpha_1)M^{\alpha_1} - \underline{\varphi}_1 \\
& - \underline{c}_{12} M \Big[\frac{1-e^{-q\mu_1}}{\mu_1} + \frac{1-e^{-(\mu_1-1)q}}{\mu_1 - 1} \Big] \Big| \,\mathrm{d}\eta.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
e^{-\mathfrak{s}} |(\Xi U)(\mathfrak{s}) - (\Xi A)(\mathfrak{s})| \ \leq\ & \|\mathcal{X} - \mathfrak{Z}\|_\lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} e^{-(\mathfrak{s}-\eta)} |\overline{\zeta_1} - \underline{c}_{11}(1+\alpha_1)M^{\alpha_1} \\
& - \underline{\varphi}_1 - \underline{c}_{12} M \Big[\frac{1-e^{-q\mu}}{\mu} + \frac{1-e^{-(\mu-1)q}}{\mu-1} \Big] \Big| \,\mathrm{d}\eta \\
<\ & \|\mathcal{X} - \mathfrak{Z}\|_\lambda.
\end{aligned}
$$

In addition,

$$
\begin{aligned}
& e^{-\mathfrak{s}} |(\Xi V)(\mathfrak{s}) - (\Xi B)(\mathfrak{s})| \\
=\ & \Big| e^{-\mathfrak{s}} \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} \Big[\mathfrak{F}_2(\eta, U(\eta), V(\eta)) - \mathfrak{F}_2(\eta, A(\eta), B(\eta)) \Big] \,\mathrm{d}\eta \Big| \\
\leq\ & \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)}\, e^{-(\mathfrak{s}-\eta)} \Big| \zeta_2(\eta) - \varphi_2(\eta) - c_{22}(\eta)(1+\alpha_2)M^{\alpha_2} \\
& + c_{12}(\eta) \int_{-q}^{0} \grave{\zeta}_2(u)\, B(\eta+u) + \int_{-q}^{0} \grave{\zeta}_2(u) c_{12}(\eta) U(\eta) e^{-u)} \Big| \, \|\mathcal{X} - \mathfrak{Z}\|_\lambda \\
\leq\ & \|\mathcal{X} - \mathfrak{Z}\|_\lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)}\, e^{-(\mathfrak{s}-\eta)} \Big| \overline{\zeta}_2 - \underline{c}_{22}(1+\alpha_2)M^{\alpha_2} - \underline{\varphi}_2 \\
& - c_{12} M \Big[\int_{-q}^{0} e^{-u\eta_2} \,\mathrm{d}u + \int_{-q}^{0} e^{-u\eta_2} e^{-u} \,\mathrm{d}u \Big] \Big| \,\mathrm{d}\eta \\
\leq\ & \|\mathcal{X} - \mathfrak{Z}\|_\lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)}\, e^{-(\mathfrak{s}-\eta)} \Big| \overline{\zeta}_2 - \underline{c}_{22}(1+\alpha_2)M^{\alpha_2} - \underline{\varphi}_2 \\
& + \bar{c}_{12} M \Big[\frac{1-e^{-q\mu_2}}{\mu_2} + \frac{1-e^{-(\mu_2-1)q}}{\mu_2 - 1} \Big] \Big| \,\mathrm{d}\eta \\
<\ & \|\mathcal{X} - \mathfrak{Z}\|_\lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\eta)^{\gamma-1}}{\Gamma(\gamma)} e^{-(\mathfrak{s}-\eta)} \,\mathrm{d}\eta \\
<\ & \|\mathcal{X} - \mathfrak{Z}\|_\lambda.
\end{aligned}
$$

Therefore, there exists a unique asymptotically λ-periodic solution $\mathcal{X}^* = (U^*, V^*)$ of system (6). □

3.3. Stability of the Solution

Theorem 2. *Assume that* (A1)-(A4) *holds. Furthermore, suppose that the following assumptions holds*

$$\begin{cases} \min_{\mathfrak{s}\in\mathbb{R}}\Big(-\zeta_1(\mathfrak{s}) + c_{11}(\mathfrak{s})m^{\alpha_1}(1+\alpha_1) - \bar{c}_{12}\int_{-q}^{0}\check{\zeta}_2(\eta)\,\mathrm{d}\eta + \varphi_1(\mathfrak{s})\Big) > 0 \\ \min_{\mathfrak{s}\in\mathbb{R}}\Big(-\zeta_2(\mathfrak{s}) + c_{22}(\mathfrak{s})m^{\alpha_2}(1+\alpha_2) - \bar{c}_{12}\int_{-q}^{0}\check{\zeta}_1(\eta)\,\mathrm{d}\eta + \varphi_2(\mathfrak{s})\Big) > 0. \end{cases} \tag{22}$$

Then, System (6) *is globally asymptotic stable.*

Proof. Let $\mathcal{X}^*(\mathfrak{s}) = (U^*(\mathfrak{s}), V^*(\mathfrak{s}))$ and $\mathcal{Y}(\mathfrak{s}) = (H(\mathfrak{s}), J(\mathfrak{s}))$ be a solution for system (6) with initial condition $(U_0^*, V_0^*), (H_0, J_0)$, respectively. Consider Laypunov's function $\mathcal{V}(\mathfrak{s}) = \mathcal{V}_1(\mathfrak{s}) + \mathcal{V}_2(\mathfrak{s})$, with

$$\begin{aligned} \mathcal{V}_1(\mathfrak{s}) &= |U^*(\mathfrak{s}) - H(\mathfrak{s})| + \bar{c}_{12}\int_{-q}^{0}\check{\zeta}_1(\eta)\int_{\mathfrak{s}+\eta}^{\mathfrak{s}}|V^*(e) - J(e)|\,\mathrm{d}e\,\mathrm{d}\eta \\ \mathcal{V}_2(\mathfrak{s}) &= |V^*(\mathfrak{s}) - J(\mathfrak{s})| + \bar{c}_{12}\int_{-q}^{0}\check{\zeta}_2(\eta)\int_{\mathfrak{s}+\eta}^{\mathfrak{s}}|U^*(e) - H(e)|\,\mathrm{d}e\,\mathrm{d}\eta. \end{aligned}$$

Under Lemma 1, the upper right derivative ${}^{C}\mathcal{D}_{\mathfrak{s}}^{\gamma}\mathcal{V}_1(\mathfrak{s})$ and ${}^{C}\mathcal{D}_{\mathfrak{s}}^{\gamma}\mathcal{V}_2(\mathfrak{s})$ along the solution of system (6), gives

$$\begin{aligned} {}^{C}\mathcal{D}_{\mathfrak{s}}^{\gamma}\mathcal{V}_1(\mathfrak{s}) &\le [\zeta_1(\mathfrak{s}) - \varphi_1(\mathfrak{s})]|U^*(\mathfrak{s}) - H(\mathfrak{s}| - c_{11}(\mathfrak{s})|(U^*)^{1+\alpha_1}(\mathfrak{s}) - H^{1+\alpha_1}(\mathfrak{s})| \\ &\quad + \bar{c}_{12}\int_{-q}^{0}\check{\zeta}_1(\eta)|V^*(\mathfrak{s}) - J(\mathfrak{s})|\,\mathrm{d}\eta, \\ {}^{C}\mathcal{D}_{\mathfrak{s}}^{\gamma}\mathcal{V}_2(\mathfrak{s}) &\le [\zeta_2(\mathfrak{s}) - \varphi_2(\mathfrak{s})]|V^*(\mathfrak{s}) - J(\mathfrak{s}| - c_{22}(\mathfrak{s})|(V^*)^{1+\alpha_1}(\mathfrak{s}) - J^{1+\alpha_1}(\mathfrak{s})| \\ &\quad + \bar{c}_{12}\int_{-q}^{0}\check{\zeta}_2(\eta)|U^*(\mathfrak{s}) - H(\mathfrak{s})|\,\mathrm{d}\eta. \end{aligned}$$

Hence

$$\begin{aligned} {}^{C}\mathcal{D}_{\mathfrak{s}}^{\gamma}\mathcal{V}(\mathfrak{s}) &\le \left[\zeta_1(\mathfrak{s}) - \varphi_1(\mathfrak{s}) + \bar{c}_{12}\int_{-q}^{0}\check{\zeta}_2(\eta)\,\mathrm{d}\eta - c_{11}(\mathfrak{s})(1+\alpha_1)m^{\alpha_1}\right]|U^*(\mathfrak{s}) - H(\mathfrak{s})| \\ &\quad + \left[\zeta_2(\mathfrak{s}) - \varphi_2(\mathfrak{s}) + \bar{c}_{12}\int_{-q}^{0}\check{\zeta}_1(\eta)\,\mathrm{d}\eta - c_{22}(\mathfrak{s})(1+\alpha_2)m^{\alpha_2}\right]|V^*(\mathfrak{s}) - J(\mathfrak{s})| \\ &\le -\left[\varphi_1(\mathfrak{s}) + c_{11}(\mathfrak{s})(1+\alpha_1)m^{\alpha_1} - \zeta_1(\mathfrak{s}) - \bar{c}_{12}\int_{-q}^{0}\check{\zeta}_2(\eta)\,\mathrm{d}\eta\right]|U^*(\mathfrak{s}) - H(\mathfrak{s})| \\ &\quad - \left[\zeta_2(\mathfrak{s}) + c_{22}(\mathfrak{s})(1+\alpha_2)m^{\alpha_2} - \zeta_2(\mathfrak{s}) - \bar{c}_{12}\int_{-q}^{0}\check{\zeta}_1(\eta)\,\mathrm{d}\eta\right]|V^*(\mathfrak{s}) - J(\mathfrak{s})|. \end{aligned}$$

By (22), let $\digamma$ be a positive constant such that

$$\begin{aligned} \digamma \ge \min_{\mathfrak{s}\in\mathbb{R}}\Big(&-\zeta_1(\mathfrak{s}) + c_{11}(\mathfrak{s})m^{\alpha_1}(1+\alpha_1) - \bar{c}_{12}\int_{-q}^{0}\check{\zeta}_2(\eta)\,\mathrm{d}\eta + \varphi_1(\mathfrak{s}), \\ &-\zeta_2(\mathfrak{s}) + c_{22}(\mathfrak{s})m^{\alpha_2}(1+\alpha_2) - \bar{c}_{12}\int_{-q}^{0}\check{\zeta}_1(\eta)\,\mathrm{d}\eta + \varphi_2(\mathfrak{s})\Big) > 0. \end{aligned}$$

Then

$${}^{C}\mathcal{D}_{\mathfrak{s}}^{\gamma}\mathcal{V}(\mathfrak{s}) \le -\digamma\,\mathcal{V}(\mathfrak{s}). \tag{23}$$

Consequently, ${}^{C}\mathcal{D}_{\mathfrak{s}}^{\gamma}V(\mathfrak{s}) < 0$, for all $\mathfrak{s} \ge 0$. Therefore, the asymptotically λ-periodic solution of (6) is globally asymptotically stable. □

3.4. An Example with Numerical Simulation

In this section, we present a few illustrative examples that guarantee our main results. Furthermore, we provide nice algorithms that help us calculate all numerical results.

Example 1. *In model (3), we consider the following Gilpin–Ayala predator-prey system with the harvesting terms as:*

$$\begin{cases} {}^{C}\mathcal{D}_{\mathfrak{s}}^{\gamma} U(\mathfrak{s}) = U(\mathfrak{s})\Big[3.25 + 2\sin(\mathfrak{s}) - \Big[\dfrac{2.75 + 0.2\sin(\mathfrak{s})}{5}\Big](\mathfrak{s})U^{1/\sqrt{5}}(\mathfrak{s}) \\ \qquad - \Big(\dfrac{2+\sqrt{3}\cos(\mathfrak{s})}{36}\Big)\displaystyle\int_{-q}^{0} \frac{1}{\eta^2+6}\, V(\mathfrak{s}+\eta)\,\mathrm{d}\eta - \Big(\dfrac{1.8+\sin(\mathfrak{s})}{15}\Big)\Big], \\ {}^{C}\mathcal{D}_{\mathfrak{s}}^{\gamma} V(\mathfrak{s}) = V(\mathfrak{s})\Big[4.25 + \cos(\mathfrak{s}) - \Big[\dfrac{\sqrt{7}+\cos(\mathfrak{s})}{6}\Big]V^{0.7}(\mathfrak{s}) \\ \qquad + \Big(\dfrac{2+\sqrt{3}\cos(\mathfrak{s})}{36}\Big)\displaystyle\int_{-q}^{0} \exp(2\eta)\, U(\mathfrak{s}+\eta)\,\mathrm{d}\eta - \Big(\dfrac{1.5+\sin(\mathfrak{s})}{10}\Big)\Big], \end{cases} \tag{24}$$

for $\mathfrak{s} \geq 0$ and for $\gamma \in \{0.13, 0.5, 0.96\}$, under initial conditions

$$U(\mathfrak{s}) = \frac{1.2+\cos(\mathfrak{s})}{12}, \quad V(\mathfrak{s}) = \frac{2.43+\sin(\mathfrak{s})}{7}, \qquad \mathfrak{s} \in \Lambda = [-1, 0]. \tag{25}$$

Without a doubt $\zeta_1(\mathfrak{s}) = 3.25 + 2\sin(\mathfrak{s})$, $c_{11} = \frac{1}{5}(2.75 + 0.2\sin(\mathfrak{s}))$, $c_{12} = \frac{1}{36}(2+\sqrt{3}\cos(\mathfrak{s}))$, $\acute{\zeta}_1(\mathfrak{s}) = \frac{1}{\mathfrak{s}^2+6}$, $\varphi_1(\mathfrak{s}) = \frac{1}{15}(1.8+\sin(\mathfrak{s}))$, $\zeta_2(\mathfrak{s}) = 4.25 + \cos(\mathfrak{s})$, $c_{22} = \frac{1}{6}(\sqrt{7}+\cos(\mathfrak{s}))$, $\acute{\zeta}_2(\mathfrak{s}) = \exp(2\mathfrak{s})$, $\varphi_2(\mathfrak{s}) = \frac{1}{10}(1.5+\sin(\mathfrak{s}))$, $\alpha_1 = \frac{1}{\sqrt{5}}$, $\alpha_2 = 0.7$, we have

$$\begin{aligned}
\underline{\zeta}_1 &= \inf_{\mathfrak{s}\in\Lambda} \zeta_1 = \inf_{\mathfrak{s}\in\Lambda}\Big(\sqrt{7} + 2\sin(\mathfrak{s})\Big) = \sqrt{7} - 2, \\
\overline{\zeta}_1 &= \sup_{\mathfrak{s}\in\Lambda} \zeta_1 = \sup_{\mathfrak{s}\in\Lambda}\Big(\sqrt{7} + 2\sin(\mathfrak{s})\Big) = \sqrt{7} + 2, \\
\underline{\zeta}_2 &= \inf_{\mathfrak{s}\in\Lambda} \zeta_2 = \inf_{\mathfrak{s}\in\Lambda}(1.1 + \cos(\mathfrak{s})) = 0.1, \\
\overline{\zeta}_2 &= \sup_{\mathfrak{s}\in\Lambda} \zeta_2 = \sup_{\mathfrak{s}\in\Lambda}(1.1 + \cos(\mathfrak{s})) = 2.1, \\
\overline{\varphi}_1 &= \sup_{\mathfrak{s}\in\Lambda} \varphi_1 = \sup_{\mathfrak{s}\in\Lambda}\Big(\frac{1}{15}(4+\sin(\mathfrak{s}))\Big) = \frac{1}{3}, \\
\underline{\varphi}_1 &= \inf_{\mathfrak{s}\in\Lambda} \varphi_1 = \inf_{\mathfrak{s}\in\Lambda}\Big(\frac{1}{15}(4+\sin(\mathfrak{s}))\Big) = \frac{1}{5}, \\
\overline{\varphi}_2 &= \sup_{\mathfrak{s}\in\Lambda} \varphi_2 = \sup_{\mathfrak{s}\in\Lambda}\Big(\frac{1}{10}(3+\sin(\mathfrak{s}))\Big) = \frac{2}{5}, \\
\underline{\varphi}_2 &= \inf_{\mathfrak{s}\in\Lambda} \varphi_2 = \inf_{\mathfrak{s}\in\Lambda}\Big(\frac{1}{10}(3+\sin(\mathfrak{s}))\Big) = \frac{1}{5}, \\
\overline{c}_{11} &= \sup_{\mathfrak{s}\in\Lambda} c_{11} = \sup_{\mathfrak{s}\in\Lambda}\Big(\frac{1}{5}(8+5\sin(\mathfrak{s}))\Big) = \frac{13}{5}, \\
\underline{c}_{11} &= \inf_{\mathfrak{s}\in\Lambda} c_{11} = \inf_{\mathfrak{s}\in\Lambda}\Big(\frac{1}{5}(8+5\sin(\mathfrak{s}))\Big) = \frac{3}{5}, \\
\overline{c}_{12} &= \sup_{\mathfrak{s}\in\Lambda} c_{12} = \sup_{\mathfrak{s}\in\Lambda}\Big(\frac{1}{9}(1.9+\sqrt{3}\cos(\mathfrak{s}))\Big) = \frac{1.9+\sqrt{3}}{9}, \\
\underline{c}_{12} &= \inf_{\mathfrak{s}\in\Lambda} c_{12} = \inf_{\mathfrak{s}\in\Lambda}\Big(\frac{1}{9}(1.9+\sqrt{3}\cos(\mathfrak{s}))\Big) = \frac{1.9-\sqrt{3}}{9},
\end{aligned}$$

$$\begin{aligned}\overline{c}_{22} &= \sup_{\mathfrak{s}\in\Lambda} c_{22} = \sup_{\mathfrak{s}\in\Lambda}\left(\frac{1}{6}(\sqrt{11}+4\cos(\mathfrak{s}))\right) = \frac{\sqrt{11}+4}{6},\\ \underline{c}_{22} &= \inf_{\mathfrak{s}\in\Lambda} c_{22} = \inf_{\mathfrak{s}\in\Lambda}\left(\frac{1}{6}(\sqrt{11}+4\cos(\mathfrak{s}))\right) = \frac{\sqrt{11}-4}{6}.\end{aligned}$$

Obviously, $\zeta_1(\mathfrak{s})$, $\zeta_2(\mathfrak{s})$, $\acute{\zeta}_1(\mathfrak{s})$, $\acute{\zeta}_2(\mathfrak{s})$, c_{11}, c_{12}, c_{22}, $\varphi_1(\mathfrak{s})$, $\varphi_2(\mathfrak{s})$ are all asymptotically λ-periodic functions with periodic $\lambda = 2\pi$ and (A1) *holds for all $\mathfrak{s}\in\Lambda$ as follows*

$$\begin{aligned}\acute{\zeta}_1(\mathfrak{s}) &= \frac{1}{\mathfrak{s}^2+1} \le \exp(\mathfrak{s}), \qquad \textit{if} \quad \mu_1 = 0.25 > 0,\\ \acute{\zeta}_1(\mathfrak{s}) &= \exp(2\mathfrak{s}) \le \exp(3\mathfrak{s}), \qquad \textit{if} \quad \mu_2 = 0.50 > 0.\end{aligned}$$

Tables 2–4 show the numerical results of all variables and (A1)-(A4).

Table 2. Numerical results of (A1)-(A4) of the Gilpin–Ayala predator-prey system with harvesting terms (24) whenever $\gamma = 0.13$.

$\mathfrak{s}$	$\gamma = 0.13$								
	$\grave{\zeta}_1(\mathfrak{s})$	$e^{\mu_1\mathfrak{s}}$	$\grave{\zeta}_2(\mathfrak{s})$	$e^{\mu_2\mathfrak{s}}$	(A2)	(A3)	(A4)-1	(A4)-2	$\ge E_\gamma$
−1.00	0.1429	0.6703	0.1353	0.4724	0.4042	0.4559	0.7870	1.8612	$6.3003\times10^{+10}$
−0.90	0.1468	0.6977	0.1653	0.5092	0.4270	0.4665	0.9468	0.2805	$3.2441\times10^{+10}$
−0.80	0.1506	0.7261	0.2019	0.5488	0.4506	0.4742	1.0912	1.2166	$1.5212\times10^{+10}$
−0.70	0.1541	0.7558	0.2466	0.5916	0.4782	0.4818	1.2124	2.6493	$63.5932\times10^{+8}$
−0.60	0.1572	0.7866	0.3012	0.6376	0.5106	0.4895	1.3123	4.0233	$23.0184\times10^{+8}$
−0.50	0.1600	0.8187	0.3679	0.6873	0.5495	0.4971	1.3926	5.3440	$6.9238\times10^{+8}$
−0.40	0.1623	0.8521	0.4493	0.7408	0.5967	0.5048	1.4552	6.6164	$1.6291\times10^{+8}$
−0.30	0.1642	0.8869	0.5488	0.7985	0.6553	0.5124	1.5015	7.8452	$27.2936\times10^{+6}$
−0.20	0.1656	0.9231	0.6703	0.8607	0.7299	0.5201	1.5330	9.0347	$2.7867\times10^{+6}$
−0.10	0.1664	0.9608	0.8187	0.9277	0.8281	0.5277	1.5509	10.1892	$13.1477\times10^{+4}$
0.00	0.1667	1.0000	1.0000	1.0000	0.9628	0.5354	1.5567	11.3123	$19.3259\times10^{+2}$

Table 3. Numerical results of (A1)–(A4) of the Gilpin–Ayala competition predator-prey system with harvesting terms (24) whenever $\gamma = 0.13$.

$\mathfrak{s}$	$\gamma = 0.50$								
	$\grave{\zeta}_1(\mathfrak{s})$	$e^{\mu_1\mathfrak{s}}$	$\grave{\zeta}_2(\mathfrak{s})$	$e^{\mu_2\mathfrak{s}}$	(A2)	(A3)	(A4)-1	(A4)-2	$\ge E_\gamma$
−1.00	0.1429	0.6703	0.1353	0.4724	0.4042	0.4559	0.7870	1.8612	$21.5243\times10^{+6}$
−0.90	0.1468	0.6977	0.1653	0.5092	0.4270	0.4665	0.9468	0.2805	$8.9914\times10^{+6}$
−0.80	0.1506	0.7261	0.2019	0.5488	0.4506	0.4742	1.0912	1.2166	$3.4485\times10^{+6}$
−0.70	0.1541	0.7558	0.2466	0.5916	0.4782	0.4818	1.2124	2.6493	$1.2008\times10^{+6}$
−0.60	0.1572	0.7866	0.3012	0.6376	0.5106	0.4895	1.3123	4.0233	$37.5030\times10^{+4}$
−0.50	0.1600	0.8187	0.3679	0.6873	0.5495	0.4971	1.3926	5.3440	$10.3696\times10^{+4}$
−0.40	0.1623	0.8521	0.4493	0.7408	0.5967	0.5048	1.4552	6.6164	$2.5060\times10^{+4}$
−0.30	0.1642	0.8869	0.5488	0.7985	0.6553	0.5124	1.5015	7.8452	$52.2701\times10^{+2}$
−0.20	0.1656	0.9231	0.6703	0.8607	0.7299	0.5201	1.5330	9.0347	$9.2586\times10^{+2}$
−0.10	0.1664	0.9608	0.8187	0.9277	0.8281	0.5277	1.5509	10.1892	$1.3395\times10^{+2}$
0.00	0.1667	1.0000	1.0000	1.0000	0.9628	0.5354	1.5567	11.3123	13.8332

Table 4. Numerical results of (A1)–(A4) of the Gilpin–Ayala competition predator-prey system with harvesting terms (24) whenever $\gamma = 0.13$.

$\mathfrak{s}$	$\gamma = 0.94$								
	$\grave{\zeta}_1(\mathfrak{s})$	$e^{\mu_1\mathfrak{s}}$	$\grave{\zeta}_2(\mathfrak{s})$	$e^{\mu_2\mathfrak{s}}$	(A2)	(A3)	(A4)-1	(A4)-2	$\ge E_\gamma$
−1.00	0.1429	0.6703	0.1353	0.4724	0.4042	0.4559	0.7870	1.8612	1.1229
−0.90	0.1468	0.6977	0.1653	0.5092	0.4270	0.4665	0.9468	0.2805	54.3416×10^{-2}
−0.80	0.1506	0.7261	0.2019	0.5488	0.4506	0.4742	1.0912	1.2166	29.4215×10^{-2}
−0.70	0.1541	0.7558	0.2466	0.5916	0.4782	0.4818	1.2124	2.6493	17.8922×10^{-2}
−0.60	0.1572	0.7866	0.3012	0.6376	0.5106	0.4895	1.3123	4.0233	12.1087×10^{-2}
−0.50	0.1600	0.8187	0.3679	0.6873	0.5495	0.4971	1.3926	5.3440	9.1835×10^{-2}
−0.40	0.1623	0.8521	0.4493	0.7408	0.5967	0.5048	1.4552	6.6164	8.3030×10^{-2}
−0.30	0.1642	0.8869	0.5488	0.7985	0.6553	0.5124	1.5015	7.8452	8.4919×10^{-2}
−0.20	0.1656	0.9231	0.6703	0.8607	0.7299	0.5201	1.5330	9.0347	8.6983×10^{-2}
−0.10	0.1664	0.9608	0.8187	0.9277	0.8281	0.5277	1.5509	10.1892	9.3835×10^{-2}
0.00	0.1667	1.0000	1.0000	1.0000	0.9628	0.5354	1.5567	11.3123	12.4923×10^{-2}

4. Conclusions

Time plays an important role in the study of any phenomena (ecology, biology, etc.) because it makes dynamic behavior more realistic. For this reason, in our research paper, we took into account the time for all the coefficients. In this paper, we have derived a classical nonlinear fractional prey-predator Gilpin–Ayala model (6) with distributed delays and control terms. The model is an important and well-known differential equation. The study of the dynamic behavior and properties of this model can provide a theoretical basis for governance and protection. First, using some inequality techniques, we obtain a priori estimates of the boundedness region of the solution. Then, sufficient criteria for the existence of asymptotic λ-periodic solutions are obtained by using the Banach fixed-point theorem. We showed that by means of control, one can control the existence and stability of our model. The results in the model can be considered with Λ-fractional differential equations [25]. We simulate the correctness of our results through a numerical example.

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