






## Article

# Sufficient Conditions for the Existence and Uniqueness of Minimizers for Variational Problems under Uncertainty

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**Abstract:** Fuzzy variational problems have received significant attention over the past decade due to a number of successful applications in fields such as optimal control theory and image segmentation. Current literature on fuzzy variational problems focuses on the necessary optimality conditions for finding the extrema, which have been studied under several differentiability conditions. In this study, we establish the sufficient conditions for the existence of minimizers for fuzzy variational problems under a weaker notion of convexity, namely preinvexity and Buckley–Feuring differentiability. We further discuss their application in a cost minimization problem.

**Keywords:** fuzzy variational problem; existence of minimizer; sufficient conditions; invex sets; preinvex functions

**MSC:** 49K21; 93C23; 94D05



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## 1. Introduction

Calculus of variations has a long history for finding the optimal solutions for given functionals, with diverse applications from different areas of statistics, engineering, and mathematics [1,2] to the areas of biomedical and materials sciences [3,4]. The classical approach of the calculus of variations with crisp functionals is limited to problems where all the variables defining the considered energy functional are crisp or precisely known [5]. However, in most practical cases, there are many uncertainties for which deterministic and classical models may not work. One of the reasons for this is that the real-world mathematical models can be complex due to the indeterminacy brought by some parameters that are usually uncertain, so representing them using real or crisp numbers may lead to an impractical presentation of the real-world [6]. For instance, in 1997, Hullermeier [7] observed that the available information about the functional relationships in a dynamical system may not be known accurately. Therefore, he utilized a fuzzy inference system and observed that the initial value taken in fuzzy numbers gave better results in terms of the semantic basis and the precision of predictions than other approaches. Many studies, varying from the detection of the objects in an image, uncertainty assessment of hydrological models, and contouring vessel structures, highlight the advantages of considering fuzzy functionals over crisp functional models [8–11].

As far as the uncertainties in variational problems are concerned, their necessary optimality conditions were first introduced by Farhadinia [12] and extended by Fard et al. [13], where they focused on the necessary optimality conditions of the fuzzy variational problem for finding its extrema by deriving its Euler–Lagrange equations. Similarly, Fard and Salehi [14] and Zhang et al. [15] addressed the fuzzy fractional variational problems based on the Caputo-type fuzzy fractional derivative. However, the current literature is based on the intuition that every extremal problem will have a solution and this assumption is

not always valid, since there is no guarantee of the existence of a solution, as observed by Verma et al. [16]. The existing studies employed various differentiability concepts to derive the necessary optimality conditions, namely the Euler–Lagrange equations, which lead to local extrema.

In addition, it can be observed that the studies on applications in image segmentation and contouring based on the conventional crisp functional models depend heavily on initial values, which are insufficient in realistically addressing the problems since the accuracy of the initial data cannot be guaranteed [8–11]. Meanwhile, fuzzy models involving fuzzy energy functionals can extract the information more accurately under the consideration of a realistic range of initial values, as in the studies by Hüllermeier [7] and Seikkala [17]. However, as discussed above, there is no sufficient result to guarantee the existence of minimizers of such problems. Hence, it is advantageous to establish the sufficient conditions for the existence and uniqueness of minimizers in the fuzzy setting [16,18].

In view of the calculus of variations, the sufficiency results are generally proven using many functional properties to guarantee the existence of minimizers. One of such properties is convexity. Though convexity might seem to be a suitable notion, it may not work well in the fuzzy environment [19]. However, it turns out that preinvexity, which is a weaker notion of convexity, may work better in the fuzzy sense [20]. Therefore, this paper considers a weaker notion of convexity, namely preinvexity, in establishing the sufficiency results.

The organization of this paper goes as follows. First, since fuzzy sets are the underlying theory of this study, we dedicate Section 2 to the basic notations and definitions that are essential throughout this paper. This is followed by Section 3, which introduces fuzzy variational problems and their necessary conditions for the existence of minimizers, and establishes the framework for the study of sufficient conditions in terms of preinvexity that will be discussed in detailed in Section 4. A result on preinvex functions in terms of  $\alpha$ -level sets is also established in Section 4. Subsequently, in Section 5, we define multi-variable preinvexity and derive the lower bound property result for differentiable preinvex functions, which is essential for our study. Finally, in Section 6, we prove the sufficient conditions for the existence and uniqueness of minimizers for the fuzzy variational problems. We then demonstrate the application of the obtained results via a cost minimization problem in Section 7 and conclude the findings of this study in Section 8.

## 2. Basic Notations and Definitions

In this section, we present some standard definitions that are essential for the study. They can be traced back to the work of Zadeh [21] and Buckley and Feuring [22], and interested readers can also refer to the books by Zimmermann [23] and Klir et al. [24].

A fuzzy set is a collection of elements along with its membership values varying from 0 to 1. In particular, a fuzzy set in the universe of discourse  $X$ , denoted by  $\tilde{A}$ , is defined as follows:

$$\tilde{A} = \{ (t, \mu_{\tilde{A}}(t)) : t \in X \},$$

where  $\mu_{\tilde{A}} : X \rightarrow [0, 1]$  is the membership function that provides a degree of membership to each element of  $X$  ranging from 0 to 1.

Such fuzzy sets do not possess addition and multiplication inverses [25], which makes it impractical to solve fuzzy equations. In response to this, some researchers have studied the properties of membership functions under the notions of convexity and upper semicontinuity so that basic arithmetic operations can be invoked [26,27], which led to the following definition of the fuzzy number.

**Definition 1** (Fuzzy number). Let  $X = \mathbb{R}$  be the universe of discourse. A fuzzy set  $\tilde{A}$  is said to be a fuzzy number if its membership function  $\mu_{\tilde{A}} : \mathbb{R} \rightarrow [0, 1]$  satisfies the following conditions:

- (i)  $\mu_{\tilde{A}}$  is normal; i.e., there exists  $t \in \mathbb{R}$  such that  $\mu_{\tilde{A}}(t) = 1$ ;
- (ii)  $\mu_{\tilde{A}}$  is fuzzy convex; i.e.,

$$\mu_{\tilde{A}}(\lambda s + (1 - \lambda)t) \geq \min\{\mu_{\tilde{A}}(s), \mu_{\tilde{A}}(t)\}$$

- for all  $\lambda \in [0, 1], s, t \in \mathbb{R}$ ;  
 (iii)  $\mu_{\tilde{A}}$  is upper semicontinuous;  
 (iv) The closure of the support of  $\mu_{\tilde{A}}$  is bounded.

We now recall the notion of  $\alpha$ -level sets that transforms a fuzzy set into its crisp level sets, which in turn allows us to extend arithmetic operations defined on crisp sets to the case of fuzzy sets [25].

**Definition 2** ( $\alpha$ -level set). Let  $\tilde{A}$  be a fuzzy number and

$$\mu_{\tilde{A}} : \mathbb{R} \rightarrow [0, 1]$$

be its membership function. The  $\alpha$ -level set of  $\tilde{A}$ , denoted by  $A[\alpha]$ , is defined as

$$A[\alpha] = \{t \in \mathbb{R} : \mu_{\tilde{A}}(t) \geq \alpha\}$$

for every  $\alpha \in [0, 1]$ .

From Definition 2, we can see that the  $\alpha$ -level set is indeed a crisp set and a closed interval in  $\mathbb{R}$  for each  $\alpha \in [0, 1]$ :

$$A[\alpha] = [A^l(\alpha), A^r(\alpha)],$$

where  $A^l$  and  $A^r$  denote the left hand and right endpoints of  $A[\alpha]$ , respectively. The properties of  $A^l$  and  $A^r$  (defined for each  $\alpha \in [0, 1]$ ) are as follows:

- (i)  $A^l : [0, 1] \rightarrow \mathbb{R}$  is an increasing and bounded function;
- (ii)  $A^r : [0, 1] \rightarrow \mathbb{R}$  is a decreasing and bounded function;
- (iii)  $A^l(\alpha) \leq A^r(\alpha)$  for every  $\alpha \in [0, 1]$ .

Since the  $\alpha$ -level set represents a closed interval in  $\mathbb{R}$  for each  $\alpha \in [0, 1]$ , the operation on  $\alpha$ -level sets follows that of interval arithmetic [28]. We now state the following concept of partial ordering on fuzzy numbers.

**Definition 3** (Partial order on  $\mathcal{F}$  [29,30]). Let  $\mathcal{F}$  be the set of all fuzzy numbers on  $\mathbb{R}$ , say,  $\tilde{A}_1$  and  $\tilde{A}_2 \in \mathcal{F}$ . Then

- (i)  $\tilde{A}_1 \preceq \tilde{A}_2$  if  $A_1^l(\alpha) \leq A_2^l(\alpha)$  and  $A_1^r(\alpha) \leq A_2^r(\alpha)$  for every  $\alpha \in [0, 1]$ ;
- (ii)  $\tilde{A}_1 \prec \tilde{A}_2$  if  $A_1^l(\alpha) < A_2^l(\alpha)$  and  $A_1^r(\alpha) < A_2^r(\alpha)$  for every  $\alpha \in [0, 1]$ ;
- (iii)  $\tilde{A}_1 \succeq \tilde{A}_2$  if  $A_1^l(\alpha) \geq A_2^l(\alpha)$  and  $A_1^r(\alpha) \geq A_2^r(\alpha)$  for every  $\alpha \in [0, 1]$ ;
- (iv)  $\tilde{A}_1 \succ \tilde{A}_2$  if  $A_1^l(\alpha) > A_2^l(\alpha)$  and  $A_1^r(\alpha) > A_2^r(\alpha)$  for every  $\alpha \in [0, 1]$ ;
- (v)  $\tilde{A}_1 \approx \tilde{A}_2$  if  $\tilde{A}_1 \preceq \tilde{A}_2$  and  $\tilde{A}_1 \succeq \tilde{A}_2$  holds for every  $\alpha \in [0, 1]$ .

To proceed further, we now present the definition of a fuzzy-valued function analogous to that of a real-valued function.

**Definition 4.** Let  $\Omega \subseteq \mathbb{R}$ . The function  $\tilde{x} : \Omega \rightarrow \mathcal{F}$  is said to be a fuzzy-valued function (or fuzzy function) if for every  $t \in \Omega$ ,  $\tilde{x}(t)$  represents a fuzzy number.

By Definition 2, the  $\alpha$ -level set of  $\tilde{x}(t)$  is represented by

$$x(t)[\alpha] = [x^l(t, \alpha), x^r(t, \alpha)],$$

where  $x^l(t, \alpha) = \min\{x(t)[\alpha]\}$  and  $x^r(t, \alpha) = \max\{x(t)[\alpha]\}$  are bounded increasing and decreasing functions of  $\alpha$ , respectively.

**Definition 5.** A fuzzy function  $\tilde{x} : \Omega \rightarrow \mathcal{F}$  is said to be

- (i) continuous at  $t \in \Omega$  if  $x^l(t, \alpha)$  and  $x^r(t, \alpha)$  are continuous functions of  $t \in \Omega$  for all  $\alpha \in [0, 1]$ ;
- (ii) differentiable at  $t \in \Omega$  if both  $x^l(t, \alpha)$  and  $x^r(t, \alpha)$  are differentiable with respect to  $t \in \Omega$  for each fixed  $\alpha \in [0, 1]$  and the interval  $\left[\frac{dx^l(t, \alpha)}{dt}, \frac{dx^r(t, \alpha)}{dt}\right]$  defines the  $\alpha$ -level set of a fuzzy number for every  $t \in \Omega$  and  $\alpha \in [0, 1]$ . It is denoted by

$$\frac{dx(t)[\alpha]}{dt} = \left[ \frac{dx^l(t, \alpha)}{dt}, \frac{dx^r(t, \alpha)}{dt} \right];$$

- (iii) integrable with respect to  $t \in \Omega$  if both  $x^l(t, \alpha)$  and  $x^r(t, \alpha)$  are Riemann integrable functions of  $t \in \Omega$  and the interval  $\left[\int_{\Omega} x^l(t, \alpha) dt, \int_{\Omega} x^r(t, \alpha) dt\right]$  defines the  $\alpha$ -level set of the corresponding fuzzy number for all  $\alpha \in [0, 1]$ . It is denoted by

$$\int_{\Omega} x(t)[\alpha] dt = \left[ \int_{\Omega} x^l(t, \alpha) dt, \int_{\Omega} x^r(t, \alpha) dt \right].$$

**Definition 6** (Anti-derivative). A fuzzy function  $\tilde{x}_1 : \Omega \rightarrow \mathcal{F}$  is said to be an anti-derivative of the fuzzy function  $\tilde{x}_2 : \Omega \rightarrow \mathcal{F}$  if

$$\frac{d\tilde{x}_1(t)[\alpha]}{dt} = \tilde{x}_2(t)[\alpha]$$

for every  $t \in \Omega$  and  $\alpha \in [0, 1]$ .

### 3. Fuzzy Variational Problems

Let  $\tilde{x} = \tilde{x}(t)$  be a fuzzy function of  $t \in [t_0, t_f] \subset \mathbb{R}$  belonging to the set of fuzzy functions whose first and second derivatives are continuous with respect to  $t \in [t_0, t_f]$ . The fuzzy variational problem (FVP) can be expressed as:

$$\begin{aligned} \text{minimize} \quad & \tilde{I}(\tilde{x}) := \int_{t_0}^{t_f} \tilde{f}(t, \tilde{x}(t), \dot{\tilde{x}}(t)) dt \\ \text{subject to} \quad & \tilde{x}(t_0) \approx \tilde{x}_0, \tilde{x}(t_f) \approx \tilde{x}_f. \end{aligned} \quad (1)$$

Here, the integrand  $\tilde{f}$  gives a fuzzy number corresponding to every point  $(t, \tilde{x}, \dot{\tilde{x}}) \in \mathbb{R} \times \mathcal{F} \times \mathcal{F}$  where  $\tilde{x}$  and  $\dot{\tilde{x}}$  are fuzzy functions of  $t \in [t_0, t_f]$ . Furthermore, we assume that  $\tilde{f}$  has continuous second partial derivative with respect to all of its variables. Let

$$\tilde{X} = \{\tilde{x} \mid \tilde{x} \text{ is twice continuously differentiable, } \tilde{x}(t_0) \approx \tilde{x}_0, \tilde{x}(t_f) \approx \tilde{x}_f\}$$

be a set of such admissible curves that satisfy the end conditions as well as being twice continuously differentiable with respect to  $t \in [t_0, t_f]$ . The necessary conditions given by the fuzzy Euler–Lagrange equations are as follows:

**Theorem 1** (Fuzzy Euler–Lagrange equations [12]). Let  $\tilde{x}_* \in \tilde{X}$  be a fuzzy function and the integrand  $\tilde{f}$  have a continuous second partial derivative with respect to all of its variables in (1). If  $\tilde{x}_*$  corresponds to a local minimum of the fuzzy functional  $\tilde{I}$  in (1) for all  $\alpha \in [0, 1]$ , then the following equations necessarily hold:

$$\frac{\partial f^l}{\partial x^l}(t, \alpha, x_*^l(t, \alpha), \dot{x}_*^l(t, \alpha)) - \frac{d}{dt} \left( \frac{\partial f^l}{\partial \dot{x}^l}(t, \alpha, x_*^l(t, \alpha), \dot{x}_*^l(t, \alpha)) \right) = 0, \quad (2)$$

and

$$\frac{\partial f^r}{\partial x^r}(t, \alpha, x_*^r(t, \alpha), \dot{x}_*^r(t, \alpha)) - \frac{d}{dt} \left( \frac{\partial f^r}{\partial \dot{x}^r}(t, \alpha, x_*^r(t, \alpha), \dot{x}_*^r(t, \alpha)) \right) = 0. \quad (3)$$

Fard et al. [13] went on to derive the fuzzy Euler–Lagrange Equations (2) and (3) under the generalized Hukuhara (gH) differentiability. The study was further expanded to fuzzy fractional variational problems by Fard and Salehi [14] and Zhang et al. [15] using the Caputo-type fuzzy fractional and Atangana–Baleanu fractional derivatives, respectively. All such studies employed different differentiability concepts from the corresponding Euler–Lagrange equations. However, these techniques were aimed at deriving the Euler–Lagrange equations in the fuzzy setting and to acknowledge the sufficient conditions for the existence of minimizer. Although Heidari et al. [31] attempted to derive some sufficiency results, the study considered the convexity notion, which is inadequate in the fuzzy setting as discussed below. Moreover, it was based on differential inclusion and not on the fuzzy setting [16].

In the study of variational problems, convexity provides a sufficient condition for the existence of a minimizer under the assumption of twice continuous differentiability of the integrand. Although the notion of convexity turns out to be sufficient in crisp variational problems, it is too strong to hold in the fuzzy sense. We can consider more relaxed conditions of convexity to enlarge the space of admissible functions. We now discuss the standard classical notion of convexity and study its fundamental property in the fuzzy setting that would pave the way to consider its relaxed notion, namely preinvexity. The standard notion of convexity for differentiable functions is as follows:

**Theorem 2 ([32]).** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function. Then,  $f$  is convex if and only if for every  $x, y \in \mathbb{R}$ , it satisfies the following:*

$$f(x) \geq f'(y)(x - y) + f(y). \quad (4)$$

The result above holds for higher dimensions. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a differentiable function. Then,  $f$  is convex if and only if for every  $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n, n \geq 1$ , it satisfies the following:

$$f(x) \geq \nabla f(y)^T(x - y) + f(y), \quad (5)$$

where  $\nabla$  denotes the gradient of  $f$ . To proceed, we present the similar property for the convex fuzzy functions [29,33,34].

**Definition 7 (Convex fuzzy function).** *Let  $\Omega \subset \mathbb{R}^n, n \geq 1$ . A fuzzy-valued function  $\tilde{f} : \Omega \rightarrow \mathcal{F}$  is said to be convex if both  $f^l(x, \alpha)$  and  $f^r(x, \alpha)$  are convex functions for all  $\alpha \in [0, 1]$ . In particular, for every  $x, y \in \Omega$  and  $\lambda \in [0, 1]$ , we have*

$$\tilde{f}((1 - \lambda)x + \lambda y) \preceq (1 - \lambda)\tilde{f}(x) + \lambda\tilde{f}(y),$$

*if and only if*

$$f^l((1 - \lambda)x + \lambda y, \alpha) \leq (1 - \lambda)f^l(x, \alpha) + \lambda f^l(y, \alpha),$$

*and*

$$f^r((1 - \lambda)x + \lambda y, \alpha) \leq (1 - \lambda)f^r(x, \alpha) + \lambda f^r(y, \alpha).$$

The functions  $f^l(x, \alpha)$  and  $f^r(x, \alpha)$  represent real-valued crisp convex functions for all  $\alpha \in [0, 1]$ . Subsequently, by Theorem 2, we obtain

$$f^l(x, \alpha) \geq \nabla f^l(y, \alpha)(x - y) + f^l(y, \alpha),$$

$$f^r(x, \alpha) \geq \nabla f^r(y, \alpha)(x - y) + f^r(y, \alpha),$$

for every  $x, y \in \Omega$  and  $\lambda \in [0, 1]$ . By Definition 7, we deduce the following equivalent expression:

$$\tilde{f}(x) \succeq \tilde{\nabla} \tilde{f}(y)(x - y) + \tilde{f}(y). \quad (6)$$

Here, we denote the gradient in the fuzzy case as  $\tilde{\nabla}$ . However, it is proven that the notion of convex fuzzy mapping in Definition 7 is restrictive enough that the fundamental property of differentiable convex function (6) may not hold in the fuzzy setting [19]. Therefore, we now consider weaker notions of convexity that are nevertheless sufficient for proving the existence results, namely, invexity and preinvexity.

#### 4. Preinvexity for Fuzzy Variational Problems

The concept of invexity was introduced by Hanson [35] in 1981, where the term ‘invex’ stands for “invariant convex” [36]. Using invex functions, Hanson formed a wider class of functions while retaining the lower bound property of convex functions. In 1994, Noor [37] extended a similar notion in the fuzzy context and studied the concepts of fuzzy preinvex functions and fuzzy invex sets. It was observed that fuzzy invex sets and preinvex functions are more relaxed than fuzzy convex sets and fuzzy convex functions, respectively [35,38]. Moreover, the KKT conditions that are used to find an optimal solution for constrained optimization problems work as sufficient and necessary for invex functions. In addition to that, the fundamental property (6) that fails to hold for fuzzy convex functions holds true in the sense of fuzzy preinvex functions [19,20]. We now state the definition of these concepts first [20,39].

**Definition 8** (Invex set). Let  $\Omega \subset \mathcal{F}$ . We say  $\Omega$  is an invex set with respect to a function  $\tilde{\eta} : \Omega \times \Omega \rightarrow \mathcal{F}$  if, for each  $\tilde{x}, \tilde{y} \in \Omega$  and  $\lambda \in [0, 1]$ ,

$$\tilde{y} + \lambda \tilde{\eta}(\tilde{x}, \tilde{y}) \in \Omega.$$

As noted by Hanson [35] and Mohan and Neogy [38], every convex set is invex, but the converse may not hold true. In general, convex sets are particular cases of invex sets. Using the above analysis, we shall now begin with a few definitions to discuss a weaker notion of fuzzy convex functions, namely the fuzzy preinvex functions [20], that enables the fuzzy functions to follow the fundamental property similar to (6).

**Definition 9** (Preinvex function). Let  $\Omega \subset \mathcal{F}$  be an invex set with respect to a function  $\tilde{\eta}$ . A fuzzy-valued function  $\tilde{f} : \Omega \rightarrow \mathcal{F}$  is said to be preinvex on  $\Omega$  if

$$\tilde{f}(\tilde{y} + \lambda \tilde{\eta}(\tilde{x}, \tilde{y})) \preceq \lambda \tilde{f}(\tilde{x}) + (1 - \lambda) \tilde{f}(\tilde{y})$$

holds for all  $\tilde{x}, \tilde{y} \in \Omega$  and  $\lambda \in [0, 1]$ . It is said to be strictly preinvex on  $\Omega$  if

$$\tilde{f}(\tilde{y} + \lambda \tilde{\eta}(\tilde{x}, \tilde{y})) \prec \lambda \tilde{f}(\tilde{x}) + (1 - \lambda) \tilde{f}(\tilde{y})$$

holds for all  $\tilde{x}, \tilde{y} \in \Omega$  and  $\lambda \in [0, 1]$ .

Writing in terms of  $\alpha$ -level sets, we have

$$\begin{aligned} f^l(y^l + \lambda \eta^l(x^l, y^l), \alpha) &\leq \lambda f^l(x^l, \alpha) + (1 - \lambda) f^l(y^l, \alpha), \\ f^r(y^r + \lambda \eta^r(x^r, y^r), \alpha) &\leq \lambda f^r(x^r, \alpha) + (1 - \lambda) f^r(y^r, \alpha), \end{aligned}$$

for all  $\lambda$  and  $\alpha \in [0, 1]$ . These inequalities express the preinvexity of  $f^l(x, \alpha)$  and  $f^r(x, \alpha)$ , respectively.

**Lemma 1.** Let  $\Omega \subset \mathcal{F}$  be an invex set with respect to a function  $\tilde{\eta}$ . Let  $\tilde{f} : \Omega \rightarrow \mathcal{F}$  be a fuzzy-valued function, represented in terms of the  $\alpha$ -level set as

$$f(x)[\alpha] = [f^l(x, \alpha), f^r(x, \alpha)]. \quad (7)$$

Then,  $\tilde{f}$  is said to be preinvex on  $\Omega$  if and only if the left and right endpoint functions  $f^l(x^l, \alpha)$  and  $f^r(x^r, \alpha)$  are preinvex with respect to  $\eta^l$  and  $\eta^r$ , respectively.



**Proof.** If we let  $\tilde{f}$  be preinvex with respect to  $\tilde{\eta}$  for all  $\alpha \in [0, 1]$ , then we can easily deduce from Definition 9 and Equation (7) that  $f^l$  and  $f^r$  are also preinvex. Now, to prove the converse, assume that the left and right endpoint functions  $f^l(x^l, \alpha)$  and  $f^r(x^r, \alpha)$  are preinvex with respect to  $\eta^l$  and  $\eta^r$ , respectively. That is,

$$f^l(y^l + \lambda\eta^l(x^l, y^l), \alpha) \leq \lambda f^l(x^l, \alpha) + (1 - \lambda)f^l(y^l, \alpha), \quad (8)$$

$$f^r(y^r + \lambda\eta^r(x^r, y^r), \alpha) \leq \lambda f^r(x^r, \alpha) + (1 - \lambda)f^r(y^r, \alpha), \quad (9)$$

for all  $x^r, y^r \in \Omega$  and  $\lambda, \alpha \in [0, 1]$ . Let us consider the  $\alpha$ -level set of  $\tilde{f}(y + \lambda\eta(x, y))$ ,

$$f(y + \lambda\eta(x, y))[\alpha] = [f^l(y^l + \lambda\eta^l(x^l, y^l), \alpha), f^r(y^r + \lambda\eta^r(x^r, y^r), \alpha)].$$

As one can observe from Inequalities (8) and (9), the left and the right endpoints of the  $\alpha$ -level set of  $\tilde{f}(y + \lambda\eta(x, y))$  are less than or equal to the left and the right endpoints of the  $\alpha$ -level set of  $\tilde{f}(\tilde{x}) + (1 - \lambda)\tilde{f}(\tilde{y})$ , respectively. Therefore, we obtain

$$\tilde{f}(\tilde{y} + \lambda\tilde{\eta}(\tilde{x}, \tilde{y})) \preceq \lambda\tilde{f}(\tilde{x}) + (1 - \lambda)\tilde{f}(\tilde{y}).$$

Hence,  $\tilde{f}$  is preinvex with respect to  $\tilde{\eta}$ .  $\square$

## 5. Multi-Variable Preinvexity

In this section, we discuss the multi-variable preinvexity, in particular, preinvexity in the last two variables. We now define the preinvexity of the integrand in (1) with respect to its last two variables.

Let  $\omega_{\mathcal{F}} \times \omega_{\mathcal{F}} \subset \mathcal{F} \times \mathcal{F}$  be invex with respect to a non-negative function  $\tilde{\eta} : (\omega_{\mathcal{F}} \times \omega_{\mathcal{F}}) \times (\omega_{\mathcal{F}} \times \omega_{\mathcal{F}}) \rightarrow \mathcal{F} \times \mathcal{F}$  such that for each  $(\tilde{x}, \dot{\tilde{x}}), (\tilde{x}_*, \dot{\tilde{x}}_*) \in \omega_{\mathcal{F}} \times \omega_{\mathcal{F}}$ ,  $\lambda \in [0, 1]$ , the following holds:

$$\left( (\tilde{x}_*, \dot{\tilde{x}}_*) + \lambda\tilde{\eta}((\tilde{x}, \dot{\tilde{x}}), (\tilde{x}_*, \dot{\tilde{x}}_*)) \right) \in \omega_{\mathcal{F}} \times \omega_{\mathcal{F}},$$

where  $\tilde{\eta}$  is defined as

$$\tilde{\eta}((\tilde{x}, \dot{\tilde{x}}), (\tilde{x}_*, \dot{\tilde{x}}_*)) = (\tilde{\eta}_1(\tilde{x}, \tilde{x}_*), \tilde{\eta}_2(\dot{\tilde{x}}, \dot{\tilde{x}}_*)),$$

where  $\tilde{\eta}_1, \tilde{\eta}_2 : \omega_{\mathcal{F}} \times \omega_{\mathcal{F}} \rightarrow \mathcal{F}$  are well defined maps. This implies that  $\tilde{\eta}$  is well defined.

Here,  $\tilde{\eta}_2$  and  $\tilde{\eta}_1$  are chosen in such a way that the anti-derivative of  $\tilde{\eta}_2$  with respect to  $t$  is equal to  $\tilde{\eta}_1$ , where  $\tilde{x}, \dot{\tilde{x}}, \tilde{x}_*, \dot{\tilde{x}}_*$  are functions of  $t$ . In other words, invexity of  $\omega_{\mathcal{F}} \times \omega_{\mathcal{F}}$  with respect to  $\tilde{\eta}_1$  and  $\tilde{\eta}_2$  for each  $\lambda \in [0, 1]$  is expressed as

$$(\tilde{x}_* + \lambda\tilde{\eta}_1(\tilde{x}, \tilde{x}_*), \dot{\tilde{x}}_* + \lambda\tilde{\eta}_2(\dot{\tilde{x}}, \dot{\tilde{x}}_*)) \in \omega_{\mathcal{F}} \times \omega_{\mathcal{F}}.$$

**Example 1.** Let  $\omega_{\mathcal{F}} \times \omega_{\mathcal{F}} \subset \mathcal{F} \times \mathcal{F}$ . Let

$$\tilde{\eta}((\tilde{x}, \dot{\tilde{x}}), (\tilde{x}_*, \dot{\tilde{x}}_*)) = (\tilde{\eta}_1(\tilde{x}, \tilde{x}_*), \tilde{\eta}_2(\dot{\tilde{x}}, \dot{\tilde{x}}_*)),$$

where

$$\tilde{\eta}_1 = \begin{cases} \tilde{x} - \tilde{x}_*, & \text{if } \tilde{x}, \tilde{x}_*, \dot{\tilde{x}}, \dot{\tilde{x}}_* \succeq \tilde{0} \text{ or } \tilde{x}, \tilde{x}_*, \dot{\tilde{x}}, \dot{\tilde{x}}_* \preceq \tilde{0}, \\ -\tilde{x}_*, & \text{otherwise.} \end{cases} \quad (10)$$

and

$$\tilde{\eta}_2 = \begin{cases} \dot{\tilde{x}} - \dot{\tilde{x}}_*, & \text{if } \dot{\tilde{x}}, \dot{\tilde{x}}_*, \tilde{x}, \tilde{x}_* \succeq \tilde{0} \text{ or } \dot{\tilde{x}}, \dot{\tilde{x}}_*, \tilde{x}, \tilde{x}_* \preceq \tilde{0}, \\ -\dot{\tilde{x}}_*, & \text{otherwise.} \end{cases} \quad (11)$$

Here, the invexity of  $\omega_{\mathcal{F}} \times \omega_{\mathcal{F}}$  with respect to  $\tilde{\eta}_1$  and  $\tilde{\eta}_2$  expressed as

$$(\tilde{x}_* + \lambda\tilde{\eta}_1(\tilde{x}, \tilde{x}_*), \dot{\tilde{x}}_* + \lambda\tilde{\eta}_2(\dot{\tilde{x}}, \dot{\tilde{x}}_*)) \in \omega_{\mathcal{F}} \times \omega_{\mathcal{F}}$$

for  $\lambda \in [0, 1]$  is satisfied.

There are vast number of functions that can be chosen for  $\eta$ , and an easy choice could be linear or constant function, as can be observed in the Example 1. Now, using this formulation, we define the preinvexity of  $\tilde{f}$  in (1) with respect to the last two variables as follows:

**Definition 10** (Preinvexity in last two variables). Let  $\omega \subset \mathbb{R}$  and  $\omega_{\mathcal{F}} \subset \mathcal{F}$ . A fuzzy-valued function  $\tilde{f} : \omega \times \omega_{\mathcal{F}} \times \omega_{\mathcal{F}} \rightarrow \mathcal{F}$  in (1) is said to be preinvex with respect to  $\tilde{\eta} : (\omega_{\mathcal{F}} \times \omega_{\mathcal{F}}) \times (\omega_{\mathcal{F}} \times \omega_{\mathcal{F}}) \rightarrow \mathcal{F} \times \mathcal{F}$  if, for each  $(\tilde{x}, \dot{\tilde{x}})$  and  $(\tilde{x}_*, \dot{\tilde{x}}_*) \in \omega_{\mathcal{F}} \times \omega_{\mathcal{F}}$ ,  $\lambda \in [0, 1]$ ,

$$\tilde{f}(t, \tilde{x}_* + \lambda \tilde{\eta}_1(\tilde{x}, \tilde{x}_*), \dot{\tilde{x}}_* + \lambda \tilde{\eta}_2(\dot{\tilde{x}}, \dot{\tilde{x}}_*)) \preceq \lambda \tilde{f}(t, \tilde{x}, \dot{\tilde{x}}) + (1 - \lambda) \tilde{f}(t, \tilde{x}_*, \dot{\tilde{x}}_*),$$

where  $\tilde{\eta}_1$  is an anti-derivative of  $\tilde{\eta}_2$ .

**Definition 11** (Strict preinvexity in last two variables). Let  $\omega \subset \mathbb{R}$  and  $\omega_{\mathcal{F}} \subset \mathcal{F}$ . A fuzzy-valued function  $\tilde{f} : \omega \times \omega_{\mathcal{F}} \times \omega_{\mathcal{F}} \rightarrow \mathcal{F}$  in (1) is said to be strictly preinvex with respect to  $\tilde{\eta} : (\omega_{\mathcal{F}} \times \omega_{\mathcal{F}}) \times (\omega_{\mathcal{F}} \times \omega_{\mathcal{F}}) \rightarrow \mathcal{F} \times \mathcal{F}$  if for each  $(\tilde{x}, \dot{\tilde{x}})$  and  $(\tilde{x}_*, \dot{\tilde{x}}_*) \in \omega_{\mathcal{F}} \times \omega_{\mathcal{F}}$ ,  $\lambda \in [0, 1]$ ,

$$\tilde{f}(t, \tilde{x}_* + \lambda \tilde{\eta}_1(\tilde{x}, \tilde{x}_*), \dot{\tilde{x}}_* + \lambda \tilde{\eta}_2(\dot{\tilde{x}}, \dot{\tilde{x}}_*)) \prec \lambda \tilde{f}(t, \tilde{x}, \dot{\tilde{x}}) + (1 - \lambda) \tilde{f}(t, \tilde{x}_*, \dot{\tilde{x}}_*),$$

where  $\tilde{\eta}_1$  is an anti-derivative of  $\tilde{\eta}_2$ .

To proceed, we recall the following Condition A introduced by Mohan and Neogy [38] that ensures the preinvexity of a function defined on an invex set.

**Condition A:** Let  $\omega_{\mathcal{F}} \subset \mathcal{F}$ . The function  $\tilde{\eta} : (\omega_{\mathcal{F}} \times \omega_{\mathcal{F}}) \times (\omega_{\mathcal{F}} \times \omega_{\mathcal{F}}) \rightarrow \mathcal{F} \times \mathcal{F}$  is said to satisfy the Condition A if, for each  $(\tilde{x}, \dot{\tilde{x}})$  and  $(\tilde{x}_*, \dot{\tilde{x}}_*) \in \omega_{\mathcal{F}} \times \omega_{\mathcal{F}}$ ,  $\lambda \in [0, 1]$

$$\tilde{\eta} \left( (\tilde{x}_*, \dot{\tilde{x}}_*), (\tilde{x}_*, \dot{\tilde{x}}_*) + \lambda \tilde{\eta}((\tilde{x}, \dot{\tilde{x}}), (\tilde{x}_*, \dot{\tilde{x}}_*)) \right) = -\lambda \tilde{\eta}((\tilde{x}, \dot{\tilde{x}}), (\tilde{x}_*, \dot{\tilde{x}}_*)), \quad (12)$$

$$\tilde{\eta} \left( (\tilde{x}, \dot{\tilde{x}}), (\tilde{x}_*, \dot{\tilde{x}}_*) + \lambda \tilde{\eta}((\tilde{x}, \dot{\tilde{x}}), (\tilde{x}_*, \dot{\tilde{x}}_*)) \right) = (1 - \lambda) \tilde{\eta}((\tilde{x}, \dot{\tilde{x}}), (\tilde{x}_*, \dot{\tilde{x}}_*)), \quad (13)$$

hold for any  $\lambda \in [0, 1]$ .

Equivalently, Condition A can be expressed in terms of  $\tilde{\eta}_1, \tilde{\eta}_2 : \omega_{\mathcal{F}} \times \omega_{\mathcal{F}} \rightarrow \mathcal{F}$  as follows:

$$\tilde{\eta}_1(\tilde{x}_*, \tilde{x}_* + \lambda \tilde{\eta}_1(\tilde{x}, \tilde{x}_*)) = -\lambda \tilde{\eta}_1(\tilde{x}, \tilde{x}_*), \quad (14)$$

$$\tilde{\eta}_1(\tilde{x}, \tilde{x}_* + \lambda \tilde{\eta}_1(\tilde{x}, \tilde{x}_*)) = (1 - \lambda) \tilde{\eta}_1(\tilde{x}, \tilde{x}_*), \quad (15)$$

and

$$\tilde{\eta}_2(\dot{\tilde{x}}_*, \dot{\tilde{x}}_* + \lambda \tilde{\eta}_2(\dot{\tilde{x}}, \dot{\tilde{x}}_*)) = -\lambda \tilde{\eta}_2(\dot{\tilde{x}}, \dot{\tilde{x}}_*), \quad (16)$$

$$\tilde{\eta}_2(\dot{\tilde{x}}, \dot{\tilde{x}}_* + \lambda \tilde{\eta}_2(\dot{\tilde{x}}, \dot{\tilde{x}}_*)) = (1 - \lambda) \tilde{\eta}_2(\dot{\tilde{x}}, \dot{\tilde{x}}_*) \quad (17)$$

for any  $\tilde{x}, \tilde{x}_*, \dot{\tilde{x}}, \dot{\tilde{x}}_* \in \omega_{\mathcal{F}}$  and  $\lambda \in [0, 1]$ .

**Theorem 3.** Let  $\Omega \subset \mathbb{R} \times \mathcal{F} \times \mathcal{F}$  be an invex set with respect to a function  $\tilde{\eta}$ , where  $\tilde{\eta}(\tilde{x}, \tilde{x}_*) \succeq \tilde{0}$  for any  $\tilde{x}, \tilde{x}_* \in \Omega$ , and satisfies Condition A. Let  $\tilde{f} : \Omega \rightarrow \mathcal{F}$  be a differentiable fuzzy-valued function on  $\Omega$ . Then,  $\tilde{f}$  is preinvex on  $\Omega$  if and only if the following holds for any  $\tilde{x}, \tilde{x}_* \in \Omega$ ,

$$\tilde{f}(\tilde{x}) \succeq \tilde{\nabla} \tilde{f}(\tilde{x}_*) \tilde{\eta}(\tilde{x}, \tilde{x}_*)^T + \tilde{f}(\tilde{x}_*). \quad (18)$$



**Proof.** Assume that  $\tilde{f}$  is differentiable and preinvex for every  $\tilde{x}, \tilde{x}_* \in \Omega$ . Using Lemma 1 we obtain

$$f^l(x_*^l + \lambda \eta^l(x^l, x_*^l), \alpha) \leq \lambda f^l(x^l, \alpha) + (1 - \lambda) f^l(x_*^l, \alpha), \quad (19)$$

and

$$f^r(x_*^r + \lambda \eta^r(x^r, x_*^r), \alpha) \leq \lambda f^r(x^r, \alpha) + (1 - \lambda) f^r(x_*^r, \alpha). \quad (20)$$

On rearranging, we obtain

$$f^l(x^l, \alpha) - f^l(x_*^l, \alpha) \geq \frac{1}{\lambda} [f^l(x_*^l + \lambda \eta^l(x^l, x_*^l), \alpha) - f^l(x_*^l, \alpha)], \quad (21)$$

and

$$f^r(x^r, \alpha) - f^r(x_*^r, \alpha) \geq \frac{1}{\lambda} [f^r(x_*^r + \lambda \eta^r(x^r, x_*^r), \alpha) - f^r(x_*^r, \alpha)]. \quad (22)$$

Since  $\tilde{f}$  is differentiable and  $\Omega$  is invex with respect to  $\tilde{\eta}(\tilde{x}, \tilde{x}_*)$ , we deduce that

$$\tilde{f}(\tilde{x}) - \tilde{f}(\tilde{x}_*) \succeq \tilde{\nabla} \tilde{f}(\tilde{x}_*) \tilde{\eta}(\tilde{x}, \tilde{x}_*)^T. \quad (23)$$

or

$$\tilde{f}(\tilde{x}) \succeq \tilde{\nabla} \tilde{f}(\tilde{x}_*) \tilde{\eta}(\tilde{x}, \tilde{x}_*)^T + \tilde{f}(\tilde{x}_*). \quad (24)$$

Hence, we obtain (18).

To prove the converse, suppose that (18) is true. Let  $\tilde{x}, \tilde{x}_* \in \Omega$ ,  $\lambda \in [0, 1]$ . Choose  $\tilde{x}^* = \tilde{x}_* + \lambda \tilde{\eta}(\tilde{x}, \tilde{x}_*) \in \Omega$ . We have

$$\tilde{f}(\tilde{x}) \succeq \tilde{\nabla} \tilde{f}(\tilde{x}^*) \tilde{\eta}(\tilde{x}, \tilde{x}^*)^T + \tilde{f}(\tilde{x}^*), \quad (25)$$

and

$$\tilde{f}(\tilde{x}_*) \succeq \tilde{\nabla} \tilde{f}(\tilde{x}^*) \tilde{\eta}(\tilde{x}_*, \tilde{x}^*)^T + \tilde{f}(\tilde{x}^*). \quad (26)$$

Using Condition A in (25) and (26), we obtain

$$\tilde{f}(\tilde{x}) \succeq (1 - \lambda) \tilde{\nabla} \tilde{f}(\tilde{x}^*) \tilde{\eta}(\tilde{x}, \tilde{x}_*)^T + \tilde{f}(\tilde{x}^*), \quad (27)$$

and

$$\tilde{f}(\tilde{x}_*) \succeq (-\lambda) \tilde{\nabla} \tilde{f}(\tilde{x}^*) \tilde{\eta}(\tilde{x}, \tilde{x}_*)^T + \tilde{f}(\tilde{x}^*). \quad (28)$$

Multiplying (27) with  $\lambda$  and (28) with  $(1 - \lambda)$  and adding them gives

$$\lambda \tilde{f}(\tilde{x}) + (1 - \lambda) \tilde{f}(\tilde{x}_*) \succeq \tilde{f}(\tilde{x}^*), \quad (29)$$

that is

$$\lambda \tilde{f}(\tilde{x}) + (1 - \lambda) \tilde{f}(\tilde{x}_*) \succeq \tilde{f}(\tilde{x}_* + \lambda \tilde{\eta}(\tilde{x}, \tilde{x}_*)). \quad (30)$$

Hence,  $\tilde{f}$  is preinvex and this completes the proof.  $\square$

Here, imposing Condition A allows one to preserve the preinvexity of a differentiable function that may not hold in general [38]. It can be noted that, using Theorem 3 and Definition 10 in (1) for the case when  $\tilde{f} : \omega \times \omega_{\mathcal{F}} \times \omega_{\mathcal{F}} \rightarrow \mathcal{F}$  is preinvex with respect to its last two variables, Expression (18) turns out to be as follows:

$$\tilde{f}(t, \tilde{x}, \dot{\tilde{x}}) \succeq \tilde{f}_{\tilde{x}}(t, \tilde{x}_*, \dot{\tilde{x}}_*) \cdot \tilde{\eta}_1(\tilde{x}, \tilde{x}_*) + \tilde{f}_{\dot{\tilde{x}}}(t, \tilde{x}_*, \dot{\tilde{x}}_*) \cdot \tilde{\eta}_2(\dot{\tilde{x}}, \dot{\tilde{x}}_*) + \tilde{f}(t, \tilde{x}_*, \dot{\tilde{x}}_*). \quad (31)$$

Using the above formulations and results, we will now proceed to establish the sufficient conditions for the existence and uniqueness of minimizers in the following section.

## 6. Sufficient Conditions for the Existence and Uniqueness of Minimizers

In the previous section, we reviewed the concept of invex sets and preinvex functions in the fuzzy sense and recalled some necessary results. Furthermore, we formulated the

definitions for the preinvexity of  $\tilde{f}$  in the last two variables. In this section, we prove our main results, namely, the sufficient conditions for the existence and uniqueness of the minimizer of the fuzzy variational problem (1).

**Theorem 4.** Let integrand  $\tilde{f}$  in (1) have a continuous second partial derivative with respect to all of its variables. Assume that  $(\tilde{x}, \dot{\tilde{x}}) \rightarrow \tilde{f}(t, \tilde{x}, \dot{\tilde{x}})$  is a preinvex function in the last two variables with respect to  $\tilde{\eta}$  for every  $t \in [t_0, t_f]$ , such that  $\tilde{\eta}((\tilde{x}(t), \dot{\tilde{x}}(t)), (\tilde{x}_*(t), \dot{\tilde{x}}_*(t))) = (\tilde{0}, \tilde{0})$  at  $t = t_0$  and  $t = t_f$  for all fuzzy functions  $\tilde{x}(t), \tilde{x}_*(t) \in \tilde{X}$ . Then,  $\tilde{x}_*$  is a minimizer of the fuzzy variational problem (1) if it satisfies (2) and (3).

**Proof.** Let  $\omega \subset \mathbb{R}$ ,  $\omega_{\mathcal{F}} \subset \mathcal{F}$  and  $\tilde{f} : \omega \times \omega_{\mathcal{F}} \times \omega_{\mathcal{F}} \rightarrow \mathcal{F}$  be as in (1). Let  $\omega_{\mathcal{F}} \times \omega_{\mathcal{F}}$  be a fuzzy invex set with respect to a non-negative function  $\tilde{\eta}$  satisfying Condition A, where  $\tilde{\eta} : (\omega_{\mathcal{F}} \times \omega_{\mathcal{F}}) \times (\omega_{\mathcal{F}} \times \omega_{\mathcal{F}}) \rightarrow \mathcal{F} \times \mathcal{F}$ , defined as follows:

$$\begin{aligned} \tilde{\eta}((\tilde{x}, \dot{\tilde{x}}), (\tilde{x}_*, \dot{\tilde{x}}_*)) &= \tilde{\eta}((\tilde{x}(t), \dot{\tilde{x}}(t)), (\tilde{x}_*(t), \dot{\tilde{x}}_*(t))) \\ &= (\tilde{\eta}_1(\tilde{x}(t), \tilde{x}_*(t)), \tilde{\eta}_2(\dot{\tilde{x}}(t), \dot{\tilde{x}}_*(t))) \\ &= (\tilde{0}, \tilde{0}) \end{aligned} \quad (32)$$

at  $t = t_0$  and  $t = t_f$  for all admissible curves. Now, since  $\tilde{f} : \omega \times \omega_{\mathcal{F}} \times \omega_{\mathcal{F}} \rightarrow \mathcal{F}$  is fuzzy preinvex in the last two variables, therefore, using Definition 10 and Expression (31), we have,

$$\tilde{f}(t, \tilde{x}, \dot{\tilde{x}}) \succeq \tilde{\nabla} \tilde{f}(t, \tilde{x}_*, \dot{\tilde{x}}_*) \tilde{\eta}((\tilde{x}, \dot{\tilde{x}}), (\tilde{x}_*, \dot{\tilde{x}}_*))^T + \tilde{f}(t, \tilde{x}_*, \dot{\tilde{x}}_*).$$

That means

$$\tilde{f}(t, \tilde{x}, \dot{\tilde{x}}) \succeq \tilde{f}(t, \tilde{x}_*, \dot{\tilde{x}}_*) \cdot \tilde{\eta}_1(\tilde{x}, \tilde{x}_*) + \tilde{f}_{\dot{\tilde{x}}}(t, \tilde{x}_*, \dot{\tilde{x}}_*) \cdot \tilde{\eta}_2(\dot{\tilde{x}}, \dot{\tilde{x}}_*) + \tilde{f}(t, \tilde{x}_*, \dot{\tilde{x}}_*),$$

for every  $\tilde{x}, \tilde{x}_* \in \tilde{X}$ . Integrating the above expression, we obtain

$$I^l(x^l) \geq I^l(x_*^l) + \int_{t_0}^{t_f} [f_{x^l}^l(t, \alpha, x_*^l, \dot{x}_*^l) \eta_1^l(x^l, x_*^l) + f_{\dot{x}^l}^l(t, \alpha, x_*^l, \dot{x}_*^l) \eta_2^l(\dot{x}^l, \dot{x}_*^l)] dt.$$

and

$$I^r(x^r) \geq I^r(x_*^r) + \int_{t_0}^{t_f} [f_{x^r}^r(t, \alpha, x_*^r, \dot{x}_*^r) \eta_1^r(x^r, x_*^r) + f_{\dot{x}^r}^r(t, \alpha, x_*^r, \dot{x}_*^r) \eta_2^r(\dot{x}^r, \dot{x}_*^r)] dt.$$

Integrating the second term in the integral by parts and using (32), we obtain

$$I^l(x^l) \geq I^l(x_*^l) + \int_{t_0}^{t_f} \left[ f_{x^l}^l(t, \alpha, x_*^l, \dot{x}_*^l) - \frac{d}{dt} (f_{\dot{x}^l}^l(t, \alpha, x_*^l, \dot{x}_*^l)) \right] \eta_1^l(x^l, x_*^l) dt,$$

and

$$I^r(x^r) \geq I^r(x_*^r) + \int_{t_0}^{t_f} \left[ f_{x^r}^r(t, \alpha, x_*^r, \dot{x}_*^r) - \frac{d}{dt} (f_{\dot{x}^r}^r(t, \alpha, x_*^r, \dot{x}_*^r)) \right] \eta_1^r(x^r, x_*^r) dt.$$

Now, using (2) and (3), we obtain

$$I^l(x^l) \geq I^l(x_*^l). \quad (33)$$

and

$$I^r(x^r) \geq I^r(x_*^r), \quad (34)$$

Hence

$$\tilde{I}(\tilde{x}) \succeq \tilde{I}(\tilde{x}_*). \quad (35)$$

This completes the proof.  $\square$

**Theorem 5.** Let integrand  $\tilde{f}$  in (1) satisfy the hypothesis of Theorem 4 and  $(\tilde{x}, \dot{\tilde{x}}) \rightarrow \tilde{f}(t, \tilde{x}, \dot{\tilde{x}})$  BE strictly preinvex function in last two variables with respect to  $\tilde{\eta}$  for every  $t \in [t_0, t_f]$ ; then, the minimizer of FVP (1), if it exists, is unique.

**Proof.** We shall prove by contradiction. Suppose that  $\tilde{f}$  is strictly preinvex and there exist two distinct solutions, namely,  $\tilde{x}_1(t)$  and  $\tilde{x}_2(t)$  of FVP (1) corresponding to the minimum value, say,  $\tilde{m} \in \mathcal{F}$ . We define

$$\begin{aligned}(\tilde{x}_3, \dot{\tilde{x}}_3) &= ((\tilde{x}_2, \dot{\tilde{x}}_2) + \lambda \tilde{\eta}((\tilde{x}_1, \dot{\tilde{x}}_1), (\tilde{x}_2, \dot{\tilde{x}}_2))) \\ &= (\tilde{x}_2 + \lambda \tilde{\eta}_1(\tilde{x}_1, \tilde{x}_2), \dot{\tilde{x}}_2 + \lambda \tilde{\eta}_2(\dot{\tilde{x}}_1, \dot{\tilde{x}}_2)).\end{aligned}$$

Now, using the strict preinvexity of  $\tilde{f}(t, \tilde{x}, \dot{\tilde{x}})$  (see Definition 10), we have

$$\lambda \tilde{f}(t, \tilde{x}_1, \dot{\tilde{x}}_1) + (1 - \lambda) \tilde{f}(t, \tilde{x}_2, \dot{\tilde{x}}_2) \succ \tilde{f}(t, \tilde{x}_3, \dot{\tilde{x}}_3)$$

and hence, on integrating

$$\tilde{m} \approx \lambda \tilde{I}(\tilde{x}_1(t)) + (1 - \lambda) \tilde{I}(\tilde{x}_2(t)) \succ \tilde{I}(\tilde{x}_3(t)) \succeq \tilde{m}.$$

Therefore, we obtain

$$\int_{t_0}^{t_f} \left[ \lambda \tilde{f}(t, \tilde{x}_1, \dot{\tilde{x}}_1) + (1 - \lambda) \tilde{f}(t, \tilde{x}_2, \dot{\tilde{x}}_2) - \tilde{f}(t, \tilde{x}_2 + \lambda \tilde{\eta}_1(\tilde{x}_1, \tilde{x}_2), \dot{\tilde{x}}_2 + \lambda \tilde{\eta}_2(\dot{\tilde{x}}_1, \dot{\tilde{x}}_2)) \right] dx \approx \tilde{0}.$$

This is a contradiction, since  $\tilde{f}$  is assumed to be strictly preinvex. Therefore,  $\tilde{x}_1(t)$  and  $\tilde{x}_2(t)$  are not distinct, which implies the uniqueness of the minimizer.  $\square$

## 7. Application of Fuzzy Variational Problem

The real world is full of uncertainties and complexities that result in ill-posed optimization problems that cannot be dealt with by classical control theory or calculus of variations [6,7]. The classical optimal control theory approach either assumes all the data and parameters to be deterministic or replaces the uncertain data with average data. However, in real problems, some naturally uncertain parameters can only be roughly estimated. The uncertainties present in real life, such as the production rate, stock level, inventory cost, and development cost cannot always be defined using crisp models. These may vary day to day [40]. For instance, if the labor department goes on strike or becomes partially available due to the weather condition, that may impact development cost, which can implicitly affect the production rate and hence, the whole working process of the model. Therefore, in such cases, it is often advantageous to define such parameters in the form of fuzzy numbers, as they can be used to model the ambiguity present in these cases [6,41,42]. In this section, we demonstrate the application of Theorem 4 to validate the existence of the minimizer for the following optimal control problem [40], which is modeled to minimize the cost function for a defective production process:

$$\begin{aligned}\tilde{I}_{t \in (t_0, t_f)} &= \int_{t_0}^{t_f} \left( \tilde{H} \tilde{x}(t) + \tilde{c}_1 + \left( \frac{c_t}{(1 - \delta)} - p \delta \right) (\dot{\tilde{x}}(t) + d_0 - a e^{-bt}) \right. \\ &\quad \left. + \frac{\beta}{(1 - \delta)^2} (\dot{\tilde{x}}(t) + d_0 - a e^{-bt})^2 \right) dt,\end{aligned}\tag{36}$$

where  $\tilde{x}(t)$  is a state variable that stands for fuzzy stock level,  $\tilde{H}$  is the fuzzy holding cost for storing per unit,  $\tilde{c}_1$  is the fuzzy development cost including labor and technology,  $\beta$  denotes the wear-tear cost,  $\delta$  denotes the defective fraction of production,  $p$  denotes the selling price of defective units,  $c_t$  is the constant material cost,  $d_0$  is the constant demand at the initial stage, and  $a$  and  $b$  are positive constants.

In [40], it was assumed that the problem has a solution, which was numerically calculated as follows:

$$\tilde{x}_* = \frac{a}{b}(1 - e^{-bt_0}) + \frac{at_0}{bt_f}(e^{-bt_f} - 1) + \frac{(1 - \delta)^2 \tilde{H}t_0}{4\beta}(t_0 - t_f). \quad (37)$$

We may observe that, using Theorem 1 for such applications, we may end up with possible local extrema. However, Theorem 4 verifies the existence of the minimizer. Moreover, whether we are interested in exact or numerical solutions, it is convenient to verify the conditions that imply the existence of the minimizer of fuzzy variational problems before proceeding with the complicated computation. It can further help reduce the computational cost for complex cases. Therefore, we would now apply Theorem 4 in the above considered optimal control problem and prove that  $\tilde{x}_*$  is a minimizer of the cost functional (36). For that, we demonstrate that the integrand of the considered problem is preinvex in the last two variables as follows:

Let  $\omega_{\mathcal{F}} \times \omega_{\mathcal{F}} \subset \mathcal{F} \times \mathcal{F}$  be a fuzzy invex set with respect to a non-negative function  $\tilde{\eta}$  satisfying Condition A, where  $\tilde{\eta} = (\omega_{\mathcal{F}} \times \omega_{\mathcal{F}}) \times (\omega_{\mathcal{F}} \times \omega_{\mathcal{F}}) \rightarrow \mathcal{F} \times \mathcal{F}$ , defined as the following:

$$\tilde{\eta}((\tilde{x}, \dot{\tilde{x}}), (\tilde{x}_*, \dot{\tilde{x}}_*)) = (\tilde{\eta}_1(\tilde{x}, \tilde{x}_*), \tilde{\eta}_2(\dot{\tilde{x}}, \dot{\tilde{x}}_*)) = (\tilde{x} - \tilde{x}_*, \dot{\tilde{x}} - \dot{\tilde{x}}_*).$$

Here,  $\tilde{\eta}$  is defined as above for all of its non-negative values and as a zero function otherwise. We now show, using Definition 10, that the integrand  $\tilde{f}$  in (36) defined on  $\Omega \subset \mathbb{R} \times \mathcal{F} \times \mathcal{F}$  is a preinvex function with respect to  $\tilde{\eta}$  in the last two variables. Here,

$$\begin{aligned} \tilde{f} &= \tilde{H}\tilde{x}(t) + \tilde{c}_1 + \left(\frac{c_t}{(1-\delta)} - p\delta\right)(\dot{\tilde{x}}(t) + d_0 - ae^{-bt}) \\ &\quad + \frac{\beta}{(1-\delta)^2}(\dot{\tilde{x}}(t) + d_0 - ae^{-bt})^2. \end{aligned}$$

We want to show the following:

$$\tilde{f}(t, \tilde{x}_* + \lambda\tilde{\eta}_1(\tilde{x}, \tilde{x}_*), \dot{\tilde{x}}_* + \lambda\tilde{\eta}_2(\dot{\tilde{x}}, \dot{\tilde{x}}_*)) \preceq \lambda\tilde{f}(t, \tilde{x}, \dot{\tilde{x}}) + (1 - \lambda)\tilde{f}(t, \tilde{x}_*, \dot{\tilde{x}}_*),$$

which is equivalent to

$$\begin{aligned} f^l(t, \alpha, x_*^l + \lambda\eta_1^l(x^l, x_*^l), \dot{x}_*^l + \lambda\eta_2^l(\dot{x}^l, \dot{x}_*^l)) \\ \leq \lambda f^l(t, \alpha, x^l, \dot{x}^l) + (1 - \lambda)f^l(t, \alpha, x_*^l, \dot{x}_*^l) \end{aligned}$$

and

$$\begin{aligned} f^r(t, \alpha, x_*^r + \lambda\eta_1^r(x^r, x_*^r), \dot{x}_*^r + \lambda\eta_2^r(\dot{x}^r, \dot{x}_*^r)) \\ \leq \lambda f^r(t, \alpha, x^r, \dot{x}^r) + (1 - \lambda)f^r(t, \alpha, x_*^r, \dot{x}_*^r) \end{aligned}$$

in terms of the  $\alpha$ -level set for all  $\alpha \in [0, 1]$ . Moreover, for simplicity, we can represent  $f^l$  as

$$f^l = H^l x^l(t) + A\dot{x}^l(t) + B(\dot{x}^l(t))^2 + C_1 e^{-bt} + C_2 e^{-2bt} + D e^{-bt} \cdot \dot{x}^l(t) + c_1^l + E$$

where  $A, B, C_1, C_2, D$ , and  $E$  are constant functions including  $c_t, \delta, \beta, d_0, a$  as follows:

$$\begin{aligned} A &= \left( \frac{c_t}{1-\delta} - p\delta \right) + \frac{2d_0\beta}{(1-\delta)^2}, \\ B &= \frac{\beta}{(1-\delta)^2}, \\ C_1 &= -a \left( \frac{c_t}{1-\delta} - p\delta \right) - \frac{2ad_0\beta}{(1-\delta)^2}, \\ C_2 &= \frac{a^2\beta}{(1-\delta)^2}, \\ D &= \frac{-2a\beta}{(1-\delta)^2}, \\ E &= \left( \frac{c_t}{1-\delta} - p\delta \right) d_0 + \frac{d_0^2\beta}{(1-\delta)^2}. \end{aligned}$$

Now, consider

$$\begin{aligned} & f^l(t, \alpha, x_*^l + \lambda\eta_1^l(x^l, x_*^l), \dot{x}_*^l + \lambda\eta_2^l(\dot{x}^l, \dot{x}_*^l)) \\ = & H^l(x_*^l(t) + \lambda(x^l(t) - x_*^l(t)), \alpha) + A(\dot{x}_*^l(t) + \lambda(\dot{x}^l(t) - \dot{x}_*^l(t)), \alpha) \\ & + B((\dot{x}_*^l(t) + \lambda(\dot{x}^l(t) - \dot{x}_*^l(t)), \alpha)^2) + C_1 e^{-bt} + C_2 e^{-2bt} \\ & + D e^{-bt}(\dot{x}_*^l(t) + \lambda(\dot{x}^l(t) - \dot{x}_*^l(t)), \alpha) + c_1^l + E. \end{aligned}$$

The function  $f^l$  is a non-negative cost function and all constants used above are also taken to be non-negative [40]. Moreover  $f^l$  is the left endpoint function of a triangular fuzzy number [40] and the exponential function is always convex. Therefore, in particular since  $\eta^l = (x - x_*, \dot{x} - \dot{x}_*)$ , so  $f^l$  is preinvex function. Hence, on rearranging, we can deduce that,

$$\begin{aligned} & f^l(t, \alpha, x_*^l + \lambda\eta_1^l(x^l, x_*^l), \dot{x}_*^l + \lambda\eta_2^l(\dot{x}^l, \dot{x}_*^l)) \\ & \leq \lambda f^l(t, \alpha, x^l, \dot{x}^l) + (1-\lambda) f^l(t, \alpha, x_*^l, \dot{x}_*^l). \end{aligned}$$

Similarly, this holds true for the right endpoint  $\alpha$ -level set of  $\tilde{f}$ ; hence,  $\tilde{f}$  in Equation (36) is a preinvex function. Moreover, the given integrand  $\tilde{f}$  satisfies Euler–Lagrange equations and has continuous second partial derivatives according to [40]; therefore, by virtue of Theorem 4, the existence of the minimizer of the above modeling problem is proven. The uniqueness could also be obtained if  $\tilde{f}$  satisfies the strict preinvexity condition in Definition 11. The applications of the fuzzy variational theory are not just restricted to production inventory control problems, but can be extended to several other areas such as image processing and prey–predator bio-mathematical models. When dealing with uncertainties, experts can use such fuzzy variational models to obtain a more accurate estimate of returns.

## 8. Conclusions & Discussion

In this paper, we established the sufficient conditions for the existence and uniqueness of minimizers for fuzzy variational problems by considering differentiability and continuity of fuzzy-valued functions in the sense of Buckley and Feuring. First, we discussed the concept of fuzzy convex functions, which turns out to be too strong for fuzzy-valued functions, therefore revealing the need to consider a weaker notion, namely preinvex functions. Several results on preinvex functions were established, leading to multi-variable preinvexity, which is essentially a sufficient condition for our main theorems. Finally, we demonstrated the application of the obtained results via a cost minimization problem.

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