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Deriving Fuzzy Weights from the Consistent Fuzzy Analytic Hierarchy Process

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Abstract: The analytic hierarchy process (AHP) is one of the most popular multi-criteria decision-making (MCDM) methods, and so is its extension fuzzy analytic hierarchy process (FAHP). However, the FAHP, unlike the AHP, easily handles the trusted weights by the consistency index (CI) or consistency ratio (CR). We need to first derive the consistent fuzzy pairwise comparison matrix (FPCM) by the transitivity axiom and then drive fuzzy weights. We also need a flexible mechanism for users to control the spread of fuzzy weights under tolerable consistency. In this paper, we propose a novel model based on mathematical programming to derive rational fuzzy weights of the FAHP and provide a parameter for decision-makers to control the spread of fuzzy weights. Three examples are used to demonstrate the proposed method and compared with others to validate and justify the proposed method.

Keywords: analytic hierarchy process; multi-criteria decision-making; consistency index; consistency ratio; mathematical programming

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1. Introduction

The analytic hierarchy process (AHP) has been one of the most popular multi-criteria decision-making (MCDM) methods in dealing with various industrial and business problems since the 1980s. With continuous development and modification, the AHP has been extended to consider different situations, e.g., fuzzy/interval, gray [1–3], and rough [4–6] environments. Among these methods, the fuzzy analytic hierarchy process (FAHP) is undoubtedly the most popular way to extend the AHP to consider the subjective uncertainty problem [7–9]. Hence, we focus on the FAHP in this paper.

Several researchers have argued that it is erroneous to fuzzify the AHP because the measurement scale itself, i.e., from one is equal to equal importance to nine is equal to extreme importance, implies the fuzzy concept [10,11] or asserts that FAHP leads to incorrect results [12]. However, they cannot stop the popularity of the FAHP and the application of the FAHP in various domains, e.g., education [13–15], and engineering [16–18]. In fact, some papers have even reported that FAHP is more stable than AHP [19] and still necessary when the pairwise comparison matrix (PCM) is vague to prevent the mistakes of the AHP [20]. The above description indicates that from one side, decision-makers do not like too vague results since it is hard to make a decision. Conversely, we still need some fuzzy techniques to conduct such fuzzy data. Therefore, the compromise strategies might be (1) to derive crisp weights or (2) to obtain the minimum spread of the fuzzy weights from a consistent fuzzy comparison matrix (FCPM).

Many papers have been proposed to fulfill the first strategy, e.g., [21–23]. However, these methods only derive crisp weights and lose the information of fuzzy weights, which is critical in some areas, e.g., banking and investment, which highlight possibility and risk. Next, the problem of finding the minimum spread of the fuzzy weights also needs to consider the issues of determining the FPCM and consistent index. That is, in the AHP, we can accept or tolerate the weights as long as the consistent index is less than the tolerable level. Hence, we still need to consider that issue to derive fuzzy weights under the acceptable inconsistent level.

The purpose here is to provide a method to derive fuzzy weights of the FAHP by considering the consistent FPCM and the tolerable parameter. The consistent FPCM ensures the fuzzy weights are entirely rational, i.e., the minimum spread of fuzzy weights. On the other hand, we add the tolerable parameter to enlarge the fuzzy weights to the acceptable level, i.e., tolerate some fuzzy weights derived from acceptably inconsistent FPCM.

The rest of the paper is organized as follows. First, we will give consideration to the problem in Section 2. In Section 3, we will introduce how past papers handled the inconsistent issue of the FAHP. In Section 4, we propose our model and demonstrate the proposed method in Section 5. The discussion and conclusion are in Sections 6 and 7.

2. The Problem Description

The problem here is to tackle the consistency of the FAHP, which means that the derived fuzzy weights of criteria should be rational with the rational judgment of a decision-maker. We will give an example to demonstrate the consistent problem of the FAHP as follows. First, let us consider a FPCM with three criteria as follows:

$$\tilde{A} = \begin{array}{c} \\ C1 \\ C2 \\ C3 \end{array} \begin{array}{c} \begin{array}{ccc} C1 & C2 & C3 \end{array} \\ \begin{bmatrix} (1,1,1) & (3,4,5) & (4,5,6) \\ \tilde{a}_{12}^{-1} & (1,1,1) & (1,2,3) \\ \tilde{a}_{13}^{-1} & \tilde{a}_{23}^{-1} & (1,1,1) \end{bmatrix} \end{array}$$

where any $\tilde{a}_{ij} = (l_{ij}, m_{ij}, r_{ij})$ denotes a triangular fuzzy number and l_{ij}, m_{ij}, and r_{ij} means the left, center, and right values of the fuzzy number, respectively. Note that we use triangular fuzzy numbers in this paper for simplicity and can easily extend to trapezoid fuzzy numbers.

Here, we use Saaty's eigenvalue method [24] to derive the crisp weights of the criteria and the corresponding consistency index (CI) and consistency ratio (CR), as shown in Table 1. Note that we use the center values of the FPCM as the data to process the eigenvalue method.

Table 1. The AHP with the crisp weights and consistent indexes.

	C1	**C2**	**C3**
Weights	0.6833	0.1998	0.1169
CI		0.0123	
CR		0.0212	

Note that $\lambda_{\max} = 3.0246$ and RI = 0.58.

As shown in Table 1, since CI and CR are less than the acceptable level of 0.1, we can conclude that the above weights are rational and trusted.

Next, we use the FAHP to derive the fuzzy weights of the criteria and illustrate how irrational fuzzy weights happen. Note that we use Huang's method [25] without considering the transitivity axiom to derive the fuzzy weights for simplicity. The fuzzy weights of the criteria and the corresponding fuzzy CI and CR are in Table 2.

Table 2. The FAHP with the fuzzy weights and consistent indexes.

	C1	C2	C3
Weights	[0.614, 0.6833, 0.732]	[0.138, 0.1998, 0.268]	[0.088, 0.1169, 0.174]
CI		[0.000, 0.098]	
CR		[0.000, 0.170]	

Here, we derive three fuzzy weights of the criteria and calculate the corresponding fuzzy CI and CR, respectively. As we can see from the fuzzy CR, some fuzzy weights result in irrational results. For example, if we take w_1 = 0.6795, w_2 = 0.2111, and w_3 = 0.1093, we can calculate CR = 0.170 and conclude that the weights are irrational. The method to handle the problem is to add the transitivity axiom of the pairwise comparison elements. Next, we will review the literature about the consistency of the FAHP.

3. Literature Review for the Consistent Issue of the FAHP

The consistency of the AHP means that the PCM fulfills both the reciprocal and transitivity axioms to ensure the decision-maker is neither random nor illogical to give pairwise comparison elements. Since the AHP is an expert opinion method, consistency and rationality are essential to derive the correct result. Generally, a consistent matrix with n criteria of the AHP means that all pairwise comparing weight ratios i to j (i.e., a_{ij}) should follow (i) the transitivity axiom:

$$a_{ij} = a_{ik} \times a_{kj}, \forall k \in n - \{i, j\} \tag{1}$$

and (ii) the reciprocal axiom:

$$a_{ji} = \frac{1}{a_{ij}} \tag{2}$$

where a_{ij} is called a consistent multiplicative reciprocal preference relation.

Next, we can derive weights of criteria by one of the AHP methods, e.g., eigenvalue method [24], least square method [26], and geometric mean method [27], and then calculate the consistency of weights to check if the weights of criteria are acceptable. Otherwise, we should revise the comparison weight matrix to avoid the inconsistent issue of the AHP. For example, we can use the eigenvalue method to obtain the weights of criteria by solving Equation (3):

$$\boldsymbol{A}\boldsymbol{w} = \lambda_{\max}\boldsymbol{w} \tag{3}$$

where $\boldsymbol{A}$ denotes the PCM, $\lambda_{\max}$ denotes the maximum eigenvalue, and $\boldsymbol{w}$ is the weight vector. Next, we can use consistent indexes to judge whether the weights are correct and rational.

Many indexes have been proposed to measure the consistency of weights derived from the AHP. For example, Let us consider an AHP problem with n criteria, and Saaty's CI and CR [24] are defined, respectively, as follows:

$$\text{CI} = \frac{\lambda_{\max} - n}{n - 1} \tag{4}$$

and

$$\text{CR} = \frac{\text{CI}}{\text{RI}} \tag{5}$$

where RI denotes the random index, and weights are considered rational when CI and CR < 0.1. In addition, Crawford and Williams [28] also proposed the geometric consistency index as:

$$GCI = \frac{1}{(n-1)(n-2)} \min \sum_{i=1}^{n} \sum_{j=1}^{n} (\log(a_{ij}) - \log(w_i) + \log(w_j))^2 \tag{6}$$

where GCI = 0.31 for n =3, GCI = 0.35 for n = 4, and GCI = 0.37 for n > 4 as the thresholds for the consistent result [29]. Surely, more consistent indexes have been proposed, e.g., [30–32], to ensure the derived weights are rational and trusted.

When the consistent issue of the AHP shifts to the FAHP, the first problem is to find the consistent FPCM. Although the consistent indexes can be easily detected by using the given PCM, it suffers many problems when extending to the fuzzy environment. Before introducing the conditions in the FAHP, we first give some basic fuzzy arithmetic operations. Let two triangular fuzzy numbers, $\widetilde{a} = (a_l, a_c, a_r) > 0$ and $\widetilde{b} = (b_l, b_c, b_r) > 0$, the basic fuzzy arithmetic operations can be described as follows:

$$\widetilde{a} + \widetilde{b} = (a_l + b_l, a_c + b_c, a_r + b_r) \tag{7}$$

$$\widetilde{a} - \widetilde{b} = (a_l - b_r, a_c - b_c, a_r - b_l) \tag{8}$$

$$\widetilde{a} \times \widetilde{b} = (a_l \times b_l, a_c \times b_c, a_r \times b_r) \tag{9}$$

$$\widetilde{a} \div \widetilde{b} = (\frac{a_l}{b_r}, \frac{a_c}{b_c}, \frac{a_r}{b_l}) \tag{10}$$

First, the reciprocal and transitivity axioms of the fuzzy comparison weight ratios mean:

$$\widetilde{a}_{ji} = \frac{1}{\widetilde{a}_{ij}} \tag{11}$$

$$\widetilde{a}_{ij} = \widetilde{a}_{ik} \times \widetilde{a}_{kj}, \forall k \in n - \{i, j\} \tag{12}$$

which is not held in the general fuzzy situation. Although we usually view Equation (11) as the postulation, we need more considerations to conduct Equation (12), i.e., the fuzzy multiplicative reciprocal property or transitivity axiom. Note that compared to the crisp AHP that uses 1 to 9 to indicate the levels from identical importance to extreme importance, we use fuzzy numbers to represent each linguistic scale. For example, we can use the fuzzy number (4,5,6) to indicate strong importance.

Several papers have been proposed to consider the consistency of the FPCM to respond to the transitivity axiom. For example, [33] suggested that the FPCM is consistent if the matrix includes the middle number. In addition, [34] defined that an interval multiplicative reciprocal matrix is consistent if the following condition holds:

$$\frac{\widetilde{a}_{ij}}{\widetilde{a}_{ik}\widetilde{a}_{kj}} = \frac{\widetilde{a}_{ik}\widetilde{a}_{kj}}{\widetilde{a}_{ij}} \tag{13}$$

Although Equation (13) seems rationable, the strict equality between fuzzy sets indicates all $\frac{\widetilde{a}_{ij}}{\widetilde{a}_{ik}\widetilde{a}_{kj}}$ values would belong to $\frac{\widetilde{a}_{ik}\widetilde{a}_{kj}}{\widetilde{a}_{ij}}$, and vice-versa. However, this is not feasible in ordinary situations.

Hence, [35] suggested that a FPCM $\tilde{A}$ is strongly transitive if

$$a_{ijc} \times a_{jkc} = a_{ikc} \tag{14}$$

and

$$\sqrt{a_{ijr} \times a_{ijl}}\sqrt{a_{jkr} \times a_{ikl}} = \sqrt{a_{ikr} \times a_{ikl}}, \forall i, j, k \tag{15}$$

where $\widetilde{a}_{ij} = (a_{ijl}, a_{ijc}, a_{ijr})$. The conditions above ensure the consistency of the derived fuzzy weights by logarithmic least square methods (LLSM).

Then, [36] mentioned that a FPCM $\tilde{A}$ with n criteria is said to be a multiplicative reciprocal matrix if it follows the following transitive condition:

$$a_{ijl}a_{ijr} = a_{ikl}a_{ikr}a_{kjl}a_{kjr}, \ \forall i, j, k = 1, 2, \ldots, n, \tag{16}$$

or as:

$$a_{ijl} = a_{ikl}a_{kjl},\ a_{ijr} = a_{ikr}a_{kjr},\ \forall i,j,k = 1,2,\ldots,n. \tag{17}$$

Furthermore, [37] also asked that a FPCM is called knowledge-based consistency if its elements satisfy:

$$\left(\underset{\forall i,j\in 1,\ldots,n|i<j}{\cap}\left(\widetilde{a}_{ik} \times \widetilde{a}_{kj}\right) \cap \widetilde{a}_{ij}\right) \neq \varnothing \tag{18}$$

The purpose of Equation (18) is to replace the strict equality between fuzzy sets by the intersection operator. We can depict the consistent comparison ratio of [37] for $\widetilde{a}_{23}$, as shown in Figure 1, and $\widetilde{c}_{23}$ can be viewed as the element of the consistent FPCM.

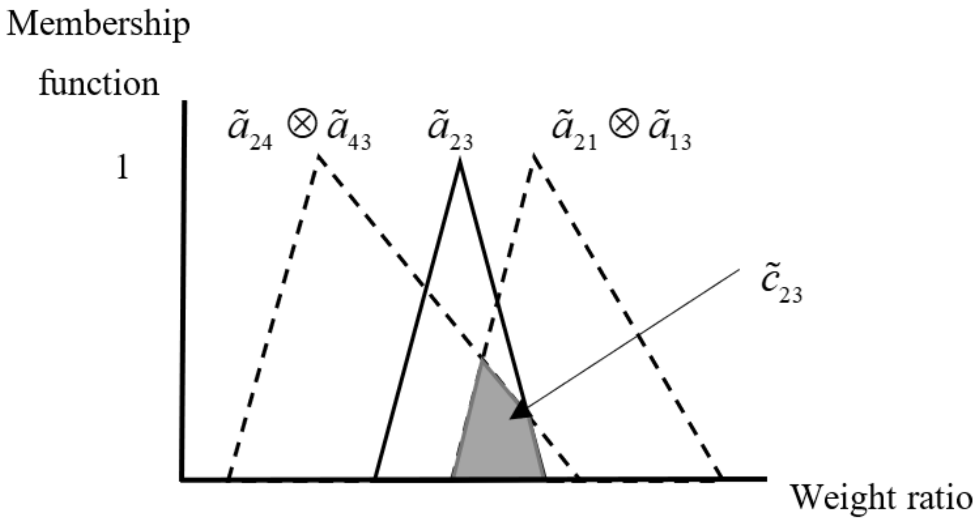

Figure 1. The consistent comparison ratio for $\widetilde{a}_{23}$.

No matter which conditions we use, once we obtain a consistent FPCM, the next step is to derive (fuzzy) weights to ensure the consistency of the FAHP. For example, we can derive weights of criteria by one of the FAHP methods from the fuzzy eigenvalue method [33,38], the logarithmic least square method [39,40], and the fuzzy geometric mean method [41,42], and then calculate the consistency ratio (CR) to check if the weights of criteria are acceptable. Usually, three kinds of methods can be used [37]: (1) mathematical programming, (2) direct fuzzification (DF) methods, and (3) fuzzy feasible region (FFR) methods. FFR methods only derive crisp weights, and DF methods usually ignore the transitivity axion to derive inconsistent results. Hence, we develop our method by optimizing a mathematical programming model. However, the distinction between the proposed method and other mathematical programming models is that our method provides the flexibility to obtain different fuzzy weights instead of only one solution.

4. The Proposed Method

This paper proposes a novel model to consider the FAHP with the following properties. First, we need to derive fuzzy weights of criteria rather than crisp weights due to the consideration of risk evaluation and sensitivity analysis. Hence, FFR methods are not suitable for the purpose here. Second, FPCM should follow the consistent multiplicative reciprocal preference relation to ensure the comparison matrix is consistent. Hence, DF methods are inappropriate since these methods do not consider extra constraints. Third, we can derive the CI/CR from the FPCM to understand the level of consistency of the FAHP. Finally, those methods only provided one solution, no matter crisp or fuzzy weights. Here, we provide a flexible solution to obtain different fuzzy weights according to the tolerable parameter determined by the decision-maker.

To achieve the above purposes, we can develop the following mathematical programming to derive the consistent fuzzy AHP as follows:

$$\min/\max w_i$$

$$\left.\begin{array}{l} \widetilde{c}_{ij} \subseteq \widetilde{a}_{ij}[\alpha] \\ \widetilde{c}_{ij} \subseteq \widetilde{a}_{ik}[\alpha] \otimes \widetilde{a}_{kj}[\alpha] + e_{ij}, \forall k \in n - \{i, j\} \end{array}\right\}\text{Consistent FPCM} \tag{19}$$

$$\left.\begin{array}{l} \widetilde{w}_{n-1} = \widetilde{c}_{n-1n}\widetilde{w}_n \\ w_{n-2} = \frac{1}{2}(\widetilde{c}_{n-2n-1}w_{n-1} + \widetilde{c}_{n-2n}w_n) \\ \quad\vdots \\ \widetilde{w}_1 = \frac{1}{n-1}(\widetilde{c}_{12}w_2 + \widetilde{c}_{13}w_3 + \cdots + \widetilde{c}_{1n}w_n) \end{array}\right\}\text{Deriving fuzzy weights} \tag{20}$$

$$\varepsilon = \sum_{i,j=1,i\neq j}^{n} e_{ij}^2 < \beta, \text{ Tolerable parameter} \tag{21}$$

where α denotes the $\alpha-$cut operation and β denotes the tolerable parameter, which controls the spread of fuzzy weights. We can view β as the relaxation of the transitivity axiom in the FAHP. Note that e_{ij} can be considered as the sensitivity parameter. If e_{ij} = 0, we can explain that $\widetilde{c}_{ij}$ cannot be enlarged to avoid the inconsistency problem of fuzzy weights.

The proposed model first use Equation (19) to derive consistent FPCM, $\widetilde{\boldsymbol{C}}$, by incorporating the transitivity axiom. However, we add the tolerable parameters to each consistent fuzzy pairwise comparison element, i.e., $\widetilde{c}_{ij}$, to enlarge fuzzy weights under the acceptable level β. If we ignore the tolerable parameters, we can only derive the fuzzy weights with CI/CR = 0. Hence, the tolerable parameters can be considered as a trade-off between the perfectly consistent and acceptably consistent FPCMs. In addition, we use Equation (21) to restrict the range of fuzzy weights in our method.

In addition, we use Equation (20) to derive fuzzy weights [25]. The main reason is that only upper triangular consistent FPCM information is needed here to derive the result. Furthermore, these conditions are linear and easily conducted by mathematical programming.

Finally, the result can be validated by the following mathematical programming:

$$\begin{array}{c} \text{maxCI or CR} \\ \widetilde{\lambda}_{\max} = \sum_{j=1}^{n}\sum_{i=1}^{n} \widetilde{c}_{ij}\widetilde{w}_j, \\ \widetilde{\text{CI}} = \frac{\widetilde{\lambda}_{\max} - n}{n-1}, \\ \widetilde{\text{CR}} = \frac{\widetilde{\text{CI}}}{\text{RI}}. \end{array} \tag{22}$$

Equations (19)–(21), where the highest values of CI and CR should be less than 0.1 to validate the derived fuzzy weights are rational and reliable.

The flowchart of the proposed method is depicted as shown in Figure 2.

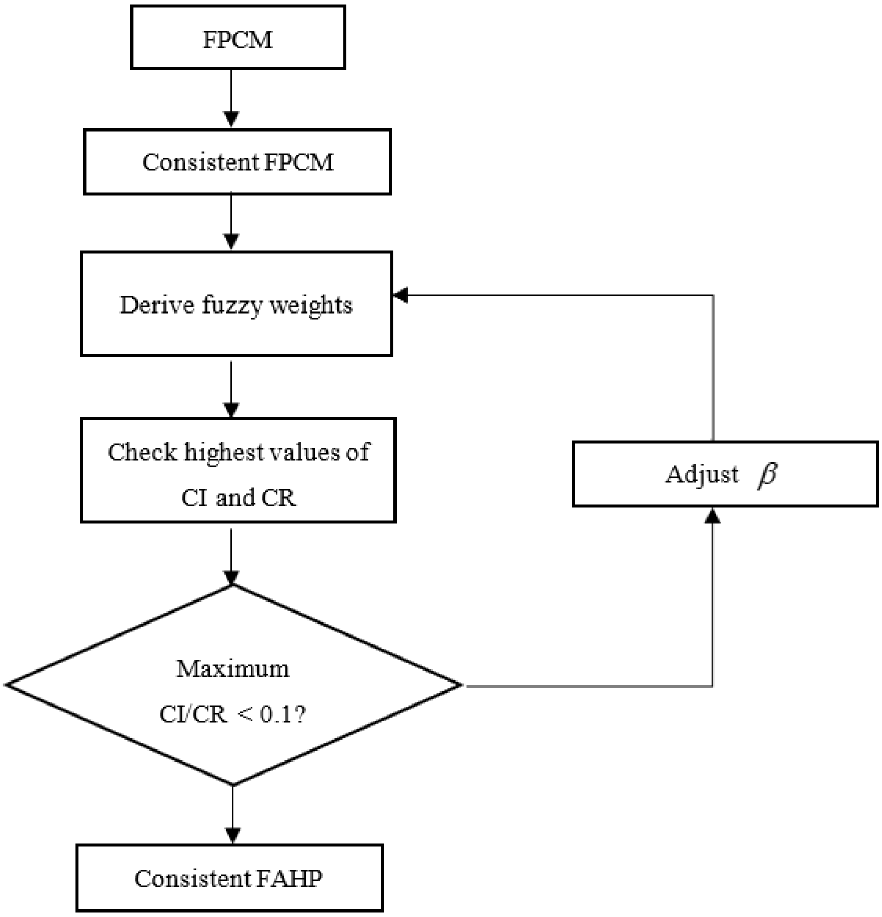

Figure 2. The flowchart of the proposed method.

5. Numerical Example

In this section, we propose three examples to demonstrate the proposed method. For simplicity, we only consider the situation with $\alpha-$cut = 0, i.e., the most spread situation of fuzzy weights. Surely, we can release the spread of fuzzy weights by using the tolerable parameter.

Example 1. *Let us first reconsider the toy example presented in* Section 2 *and derive the fuzzy weights based on the proposed method, as shown in* Table 3. *Note that we use the tolerable parameter* $\beta = 0$ *here.*

Table 3. Fuzzy weights derived from the consistent FPCM in Example 1.

	C1	C2	C3	Max CI	Max CR
The Proposed	[0.632, 0.686]	[0.143, 0.222]	[0.111, 0.167]	0.000	0.000
Without transitivity	[0.614, 0.732]	[0.138, 0.268]	[0.088, 0.174]	0.098	0.170

Compared with the result of the FAHP without considering the transitivity axiom, the proposed method can solve the consistent problem of the conventional FAHP that ensures the maximum CI or CR is less than 0.1. However, at some point, we still hope to enlarge the spread of fuzzy weights by releasing CI and CR with some tolerable levels to understand the sensitive situation or fluctuated possibilities. Therefore, we can add the tolerable parameter β in the following examples to consider more situations of fuzzy weights from different consistent FPCMs.

Example 2 ([43])**.** *This example is also used by Kubler et al.* [37] *to demonstrate their method. It contains four criteria and assumes that an expert gave the FPCM as follows*:

$$\tilde{A} = \begin{array}{c} \\ C1 \\ C2 \\ C3 \\ C4 \end{array} \begin{array}{c} \begin{array}{cccc} C1 & C2 & C3 & C4 \end{array} \\ \begin{bmatrix} (1,1,1) & (3/2,2,5/2) & (2/3,1,2) & (1,3/2,2) \\ \tilde{a}_{12}^{-1} & (1,1,1) & (2/3,1,2) & (1/2,2/3,1) \\ \tilde{a}_{13}^{-1} & \tilde{a}_{23}^{-1} & (1,1,1) & (1/2,2/3,1) \\ \tilde{a}_{14}^{-1} & \tilde{a}_{24}^{-1} & \tilde{a}_{34}^{-1} & (1,1,1) \end{bmatrix} \end{array}$$

Here, we first derive the fuzzy weights using the proposed method and compare the result with Kubler et al. [37], as shown in Table 4. In addition, we also show the fuzzy weights, which are derived without adding transitivity conditions.

Table 4. Different FAHP results in Example 2.

Weights	C1	C2	C3	C4
The proposed (β = 0) Max CI	[0.2727, 0.4167] 0.00	[0.1428, 0.2353]	[0.1579, 0.2727] Max CR	[0.2000, 0.3333] 0.00
The proposed (β = 0.02) Max CI	[0.2548, 0.4167] 0.0356	[0.1389, 0.2476]	[0.1532, 0.2818] Max CR	[0.1993, 0.3521] 0.0396
The proposed (β = 0.04) Max CI	[0.2482, 0.4167] 0.0485	[0.1373, 0.2524]	[0.1512, 0.2852] Max CR	[0.1989, 0.3594] 0.0538
The proposed (β = 0.06) Max CI	[0.2433, 0.4167] 0.0577	[0.1361, 0.2559]	[0.1497, 0.2877] Max CR	[0.1986, 0.3648] 0.0641
The proposed (β = 0.08) Max CI	[0.2393, 0.4167] 0.0651	[0.1351, 0.2589]	[0.1483, 0.2898] Max CR	[0.1983, 0.3692] 0.0723
The proposed (β = 0.10) Max CI	[0.2359, 0.4167] 0.0713	[0.1342, 0.2615]	[0.1471, 0.2915] Max CR	[0.1980, 0.3730] 0.0792
Without transitivity Max CI	[0.2470, 0.4247] 0.1143	[0.1305, 0.3121]	[0.1154, 0.2915] Max CR	[0.1644, 0.3892] 0.1270
Gogus & Boucher [35] Max CI	[0.2727, 0.4167] 0.00	[0.1428, 0.2353]	[0.1579, 0.2727] Max CR	[0.2000, 0.3333] 0.00
Wang [36] Max CI	[0.2727, 0.4167] 0.0392	[0.1322, 0.2553]	[0.1333, 0.2727] Max CR	[0.2000, 0.3333] 0.0426
Kubler et al. [37] Max CI	0.332 0	0.186	0.220 Max CR	0.262 0

The results of Example 2 indicate that the fuzzy weights derived by the proposed method with $\beta = 0$ are perfectly rational, i.e., CI and CR are zeros. The result is the same as Gogus & Boucher's method [35] and indicates that the proposed method is correct and rational. Although Kubler et al.'s weights can also obtain the perfect consistency, their weights are crisp rather than fuzzy, which is required in this paper. In addition, Wang's method [36] can also obtain rational fuzzy weights with tolerable CI and CR levels. Finally, suppose we ignore the transitivity axiom of the FPCM, just as in these DF methods. In that case, we will obtain fuzzy weights, which do not meet the acceptable level of the consistent index, i.e., CI or CR is less than 0.1. For example, when $w_1 = 0.3810$, $w_2 = 0.1905$, $w_3 = 0.1429$, and $w_4 = 0.2857$, the corresponding CR will be 0.1270, which indicates the weights are not trusted.

Example 3. *In this example, we consider a given FPCM with a six criteria problem, which is proposed by Wang & Chen* [44]*, as follows:*

$$
\tilde{A} = \begin{array}{c} \\ C1 \\ C2 \\ C3 \\ C4 \\ C5 \\ C6 \end{array}
\begin{array}{cccccc} C1 & C2 & C3 & C4 & C5 & C6 \\
(1,1,1) & (2/3,1,2) & (2/5,1/2,2/3) & (2/5,1/2,2/3) & (1/2,2/3,1) & (2/5,1/2,2/3) \\
\tilde{a}_{12}^{-1} & (1,1,1) & (1,1,1) & (1/2,1,3/2) & (1/2,1,3/2) & (1,1,1) \\
\tilde{a}_{13}^{-1} & \tilde{a}_{23}^{-1} & (1,1,1) & (1/2,1,3/2) & (1,3/2,2) & (1/2,2/3,1) \\
\tilde{a}_{14}^{-1} & \tilde{a}_{24}^{-1} & \tilde{a}_{34}^{-1} & (1,1,1) & (2/3,1,2) & (1/2,2/3,1) \\
\tilde{a}_{15}^{-1} & \tilde{a}_{25}^{-1} & \tilde{a}_{35}^{-1} & \tilde{a}_{45}^{-1} & (1,1,1) & (2/3,1,2) \\
\tilde{a}_{16}^{-1} & \tilde{a}_{26}^{-1} & \tilde{a}_{36}^{-1} & \tilde{a}_{45}^{-1} & \tilde{a}_{56}^{-1} & (1,1,1)
\end{array}
$$

Here, we consider two situations, i.e., $\beta = 0$ and $\beta = 1$, in our method, and compare the results with others, as shown in Table 5. Note that although Wang and Chen did not report the CI/CR in their paper, we calculate the corresponding maximum CI and CR.

Table 5. Different FAHP results in Example 3.

Weights				Max CI	Max CR
	C1	**C2**	**C3**		
The proposed (β = 0)	[0.1176, 0.1250]	[0.1765, 0.1875]	[0.1765, 0.1875]	0.0000	0.0000
	C4	**C5**	**C6**		
	[0.1765, 0.1875]	[0.1250, 0.1765]	[0.1765, 0.1875]		
	C1	**C2**	**C3**		
The proposed (β = 1)	[0.0898, 0.1356]	[0.1261, 0.2090]	[0.1231, 0.2319]	0.0298	0.0241
	C4	**C5**	**C6**		
	[0.1169, 0.2382]	[0.1035, 0.2380]	[0.1429, 0.2715]		
	C1	**C2**	**C3**		
Gogus & Boucher [35]	[0.1176, 0.1176]	[0.1765, 0.1765]	[0.1765, 0.1765]	0.0000	0.0000
	C4	**C5**	**C6**		
	[0.1765, 0.1765]	[0.1765, 0.1765]	[0.1765, 0.1765]		
	C1	**C2**	**C3**		
Wang [36]	[0.1176, 0.1176]	[0.1765, 0.1765]	[0.1765, 0.1765]	0.0000	0.0000
	C4	**C5**	**C6**		
	[0.1765, 0.1765]	[0.1765, 0.1765]	[0.1765, 0.1765]		
	C1	**C2**	**C3**		
Without transitivity	[0.084, 0.180]	[0.120, 0.228]	[0.107, 0.270]	0.2350	0.1895
	C4	**C5**	**C6**		
	[0.095, 0.291]	[0.101, 0.317]	[0.075, 0.294]		
	C1	**C2**	**C3**		
Wang & Chen [42]	[0.06, 0.21]	[0.09, 0.26]	[0.11, 0.31]	0.4598	0.3708
	C4	**C5**	**C6**		
	[0.10, 0.37]	[0.08, 0.32]	[0.12, 0.35]		

From the presented results in Table 5, we can see that [35,36] only derive crisp weights, despite the fact that these methods should obtain fuzzy weights. The main reason is that with the increase in criteria, it is hard for these methods to find fuzzy weights under the strict equality constraints, i.e., their transitivity axioms. However, if we do not consider the transitivity axiom of the FPCM, the derived fuzzy weights are irrational, i.e., the maximum CI or CR is larger than 0.1. This inconsistent situation also happens in [44]. In contrast, the proposed method is the only method to derive fuzzy and rational weights in this example.

6. Discussion

Obtaining rational fuzzy weights is an important issue in the AHP to ensure the given PCM is rational and the result is trusted. Therefore, many consistency indexes have been proposed to verify this issue. However, the problem becomes more complicated when we extend the AHP to FAHP since the transitivity axiom is hard to apply in fuzzy sets. Although several papers have proposed the transitivity axiom to handle FPCM, there are still many issues and open questions. First, the transitivity axiom is used to derive the consistent FPCM. However, the past methods usually used strict equality to handle the fuzzy sets and restrict the flexible solution region. Take Example 3, for example. The methods of [35,36] can only derive crisp weights rather than fuzzy weights. In addition, the result of Example 2 also indicates that we can consider the past methods as one special case of the proposed method.

Next, in terms of practical consideration, we need to understand the possible range of fuzzy weights, since this can be realized as sensitivity analysis or risk level. In addition, our method considers different consistency levels of fuzzy weights. In the AHP, we derive the crisp weights and the corresponding CI and CR. Then we use the values of CI and CR to conclude if the weights are rational. However, in the fuzzy environment, since weights, CI, and CR are fuzzy, the ranges of fuzzy weights get large as CI and CR increase. As long as the maximum CI or CR is less than 0.1, the fuzzy weights are considered to be rational. Hence, to consider risk, we can release the tolerable parameter to understand the possibility of a fuzzy weight. In contrast, past methods only derive one of the solutions. We should highlight the difference between the $\alpha-$cut operation and the use of the tolerable parameter, β, in this paper. As we know, the $\alpha-$cut operation reflects the level of subjective uncertainty and $\alpha-$cut is reduced if the decision-maker has smaller subjective uncertainty. On the other hand, the tolerable parameter is used to release fuzzy weights to tolerate some levels of inconsistency of the FPCM.

However, our method also triggers another problem. That is, how to find the appropriate tolerable parameter. From our tests, we know that the setting of β depends on the number of criteria. That is, we can increase the value of β when conducting more criteria and vice versa. Although we cannot determine β in advance, we can reduce β if the maximum CI or CR exceeds the acceptable setting level and vice versa. Without doubt, this also can be further researched.

7. Conclusions

In this paper, we developed a novel model to derive rational fuzzy weights of the FAHP. Compared with other methods, the proposed method is more flexible and ensures rational fuzzy weights. In addition, we used three examples to demonstrate the proposed method with different tolerable parameters and compare the results with others. The results justified the rationality of the proposed method.

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