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The Exact Solutions for Several Partial Differential-Difference Equations with Constant Coefficients

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Abstract: This article is concerned with the description of the entire solutions of several Fermat type partial differential-difference equations (PDDEs) $[\mu f(z)+\lambda f_{z_1}(z)]^2+[\alpha f(z+c)-\beta f(z)]^2=1$, and $[\mu f(z)+\lambda_1 f_{z_1}(z)+\lambda_2 f_{z_2}(z)]^2+[\alpha f(z+c)-\beta f(z)]^2=1$, where $f_{z_1}(z)=\frac{\partial f}{\partial z_1}$ and $f_{z_2}(z)=\frac{\partial f}{\partial z_2}$, $c=(c_1,c_2)\in\mathbb{C}^2$, $\alpha,\beta,\mu,\lambda,\lambda_1,\lambda_2,c_1,c_2$ are constants in $\mathbb{C}$. Our theorems in this paper give some descriptions of the forms of transcendental entire solutions for the above equations, which are some extensions and improvement of the previous theorems given by Xu, Cao, Liu, and Yang. In particular, we exhibit a series of examples to explain that the existence conditions and the forms of transcendental entire solutions with a finite order of such equations are precise.

Keywords: Nevanlinna theory; entire solution; partial differential-difference equation

MSC: 30D35; 35M30; 39A45

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1. Introduction and Some Basic Results

As is well known, the classical result of the Fermat type functional equation

$$f^2+g^2=1 \tag{1}$$

is that the entire solutions are $f=\cos\zeta(z), g=\sin\zeta(z)$, where $\zeta(z)$ is an entire function, which was given by Gross [1]. Actually, the study of this functional equation can be tracked back to more than sixty years ago or even earlier (see [1–3]). Moreover, there are important and famous results on the Fermat type equation (see [4,5]). In recent years, replying on the rapid development of Nevanlinna theory in many fields including functional equations and difference of meromorphic function with one and several variables ([6–12]), there were lots of references focusing on the solutions of the Fermat type equation; when the function f has a special relationship with g, readers can refer to [13–17].

Around 2012, for the case $f\in\mathbb{C}$, Liu and his colleagues paid considerable attention to the solutions of a series of Fermat type functional equations when g is replaced by $f', f(z+c), f(z+c)-f(z)$ in Equation (1) (see [18–20]), they proved that the form of the finite order transcendental entire solution of $f'(z)^2+f(z+c)^2=1$ must be $f(z)=\sin(z\pm Bi)$, and the form of the finite order transcendental entire solution of $f'(z)^2+[f(z+c)-f(z)]^2=1$ must be $f(z)=12\sin(2z+Bi)$, where B is a constant. Later, Han and Lü [21]. Liu and Gao [22] investigated the existence of solutions of several deformations of Equation (1) such as $f(z)^2 + f'(z)^2 = e^{\alpha z+\beta}$, $f''(z)^2 + f(z+c)^2 = Q(z)$, where α, β are constants and $Q(z)$ is a polynomial.

For the case $f \in \mathbb{C}^n, n \geq 2$, Khavinson [14] in 1995 pointed out that any entire solutions of the partial differential equations $(f_{z_1})^2 + (f_{z_2})^2 = 1$ in $\mathbb{C}^2$ are necessarily linear. In 1999 and 2004, Saleeby [23,24] further studied the forms of the entire and meromorphic solutions of some partial differential equations, and obtained

Theorem 1 (see [23] Theorem 1)**.** *If f is an entire solution of*

$$f_{z_1}(z)^2 + f_{z_2}(z)^2 = 1, \tag{2}$$

in $\mathbb{C}^2$, then $f(z_1, z_2) = c_1 z_1 + c_2 z_2 + \eta$, where $\eta, c_1, c_2 \in \mathbb{C}$ and $c_1^2 + c_2^2 = 1$.

In 2012, Chang and Li [25] investigated the entire solutions of

$$X_1(f)^2 + X_2(f)^2 = 1, \tag{3}$$

where

$$X_1 = p_1 \frac{\partial}{\partial z_1} + p_2 \frac{\partial}{\partial z_2}. \quad X_1 = p_3 \frac{\partial}{\partial z_1} + p_4 \frac{\partial}{\partial z_2},$$

are linearly independent operators with p_j being polynomials in $\mathbb{C}^2$ and obtained:

Theorem 2 (see [25] Corollary 2.2)**.** *Let f be an entire solution of the Equation (3). Then, f satisfies*

$$\frac{\partial f}{\partial z_1} = \frac{1}{D}(p_4 \cos h - p_2 \sin h), \quad \frac{\partial f}{\partial z_2} = \frac{1}{D}(p_1 \sin h - p_3 \cos h),$$

where $D = p_1 p_4 - p_2 p3$, h is a constant or a nonconstant polynomial satisfying

$$\frac{\partial h}{\partial z_1} = \frac{a p_2 + b p_4}{D^2}, \quad \frac{\partial h}{\partial z_2} = \frac{-a p_1 + b p_3}{D^2},$$

and

$$a = D\frac{\partial p_2}{\partial z_2} - p_2 \frac{\partial D}{\partial z_1} + D\frac{\partial p_1}{\partial z_1} - p_1 \frac{\partial D}{\partial z_1}.$$
$$b = D\frac{\partial p_3}{\partial z_1} - p_3 \frac{\partial D}{\partial z_1} + D\frac{\partial p_4}{\partial z_2} - p_4 \frac{\partial D}{\partial z_2}.$$

In fact, Li [16,26] also discussed a series of partial differential equations with more general forms including $(f_{z_1})^2 + (f_{z_2})^2 = e^g$, $(f_{z_1})^2 + (f_{z_2})^2 = p$, etc., where g, p are polynomials in $\mathbb{C}^2$. Recently, by using the characteristic equations for quasi-linear PDEs, and the Nevanlinna theory in $\mathbb{C}^n, n \geq 2$, Chen, Han, and Lü. Xu and his colleagues, etc. [27–36] investigated the entire and meromorphic solutions of the nonlinear partial differential equations; for example, Chen and Han [36] discussed the entire solutions of equation $(f^l f_{z_1})^m (f^l f_{z_2})^n = p(z_1) e^{g(z)}$, where $l \geq 0, m, n \geq 1$ are integers, $p(z_1)$ is a polynomial in $\mathbb{C}$ and $g(z)$ is a polynomial in $\mathbb{C}^2$, Lü [28] studied the entire solution of equation $f_{z_1}^2 + 2B f_{z_1} f_{z_2} + f_{z_2}^2 = e^g$, where B is a constant and g is a polynomial or an entire function in $\mathbb{C}^2$, etc., and they generalized and improved the previous results given by Li [15].

Based on the establishment of Nevanlinna difference theory in $\mathbb{C}^n, n \geq 2$ (can be found in [6,37]), Xu and Cao [38] in 2018 and 2020 studied the solutions of some Fermat type partial differential-difference equations (PDDEs) and obtained:

Theorem 3 (see [38] Theorem 1.2)**.** *Let $c = (c_1, c_2) \in \mathbb{C}^2$. Then, any transcendental entire solutions with a finite order of the partial differential-difference equation*

$$f_{z_1}(z)^2 + f(z+c)^2 = 1 \tag{4}$$

has the form of $f(z_1, z_2) = \sin(Az_1 + B)$*, where* A *is a constant on* $\mathbb{C}$ *satisfying* $Ae^{iAc_1} = 1$*, and* B *is a constant on* $\mathbb{C}$*; in the special case whenever* $c_1 = 0$*, we have* $f(z_1, z_2) = \sin(z_1 + B)$*.*

Remark 1. *In general,* f *is called as a transcendental entire solution of the equation if* f *is a transcendental entire function and also the solution of this equation, here a meromorphic function* $f(z)$ *is transcendental if and only if*

$$\limsup_{r\to+\infty} \frac{T(r,f)}{\log r} = \infty,$$

this definition can be found in [17].

Inspired by the above results, this article concerns the entire solutions of the following PDDEs

$$[\mu f(z) + \lambda f_{z_1}(z)]^2 + [\alpha f(z+c) - \beta f(z)]^2 = 1, \tag{5}$$

and

$$[\mu f(z) + \lambda_1 f_{z_1}(z) + \lambda_2 f_{z_2}(z)]^2 + [\alpha f(z+c) - \beta f(z)]^2 = 1, \tag{6}$$

where $z = (z_1, z_2)$, $c = (c_1, c_2)$, and $\alpha, \beta, \mu, \lambda, \lambda_1, \lambda_2, c_1, c_2$ are constants in $\mathbb{C}$. Obviously, we can see that (5) and (6) are some deformation Equations of (1) and (4).

The details theorems on the properties of transcendental entire solutions of the partial differential-difference Equations (5) and (6) are be shown in Section 2, and the proofs are given in Sections 4 and 5. The results obtained in the paper are motivated by and benefit from the factorization theory of meromorphic functions and Nevanlinna theory in several complex variables. In particular, we will assume that the reader is familiar with the basics of Nevanlinna theory in several complex variables.

2. Results and Examples

The first main theorem is about the existence and the forms of the solutions for Equation (5).

Theorem 4. *Let* $c = (c_1, c_2) \in \mathbb{C}^2$*,* $c_2 \neq 0$*, and* $\alpha, \beta, \mu, \lambda$ *be nonzero constants in* $\mathbb{C}$*. Let* $f(z_1, z_2)$ *be a finite order transcendental entire solution of Equation (5). Then,* $f(z_1, z_2)$ *must satisfy one of the following cases:*

(i) if $\mu f(z) + \lambda f_{z_1}(z)$ *is a constant, then*

$$f(z_1, z_2) = \frac{\eta_1}{\mu} - \frac{1}{\mu} e^{-\frac{\mu}{\lambda} z_1 + Az_2 + B},$$

where $\eta_1, A, B \in \mathbb{C}$ *satisfy* $\eta_1^2 = \frac{\mu^2}{\mu^2 + (\alpha-\beta)^2}$ *and* $e^{Ac_2} = \frac{\beta}{\alpha} e^{\frac{\mu}{\lambda} c_1}$*;*

(ii) if $A_1 \neq \pm\frac{\mu}{\lambda}$*, then*

$$f(z_1, z_2) = \frac{1}{2(\lambda A_1 + \mu)} e^{A_1 z_1 + A_2 z_2 + B} - \frac{1}{2(\lambda A_1 - \mu)} e^{-A_1 z_1 - A_2 z_2 - B} + \vartheta(z_2) e^{-\frac{\mu}{\lambda} z_1},$$

where $\vartheta(z_2)$ *is a finite order entire function,* $A_1, A_2, B \in \mathbb{C}$ *satisfy*

$$(\lambda A_1 + \beta i)^2 = \mu^2 - \alpha^2, \quad e^{2(A_1 c_1 + A_2 c_2)} = \frac{\lambda A_1 + \mu + \beta i}{\lambda A_1 - \mu + \beta i},$$

and

$$\frac{\vartheta(z_2 + c_2)}{\vartheta(z_2)} = \frac{\beta}{\alpha} e^{\frac{\mu}{\lambda} c_1}; \tag{7}$$

(iii) if $A_1 = \frac{\mu}{\lambda}$, *then*

$$f(z_1, z_2) = \frac{1}{4\mu} e^{A_1 z_1 + A_2 z_2 + B} + \frac{z_1}{2\lambda} e^{-A_1 z_1 - A_2 z_2 - B} + \vartheta(z_2) e^{-\frac{\mu}{\lambda} z_1},$$

where $\vartheta(z_2)$ *is a finite order entire function satisfying* (7). $A_1, A_2, B \in \mathbb{C}$ *satisfy*

$$2\mu\beta i = \beta^2 - \alpha^2, \;\; \beta c_1 = \lambda i, \;\; e^{2(A_1 c_1 + A_2 c_2)} = 1 - \frac{2\mu}{\beta} i; \tag{8}$$

(iv) if $A_1 = -\frac{\mu}{\lambda}$, *then*

$$f(z_1, z_2) = \frac{z_1}{2\lambda} e^{A_1 z_1 + A_2 z_2 + B} + \frac{1}{4\mu} e^{-A_1 z_1 - A_2 z_2 - B} + \vartheta(z_2) e^{-\frac{\mu}{\lambda} z_1},$$

where $\vartheta(z_2)$ *is a finite order entire function satisfying* (7). $A_1, A_2, B \in \mathbb{C}$ *satisfy*

$$2\mu\beta i = \alpha^2 - \beta^2, \;\; \beta c_1 = -\lambda i, \;\; e^{-2(A_1 c_1 + A_2 c_2)} = 1 + \frac{2\mu}{\beta} i. \tag{9}$$

The following examples show the existence of transcendental entire solutions of Equation (5).

Example 1. *Let* $\eta_1^2 = \frac{1-\sqrt{3}+i}{4-2\sqrt{3}+1}$ *and*

$$f(z_1, z_2) = \eta_1 - e^{-\frac{1}{2} z_1 + z_2}.$$

Thus, $f(z_1, z_2)$ *is a transcendental entire solution of* (5) *with* $\alpha = e^{-\frac{\pi}{6} i}$, $\beta = e^{\frac{\pi}{3} i}$, $\lambda = 2$, $\mu = 1$, $(c_1, c_2) = (\pi i, \pi i)$ *and* $\rho(f) = 1$. *This shows that the form of solution in the conclusion (i) of Theorem 4 is precise.*

Example 2. *Let* $A_2 = \frac{1}{2\pi i} \ln \cot \frac{\pi}{12} - \frac{1}{3} - \frac{\sqrt{3}}{2}$ *and*

$$f(z_1, z_2) = \frac{1}{2(\sqrt{3}+1)} e^{\frac{\sqrt{3}}{2} z_1 + A_2 z_2} - \frac{1}{2(\sqrt{3}-1)} e^{-\frac{\sqrt{3}}{2} z_1 - A_2 z_2} - \cos(2z_2) e^{-\frac{1}{2} z_1 + z_2}.$$

Thus, $f(z_1, z_2)$ *is a transcendental entire solution of* (5) *with* $\alpha = e^{-\frac{\pi}{6} i}$, $\beta = e^{\frac{\pi}{3} i}$, $\lambda = 2$, $\mu = 1$, $(c_1, c_2) = (\pi i, \pi i)$ *and* $\rho(f) = 1$. *This shows that the form of solution in the conclusion (ii) of Theorem 4 is precise.*

Example 3. *Let* $D = -\frac{1}{4} \ln 2 + \frac{\pi}{8} i + \frac{1}{2} i$, $A_2 = \frac{1}{4} \ln 2 - \frac{\pi}{8} i - \frac{1}{2} i$ *and*

$$f(z_1, z_2) = \frac{1}{4} e^{\frac{i}{2} z_1 + A_2 z_2} - \frac{i z_1}{4} e^{-\frac{i}{2} z_1 - A_2 z_2} - e^{(D + 2\pi i) z_2} e^{-\frac{i}{2} z_1}.$$

Thus, $f(z_1, z_2)$ *is a transcendental entire solution of* (5) *with* $\alpha = 2^{\frac{5}{4}} e^{-\frac{\pi}{8} i}$, $\beta = 2$, $\lambda = -2i$, $\mu = 1$, $(c_1, c_2) = (1, 1)$ *and* $\rho(f) = 1$. *This shows that the form of solution in the conclusion (iii) of Theorem 4 is precise.*

From Theorem 4, letting $\lambda = \mu = 1$ and $\alpha = \beta = 1$, one can obtain the following result:

Corollary 1. *Let* $c = (c_1, c_2) \in \mathbb{C}^2$ *and* $c_2 \neq 0$. *If* $f(z_1, z_2)$ *is a finite order transcendental entire solution of equation,*

$$[f(z) + f_{z_1}(z)]^2 + [f(z+c) - f(z)]^2 = 1,$$

then $f(z_1, z_2)$ must be of the form

$$f(z_1, z_2) = \pm 1 - e^{-z_1 + Az_2 + B},$$

where A, B are constants and

$$Ac_2 = c_1 + 2k\pi i, \ \ k \in \mathbb{Z};$$

or

$$f(z_1, z_2) = \frac{1}{2(1-i)} e^{-iz_1 + A_2 z_2 + B} + \frac{1}{2(1+i)} e^{iz_1 - A_2 z_2 - B} + \vartheta(z_2) e^{-z_1},$$

where $\vartheta(z_2)$ is a finite order entire function, a_2, b are constants and

$$e^{2(-ic_1 + A_2 c_2)} = -1, \ \ \frac{\vartheta(z_2 + c_2)}{\vartheta(z_2)} = -e^{c_1},$$

From Theorem 4, letting $\alpha = 1$ and $\beta = 0$, one can obtain the following corollary:

Corollary 2. *Let $c = (c_1, c_2) \in \mathbb{C}^2$, $c_2 \neq 0$, and λ, μ be nonzero constants. If $f(z_1, z_2)$ is a finite order transcendental entire solution of equation*

$$[\mu f(z) + \lambda f_{z_1}(z)]^2 + f(z+c)^2 = 1,$$

then $f(z_1, z_2)$ must be of the form

$$f(z_1, z_2) = \frac{1}{2(\mu + \lambda a_1)} e^{A_1 z_1 + A_2 z_2 + B} + \frac{1}{2(\mu - \lambda A_1)} e^{-A_1 z_1 - A_2 z_2 - B},$$

where A_1, A_2, B are constants and satisfying

$$A_1^2 = \frac{\mu^2 - 1}{\lambda^2}, \ \ e^{2(A_1 c_1 + A_2 c_2)} = \frac{\lambda A_1 + \mu}{\lambda A_1 - \mu}.$$

Remark 2. *In view of the form of $f(z)$ in Corollary 2, one can see that the order of f must be 1. However, the following example shows that the equation can admit the transcendental entire solution of the order greater than one if we remove the condition $c_2 \neq 0$. Let*

$$f(z) = \frac{1}{2i(\sqrt{3}+2)} e^{z_1 + z_2 + z_2^2} + \frac{1}{2i(\sqrt{3}-2)} e^{-z_1 - z_2 - z_2^2}.$$

Then, $\rho(f) = 2$ and f is a transcendental entire solution of equation

$$\left[\sqrt{3} i f(z_1, z_2) + 2i f_{z_1}(z_1, z_2)\right]^2 + f(z_1 - \ln(2 - \sqrt{3}), z_2 + 0)^2 = 1,$$

For $\alpha = 0$ and $\beta = 1$ in Equation (5), we have

Corollary 3. *Let λ, μ be two nonzero constants. Then, the following partial differential equation*

$$[\mu f(z) + \lambda f_{z_1}(z)]^2 + f(z)^2 = 1 \tag{10}$$

does not admit any finite order transcendental entire solution.

Proof. Assume that $f(z)$ is a finite order transcendental entire solution of Equation (10). By using the same argument as in the proof of Theorem 4, there exists a nonconstant polynomial $p(z) \in \mathbb{C}^2$ satisfying

$$\mu f(z) + \lambda f_{z_1}(z) = \frac{1}{2}(e^p + e^{-p}), \ \ f(z) = \frac{1}{2i}(e^p - e^{-p}).$$

Thus, it follows that

$$(\mu + \lambda p_{z_1}(z) - i)e^{2p} = \mu - \lambda p_{z_1} + i. \tag{11}$$

Noting that p is a nonconstant polynomial, we can deduce that

$$\mu + \lambda p_{z_1} - i \equiv 0, \;\; \mu - \lambda p_{z_1} + i \equiv 0. \tag{12}$$

Otherwise, the left-side of Equation (11) is transcendental and the right is polynomial; this is a contradiction. In view of (12), it follows that $\mu = 0$, which is a contradiction.

Therefore, this proves the conclusion of Corollary 3. □

For Equation (6), we obtain the following results about the existence and the forms of transcendental entire solutions of such equation.

Theorem 5. *Let $c = (c_1, c_2) \in \mathbb{C}^2$, $\alpha, \beta, \mu, \lambda_1, \lambda_2$ be nonzero constants in $\mathbb{C}$, $s_1 := \lambda_2 z_1 - \lambda_1 z_2$ and $s_0 := \lambda_2 c_1 - \lambda_1 c_2 \neq 0$. Let $f(z_1, z_2)$ be a finite order transcendental entire solution of Equation (6). Then, $f(z)$ must satisfy one of the following cases:*

(i) if $\mu f(z) + \lambda_1 f_{z_1}(z) + \lambda_2 f_{z_2}(z)$ is a constant, then

$$f(z_1, z_2) = \frac{\eta_1}{\mu} - \frac{1}{\mu} e^{-\frac{\mu}{\lambda_1} z_1 + A(\lambda_2 z_1 - \lambda_1 z_2) + B},$$

where $\eta_1, A, B \in \mathbb{C}$ satisfy $\eta_1^2 = \frac{\mu^2}{\mu^2 + (\alpha - \beta)^2}$ and $e^{A(\lambda_2 c_1 - \lambda_1 c_2)} = \frac{\beta}{\alpha} e^{\frac{\mu}{\lambda_1} c_1}$;

(ii) if $\mu^2 \neq (\lambda_1 A_1 + \lambda_2 A_2)^2$, then

$$\begin{aligned} f(z_1, z_2) = & \frac{1}{2(\lambda_1 A_1 + \lambda_2 A_2 + \mu)} e^{A_1 z_1 + A_2 z_2 + B} - \frac{1}{2(\lambda_1 A_1 + \lambda_2 A_2 - \mu)} e^{-A_1 z_1 - A_2 z_2 - B} \\ & + \vartheta(s_1) e^{-\frac{\mu}{\lambda_1} z_1}, \end{aligned}$$

where $\vartheta(s_1)$ is a finite order entire function in s_1 satisfying

$$\frac{\vartheta(s_1 + s_0)}{\vartheta(s_1)} = \frac{\beta}{\alpha} e^{\frac{\mu}{\lambda_1} s_0}, \tag{13}$$

and $A_1, A_2, B \in \mathbb{C}$ satisfy

$$(\lambda_1 A_1 + \lambda_2 A_2 + \beta i)^2 = \mu^2 - \alpha^2, \quad e^{2(A_1 c_1 + A_2 c_2)} = \frac{\lambda_1 A_1 + \lambda_2 A_2 + \beta i + \mu}{\lambda_1 A_1 + \lambda_2 A_2 + \beta i - \mu}; \tag{14}$$

(iii) if $\mu = \lambda_1 A_1 + \lambda_2 A_2$, then

$$f(z_1, z_2) = \frac{1}{4\mu} e^{A_1 z_1 + A_2 z_2 + B} + \frac{z_1}{2\lambda_1} e^{-A_1 z_1 - A_2 z_2 - B} + \vartheta(s_1) e^{-\frac{\mu}{\lambda_1} z_1},$$

where $\vartheta(s_1)$ is a finite order entire function satisfying (13) and $A_1, A_2, B \in \mathbb{C}$ satisfy

$$2\mu\beta i = \beta^2 - \alpha^2, \;\; \beta c_1 = \lambda_1 i, \;\; e^{2(A_1 c_1 + A_2 c_2)} = 1 - \frac{2\mu}{\beta} i; \tag{15}$$

(iv) if $\mu = -(\lambda_1 A_1 + \lambda_2 A_2)$, then

$$f(z_1, z_2) = \frac{z_1}{2\lambda_1} e^{A_1 z_1 + A_2 z_2 + B} + \frac{1}{4\mu} e^{-A_1 z_1 - A_2 z_2 - B} + \vartheta(s_1) e^{-\frac{\mu}{\lambda_1} z_1},$$

where $\vartheta(s_1)$ is a finite order entire function satisfying (13) and $A_1, A_2, B \in \mathbb{C}$ satisfy

$$-2\mu\beta i = \beta^2 - \alpha^2, \ \ \beta c_1 = -\lambda_1 i, \ \ e^{-2(A_1c_1+A_2c_2)} = 1 + \frac{2\mu}{\beta} i. \tag{16}$$

The following examples show the existence of transcendental entire solutions of (6).

Example 4. *Let $\eta_1^2 = \frac{1}{1-2i}$ and*

$$f(z_1, z_2) = \eta_1 - e^{-3z_1+z_2}.$$

Thus, $f(z_1, z_2)$ is a transcendental entire solution of (6) with $\alpha = 1$, $\beta = i$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\mu = 1$, $(c_1, c_2) = (\frac{\pi}{2} i, 2\pi i)$ and $\rho(f) = 1$. This shows that the form of solution in the conclusion (i) of Theorem 5 is precise.

Example 5. *Let*

$$\begin{aligned} f(z_1, z_2) =& \frac{1}{2(\sqrt{2}+1-i)} e^{A_1z_1+A_2z_2} + \frac{1}{2(\sqrt{2}-1+i)} e^{-A_1z_1-A_2z_2} \\ & - \frac{1}{\sqrt{2}} \sin[2\pi i(2z_1 - z_2)] e^{-\sqrt{2}z_1+\sqrt{2}(2z_1-z_2)}. \end{aligned}$$

where $A_1 = 2 - 2\log(\sqrt{2}+1) - (2-\pi)i$ and $A_2 = \log(\sqrt{2}+1) - 1 + (\frac{\pi}{2} - 1)i$. Thus, $f(z_1, z_2)$ is a transcendental entire solution of (6) with $\alpha = 1$, $\beta = 1$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\mu = \sqrt{2}$, $(c_1, c_2) = (1, 3)$ and $\rho(f) = 1$. This shows that the form of solution in the conclusion (ii) of Theorem 5 is precise.

From Theorem 5, we have

Corollary 4. *Let $c = (c_1, c_2) \in \mathbb{C}^2$ and $c_1 \neq c_2$. If $f(z_1, z_2)$ is a finite order transcendental entire solution of equation*

$$[f(z) + f_{z_1}(z) + f_{z_2}(z)]^2 + [f(z+c) - f(z)]^2 = 1,$$

then $f(z_1, z_2)$ must be of the form

$$f(z_1, z_2) = \pm 1 - e^{-z_1+A(z_2-z_1)+B},$$

where A, B are constants and

$$A(c_2 - c_1) = c_1 + 2k\pi i, k \in \mathbb{Z};$$

or

$$f(z_1, z_2) = \frac{1}{2(1-i)} e^{-iz_1+A_2z_2+B} + \frac{1}{2(1+i)} e^{iz_1-A_2z_2-B} + \vartheta(z_1 - z_2) e^{-z_1},$$

where $\vartheta(z_1 - z_2)$ is a finite order entire function, A_2, B are constants and

$$e^{2(-ic_1+A_2c_2)} = -1, \frac{\vartheta(z_1 - z_2 + c_1 - c_2)}{\vartheta(z_1 - z_2)} = e^{c_1-c_2},$$

When $\alpha = 1$ and $\beta = 0$ in Equation (6), we obtain

Corollary 5. *Let $c = (c_1, c_2) \in \mathbb{C}^2$ and $\lambda_1, \lambda_2, \mu$ be nonzero constants such that $\lambda_1 c_2 - \lambda_2 c_1 \neq 0$. If $f(z_1, z_2)$ is a finite order transcendental entire solution of equation*

$$[\mu f(z) + \lambda_1 f_{z_1}(z) + \lambda_2 f_{z_2}(z)]^2 + f(z+c)^2 = 1, \tag{17}$$

then $f(z_1, z_2)$ must be of the form

$$f(z_1,z_2) = \frac{1}{2(\mu+\lambda_1 A_1+\lambda_2 A_2)} e^{A_1 z_1 + A_2 z_2 + B} + \frac{1}{2(\mu-\lambda_1 A_1-\lambda_2 A_2)} e^{-A_1 z_1 - A_2 z_2 - B},$$

where a_1, a_2, b are constants and satisfying

$$(\lambda_1 A_1 + \lambda_2 A_2)^2 = \mu^2 - 1, \quad e^{2(A_1 c_1 + A_2 c_2)} = \frac{\lambda_1 A_1 + \lambda_2 A_2 + \mu}{\lambda_1 A_1 + \lambda_2 A_2 - \mu}, \tag{18}$$

Remark 3. *From Corollary 5, the order of f must be 1. However, we can find the transcendental entire solutions of Equation (17) of the order greater than one if $\lambda_1 c_2 - \lambda_2 c_1 = 0$. For example, let*

$$f(z) = \frac{1}{2i(\sqrt{3}+2)} e^{z_1+z_2+[(2i-1)z_1-z_2]^3} + \frac{1}{2i(\sqrt{3}-2)} e^{-z_1-z_2-[(2i-1)z_1-z_2]^3},$$

then $\rho(f) = 3$ and f is a transcendental entire solution of equation

$$\left[\sqrt{3} i f(z_1,z_2) + f_{z_1} + (2i-1) f_{z_2}\right]^2 + f\left(z_1 + \frac{1}{2i}\ln(2+\sqrt{3}), z_2 + \frac{2i-1}{2i}\ln(2+\sqrt{3})\right)^2 = 1.$$

When $\alpha = 0$ and $\beta = 1$ in Equation (6), similar to the argument as in the proof of Corollary 3, we have

Corollary 6. *Let $\lambda_1, \lambda_2, \mu$ be two nonzero constants. Then, the following partial differential equation*

$$[\mu f(z) + \lambda_1 f_{z_1}(z) + \lambda_2 f_{z_2}(z)]^2 + f(z)^2 = 1$$

does not admit any finite order transcendental entire solution.

3. Some Lemmas

The following lemma plays the key role in proving our results.

Lemma 1 ([39] Lemma 3.1)**.** *Let $f_j(\not\equiv 0), j = 1,2,3$ be meromorphic functions on $\mathbb{C}^m$ such that f_1 is not constant, and $f_1 + f_2 + f_3 = 1$, and such that*

$$\sum_{j=1}^{3}\left\{N_2(r, \frac{1}{f_j}) + 2\overline{N}(r, f_j)\right\} < \lambda T(r, f_1) + O(\log^+ T(r, f_1)),$$

for all r outside possibly a set with finite logarithmic measure, where $\lambda < 1$ is a positive number. Then, either $f_2 = 1$ or $f_3 = 1$.

Remark 4. *Here, $N_2(r, \frac{1}{f})$ is the counting function of the zeros of f in $|z| \le r$, where the simple zero is counted once, and the multiple zero is counted twice.*

4. The Proof of Theorem 4

Proof. Suppose that f is a transcendental entire solution of Equation (5) with finite order. Now, we will divide into two cases below.

(i) If $\lambda f_{z_1} + \mu f$ is a constant, let

$$\lambda f_{z_1} + \mu f = \eta_1, \tag{19}$$

and

$$\alpha f(z+c) - \beta f(z) = \eta_2, \tag{20}$$

where η_1, η_2 are constants in $\mathbb{C}$. In view of (5), it follows that

$$\eta_1^2 + \eta_2^2 = 1, \tag{21}$$

Solving Equation (19), we have

$$f(z) = \frac{\eta_1 - e^{-\frac{\mu}{\lambda}z_1 + \phi(z_2)}}{\mu}, \tag{22}$$

where $\phi(z_2)$ is an entire function in z_2. Substituting (22) into (20), it yields

$$\alpha \frac{\eta_1 - e^{-\frac{\mu}{\lambda}(z_1+c_1) + \phi(z_2+c_2)}}{\mu} - \beta \frac{\eta_1 - e^{-\frac{\mu}{\lambda}z_1 + \phi(z_2)}}{\mu} = \eta_2, \tag{23}$$

Thus, it follows from (23) that

$$(\alpha - \beta)\eta_1 = \mu\eta_2, \quad -\frac{\alpha}{\mu}e^{-\frac{\mu}{\lambda}(z_1+c_1)+\phi(z_2+c_2)} + \frac{\beta}{\mu}e^{-\frac{\beta}{\lambda}z_1+\phi(z_2)} = 0,$$

that is,

$$\eta_1^2 = \frac{\mu^2}{\mu^2 + (\alpha-\beta)^2}, \quad e^{\phi(z_2+c_2)-\phi(z_2)} = \frac{\beta}{\alpha}e^{\frac{\mu}{\lambda}c_1},$$

Hence, we have

$$\phi(z_2) = Az_2 + b, \quad e^{Ac_2} = \frac{\beta}{\alpha}e^{\frac{\mu}{\lambda}c_1}. \tag{24}$$

Thus, the conclusion (i) of Theorem 4 is proved from (22) and (24).

(ii) If $\lambda f_{z_1} + \mu f$ is not a constant, we can rewrite (5) as the form

$$[\mu f(z) + \lambda f_{z_1}(z) + i(\alpha f(z+c) - \beta f(z))][\mu f(z) + \lambda f_{z_1}(z) - i(\alpha f(z+c) - \beta f(z))] = 1.$$

Since f is an entire function, it follows that $\mu f(z) + \lambda f_{z_1}(z) + i(\alpha f(z+c) - \beta f(z))$ and $\mu f(z) + \lambda f_{z_1}(z) - i(\alpha f(z+c) - \beta f(z))$ do not exist zeros and poles. Thus, by virtue of Refs. [3,10,11], there exists a nonconstant polynomial $p(z)$ in $\mathbb{C}^2$ such that

$$\mu f(z) + \lambda f_{z_1}(z) + i(\alpha f(z+c) - \beta f(z)) = e^{p(z)},$$
$$\mu f(z) + \lambda f_{z_1}(z) - i(\alpha f(z+c) - \beta f(z)) = e^{-p(z)}.$$

The above equations lead to

$$\mu f(z) + \lambda f_{z_1}(z) = \frac{1}{2}(e^p + e^{-p}), \tag{25}$$

$$\alpha f(z+c) - \beta f(z) = \frac{1}{2i}(e^p - e^{-p}). \tag{26}$$

In view of (25) and (26), we can deduce that

$$\beta\mu f(z) + \beta\lambda f_{z_1}(z) = \frac{\alpha}{2}(e^{p(z+c)} + e^{-p(z+c)}) - \frac{\lambda p_{z_1} + \mu}{2i}e^{p(z)} - \frac{\lambda p_{z_1} - \mu}{2i}e^{-p(z)}. \tag{27}$$

Thus, it yields from (25) and (27) that

$$\frac{\lambda p_{z_1} + \mu + \beta i}{\alpha i}e^{p(z)+p(z+c)} + \frac{\lambda p_{z_1} - \mu + \beta i}{\alpha i}e^{p(z+c)-p(z)} - e^{2p(z+c)} \equiv 1. \tag{28}$$

Noting that $\mu \neq 0$, we thus have that $\frac{\lambda p_{z_1}+\mu+\beta i}{\alpha i} \equiv 0$ and $\frac{\lambda p_{z_1}-\mu+\beta i}{\alpha i} \equiv 0$ can not hold at the same time. Otherwise, it follows from (28) that $e^{2p(z+c)} = 1$, that is, $p(z) \equiv 0$, which is a contradiction with $p(z)$ being a nonconstant polynomial.

If $\frac{\lambda p_{z_1}-\mu+\beta i}{\alpha i} \equiv 0$, it follows that $\frac{\lambda p_{z_1}+\mu+\beta i}{\alpha i} \not\equiv 0$ from (28) and that

$$\frac{\lambda p_{z_1}+\mu+\beta i}{\alpha i}e^{p(z)+p(z+c)} - e^{2p(z+c)} \equiv 1. \tag{29}$$

By using the second basic theorem for the function $e^{2p(z+c)}$, we have from (29) that

$$\begin{aligned} T(r, e^{2p(z+c)}) &\leq N\left(r, \frac{1}{e^{2p(z+c)}}\right) + N\left(r, \frac{1}{e^{2p(z+c)}+1}\right) + S(r, e^{2p(z+c)}) \\ &\leq N\left(r, \frac{1}{\frac{\lambda p_{z_1}+\mu+\beta i}{\alpha i}e^{p(z)+p(z+c)}}\right) + S(r, e^{2p(z+c)}) \\ &\leq O(\log r) + S(r, e^{2p(z+c)}); \end{aligned}$$

this is impossible. If $\frac{\lambda p_{z_1}+\mu+\beta i}{\alpha i} \equiv 0$, similar to the argument as in the above, we also obtain a contradiction. Hence, we have $\frac{\lambda p_{z_1}+\mu+\beta i}{\alpha i} \not\equiv 0$ and $\frac{\lambda p_{z_1}-\mu+\beta i}{\alpha i} \not\equiv 0$.

By Lemma 1, we have

$$\frac{\lambda p_{z_1}+\mu+\beta i}{\alpha i}e^{p(z)+p(z+c)} \equiv 1, \quad or \quad \frac{\lambda p_{z_1}-\mu+\beta i}{\alpha i}e^{p(z+c)-p(z)} \equiv 1.$$

If

$$\frac{\lambda p_{z_1}+\mu+\beta i}{\alpha i}e^{p(z)+p(z+c)} \equiv 1,$$

then $p(z)+p(z+c)$ should be constant; this is a contradiction.

If

$$\frac{\lambda p_{z_1}-\mu+\beta i}{\alpha i}e^{p(z+c)-p(z)} \equiv 1, \tag{30}$$

then $p(z+c)-p(z)$ is a constant. Thus, we have $p(z) = L(z) + H(c_2z_1 - c_1z_2) + b$, where $L(z) = A_1z_1 + A_2z_2$, $H(s)$ is a polynomial in $s = c_2z_1 - c_1z_2$, A_1, A_2, B are constants in $\mathbb{C}$. By combining (30) with (28), we have

$$\frac{\lambda p_{z_1}+\mu+\beta i}{\alpha i}e^{p(z)-p(z+c)} \equiv 1. \tag{31}$$

Substituting $p(z)$ into (30) and (31), it follows that

$$\frac{\lambda(A_1+c_2H')-\mu+\beta i}{\alpha i}e^{L(c)} \equiv 1, \quad \frac{\lambda(A_1+c_2H')+\mu+\beta i}{\alpha i}e^{-L(c)} \equiv 1, \tag{32}$$

where $L(c) = A_1c_1 + A_2c_2$. Thus, we have that c_2H' is a constant, which implies $\deg_s H \leq 1$ as $c_2 \neq 0$. This shows that $L(z) + H(c_2z_1 - c_1z_2) + B$ is a linear form of z_1, z_2. For convenience, we still denote it to be $p(z) = L(z) + B$. Thus, it follows from (32) that

$$\frac{\lambda A_1-\mu+\beta i}{\alpha i}e^{L(c)} \equiv 1, \quad \frac{\lambda A_1+\mu+\beta i}{\alpha i}e^{-L(c)} \equiv 1. \tag{33}$$

This leads to

$$(\lambda A_1 + \beta i)^2 = \mu^2 - \alpha^2, \quad e^{2L(c)} = \frac{\lambda A_1+\mu+\beta i}{\lambda A_1-\mu+\beta i}. \tag{34}$$

Substituting $p(z) = A_1z_1 + A_2z_2 + B$ into the Equation (25), it follows

$$\mu f(z) + \lambda f_{z_1} = \frac{1}{2}(e^{A_1z_1+A_2z_2+B} + e^{-A_1z_1-A_2z_2-B}). \tag{35}$$

If $A_1 \neq \pm\frac{\mu}{\lambda}$, solving Equation (35), we have

$$f(z_1,z_2) = \frac{1}{2(\lambda A_1+\mu)}e^{A_1z_1+A_2z_2+B} - \frac{1}{2(\lambda A_1-\mu)}e^{-A_1z_1-A_2z_2-B} + \vartheta(z_2)e^{-\frac{\mu}{\lambda}z_1}, \tag{36}$$

where $\vartheta(z_2)$ is a finite order entire function. Substituting (36) into (26), and combining with (34), we have

$$(\lambda A_1+\beta i)^2 = \mu^2-\alpha^2, \quad e^{2(A_1c_1+A_2c_2)} = \frac{\lambda A_1+\mu+\beta i}{\lambda A_1-\mu+\beta i},$$

and

$$\frac{\vartheta(z_2+c_2)}{\vartheta(z_2)} = \frac{\beta}{\alpha}e^{\frac{\mu}{\lambda}c_1}. \tag{37}$$

If $A_1 = \frac{\mu}{\lambda}$, similar to the above argument, we have

$$f(z_1,z_2) = \frac{1}{4\mu}e^{A_1z_1+A_2z_2+B} + \frac{z_1}{2\lambda}e^{-A_1z_1-A_2z_2-B} + \vartheta(z_2)e^{-\frac{\mu}{\lambda}z_1}, \tag{38}$$

where $\vartheta(z_2)$ is a finite order entire function satisfying (37). Substituting (38) into (26), and combining with (34), it follows that $f(z_1,z_2)$ satisfies (8).

If $A_1 = -\frac{\mu}{\lambda}$, similar to the above argument, we have

$$f(z_1,z_2) = \frac{z_1}{2\lambda}e^{A_1z_1+A_2z_2+B} + \frac{1}{4\mu}e^{-A_1z_1-A_2z_2-B} + \vartheta(z_2)e^{-\frac{\mu}{\lambda}z_1}, \tag{39}$$

where $\vartheta(z_2)$ is a finite order entire function satisfying (37). Substituting (39) into (26), and combining with (34), it follows that $f(z_1,z_2)$ satisfies (9).

Therefore, this completes the proof of Theorem 4. □

5. The Proof of Theorem 5

Proof. Suppose that f is a transcendental entire solution of Equation (6) with a finite order. Two cases will be considered below.

(i) If $\lambda_1 f_{z_1} + \lambda_2 f_{z_2} + \mu f$ is a constant, let

$$\lambda_1 f_{z_1} + \lambda_2 f_{z_2} + \mu f = \eta_1, \tag{40}$$

and

$$\alpha f(z+c) - \beta f(z) = \eta_2, \tag{41}$$

where η_1, η_2 are constants in $\mathbb{C}$ satisfying (21) from (6). The characteristic equations of (40) are

$$\frac{dz_1}{dt} = \lambda_1, \quad \frac{dz_2}{dt} = \lambda_2, \quad \frac{df}{dt} = \eta_1 - \mu f.$$

Using the initial conditions: $z_1 = 0, z_2 = s_1$, and $f = f(0,s_1)$ with a parameter s. Thus, we obtain the following parametric representation for the solutions of the characteristic equations: $z_1 = \lambda_1 t$, $z_2 = \lambda_2 t + s_1$, and

$$f(z_1,z_2) = \frac{\eta_1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu}{\lambda_1}z_1+\varphi(s_1)}, \tag{42}$$

where $\varphi(s_1)$ is an entire function in $s_1 := \lambda_1 z_2 - \lambda_2 z_1$. Substituting (42) into (41), we have

$$\alpha\left(\frac{\eta_1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu}{\lambda_1}(z_1+c_1)+\varphi(s_1+s_0)}\right) - \beta\left(\frac{\eta_1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu}{\lambda_1}z_1+\varphi(s_1)}\right) = \eta_2,$$

which implies that

$$\frac{\alpha\eta_1}{\mu}-\frac{\beta\eta_1}{\mu}\equiv\eta_2,\quad \frac{\alpha}{\mu}e^{-\frac{\mu}{\lambda_1}(z_1+c_1)+\varphi(s_1+s_0)}-\frac{\beta}{\mu}e^{-\frac{\mu}{\lambda_1}z_1+\varphi(s_1)}\equiv 0,$$

where $s_0:=\lambda_2c_1-\lambda_1c_2$. In view of (21), we have

$$\eta_1^2=\frac{\mu^2}{\mu^2+(\alpha-\beta)^2}, \tag{43}$$

and

$$e^{\varphi(s_1+s_0)-\varphi(s_1)}=\frac{\beta}{\alpha}e^{\frac{\mu}{\lambda_1}c_1}.$$

Thus, it follows that $\varphi(s_1)=As_1+b$ where A,b are constants satisfying

$$e^{As_0}=e^{A(\lambda_2c_1-\lambda_1c_2)}=\frac{\beta}{\alpha}e^{\frac{\mu}{\lambda_1}c_1}. \tag{44}$$

In view of (42)–(44), we have

$$f(z_1,z_2)=\frac{\eta_1}{\mu}-\frac{1}{\mu}e^{-\frac{\mu}{\lambda_1}z_1+A(\lambda_2z_1-\lambda_1z_2)+b}, \tag{45}$$

where $\alpha,\beta,\mu,\lambda_1,\lambda_2,\eta_1,c_1,c_2,A$ are constants and satisfying (43) and (44). Therefore, this proves the conclusion (i) of Theorem 5.

(ii) If $\lambda_1f_{z_1}+\lambda_2f_{z_2}+\mu f$ is not a constant, we can rewrite (6) as the form

$$\begin{aligned}&[\mu f(z)+\lambda_1f_{z_1}(z)+\lambda_2f_{z_2}(z)+i(\alpha f(z+c)-\beta f(z))]\times\\&\qquad[\mu f(z)+\lambda_1f_{z_1}(z)+\lambda_2f_{z_2}(z)-i(\alpha f(z+c)-\beta f(z))]=1.\end{aligned}$$

Since f is an entire function, it follows that $\mu f(z)+\lambda_1f_{z_1}(z)+\lambda_2f_{z_2}(z)+i(\alpha f(z+c)-\beta f(z))$ and $\mu f(z)+\lambda_1f_{z_1}(z)+\lambda_2f_{z_2}(z)-i(\alpha f(z+c)-\beta f(z))$ do not exist zeros and poles. Thus, by virtue of Refs. [3,10,11], there exists a nonconstant polynomial $p(z)$ in $\mathbb{C}^2$ such that

$$\begin{aligned}\mu f(z)+\lambda_1f_{z_1}(z)+\lambda_2f_{z_2}(z)+i(\alpha f(z+c)-\beta f(z))&=e^{p(z)},\\ \mu f(z)+\lambda_1f_{z_1}(z)+\lambda_2f_{z_2}(z)-i(\alpha f(z+c)-\beta f(z))&=e^{-p(z)}.\end{aligned}$$

The above equations lead to

$$\mu f(z)+\lambda_1f_{z_1}(z)+\lambda_2f_{z_2}(z)=\frac{1}{2}(e^p+e^{-p}), \tag{46}$$

$$\alpha f(z+c)-\beta f(z)=\frac{1}{2i}(e^p-e^{-p}). \tag{47}$$

In view of (46) and (47), we can deduce that

$$\begin{aligned}&\alpha\mu f(z+c)+\beta[\lambda_1f_{z_1}(z)+\lambda_2f_{z_2}(z)]\\=&\frac{\alpha}{2}(e^{p(z+c)}+e^{-p(z+c)})-\frac{\lambda_1p_{z_1}+\lambda_2p_{z_2}}{2i}(e^{p(z)}+e^{-p(z)}).\end{aligned} \tag{48}$$

From (48) and (46), we have

$$\alpha\mu f(z+c)-\beta\mu f(z)=\frac{\alpha}{2}(e^{p(z+c)}+e^{-p(z+c)})-\frac{\lambda_1p_{z_1}+\lambda_2p_{z_2}+\beta i}{2i}(e^{p(z)}+e^{-p(z)}). \tag{49}$$

Thus, it yields from (46) and (49) that

$$\frac{\lambda_1 p_{z_1} + \lambda_2 p_{z_2} + \mu + \beta i}{\alpha i} e^{p(z)+p(z+c)} + \frac{\lambda_1 p_{z_1} + \lambda_2 p_{z_2} - \mu + \beta i}{\alpha i} e^{p(z+c)-p(z)} - e^{2p(z+c)} \equiv 1. \quad (50)$$

By using the same argument as in the proof of Theorem 4, we have $\frac{\lambda_1 p_{z_1}+\lambda_2 p_{z_2}+\mu+\beta i}{\alpha i} \not\equiv 0$ and $\frac{\lambda_1 p_{z_1}+\lambda_2 p_{z_2}-\mu+\beta i}{\alpha i} \not\equiv 0$. By Lemma 1 and (50), we have

$$\frac{\lambda_1 p_{z_1} + \lambda_2 p_{z_2} + \mu + \beta i}{\alpha i} e^{p(z)+p(z+c)} \equiv 1, \quad or \quad \frac{\lambda_1 p_{z_1} + \lambda_2 p_{z_2} - \mu + \beta i}{\alpha i} e^{p(z+c)-p(z)} \equiv 1.$$

If

$$\frac{\lambda_1 p_{z_1} + \lambda_2 p_{z_2} + \mu + \beta i}{\alpha i} e^{p(z)+p(z+c)} \equiv 1,$$

then $p(z) + p(z+c)$ should be constant; this is a contradiction.

If

$$\frac{\lambda_1 p_{z_1} + \lambda_2 p_{z_2} - \mu + \beta i}{\alpha i} e^{p(z+c)-p(z)} \equiv 1, \quad (51)$$

then $p(z+c) - p(z)$ is a constant. Thus, we have $p(z) = L(z) + H(c_2 z_1 - c_1 z_2) + b$, where $L(z) = A_1 z_1 + A_2 z_2$, $H(s)$ is a polynomial in $s = c_2 z_1 - c_1 z_2$, A_1, A_2, B are constants in $\mathbb{C}$. By combining (51) with (50), we have

$$\frac{\lambda_1 p_{z_1} + \lambda_2 p_{z_2} + \mu + \beta i}{\alpha i} e^{p(z)-p(z+c)} \equiv 1. \quad (52)$$

Substituting $p(z)$ into (51) and (52), it follows that

$$\frac{\lambda_1 A_1 + \lambda_2 A_2 + (\lambda_1 c_2 - \lambda_2 c_1)H' - \mu + \beta i}{\alpha i} e^{L(c)} \equiv 1, \quad (53)$$

$$\frac{\lambda_1 A_1 + \lambda_2 A_2 + (\lambda_1 c_2 - \lambda_2 c_1)H' + \mu + \beta i}{\alpha i} e^{-L(c)} \equiv 1. \quad (54)$$

Thus, we have that $(\lambda_1 c_2 - \lambda_2 c_1)H'$ is a constant, which implies $\deg_s H \leq 1$ as $\lambda_1 c_2 - \lambda_2 c_1 \neq 0$. This shows that $L(z) + H(c_2 z_1 - c_1 z_2) + B$ is a linear form of z_1, z_2. For convenience, we still assume that $p(z) = L(z) + B$. Thus, it follows from (53) and (54) that

$$\frac{\lambda_1 A_1 + \lambda_2 A_2 - \mu + \beta i}{\alpha i} e^{L(c)} \equiv 1, \quad \frac{\lambda_1 A_1 + \lambda_2 A_2 + \mu + \beta i}{\alpha i} e^{-L(c)} \equiv 1. \quad (55)$$

This leads to

$$(\lambda_1 A_1 + \lambda_2 A_2 + \beta i)^2 = \mu^2 - \alpha^2, \quad e^{2L(c)} = \frac{\lambda_1 A_1 + \lambda_2 A_2 + \mu + \beta i}{\lambda_1 A_1 + \lambda_2 A_2 - \mu + \beta i}. \quad (56)$$

Substituting $p(z) = A_1 z_1 + A_2 z_2 + B$ into the Equation (46), it follows

$$\mu f(z) + \lambda_1 f_{z_1} + \lambda_2 f_{z_2} = \frac{1}{2}(e^{A_1 z_1 + A_2 z_2 + B} + e^{-A_1 z_1 - A_2 z_2 - B}). \quad (57)$$

If $\mu^2 \neq (\lambda_1 A_1 + \lambda_2 A_2)^2$, solving Equation (57), we have

$$f(z_1, z_2) = \frac{1}{2(\lambda_1 A_1 + \lambda_2 A_2 + \mu)} e^{A_1 z_1 + A_2 z_2 + B} - \frac{1}{2(\lambda_1 A_1 + \lambda_2 A_2 - \mu)} e^{-A_1 z_1 - A_2 z_2 - B} + \vartheta(s_1) e^{-\frac{\mu}{\lambda_1} z_1}, \quad (58)$$

where ϑ is a finite order entire function in s_1. Substituting (58) into (47), and combining with (55), we have

$$\frac{\vartheta(s_1+s_0)}{\vartheta(s_1)} = \frac{\beta}{\alpha} e^{\frac{\mu}{\lambda_1} s_0}, \tag{59}$$

where $s_0 = \lambda_2 c_1 - \lambda_1 c_2$. Therefore, in view of (56), (58), and (59), we have

$$\begin{aligned} f(z_1,z_2) =& \frac{1}{2(\lambda_1 A_1 + \lambda_2 A_2 + \mu)} e^{A_1 z_1 + A_2 z_2 + B} - \\ & \frac{1}{2(\lambda_1 A_1 + \lambda_2 A_2 - \mu)} e^{-A_1 z_1 - A_2 z_2 - B} + \vartheta(s_1) e^{-\frac{\mu}{\lambda_1} z_1}, \end{aligned} \tag{60}$$

where ϑ is a finite order entire function in s_1, and A_1, A_2, B are constants satisfying (56) and (59).

If $\mu = \lambda_1 A_1 + \lambda_2 A_2$, solving Equation (56), similar to the above argument, we have

$$f(z_1,z_2) = \frac{1}{4\mu} e^{A_1 z_1 + A_2 z_2 + B} + \frac{z_1}{2\lambda_1} e^{-A_1 z_1 - A_2 z_2 - B} + \vartheta(s_1) e^{-\frac{\mu}{\lambda_1} z_1}, \tag{61}$$

where $\vartheta(s_1)$ is a finite order entire function satisfying (59). Substituting (61) into (47), and combining with (55), we can obtain (15).

If $\mu = -(\lambda_1 A_1 + \lambda_2 A_2)$, solving Equation (56), similar to the above argument, we have

$$f(z_1,z_2) = \frac{z_1}{2\lambda_1} e^{A_1 z_1 + A_2 z_2 + B} + \frac{1}{4\mu} e^{-A_1 z_1 - A_2 z_2 - B} - \vartheta(s_1) e^{-\frac{\mu}{\lambda_1} z_1}, \tag{62}$$

where $\vartheta(s_1)$ is a finite order entire function satisfying (59). Substituting (62) into (47), and combining with (55), we can obtain (16).

Therefore, we complete the proof of Theorem 5. □

6. Conclusions

From Theorems 4 and 5, we investigate the transcendental entire solutions of two classes of partial differential-difference equations with constant coefficients, which are more general than the previous equations given by [19,20,38]. We describe the forms of the finite order transcendental entire solutions of these equations under the different conditions of the coefficients, and we also give several examples to demonstrate that every form of the solutions of these equations are precise. By comparing previous relevant references, we can find that our results are some improvements and generalizations of the previous theorems [19,20,38].

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