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A Game Theoretic Model of Struggle with Corruption in Auctions: Computer Simulation

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Abstract: There is a great deal of literature devoted to mathematical models of corruption, including corruption in auctions. However, the relationship between the seller and the auctioneer is not studied sufficiently. The research aim is to analyze such relations in a game theoretic setup. We built a difference game theoretic model in normal form that describes possible collusion between an auctioneer and participants of an auction. The auctioneer acts on behalf of a seller. The seller can control possible collusions by administrative and economic mechanisms. The probability of detection depends on audit cost. We consider four cases of absence/presence of the collusion and those of the audit. The model is investigated numerically by simulation modeling using an original method of qualitatively representative scenarios. Several conclusions are made: factors of corruption are low probability of detection, small penalty, and big corruption gain of the auctioneer.

Keywords: auctions; corruption; game theory; games in normal form; simulation modeling

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1. Introduction

Corruption is traditionally defined as the "misuse of a position of trust for dishonest gain". Most papers devoted to the modeling of corruption use Gary Becker's idea [1] that struggle with any crime is effective when the utility of prevention of the crime is greater than the respective costs. This economic approach was adapted to corruption by Susan Rose-Ackerman and other authors [2–5]. Good reviews are proposed in [6–8]. Most papers use static game theoretic models in normal form or multi-step games; some of them are based on dynamic control models with one or several agents [9].

Interesting modern mathematical models of corruption are based on multi-agent approach and mean field game theory [10–12]. Games of inspection and corruption are described in the literature [13,14]. Kolokoltsov [12] develops an evolutionary games approach further in a setting of the two-level hierarchy, where a local inspector can be corrupted and is further controlled by the higher authority.

A big stream of literature is concerned with corruption in auctions. In the general case, the players are a seller (principal), an auctioneer who acts on behalf of the seller (supervisor), and auction participants (agents). An illustrative historical example with a famous personality, Goethe, is described in [15]. Many modern cases from real life are also documented, especially in construction and procurement auctions [16].

Most papers discuss collusion among agents when the interests of the seller and the auctioneer coincide [17–22]. Graham and Marshall [17] studied collusive bidder behavior at single-object second price and English auctions. Mailath and Zemsky [18] considered heterogeneous bidders, and McAfee and McMillan [19] formalized bidding rings. Repeated auctions are analyzed in [20,22]. Athey et al. [21] presented evidence from timber auctions.

A smaller amount of literature focuses on collusion between the auctioneer and the agents. This kind of corruption can only occur if the seller delegates the sale to an auctioneer, and their interests may not coincide. This literature can be subdivided into three directions [16].

The first one was introduced in a seminal paper by Laffont and Tirole [23]. They assumed that the auctioneer could assess complex multidimensional bids and was predisposed to a particular agent. That framework was developed in [24,25]. The authors showed that corruption, counter-intuitively, may increase if the number of competing agents increases. Additionally, corruption may entail inefficiency proportional to the manipulation by an auctioneer.

A second branch investigates a particular form of bid rigging when the auctioneer grants a "right of first refusal" to a favored agent. Then, in a first-price auction, the favored agent plays in fact a second-price auction, whereas the other agents pay their bid if they win. A positive feature of that approach is that it explains how corruption destroys efficiency. However, this advantage is lost when the selection of the favored agent becomes endogenous. Besides, the corrupt auctioneer should contact all agents to select the favored one. Naturally, the risk of detection and punishment increases [26,27].

The third branch of the literature supposes that the auctioneer arranges bid rigging after he has observed all the bids. This permits him to contact only one–two agents and select the one whose collaboration promises the highest profit [16,28].

Serious attention to the modeling of corruption is analyzed by the theory of sustainable management [29,30]. The author's approach is presented in detail in [31]. A method of qualitatively representative scenarios in simulation modeling is described in [32]. Different game theoretic models of corruption and their computer simulation are analyzed in [33–35].

In this paper, we follow the third branch of literature characterized above and study a possible collusion between an auctioneer and the agents. However, the players in a built game theoretic model are the auctioneer and the seller who uses administrative and economic mechanisms to control the corruption. The research aim is to analyze such relations in a game theoretic setup.

The contribution of this paper is as follows:

- We build a dynamic game theoretic model of collusion in auctions.
- We formalize administrative and economic control mechanisms used by a seller to detect the collusion.
- Several model examples for different model parameters are numerically simulated. For this purpose, the author's method of qualitatively representative scenarios [31] is used.
- Some dependences between model parameters and possible corruption behavior of an auctioneer are elicited.

Section 2 introduces the game theoretic model and its information structure. Section 3 presents the numerical simulation results for different model parameters. Section 4 summarizes the results and outlines further research.

2. Game Theoretic Model

Assume that a set of agents $N = \{1, 2, \ldots, n\}$ takes part in different auctions during several years $t = \{1, 2, \ldots, T\}$. Denote by b^t_{ij} a bid of the agent i in the auction j in the year t. A matrix $B^t = ||b^t_{ij}||$ and a vector $m(t)$, of which components describe a number of auctions in the year, t are given by a scenario as structural model parameters, as well as values n and T.

A seller S (she) and an auctioneer A (he) are the players. In each auction, the seller entrusts the auctioneer to sell by her name a unique indivisible good (a lot) for a maximal possible price. It is assumed that in each auction, j in each year t the auctioneer can propose to the agent with a maximal bid $b^t_{\max,j} = \max\limits_{1 \le i \le n} b^t_{ij}$ to win in the auction with the next bid $b^t_{*j} = \max\limits_{i \ne \max} b^t_{ij}$. In this case, the agent always accepts the proposal (enters a collusion). Then, the auctioneer receives a share α from the difference $\Delta_j^t = b_{\max,j}^t - b_{*j}^t$ (corruption gain), and the winner agent takes the rest.

Introduce an indicator of corruption (collusion) for an auction j in the year t

$$C_j^t = \begin{cases} 1, & \textit{there is a collusion}, \\ 0, & \textit{collusion is absent}. \end{cases}$$

For the struggle with corruption, the seller uses administrative and economic control mechanisms. The administrative mechanism consists in sample audits of a possible collusion. A probability of detection p_j^t of the collusion is a function of the audit cost c_j^t: $p_j^t = p(c_j^t)$. The function p increases monotonically, $p(0) = p_0$, $0 < p_0 << 1$, $\lim_{c\to\infty} p(c) = 1$. In the case of detection, the auction is cancelled, and the auctioneer pays a very big penalty M >> 1. This penalty makes his economic activity senseless, and in fact, the auctioneer cannot organize auctions anymore. However, if a collusion is not detected, then the seller loses her expenditure for the audit.

Introduce an indicator of audit for an auction j in the year t

$$I_j^t = \begin{cases} 1, & \textit{audit is conducted}, \\ 0, & \textit{audit is not conducted}. \end{cases}$$

The economic control mechanism consists in that the seller proposes to the auctioneer a share in the profit of sale (a constant reward of the auctioneer is not considered in this model). Thus, if the collusion is absent, then the auctioneer receives $s_j^t b_{\max,j}^t$, and if the collusion is present, then he receives $s_j^t b_{*j}^t + \alpha\Delta_j^t = \alpha b_{\max,j}^t + (s_j^t - \alpha) b_{*j}^t$.

So, for an auction, the conditions of advantages for the auctioneer to be honest are

$$s_j^t b_{\max,j}^t \;\geq\; \alpha b_{\max,j}^t \;+\; (s_j^t - \alpha) b_{*j}^t, \text{ or } s_j^t \;\geq\; \alpha \text{because } b_{\max,j}^t \;>\; b_{*j}^t.$$

Given the made assumptions, the payoffs of both players at an auction j in the year t for different values of the indicators of corruption and audit are the following:

$$I_j^t = C_j^t = 1$$

$$S_{jt}^{11} = p_j^t (M - c_j^t) + (1 - p_j^t)[(1 - s_j^t) b_{*j}^t - c_j^t]$$

$$A_{jt}^{11} = -M p_j^t + (1 - p_j^t)(s_j^t b_{*j}^t + \alpha\Delta_j^t)$$

$$I_j^t = 1, C_j^t = 0$$

$$S_{jt}^{10} = (1 - s_j^t) b_{\max,j}^t - c_j^t$$

$$A_{jt}^{10} = s_j^t b_{\max,j}^t$$

$$I_j^t = 0, C_j^t = 1$$

$$S_{jt}^{01} = (1 - s_j^t) b_{*j}^t$$

$$A_{jt}^{01} = s_j^t b_{*j}^t + \alpha\Delta_j^t$$

$$I_j^t = C_j^t = 0$$

$$S_{jt}^{00} = (1 - s_j^t) b_{\max,j}^t$$

$$A_{jt}^{00} = s_j^t b_{\max,j}^t$$

Let us remind some notions from game theory. A game in normal form is defined as a triple (N, X, u), where $N = \{1, \ldots, n\}$ is a finite set of players; $X = X_1 \times \ldots \times X_n$ is a set of feasible outcomes; $x \in X$ is an outcome; $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$; X_i is a set of feasible strategies of the i-th player; $x_i \in X_i$ is a feasible strategy; $u = (u_1, \ldots, u_n)$; $u_i : X \to R$ is a payoff function of the i-th player. An outcome x^{NE} is a Nash equilibrium if $\forall i \in N \, \forall x_i \in X_i \; u_i(x^{NE}) \geq u_i(x_i, x_{-i}^{NE})$.

Thus, we receive a game in normal form between the seller and the auctioneer:

$$J_S = \sum_{t=1}^{T} \sum_{j=1}^{m(t)} e^{-\rho t} \left\{ I_j^t [C_j^t S_{jt}^{11} + (1 - C_j^t) S_{jt}^{10}] + (1 - I_j^t)[C_j^t S_{jt}^{01} + (1 - C_j^t) S_{jt}^{00}] \right\} \to \max$$

$$I_j^t \in \{0, 1\}, \; j = 1, \ldots, m(t), \; t = 1, \ldots, T;$$

$$J_A = \sum_{t=1}^{T} \sum_{j=1}^{m(t)} e^{-\rho t} \left\{ C_j^t [I_j^t A_{jt}^{11} + (1 - I_j^t) A_{jt}^{10}] + (1 - C_j^t)[I_j^t A_{jt}^{01} + (1 - I_j^t) A_{jt}^{00}] \right\} \to \max$$

$$C_j^t \in \{0, 1\}, \; j = 1, \ldots, m(t), \; t = 1, \ldots, T.$$

The information structure of this game is the following.

1. The values of n and T, vector $m(t)$, matrices $B^t = ||b_{ij}^t||$, $S^t = ||s_j^t||$, $CC^t = ||c_j^t||$, $j = 1, \ldots, m(t), t = 1, \ldots, T$, and the function $p(c)$ are given.
2. The seller and the auctioneer choose simultaneously and independently the matrices $||I_j^t||$ and $||C_j^t||$, $j = 1, \ldots, m(t), t = 1, \ldots, T$.
3. The payoffs J_S and J_A are calculated.

The research task is to implement a computer simulation with the built game theoretic model for a comparative analysis of the efficiency of strategies I and C for different values of the structural parameters $n, T, B^t, m(t)$, and the control parameters s_j^t, c_j^t. For planning the simulation experiments, we use the author's method of qualitatively representative scenarios [32].

3. Simulation Modeling

We used an original method of qualitatively representative scenarios (QRS) in simulation modeling [32]. An initial QRS set contains a small number of scenarios. It is assumed that if they are chosen well, then it is enough to receive an acceptable forecast of the controlled system dynamics. To prove the representativeness, we check the conditions of internal and external stability of the QRS set. Internal stability means that for any two scenarios from the QRS set, the respective payoffs of the players differ essentially. External stability means that for any scenario that does not belong to the QRS set, there is a scenario from this set such that the difference of the payoffs is small. The QRS set is corrected until both conditions are satisfied.

From now on, we fix the discount factor $\rho = 0.1$, the number of agents $n = 3$, the game length $T = 3$, and suppose that the value $p_j^t = \frac{c_j^t}{b_{maxj}^t} \in [0.1, 0.9]$.

Example 1. In the first example, we consider a simple case when the bids are the same for any auction and year. Namely,

$$B^1 = \begin{pmatrix} 10 & 20 & 40 \\ 10 & 20 & 40 \\ 10 & 20 & 40 \end{pmatrix}, \; B^2 = \begin{pmatrix} 10 & 20 & 40 \\ 10 & 20 & 40 \\ 10 & 20 & 40 \end{pmatrix}, \; B^3 = \begin{pmatrix} 10 & 20 & 40 \\ 10 & 20 & 40 \\ 10 & 20 & 40 \end{pmatrix},$$

i.e., the first agent bids \$10, the second agent bids \$20, and the third agent bids \$40 in each auction and each year.

Additionally, fix $m(t) \equiv 3$, M = 10, and

$$S = \begin{pmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{pmatrix}, \; c = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

For these parameters independently of α for $I_{ij} = 0$ and $C_{ij} = 0$, we receive $J_S = 294.145$, and $J_A = 32.683$. The results for $C_{ij} = 1, I_{ij} = 1$ are given in Table 1.

Table 1. Players' payoffs for the chosen strategies $C_{ij} = 1, I_{ij} = 1$.

α	J_S	J_A
0.1	137.268	29.823
0.2	137.268	45.756
0.3	137.268	61.689
0.4	137.268	77.622
0.5	137.268	93.555
0.6	137.268	109.487
0.7	137.268	125.420
0.8	137.268	141.353
0.9	137.268	157.286

Example 2. In this example, we fix the following parameters and find the optimal strategies of the players and the respective payoffs in dependence of the penalty value are presented in Tables 2–4. Suppose that $m(t) = 3$, $\alpha = 0.5$.

$$B^1 = \begin{pmatrix} 10 & 14 & 16 \\ 8 & 10 & 12 \\ 6 & 8 & 16 \end{pmatrix}, B^2 = \begin{pmatrix} 12 & 10 & 14 \\ 14 & 16 & 8 \\ 12 & 6 & 10 \end{pmatrix}, B^3 = \begin{pmatrix} 20 & 18 & 16 \\ 12 & 16 & 20 \\ 20 & 18 & 16 \end{pmatrix},$$

$$S = \begin{pmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{pmatrix}, \qquad c = \begin{pmatrix} 1 & 2 & 4 \\ 6 & 8 & 10 \\ 12 & 14 & 16 \end{pmatrix}.$$

Table 2. Players' optimal strategies and payoffs in dependence of the penalty value M.

M	J_S	J_A	I	C
1	89.180	75.243	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
10	148.984	66.546	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
100	65.564	64.323	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
1000	65.564	64.323	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Table 3. Players' optimal strategies and payoffs in dependence of the value of α.

α	J_S	J_A	I	C
0.1	66.016	57.971	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
0.3	65.326	58.144	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
0.5	65.326	59.324	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
0.9	65.326	64.323	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Table 4. Players' optimal strategies and payoffs in dependence of the audit cost.

No	c	J_S	J_A	I	C
1	$c_j^t = 0$	47,619	12,658	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
2	$c_1^1 = 1 \ otherwise \ c_j^t = 0$	47.326	15.14	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
3	$c_j^t = 1$	46,231	52.91	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
4	$c_j^t : p_j^t = 0.1$	45.387	14.542	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Thus, there is a maximal value of the penalty that encourages the auctioneer to enter a collusion. If the penalty is greater that this value, then it is more advantageous to be honest.

Example 3. In this example, we fix the same parameter values as in the previous one but also assume that M = 10. Now we vary the parameter α from 0.1 till 0.9.

In dependence of the value of α, it becomes advantageous for the auctioneer to enter a collusion with agents having certain parameters. Therefore, with an increase in the share of the auctioneer in the payoff from a corrupt collusion, the total gain of the auctioneer and the likelihood of corruption increase. Further, this statement makes sense if the seller spends quite a bit to check collusion at the next auction.

Example 4. In this example, we examine a dependence of I and C on the audit cost. Suppose that $m(t)$ = 3, $\alpha = 0.5$,

$$B^1 = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 5 & 4 \\ 6 & 3 & 2 \end{pmatrix}, \ B^2 = \begin{pmatrix} 6 & 3 & 5 \\ 4 & 8 & 3 \\ 2 & 5 & 6 \end{pmatrix}, \ B^3 = \begin{pmatrix} 3 & 7 & 2 \\ 5 & 5 & 7 \\ 3 & 4 & 3 \end{pmatrix}, \ S = \begin{pmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{pmatrix}.$$

Naturally, if the seller has zero audit cost, then she detects each auction in each year. For the given bids, a collusion is not advantageous for the auctioneer. If probability of detection is equal to 0.1, then the conclusion is more interesting: both collusion and detection depend directly on the difference between the maximal bid and the next one.

Example 5. Let us vary the share of the auctioneer $s_j^i : 0 \leq s_j^i < 1$ for each auction and in each year. In this case, the bids of the auction participants are not so important to us, so we will use the values of the parameters from example 1 and try to find how the payoffs and strategies will change. You can find the results in Table 5 below.

From the calculations in Table 5, we see that, predictably, along with an increase in the share of the auctioneer, his payoff increases, but along with this, it can also be seen that s_j^i affects the strategies of the auction participants. So, with an increase in the share of the auctioneer in payoffs, the share of the seller decreases, along with this, it becomes less profitable for her to spend money on checking collusion in the next auction. We can notice that starting from a share of 0.2, the auctioneer colludes with the players, and the number of collusions only increases with the increase in the share of the auctioneer. As a result, with a share of 0.9, the auctioneer colludes with the players in each auction and in each year. That is, with the selected parameters, the auction–seller economic control method works in the opposite direction. Since the penalty remains unchanged in this example, it becomes profitable for the auctioneer to risk colluding with the bidder in the hope of making even more profit as the profit from the share in the sale increases. Therefore, with an increase in the share of the auctioneer and his profit, it makes sense to increase the penalty for collusion.

Table 5. Players' optimal strategies and payoffs in dependence of the auctioneer share.

No	s_j^i	J_S	J_A	I	C
1	0	314.4	23.43	$\begin{pmatrix}1&1&1\\1&1&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$
2	0.1	287.15	44.68	$\begin{pmatrix}1&1&1\\1&1&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$
3	0.2	258.46	69.37	$\begin{pmatrix}1&1&1\\1&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&0\\0&0&0\\0&0&0\end{pmatrix}$
4	0.3	228.78	98.05	$\begin{pmatrix}1&1&1\\1&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$
5	0.4	196.1	130.73	$\begin{pmatrix}1&1&1\\1&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$
6	0.5	163.41	172.13	$\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&0&0\\0&0&0\end{pmatrix}$
7	0.6	130.73	196.1	$\begin{pmatrix}1&1&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&0\\0&0&0\end{pmatrix}$
8	0.7	65.37	261.462	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\0&0&0\end{pmatrix}$
9	0.8	32.68	294.15	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&0&0\end{pmatrix}$
10	0.9	2.97	326.83	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$

Example 6. Let us now look at the dependence of the strategies and payoffs of the players depending on the discount factor ρ. For the calculation, we will use the parameters from example 1. The calculation results are presented in Table 6.

Table 6. Players' optimal strategies and payoffs in dependence of the discount factor ρ..

No	ρ	J_S	J_A	I	C
1	0.1	215.89	130.73	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$
2	0.2	197.12	119.47	$\begin{pmatrix}1&1&1\\1&1&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&0&0\end{pmatrix}$
3	0.3	181.17	109.9	$\begin{pmatrix}1&1&1\\1&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&0\\0&0&0\end{pmatrix}$
4	0.4	167.57	101.74	$\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$
5	0.5	155.95	94.77	$\begin{pmatrix}1&1&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&0\\0&0&0\\0&0&0\end{pmatrix}$
6	0.6	146.0	88.8	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$
7	0.7	137.45	83.67	(0)	(0)
8	0.8	130.1	79.26	(0)	(0)
9	0.9	118.26	72.15	(0)	(0)

Given such parameters, with an increase in the discount factor, the gains of the seller and the auctioneer fall. It becomes less profitable for the seller to spend money on checking the next auction, and for the auctioneer to collude because of the decreasing benefits compared to the fine. At the maximum values of the discount coefficient, it becomes not at all profitable for the seller to spend funds on checking auctions, and the auctioneer to collude to obtain additional profit.

Example 7. Let us study the dependence of the winnings and strategies of the auction participants on the size of the players' bids. To do this, we take the initial values of the bets of each player as it follows:

$$B^1 = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \\ 1 & 2 & 4 \end{pmatrix},\ B^2 = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \\ 1 & 2 & 4 \end{pmatrix},\ B^3 = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \\ 1 & 2 & 4 \end{pmatrix}.$$

We are interested to see how, with fixed parameters, and how the strategies of the seller and the auctioneer will change with a proportional increase in the difference in the bids of the auction participants. Therefore, we introduce a multiplier $\nu : \nu = 1, \ldots, 10$ for the players' rates νB^i and calculate the optimal strategies and payoffs of the auctioneer and the seller for each $\nu B^i,\ i = 1,\ \ldots,\ 3$. The calculation results are presented in Table 7. We suppose that $m(t) = 3$, $\alpha = 0.3$ and $S = \begin{pmatrix} 0.5 & 0.2 & 0.1 \\ 0.4 & 0.5 & 0.2 \\ 0.4 & 0.4 & 0.1 \end{pmatrix}, c = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$.

Table 7. Optimal strategies of auction participants and their payoffs depending on the bid size.

ν	J_S	J_A	I	C
1	19.61	13.07	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
2	39.22	26.15	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
3	57.83	40.42	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
4	77.44	53.89	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
5	95.32	69.09	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
6	113.29	84.87	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
7	132.91	99.01	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
8	149.81	116.06	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$
9	169.41	130.57	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$
10	189.021	145.072	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

From the results in Table 7, we see that with an increase in rates, that is, an increase in the value of an item sold at an auction, the likelihood of corruption and the expediency of inspections increase. You can also notice that as the rates increase, the seller begins to check the auctions for which the cost of checking is minimal, and the share of the auctioneer is maximal. The auctioneer, in turn, colludes in those cases when his share is maximal in the auctions in which it is not profitable for the seller to conduct an audit. As a result, we get a situation in which it is beneficial for the seller to check those auctions in which the auctioneer does not collude with the auction participants. On the other hand, if the seller had not carried out these checks, then the auctioneer would have colluded in these competitions and would have received a large profit, while the seller would have lost more than he would spend on checking the auctions.

Example 8. Now let us see how the strategies of the players change depending on the probability of being caught, which is a function of the costs of its implementation. For calculations, we will use the parameters from example 2. The calculation results are presented in Table 8.

Table 8. Optimal strategies of auction participants and their payoffs depending on the probability of capture.

No	$p(c_j^t)$	J_S	J_A	I	C
1	$\left(\frac{c_j^t+0.1}{c_j^t+1}\right)^{1/10}$	161.06	41.12	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$
2	$\left(\frac{c_j^t+0.1}{c_j^t+1}\right)^{1/2}$	153.4	45.91	$\begin{pmatrix}0&0&1\\1&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$
3	$\frac{c_j^t+0.1}{c_j^t+1}$	145.92	66.55	$\begin{pmatrix}0&0&0\\1&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$
4	$\left(\frac{c_j^t+0.1}{c_j^t+1}\right)^{2}$	131.71	71.08	$\begin{pmatrix}0&0&0\\0&0&0\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\0&0&0\end{pmatrix}$
5	$\left(\frac{c_j^t+0.1}{c_j^t+1}\right)^{10}$	122.21	82.31	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$

The seller and the auctioneer change their strategies depending on the growth rate of the catch probability function. So, if the probability of being caught is high enough and the cost of checking auctions is relatively low, then it is beneficial for the seller to check each auction (note that the cost of checking in our example increases with each auction held). In the last line in Table 8, we see that the seller does not conduct any checks at all, because in order to achieve the same probability of capture as for the function from line 1 of Table 8, the seller will have to spend a multiple of more resources, which is not comparable with respect to the potential win.

So, we have considered the main examples for the auction corruption model. We did not increase the number of auctions per year and the number of years considered, because for such calculations, it would take much more time and serious revision of the program algorithm, because the number of considered cases increases exponentially. It makes sense to reduce the number of participants in the competition to two to simplify the computational process and at the same time, to vary several parameters with more auctions in one year. In addition, the above examples allow us to understand and evaluate the model of corruption at the auction and draw some conclusions for further research.

4. Conclusions and Future Work

In every auction, there is a maximal value of penalty M, depending on the model parameters, for which it is advantageous for the auctioneer to organize a collusion for any strategy of the agents. If the penalty is greater than M, then it is more advantageous for him to be honest. Additionally, the greater is α (in fact, a share of the auctioneer in the difference of the first and the second bid), the more advantageous for him to enter a collusion. Thus, a probability of collusion is proportional to the corruption gain of the auctioneer. Depending on the share of the auctioneer, he may collude with the bidders for more profit if the profit from such collusion is greater than the expected penalty for the collusion itself. Therefore, increasing only the share of the auctioneer in the hope that he will have enough profit will not protect against the emergence of corruption. With an increase in the share of the auctioneer, it is also necessary to increase the cost of checking corruption and the amount of the fine in case of capture.

Thus, factors of corruption are low probability of detection, small penalty, and big corruption gain of the auctioneer. This result is quite expected but it confirms the model adequacy and allows for its development.

Certainly, the used methodology has some limitations and constraints. The method of qualitatively representative scenarios in simulation modeling is rather a heuristic one. There are still no strict estimates of its precision. Additionally, the external stability cannot be checked completely, and the number of required tests is also unknown.

The following areas of further research seem interesting.

1. To conduct more simulation experiments and to elicit additional connections between the model parameters.
2. To prove internal and external stability (qualitative representativeness) of the sets of simulation scenarios [32].
3. To investigate analytically several specific cases of the considered game theoretic model.

The results of investigation can be used by organizers of different auctions and tenders, especially in public procurement.

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