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Decidability Preservation and Complexity Bounds for Combined Logics

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Abstract: Transfer theorems for combined logics provide essential tools and insight for reasoning about complex logical systems. In this paper, we present the first sufficient criterion *(contextual extensibility*) for decidability to be preserved through combination of propositional logics, and we study the complexity upper bounds induced by the method. In order to assess the scope and usability of our criterion, we illustrate its use in re-obtaining two standard important (though partial) results of the area: the preservation of decidability for disjoint combinations of logics, and the preservation of decidability for fusions of modal logics. Due to the very abstract nature and generality of the idea underlying contextual extensibility, we further explore its applicability beyond propositional logics. Namely, we explore the particular case of 2-deductive systems, and as a byproduct, we obtain the preservation of decidability for disjoint combinations of equational logics and discuss the relationship of this result and of our criterion with several related results with meaningful applications in satisfiability modulo theories.

Keywords: combined logics; combined theories; decidability; complexity

MSC: 03B62; 03B22; 03B35

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1. Introduction

Combining logics is a powerful and appealing idea—namely when coupled with powerful results that may allow the transfer of useful properties from the simpler logics being combined to the more complex resulting logic—proposed in its general form by Dov Gabbay in [1,2]. Given its fundamental character, the understanding of combined logics is a key ingredient of the general theory of universal logic [3,4] as well as a valuable tool for the construction and analysis of complex logics, a subject of growing importance in application fields such as software engineering and artificial intelligence (see, for instance, the FroCoS series of events and publications in [5]).

Despite the long track of work on combined logics, leading to a substantial understanding of their semantics and proof-theory (see [6–12]), automated support for combined logics is still lacking. This happens, in particular, because decidability-preservation results are scarce. Namely, we know from [13,14] that decidability is preserved by disjoint combinations of propositional logics, a result that is still far from most interesting practical uses. The only general result related to (but still distant from) decidability is [15], where the preservation of the semantic notion of *finite model property* is studied. There is also a proof of decidability preservation for fusions of modal logics [16,17], but which uses ideas and results from modal semantics that cannot be easily generalized. In a related, but somewhat different vein, there are a number of interesting and important decidability results about combined theories of equational and first-order logic [18–27] that explore similar ideas but that do not exactly fit the same purpose.

It is tempting to try to use semantic arguments to address the decidability of combined logics. However, only recently have we obtained usable general denotational semantics for combined propositional logics [12], which naturally builds models of the combined logic from models of the component logics. In this paper, we shall not use semantics, but we will take advantage, in an essential way, the breakthroughs allowed by this gained understanding beyond the disjoint case, in order to formulate a natural and abstract criterion for decidability preservation when combining logics in *context extensibility*. It is worth noting that, in general (see, for instance, [14]), decidability is not preserved nor reflected by combination, and that our criterion is a sufficient condition for decidability to be transferrable.

The main contributions of this paper are: (i) the definition, for the first time, of such an abstract criterion for decidability preservation, in the form of an extensibility condition with respect to a contextual syntactic function (Section 2.4); (ii) the fact that our proof of decidability preservation is constructive, which allows us to show how to put up decision procedures for the combined logic using decision procedures for the components and to study the resulting complexity upper bounds (Theorem 2); (iii) the illustrations provided show that the notion of contextual extensibility is not too strong, as it can be used to recover two previous results in the area, namely, the preservation of decidability for disjoint combinations of logics [14] and the preservation of decidability for fusions of modal logics [16] (Section 2.5); and (iv) finally, but notably, the smooth extension of our results beyond the propositional case, and in particular to the setting of 2-deductive systems (Section 3) allows us to prove the preservation of decidability for disjoint combinations of equational logics (Section 3.5.2). This last application is definitely related to the literature on decidability of combined theories already mentioned and as we discuss further, and somehow opens the way for a hopefully fruitful track of future results.

The rest of the paper consists, essentially, of two similar parts, namely comprising Sections 2 and 3. The first of these is dedicated to combined propositional logics, while the other extends all the results to the setting of 2-deductive systems, but both essentially follow the same structure, including a characterization of combined logics, our criterion for decidability preservation, and then meaningful illustrations. The paper closes in Section 4 with a summary of the results achieved and an outlook of future research.

2. Logics and Their Combination

We first study Tarski-style consequence relations (propositional logics), their combination, and the transference of decidability from the logics being combined to the resulting combined logic.

2.1. Syntax

The *syntax* of a *(propositional) logic* is defined, as usual, by means of a *signature*, an indexed family $\Sigma = \{\Sigma^{(n)}\}_{n \in \mathbb{N}_0}$ of denumerable sets, where each $\Sigma^{(n)}$ contains all allowed n-place connectives, and an infinite denumerable set P of *variables* (which we consider fixed once and for all). As standard, $L_\Sigma(P)$ denotes the set of all *formulas* constructed from the variables in P using the connectives in Σ. We will use $p, q, r, \ldots$ to denote variables, $A, B, C, \ldots$ to denote formulas, and $\Gamma, \Delta, \Theta, \ldots$ to denote sets of formulas, in all cases possibly with annotations.

We use $\mathsf{var}(A)$, $\mathsf{sub}(A)$, $\mathsf{hd}(A)$ to denote, respectively, the set of *variables occurring in* A, the set of *subformulas of* A, and the *head constructor of* A, given a formula $A \in L_\Sigma(P)$. These notations have simple inductive definitions: $\mathsf{var}(p) = \mathsf{sub}(p) = \{p\}$, and $\mathsf{hd}(p) = p$ for $p \in P$; and if $© \in \Sigma^{(n)}$ and $A_1, \ldots, A_n \in L_\Sigma(P)$, $\mathsf{var}(©(A_1, \ldots, A_n)) = \bigcup_{i=1}^{n} \mathsf{var}(A_i)$, $\mathsf{sub}(©(A_1, \ldots, A_n)) = \{©(A_1, \ldots, A_n)\} \cup \bigcup_{i=1}^{n} \mathsf{sub}(A_i)$, and $\mathsf{hd}(©(A_1, \ldots, A_n)) = c$. All these notations extend to sets of formulas in the obvious way.

As we are considering combined logics with mixed syntax, we need to consider different signatures, as well as relations and operations between signatures. Signatures being families of sets, the usual set-theoretic notions can be smoothly extended to signatures.

We sometimes abuse notation and confuse a signature Σ with the set $(\biguplus_{n\in\mathbb{N}_0}\Sigma^{(n)})$ of all its connectives, and write $© \in \Sigma$ when $©$ is some n-place connective $© \in \Sigma^{(n)}$. For this reason, the *empty signature*, with no connectives at all, will be simply denoted by $\emptyset$.

Let Σ, Σ_0 be two signatures. We say that Σ_0 is a *subsignature* of Σ and write $\Sigma_0 \subseteq \Sigma$ whenever $\Sigma_0^{(n)} \subseteq \Sigma^{(n)}$ for every $n \in \mathbb{N}_0$. Expectedly, given signatures Σ_1, Σ_2, we can also define the *shared subsignature* $\Sigma_1 \cap \Sigma_2 = \{\Sigma_1^{(n)} \cap {\Sigma_2}^{(n)}\}_{n\in\mathbb{N}_0}$, the *combined signature* $\Sigma_1 \cup \Sigma_2 = \{\Sigma_1^{(n)} \cup {\Sigma_2}^{(n)}\}_{n\in\mathbb{N}_0}$, and the *difference signature* $\Sigma_1 \setminus \Sigma_2 = \{{\Sigma_1}^{(n)} \setminus \Sigma_2^{(n)}\}_{n\in\mathbb{N}_0}$. Clearly, $\Sigma_1 \cap \Sigma_2$ is the largest subsignature of both Σ_1 and Σ_2 and contains the connectives shared by both. When there are no shared connectives, we have that $\Sigma_1 \cap \Sigma_2 = \emptyset$. Analogously, $\Sigma_1 \cup \Sigma_2$ is the smallest signature that has both Σ_1 and Σ_2 as subsignatures, and it features all the connectives from both Σ_1 and Σ_2 in a combined signature. Furthermore, $\Sigma_1 \setminus \Sigma_2$ is the largest subsignature of Σ_1 that does not share any connectives with Σ_2.

A *substitution* is a function $\sigma : P \to L_\Sigma(P)$, which, of course, extends freely to a function $\sigma : L_{\Sigma_0}(P) \to L_\Sigma(P)$ for every $\Sigma_0 \subseteq \Sigma$. As usual, we use A^σ to denote the formula that results from $A \in L_{\Sigma_0}(P)$ by uniformly replacing each variable $p \in \mathsf{var}(A)$ with $\sigma(p)$, and $\Gamma^\sigma = \{A^\sigma : A \in \Gamma\}$ for each $\Gamma \subseteq L_\Sigma(P)$.

Note that if $\Sigma_0 \subseteq \Sigma$, then $L_{\Sigma_0}(P) \subseteq L_\Sigma(P)$. Still, $L_{\Sigma_0}(P)$ and $L_\Sigma(P)$ are both infinite denumerable. In fact, the pair can be endowed with a very useful bijection capturing the view of an arbitrary $L_\Sigma(P)$ formula from the point of view of Σ_0, the *skeleton* function $\mathsf{skel}_{\Sigma_0} : L_\Sigma(P) \to L_{\Sigma_0}(P)$ (or simply skel_0, or even skel), for which the underlying idea we borrow from [28]. Note that given $A \in L_\Sigma(P)$, $\mathsf{hd}(A)$ may be in $\Sigma \setminus \Sigma_0$, in which case we dub A a Σ_0-*monolith* or simply a *monolith*. The idea is simply to replace monoliths with dedicated variables, just renaming the original variables. Let $\mathsf{Mon}(\Sigma_0, \Sigma)$ be the set of all monoliths. It is easy to see that $\mathsf{Mon}(\Sigma_0, \Sigma)$ is always denumerable, though it can be finite when $\Sigma \setminus \Sigma_0$ contains nothing but a finite set of 0-place connectives. In any case, $\mathsf{Mon}(\Sigma_0, \Sigma) \cup P$ is always infinite denumerable because P is, and thus we can fix a bijection $\eta : \mathsf{Mon}(\Sigma_0, \Sigma) \cup P \to P$. The skel bijection is now easily definable from η, inductively, by letting $\mathsf{skel}(p) = \eta(p)$ for $p \in P$, and for $© \in \Sigma^{(n)}$ and $A_1, \dots, A_n \in L_\Sigma(P)$, $\mathsf{skel}(©(A_1, \dots, A_n)) = ©(\mathsf{skel}(A_1), \dots, \mathsf{skel}(A_n))$ if $© \in \Sigma_0$, and $\mathsf{skel}(©(A_1, \dots, A_n)) = \eta(©(A_1, \dots, A_n))$ if $© \in \Sigma \setminus \Sigma_0$.

The skel bijection thus defined can be easily inverted by means of the substitution $\mathsf{unskel}_{\Sigma_0} : P \to L_\Sigma(P)$ (or simply unskel_0, or even unskel) defined by $\mathsf{unskel}(p) = \eta^{-1}(p)$. Note, namely, that $\mathsf{skel}(A)^{\mathsf{unskel}} = A$ for every $A \in L_\Sigma(P)$. Note also that the restriction of skel to P, $\mathsf{skel} : P \to L_{\Sigma_0}(P)$ (with a slight abuse of notation, we will use the same name) is a substitution, and $\mathsf{skel}(A) = A^{\mathsf{skel}}$ for every $A \in L_{\Sigma_0}(P)$.

2.2. Propositional Logics and Theories

Definition 1. *A logic is a pair $\langle \Sigma, \vdash \rangle$ where Σ is a signature and $\vdash \subseteq \wp(L_\Sigma(P)) \times L_\Sigma(P)$ is a relation satisfying:*

(R) $\Gamma \vdash A$ *whenever* $A \in \Gamma$ *(reflexivity);*
(M) $\Gamma \vdash A$ *whenever* $\Gamma' \vdash A$ *for* $\Gamma' \subseteq \Gamma$ *(monoticity);*
(T) $\Gamma \vdash A$ *whenever* $\Theta \vdash A$ *and* $\Gamma \vdash B$ *for every* $B \in \Theta$ *(transitivity);*
(S) $\Gamma \vdash A$ *implies* $\Gamma^\sigma \vdash A^\sigma$ *for any substitution* σ *(subst. invariance).*

We further say that $\langle \Sigma, \vdash \rangle$ is *compact* whenever it further satisfies:

(F) $\Gamma \vdash A$ implies there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash A$.

The compact part of a logic $\langle \Sigma, \vdash \rangle$ is also a logic $\langle \Sigma, \vdash_{\mathrm{fin}} \rangle$ where $\Gamma \vdash_{\mathrm{fin}} A$ if and only if there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash A$. Of course, $\langle \Sigma, \vdash \rangle$ is compact if and only if $\vdash = \vdash_{\mathrm{fin}}$.

We say that a set of *rules* $\mathsf{R} \subseteq \wp(L_\Sigma(P)) \times L_\Sigma(P)$ axiomatizes $\langle \Sigma, \vdash \rangle$ whenever $\vdash$ is the closure of R by (R), (M), (T), and (S), and write $\vdash = \vdash_{\mathsf{R}}$. Clearly, $\langle \Sigma, \vdash \rangle$ is compact if and only if it is axiomatized by a set of finitary rules R; that is, Γ is finite for every rule $(\Gamma, A) \in \mathsf{R}$, also simply denoted by $\frac{\Gamma}{A}$.

A set of formulas Γ is said to be a *theory of* $\langle\Sigma,\vdash\rangle$ when for every $A \in L_\Sigma(P)$, if $\Gamma \vdash A$, then $A \in \Gamma$. Given a set Γ, $\Gamma^\vdash$ is the least theory that contains Γ. We write $\mathsf{Th}(\langle\Sigma,\vdash\rangle) = \{\Gamma \subseteq L_\Sigma(P) : \Gamma^\vdash = \Gamma\}$ for the set of theories of $\langle\Sigma,\vdash\rangle$. We always have that $L_\Sigma(P) \in \mathsf{Th}(\langle\Sigma,\vdash\rangle)$. Every logic can be recovered from its set of theories, as $\Gamma \vdash A$ if and only if $A \in \Delta$ whenever $\Gamma \subseteq \Delta$ for every $\Delta \in \mathsf{Th}(\langle\Sigma,\vdash\rangle)$. Furthermore, from (S), it immediately follows that the set $\mathsf{Th}(\langle\Sigma,\vdash\rangle)$ is closed for inverse substitutions; that is, given $\sigma : P \to L_\Sigma(P)$, $\Delta \in \mathsf{Th}(\langle\Sigma,\vdash\rangle)$ implies that $\sigma^{-1}(\Delta) \in \mathsf{Th}(\langle\Sigma,\vdash\rangle)$. Further, given logics $\langle\Sigma,\vdash_1\rangle$ and $\langle\Sigma,\vdash_2\rangle$, we have that $\vdash_1 \subseteq \vdash_2$ if and only if $\mathsf{Th}(\langle\Sigma,\vdash_2\rangle) \subseteq \mathsf{Th}(\langle\Sigma,\vdash_1\rangle)$.

Example 1. *The smallest logic over a given signature* Σ $\langle\Sigma,\vdash_{sml}\rangle$ *is given by* $\Gamma \vdash_{sml} A$ *if and only if* $A \in \Gamma$. *It is easy to see that* $\langle\Sigma,\vdash_{sml}\rangle$ *is compact logic and is axiomatized by the empty set of rules. We have that* $Th(\langle\Sigma,\vdash_{sml}\rangle) = \wp(L_\Sigma(P))$.

It is relatively straightforward to check that intersections of consequence relations are consequence relations (see [29]). These facts make it relatively easy to enrich the signature of a logic. Namely, if $\Sigma_0 \subseteq \Sigma$ and $\langle\Sigma_0,\vdash_0\rangle$ is a logic, then the *extension of* $\langle\Sigma_0,\vdash_0\rangle$ *to* Σ, denoted by $\langle\Sigma,\vdash_0^\Sigma\rangle$, is the least logic with signature Σ such that $\vdash_0 \subseteq \vdash_0^\Sigma$. The following is a useful alternative definition of such an extension.

Proposition 1. *For* $\Gamma \cup \{A\} \subseteq L_\Sigma(P)$, *we have that* $\Gamma \vdash_0^\Sigma A$ *if and only if* $\mathsf{skel}(\Gamma) \vdash_0 \mathsf{skel}(A)$. *Hence,* $Th(\langle\Sigma,\vdash_0^\Sigma\rangle) = \mathsf{unskel}(Th(\langle\Sigma_0,\vdash_0\rangle))$.

Proof. Just note that for each substitution $\sigma : P \to L_\Sigma(P)$, we have that $(\mathsf{skel} \circ \sigma) : P \to L_{\Sigma_0}(P)$ is also a substitution, and also $\mathsf{unskel} \circ (\mathsf{skel} \circ \sigma) = \sigma$. We immediately obtain that $\vdash_0 \subseteq \vdash_0^\Sigma$, and also that if $\vdash_0 \subseteq \vdash$ and $\vdash$ satisfies (S), then $\vdash_0^\Sigma \subseteq \vdash$. Since $\mathsf{skel}, \mathsf{unskel}$ are bijections, it is straightforward to show that $\vdash_0^\Sigma$ satisfies properties (R), (M), (T), and (S). □

Thus, equivalently, $\Gamma \vdash_0^\Sigma A$ if and only if there exists $\Gamma_0 \cup \{A_0\} \subseteq L_{\Sigma_0}(P)$ and $\sigma : P \to L_\Sigma(P)$ such that $\Gamma_0^\sigma \subseteq \Gamma$, $A_0^\sigma = A$, and $\Gamma_0 \vdash_0 A_0$.

2.3. Combining Logics

Consider fixed signatures Σ_1 and Σ_2 and let $\Sigma_{12} = \Sigma_1 \cup \Sigma_2$. It is quite natural to formulate the *combination of logics* (also known as *fibring*) as follows.

Definition 2. *The* combination *of logics* $\langle\Sigma_1,\vdash_1\rangle$, $\langle\Sigma_2,\vdash_2\rangle$, *which we denote by* $\langle\Sigma_1,\vdash_1\rangle \bullet \langle\Sigma_2,\vdash_2\rangle$, *is the least logic* $\langle\Sigma_{12},\vdash_{12}\rangle$ *such that* $\vdash_1, \vdash_2 \subseteq \vdash_{12}$. *The combination is said to be* disjoint *if* $\Sigma_1 \cap \Sigma_2 = \emptyset$.

We immediately obtain that $\langle\Sigma_1,\vdash_1\rangle \bullet \langle\Sigma_{12},\vdash_2\rangle$ is the smallest logic over Σ_{12} that contains both $\langle\Sigma_1,\vdash_1^{\Sigma_{12}}\rangle$ and $\langle\Sigma_2,\vdash_2^{\Sigma_{12}}\rangle$. Note that it also follows easily that the combination of compact logics is necessarily compact. Namely, if $\langle\Sigma_1,\vdash_1\rangle$, $\langle\Sigma_2,\vdash_2\rangle$ are compact, then the least logic such that $\vdash_1, \vdash_2 \subseteq \vdash_{12}$ is also the least logic such that $\vdash_{1,\text{fin}}, \vdash_{2,\text{fin}} \subseteq \vdash_{12}$. Since it is clear that $\vdash_{1,\text{fin}}, \vdash_{2,\text{fin}} \subseteq \vdash_{12,\text{fin}}$, it follows that $\vdash_{12} = \vdash_{12,\text{fin}}$. Similarly, we have that if $\vdash_1 = \vdash_{\mathsf{R}_1}$ and $\vdash_2 = \vdash_{\mathsf{R}_2}$, then $\vdash_{12} = \vdash_{\mathsf{R}_1 \cup \mathsf{R}_2}$.

Example 2. *Let* $\langle\Sigma_1,\vdash_1\rangle$ *be a logic and* Σ_2 *a signature. Consider* $\langle\Sigma_2,\vdash_{sml}\rangle$ *to be the smallest logic over* Σ_2, *as in Example 1. Since* $\vdash_{sml}^{\Sigma_{12}} \subseteq \vdash_1^{\Sigma_{12}}$, *we obtain that* $\langle\Sigma_{12},\vdash_1^{\Sigma_{12}}\rangle = \langle\Sigma_1,\vdash_1\rangle \bullet \langle\Sigma_2,\vdash_{sml}\rangle$. *Note, in particular, that*

$$Th(\langle\Sigma_{12},\vdash_1^{\Sigma_2}\rangle) = Th(\langle\Sigma_{12},\vdash_1^{\Sigma_{12}}\rangle) \cap Th(\langle\Sigma_{12},\vdash_{\mathsf{sml}}^{\Sigma_{12}}\rangle)$$

as $Th(\langle\Sigma_{12},\vdash_{\mathsf{sml}}^{\Sigma_{12}}\rangle) = \mathsf{unskel}_{\Sigma_1}(Th(\langle\Sigma_2,\vdash_{\mathsf{sml}}\rangle)) = \mathsf{unskel}_{\Sigma_1}(\wp(L_{\Sigma_2}(P))) = \wp(L_{\Sigma_{12}}(P))$. *The next result shows that this equality holds in general.*

We can now provide an explicit characterization of $\vdash_{12}$ using only $\vdash_1$ and $\vdash_2$ or, more concretely, $\vdash_1^{\Sigma_2}$ and $\vdash_2^{\Sigma_1}$.

Theorem 1. *Let $\langle \Sigma_{12}, \vdash_{12} \rangle = \langle \Sigma_1, \vdash_1 \rangle \bullet \langle \Sigma_2, \vdash_2 \rangle$. For every $\Gamma \cup \{A\} \subseteq L_{\Sigma_{12}}(P)$, we have:*

$$\Gamma \vdash_{12} A$$

if and only if

$$A \in \Delta \textit{ for every } \Delta \in \mathit{Th}(\langle \Sigma_{12}, \vdash_1^{\Sigma_{12}} \rangle) \cap \mathit{Th}(\langle \Sigma_{12}, \vdash_2^{\Sigma_{12}} \rangle) \textit{ with } \Gamma \subseteq \Delta.$$

Hence, $\Gamma^{\vdash_{12}}$ is the smallest element of both $\mathit{Th}(\langle \Sigma_{12}, \vdash_1^{\Sigma_{12}} \rangle)$ and $\mathit{Th}(\langle \Sigma_{12}, \vdash_2^{\Sigma_{12}} \rangle)$ that contains Γ, and

$$\mathit{Th}(\langle \Sigma_1, \vdash_1 \rangle \bullet \langle \Sigma_2, \vdash_2 \rangle) = \mathit{Th}(\langle \Sigma_{12}, \vdash_1^{\Sigma_{12}} \rangle) \cap \mathit{Th}(\langle \Sigma_{12}, \vdash_2^{\Sigma_{12}} \rangle).$$

Proof. Let $\langle \Sigma_{12}, \vdash \rangle$ be defined by $\Gamma \vdash A$ if and only if $A \in \Delta$ for every $\Gamma \subseteq \Delta \subseteq L_{\Sigma_{12}}(P)$ with $\Delta \in \mathtt{Th}(\langle \Sigma_{12}, \vdash_1^{\Sigma_{12}} \rangle) \cap \mathtt{Th}(\langle \Sigma_{12}, \vdash_2^{\Sigma_{12}} \rangle)$.

Let us first show that $\langle \Sigma_{12}, \vdash \rangle$ is a logic. Clearly, $\vdash$ satisfies (R) and (M); let us show it satisfies (T) and (S) also.

(T) Assume that $\Theta \vdash A$ and $\Gamma \vdash B$ for every $B \in \Theta$. This means that $A \in \Delta_1$ for every $\Theta \subseteq \Delta_1 \in \mathtt{Th}(\langle \Sigma_{12}, \vdash_1^{\Sigma_{12}} \rangle) \cap \mathtt{Th}(\langle \Sigma_{12}, \vdash_2^{\Sigma_{12}} \rangle)$, and $\Theta \subseteq \Delta_2$ for every $\Gamma \subseteq \Delta_2 \in \mathtt{Th}(\langle \Sigma_{12}, \vdash_1^{\Sigma_{12}} \rangle) \cap \mathtt{Th}(\langle \Sigma_{12}, \vdash_2^{\Sigma_{12}} \rangle)$. By (M) for each $\vdash_i^{\Sigma_{12}}$, we conclude that $A \in \Theta_3$ for for every $\Gamma \subseteq \Theta_3 \in \mathtt{Th}(\langle \Sigma_{12}, \vdash_1^{\Sigma_{12}} \rangle) \cap \mathtt{Th}(\langle \Sigma_{12}, \vdash_2^{\Sigma_{12}} \rangle)$. Therefore, $\Gamma \vdash A$.

(S) Assume that $\Gamma \vdash A$, and thus $A \in \Delta_1$ for every $\Gamma \subseteq \Delta_1 \in \mathtt{Th}(\langle \Sigma_{12}, \vdash_1^{\Sigma_{12}} \rangle) \cap \mathtt{Th}(\langle \Sigma_{12}, \vdash_2^{\Sigma_{12}} \rangle)$. By (S), for each $\vdash_i^{\Sigma_{12}}$, we conclude that $A^\sigma \in \Delta_2$ for every $\Gamma^\sigma \subseteq \Delta_2 \in \mathtt{Th}(\langle \Sigma_{12}, \vdash_1^{\Sigma_{12}} \rangle) \cap \mathtt{Th}(\langle \Sigma_{12}, \vdash_2^{\Sigma_{12}} \rangle)$. Therefore, $\Gamma^\sigma \vdash A^\sigma$.

By definition, $\mathtt{Th}(\langle \Sigma_{12}, \vdash \rangle) = \mathtt{Th}(\langle \Sigma_{12}, \vdash_1^{\Sigma_{12}} \rangle) \cap \mathtt{Th}(\langle \Sigma_{12}, \vdash_2^{\Sigma_{12}} \rangle)$. Now, given $\langle \Sigma_{12}, \vdash' \rangle$ with $\vdash_1^{\Sigma_{12}}, \vdash_2^{\Sigma_{12}} \subseteq \vdash' \subseteq \vdash$, we obtain that

$$\mathtt{Th}(\langle \Sigma_{12}, \vdash \rangle) \subseteq \mathtt{Th}(\langle \Sigma_{12}, \vdash' \rangle) \subseteq \mathtt{Th}(\langle \Sigma_{12}, \vdash_1^{\Sigma_{12}} \rangle) \cap \mathtt{Th}(\langle \Sigma_{12}, \vdash_2^{\Sigma_{12}} \rangle).$$

Hence , we can finally conclude that $\mathtt{Th}(\langle \Sigma_{12}, \vdash \rangle) = \mathtt{Th}(\langle \Sigma_{12}, \vdash' \rangle)$ and $\vdash' = \vdash = \vdash_{12}$. □

2.4. Contextual Extensibility and Decidability Preservation

We say that a logic $\langle \Sigma, \vdash \rangle$ is *decidable* if there exists an algorithm $\mathtt{D}$, which terminates when given any finite set $\Gamma \subseteq L_\Sigma(P)$ and formula $A \in L_\Sigma(P)$ as input, and outputs $\mathtt{D}(\Gamma, A) = \mathtt{yes}$ if $\Gamma \vdash A$, and $\mathtt{D}(\Gamma, A) = \mathtt{no}$ if $\Gamma \nvdash A$. We will henceforth assume without loss of generality that the logic at hand is compact, as this definition is equivalent to deciding the compact version $\langle \Sigma, \vdash_{\mathrm{fin}} \rangle$ of the logic.

Theorem 1 is quite appealing, and mathematically clean, but a decision procedure for $\langle \Sigma_1, \vdash_1 \rangle \bullet \langle \Sigma_2, \vdash_2 \rangle$ based on it would require (potentially) running through all common theories of the given logics containing a given set of premisses. One may try to obtain a more usable version which, instead, may only need to go through fragments of theories of the given logics which agree on a suitable, possibly finite, set of formulas. For the purpose, we introduce the notion of *context*, as a function $\mathsf{ctx} : \wp(L_{\Sigma_{12}}(P)) \to \wp(L_{\Sigma_{12}}(P))$ such that $\Omega \subseteq \mathsf{ctx}(\Omega)$. Aiming at decidability preservation, of course, we will further require that $\mathsf{ctx}(\Omega)$ is finite for finite $\Omega \subseteq L_{\Sigma_{12}}(P)$.

Definition 3. *For a fixed context function* ctx, *we say that two logics $\langle \Sigma_1, \vdash_1 \rangle, \langle \Sigma_2, \vdash_2 \rangle$ are* ctx-extensible *when for every $\Omega \subseteq L_{\Sigma_{12}}(P)$ and theories Δ_i of $\langle \Sigma_{12}, \vdash_i^{\Sigma_{12}} \rangle$ for $i \in \{1, 2\}$,*

if

$$\Delta_1 \cap \mathsf{ctx}(\Omega) = \Delta_2 \cap \mathsf{ctx}(\Omega)$$

then there exists a theory Δ *of* $\langle \Sigma_{12}, \vdash_{12} \rangle$ *such that*

$$\Delta \cap \Omega = \Delta_1 \cap \Omega = \Delta_2 \cap \Omega.$$

That is, two logics are ctx-extensible if any theories of the given logics that agree on the formulas in $\mathsf{ctx}(\Omega)$ can be extended to a theory of the combined logic that agrees with the given theories on the formulas in Ω.

Lemma 1. *Let* $\langle \Sigma_1, \vdash_1 \rangle$, $\langle \Sigma_2, \vdash_2 \rangle$ *be* ctx*-extensible logics;* $\langle \Sigma_1, \vdash_1 \rangle \bullet \langle \Sigma_2, \vdash_1 \rangle = \langle \Sigma_{12}, \vdash_{12} \rangle$ *is their combination. For every* $\Gamma \cup \{A\} \subseteq L_{\Sigma_{12}}(P)$*, we have:*

$$\Gamma \vdash_{12} A$$

if and only if

$$A \in \Omega \textit{ for every } \Omega = (\Omega^{\vdash_1^{\Sigma_{12}}} \cup \Omega^{\vdash_2^{\Sigma_{12}}}) \cap \mathsf{ctx}(\Gamma \cup \{A\}) \textit{ with } \Gamma \subseteq \Omega.$$

Proof. Using Theorem 1, if $\Gamma \nvdash_{12} A$, then there exists $\Gamma \subseteq \Delta \not\ni A$ such that Δ is a theory of both $\langle \Sigma_{12}, \vdash_1^{\Sigma_{12}} \rangle$ and $\langle \Sigma_{12}, \vdash_2^{\Sigma_{12}} \rangle$. Easily then, one has

$$(\Delta \cap \mathsf{ctx}(\Gamma \cup \{A\}))^{\vdash_i^{\Sigma_{12}}} \cap \mathsf{ctx}(\Gamma \cup \{A\}) = \Delta \cap \mathsf{ctx}(\Gamma \cup \{A\})$$

for each $i \in \{1, 2\}$, and $\Gamma \subseteq \Delta \cap \mathsf{ctx}(\Gamma \cup \{A\}) = \Omega \not\ni A$.

Reciprocally, if there is $\Gamma \subseteq \Omega \subseteq \mathsf{ctx}(\Gamma \cup \{A\})$ such that $A \notin \Omega$, but with $\Omega^{\vdash_i^{\Sigma_{12}}} \cap \mathsf{ctx}(\Gamma \cup \{A\}) \subseteq \Omega$ for each $i \in \{1, 2\}$, then it follows that $\Delta_1 = \Omega^{\vdash_1^{\Sigma_{12}}}$ and $\Delta_2 = \Omega^{\vdash_2^{\Sigma_{12}}}$ are theories, such that $\Delta_1 \cap \mathsf{ctx}(\Gamma \cup \{A\}) = \Delta_2 \cap \mathsf{ctx}(\Gamma \cup \{A\})$. Thus, directly from ctx-extensibility, we can conclude that there exists a theory Δ of $\langle \Sigma_{12}, \vdash_{12} \rangle$ such that $\Delta \cap (\Gamma \cup \{A\}) = \Delta_1 \cap (\Gamma \cup \{A\}) = \Delta_2 \cap (\Gamma \cup \{A\})$. It follows that $\Gamma \subseteq \Delta \not\ni A$, and so $\Gamma \nvdash_{12} A$. $\square$

In order to apply these ideas toward decidability preservation, namely with the aim of analyzing the complexity of the underlying decision problems, we assume that the context function ctx is computable in $\mathbf{TIME}(c(n))$ and $\mathbf{SPACE}(d(n))$, obviously with $d(n) \leq c(n)$.

Theorem 2. *Let* $\langle \Sigma_1, \vdash_1 \rangle$, $\langle \Sigma_2, \vdash_2 \rangle$ *be* ctx*-extensible logics. If the decision problems for* $\langle \Sigma_1, \vdash_1 \rangle$, $\langle \Sigma_2, \vdash_2 \rangle$ *are both in complexity class* $\mathbf{C}$*, then the decision problem for* $\langle \Sigma_{12}, \vdash_{12} \rangle$ *is in complexity class* $\mathbf{C}'$*, as given by Table 1.*

Table 1. Complexity bounds for deciding the combination of ctx-extensible logics.

$\mathbf{C}$	$\mathbf{C}'$
$\mathbf{TIME}(t(n))$	$\mathbf{TIME}(c(n) + d(n) \times t(d(n)))$
$\mathbf{SPACE}(s(n))$	$\mathbf{SPACE}(d(n) + s(d(n)))$
$\mathbf{coNTIME}(t'(n))$	$\mathbf{coNTIME}(c(n) + d(n) \times t'(d(n)))$

Proof. Let $\mathtt{D}_1, \mathtt{D}_2$ be deterministic algorithms deciding $\langle \Sigma_1, \vdash_1 \rangle, \langle \Sigma_2, \vdash_2 \rangle$, respectively, both running in time bounded by $\mathcal{O}(t(n))$ and space bounded by $\mathcal{O}(s(n))$. To decide $\langle \Sigma_{12}, \vdash_{12} \rangle$, consider the following deterministic algorithm $\mathtt{D}$.

$$
\begin{array}{lll}
\mathtt{D} & : & \mathtt{input}\ \Gamma, A \\
& & \mathtt{set}\ \Theta := \mathrm{ctx}(\Gamma \cup \{A\}) \\
& & \mathtt{set}\ \Omega_1, \Omega_2 := \Gamma \\
& & \mathtt{while}\ A \notin \Omega_1 \cup \Omega_2 \\
& & \quad \mathtt{set}\ \Omega := \Omega_1 \cup \Omega_2 \\
& & \quad \mathtt{set}\ \Omega_i := \{B \in \Theta : \mathtt{D_i}(\Omega, B) = \mathtt{yes}\}\ \mathtt{for\ i} = 1,2 \\
& & \quad \mathtt{if}\ \ \Omega = \Omega_1 \cup \Omega_2 \\
& & \qquad \mathtt{output\ no} \\
& & \mathtt{output\ yes}
\end{array}
$$

The correctness of $\mathtt{D}$ is an immediate consequence of Lemma 1, as the algorithm builds precisely the least set Ω such that $\Omega = (\Omega^{\vdash_1^{\Sigma_{12}}} \cup \Omega^{\vdash_2^{\Sigma_{12}}}) \cap \mathrm{ctx}(\Gamma \cup \{A\})$ and $\Gamma \subseteq \Omega$. The $\mathtt{no}$ output happens when a fixed point is reached, meaning that $A \notin \Omega$, and thus $A \notin \Gamma^{\vdash_{12}}$. When the yes output is reached, we are sure that $A \in \Gamma^{\vdash_{12}}$, as A was reached by departing from $\Omega = \Gamma$ and iteratively adding formulas in $\mathrm{ctx}(\Gamma \cup \{A\})$ to Ω if they follow either $\vdash_1^{\Sigma_{12}}$ or $\vdash_2^{\Sigma_{12}}$.

Let n be the size of $\Gamma \cup \{A\}$. We know that $\Theta = \mathrm{ctx}(\Gamma \cup \{A\})$ is computed in time bounded by $\mathcal{O}(c(n))$, and also that the number of formulas in Θ, as well as the size of each such formulas, is bounded by $\mathcal{O}(d(n))$. Therefore, the cycle is repeated $\mathcal{O}(d(n))$ times, each time on inputs of size $\mathcal{O}(d(n))$, and $\mathtt{D}$ runs in time bounded by $\mathcal{O}(c(n) + d(n) \times (2.t(d(n)))) = \mathcal{O}(c(n) + d(n) \times t(d(n)))$.

Spacewise, we need to count the space used by each of Ω_1, Ω_2, but we can assume that the independent calls to $\mathtt{D}_1, \mathtt{D}_2$ reuse space, and hence $\mathtt{D}$ runs in space $\mathcal{O}(d(n) + 2.d(n) + s(d(n))) = \mathcal{O}(d(n) + s(d(n)))$.

Assume now that $\mathtt{N}_1, \mathtt{N}_2$ are non-deterministic algorithms deciding the complementary problems $\langle \Sigma_1, \nvdash_1 \rangle, \langle \Sigma_2, \nvdash_2 \rangle$, respectively, both running in time bounded by $\mathcal{O}(t'(n))$. To decide $\langle \Sigma_{12}, \nvdash_{12} \rangle$, consider the non-deterministic algorithm $\mathtt{N}$.

$$
\begin{array}{lll}
\mathtt{N} & : & \mathtt{input}\ \Gamma, A \\
& & \mathtt{set}\ \Theta := \mathrm{ctx}(\Gamma \cup \{A\}) \\
& & \mathtt{guess\ non\text{-}deterministically}\ A \notin \Omega \subseteq \Theta \\
& & \mathtt{for\ each}\ B \in \Theta \setminus \Omega \\
& & \quad \mathtt{if}\ \ \mathtt{N}_1(\Omega, B) = \mathtt{no\ or}\ \mathtt{N}_2(\Omega, B) = \mathtt{no} \\
& & \qquad \mathtt{output\ no} \\
& & \mathtt{output\ yes}
\end{array}
$$

The correctness of $\mathtt{N}$ is again a direct consequence of Lemma 1, as the algorithm guesses a set Ω such that $\Omega \subseteq \Theta$ and $A \notin \Omega$, and then answers according to whether $\Omega = (\Omega^{\vdash_1^{\Sigma_{12}}} \cup \Omega^{\vdash_2^{\Sigma_{12}}}) \cap \mathrm{ctx}(\Gamma \cup \{A\})$. Easily, $\mathtt{N}$ answers yes precisely when $\Gamma \nvdash_{12} A$ by guessing correctly a set Ω for which $\Omega \nvdash_1 B$ and $\Omega \nvdash_2 B$, and, hence, with $\mathtt{N}_1(\Omega, B) = \mathtt{yes}$ and $\mathtt{N}_2(\Omega, B) = \mathtt{yes}$ for every $B \in \Theta \setminus \Omega$.

Similarly, the running time of $\mathtt{N}$ is bounded by $\mathcal{O}(c(n) + d(n) \times (2.t'(d(n)))) = \mathcal{O}(c(n) + d(n) \times t'(d(n)))$. $\square$

We conclude that whenever the context function ctx is computable in polynomial time ($c(n)$ being a polynomial), then the combined logic often retains the same complexity upper bound of the logics being combined, notably in case **C** is **P**, **coNP**, **PSPACE**, **EXPTIME**, and beyond. When ctx is at least computable in polynomial space ($d(n)$ being a polynomial), the combined logic still retains the same space complexity class of the logics being combined, above polynomial space, namely when **C** is **PSPACE**, **EXPSPACE**.

2.5. Applications

We now illustrate the results with some particular applications of Theorem 2. These illustrations are crucial in order to assess that our sufficiency criterion is not too strong to be usable in concrete cases.

2.5.1. Combining Logics with Disjoint Signatures

First of all, we obtain a much simpler proof, using Theorem 2, of the major result of [14]: the preservation of decidability for disjoint combinations of logics. Assume that both $\langle\Sigma_1, \vdash_1\rangle$, $\langle\Sigma_2, \vdash_2\rangle$ are decidable, and $\Sigma_1 \cap \Sigma_2 = \emptyset$. In order to prove that $\langle\Sigma_{12}, \vdash_1 \bullet \vdash_2\rangle$ is decidable and obtain a complexity upper bound for deciding it, it is enough to show the following proposition.

Proposition 2. *Assuming $\Sigma_1 \cap \Sigma_2 = \emptyset$, we have that $\langle\Sigma_1, \vdash_1\rangle$ and $\langle\Sigma_2, \vdash_2\rangle$ are* ctx*-extensible for some context function* ctx *computable in polynomial time.*

Proof. Let $\underline{X} = \{X\}$ be a singleton containing a theorem of either $\vdash_1$ or $\vdash_2$; that is, $\emptyset \vdash_i X$ for some $i \in \{1, 2\}$, if such a theorem exists. When none of the component logics has a theorem, then $\underline{X} = \emptyset$. We consider the context function

$$\mathsf{ctx}(\Omega) = \mathsf{sub}(\Omega \cup \underline{X}),$$

which can clearly be computed in quadratic time on $\mathsf{size}(\Omega)$.

Suppose now that for some Ω, there are $\Delta_i \in \mathsf{Th}(\langle\Sigma_{12}, \vdash_i^{\Sigma_{12}}\rangle)$ for $i = 1, 2$ such that $\Omega_1 \cap \mathsf{ctx}(\Omega) = \Omega_2 \cap \mathsf{ctx}(\Omega)$.

On the one hand, if $\Delta_1 \cap \mathsf{ctx}(\Omega) = \Delta_2 \cap \mathsf{ctx}(\Omega) = \emptyset$, then none of the component logics has a theorem; $\underline{X} = \emptyset$. Hence, $\emptyset \in \mathsf{Th}(\langle\Sigma_{12}, \vdash_1^{\Sigma_{12}}\rangle) \cap \mathsf{Th}(\langle\Sigma_{12}, \vdash_2^{\Sigma_{12}}\rangle)$, and by Theorem 1, we obtain $\emptyset \in \mathsf{Th}(\langle\Sigma_{12}, \vdash_{12}\rangle)$.

On the other hand, if $\Delta_1 \cap \mathsf{ctx}(\Omega) = \Delta_2 \cap \mathsf{ctx}(\Omega) = \mathsf{ctx}(\Omega)$, then we can simply pick the largest theory of any logic containing every formula in its language $L_{\Sigma_{12}}(P) \in \mathsf{Th}(\langle\Sigma_{12}, \vdash_{12}\rangle)$.

Thus, we proceed knowing that we can fix formulas $F_-, F_+ \in \mathsf{ctx}(\Omega)$ such that $F_- \notin \Delta_i$ and $F_+ \in \Delta_i$, where $i \in \{1, 2\}$ for the remainder of the proof. We now build $\Delta \in \mathsf{Th}(\langle\Sigma_{12}, \vdash_{12}\rangle)$ such that $\Delta \cap \Omega = \Delta_i \cap \Omega$.

First, we modify Δ_1, Δ_2 so that they also agree on $P \setminus \mathsf{var}(\mathsf{ctx}(\Omega))$ (for simplicity, we chose to include none in Δ). Consider the substitution $\sigma : P \to L_{\Sigma_1 \cup \Sigma_2}(P)$ such that

$$\sigma(p) = \begin{cases} p & \text{if } p \in \mathsf{var}(\mathsf{ctx}(\Omega)) \\ F_- & \text{otherwise} \end{cases}$$

Clearly, $\Delta_i^0 = \Delta_i^\sigma \in \mathsf{Th}(\langle\Sigma_{12}, \vdash_i^{\Sigma_{12}}\rangle)$ and $\Delta_1^\sigma \cap \Omega_0 = \Delta_2^\sigma \cap \Omega_0$, where $\Omega_0 = P \cup \mathsf{ctx}(\Omega)$.

Let $\Omega_{k+1} = \Omega_k \cup \{©(A_1, \ldots, A_n) : © \in \Sigma_{12}^{(k)}, A_1, \ldots, A_n \in \Omega_k\}$ and obtain theories $\Delta_1^{k+1} \in \mathsf{Th}(\langle\Sigma_{12}, \vdash_i^{\Sigma_{12}}\rangle)$ and $\Delta_2^{k+1} \in \mathsf{Th}(\langle\Sigma_{12}, \vdash_i^{\Sigma_{12}}\rangle)$ coinciding with Δ_i^k in formulas of Ω_k and further agreeing on Ω_{k+1}. For each formula $A \in \Omega_{k+1} \setminus \Omega_k$ with $\mathsf{hd}(A) \notin \Sigma_i$ we check if $A \in \Delta_{3-i}^k$ and modify the skeleton variable $\mathsf{skel}_i(A)$ accordingly when building Δ_i^{k+1}. Hence, consider for each $i \in \{1, 2\}$ the substitution $\sigma_i^k : P \to L_{\Sigma_i}(P)$ such that

$$\sigma_i^k(p) = \begin{cases} \mathsf{skel}_i(F_0) & \text{if } p = \mathsf{skel}_i(A), A \in \Omega_{k+1} \setminus \Omega_k, \mathsf{hd}(A) \notin \Sigma_i, A \notin \Delta_{3-i}^k, \\ \mathsf{skel}_i(F_1) & \text{if } p = \mathsf{skel}_i(A), A \in \Omega_{k+1} \setminus \Omega_k, \mathsf{hd}(A) \notin \Sigma_i, A \in \Delta_{3-i}^k, \\ p & \text{otherwise.} \end{cases}$$

Then, set $\Delta_i^{k+1} = \mathsf{unskel}_i \circ ((\mathsf{skel}_i(\Delta_i^k))^{\sigma_i^k})$. By Proposition 1, we know that $\Delta_i^{k+1} \in \mathsf{Th}(\langle \Sigma_{12}, \vdash_i^{\Sigma_{12}} \rangle)$. Easily, if $A \in \Omega_k$, or $A \in \Omega_{k+1} \setminus \Omega_k$ and $\mathsf{hd}(A) \in \Sigma_i$, then $\mathsf{skel}_i(A)^{\sigma_i^k} = \mathsf{skel}_i(A)$. Thus, we define sequences for $k \in \mathbb{N}$, $\Delta_1^k \in \mathsf{Th}(\vdash_1^{\Sigma_{12}})$ and $\Delta_2^k \in (\vdash_2^{\Sigma_{12}})$ satisfying $i = 1,2$:

$$\Delta_i^{k+1} \cap \Omega_k = \Delta_1^k \cap \Omega_k = \Delta_2^k \cap \Omega_k$$

Let $\Gamma_+^k = \Delta_1^k \cap \Omega_k = \Delta_2^k \cap \Omega_k$ for $k \in \mathbb{N}_0$. It is clear that $\Gamma_+^k \subseteq \Gamma_+^{k+1}$. Make $\Gamma_+ = \bigcup_{k \in \mathbb{N}_0} \Gamma_+^k$. For any formula $A \notin \Gamma_+^k$, we have that $\Gamma_+^\ell \nvdash_i^{\Sigma_{12}} A$ for every $\ell \geq k$. By compactness, we conclude that $\Gamma_+ \nvdash_i^{\Sigma_{12}} A$. Hence, $\Delta = \Gamma_+ \in \mathsf{Th}(\langle \Sigma_{12}, \vdash_i^{\Sigma_{12}} \rangle)$ for $i \in \{1,2\}$.

Thus, $\Delta \in \mathsf{Th}(\langle \Sigma_{12}, \vdash_{12} \rangle) = \mathsf{Th}(\langle \Sigma_{12}, \vdash_1^{\Sigma_{12}} \rangle) \cap \mathsf{Th}(\langle \Sigma_{12}, \vdash_2^{\Sigma_{12}} \rangle)$ (using Theorem 1) and agrees with Δ_1, Δ_2 on $\mathsf{ctx}(\Omega) \supseteq \Omega$, which concludes the argument. $\square$

It immediately follows from Theorem 2 that combining logics with disjoint signatures preserves the decision complexity classes **P**, **coNP**, **PSPACE**, **EXPTIME**, and **EXPSPACE**.

2.5.2. Fusion of Modal Logics

One of the seminal examples of transfer theorem for combined logics is the preservation of decidability for the *fusion of modal logics*. We show here that this result can also be recovered using Theorem 2. The technicalities of our proof follow along the lines of the proofs in [16,17], where the reader can find further details. We also obtain an upper bound for the complexity of the fusion of two logics depending on their complexity.

Let $\langle \Sigma_{\mathrm{cls}}, \vdash_{\mathrm{cls}} \rangle$ stand for classical propositional logic, where Σ_{cls} is a signature containing the usual classical connectives, namely $\neg \in \Sigma_{\mathrm{cls}}^{(1)}$ and $\wedge, \vee, \Rightarrow, \Leftrightarrow \in \Sigma_{\mathrm{cls}}^{(2)}$. For $i \in \{1,2\}$, consider finite signatures Σ_i such that $\Sigma_{\mathrm{cls}} = \Sigma_1 \cap \Sigma_2$ is the shared signature. Every other connective $\Box \in \Sigma_i^{(n)} \setminus \Sigma_{3-i}^{(n)}$ is understood as an n-place modal operator of the Σ_i signature. Thus, we assume that each $\langle \Sigma_i, \vdash_i \rangle$ is a *modal logic* (see, for instance, [16]), which in particular is classically based; that is, $\Gamma \vdash_i A$ if and only if $\Gamma \vdash_{\mathrm{cls}} A$ for $\Gamma \cup \{A\} \subseteq L_{\Sigma_{\mathrm{cls}}}(P)$, and for every n-place modal operator, $\Box \in \Sigma_i^{(n)}$ satisfies $\{p_1 \Leftrightarrow q_1, \ldots, p_n \Leftrightarrow q_n\} \vdash_i \Box(p_1, \ldots, p_n) \Leftrightarrow \Box(q_1, \ldots, q_n)$.

Proposition 3. *Any two modal logics $\langle \Sigma_1, \vdash_1 \rangle$ and $\langle \Sigma_2, \vdash_2 \rangle$ with $\Sigma_1 \cap \Sigma_2 = \Sigma_{cls}$ are* ctx*-extensible for some context function* ctx *computable in exponential time and space.*

Proof. For each finite set $\Omega \subseteq L_{\Sigma_{12}}(P)$, let $\Omega^+ = \{A \in \mathsf{sub}(\Omega) : \mathsf{hd}(A) \notin \Sigma_{\mathrm{cls}}\}$, and given $\underline{\Omega} \subseteq \Omega^+$, let $\neg\overline{\Omega} = \{\neg A : A \in \Omega^+ \setminus \underline{\Omega}\}$, and consider the formula defined by $C_{\underline{\Omega}} = (\bigwedge \underline{\Omega}) \wedge (\bigwedge \neg\overline{\Omega})$. Further, let $\Omega^\sharp = \{C_{\underline{\Omega}} : \underline{\Omega} \subseteq \Omega^+\}$. We show that the logics $\langle \Sigma_1, \vdash_1 \rangle$ and $\langle \Sigma_2, \vdash_2 \rangle$ are ctx-extensible, with

$$\mathsf{ctx}(\Omega) = \mathsf{sub}(\{\neg C : C \in \Omega^\sharp\}).$$

It is clear that $\mathsf{ctx}(\Omega)$ is exponentially larger than Ω and computable in exponential time.

Let $\Delta_1 \in \mathsf{Th}(\langle \Sigma_{12}, \vdash_1^{\Sigma_{12}} \rangle), \Delta_2 \in \mathsf{Th}(\langle \Sigma_{12}, \vdash_2^{\Sigma_{12}} \rangle)$ be such that $\Delta_1 \cap \mathsf{ctx}(\Omega) = \Delta_2 \cap \mathsf{ctx}(\Omega)$. If $\Delta_1 \cap \Omega = \Delta_2 \cap \Omega = \Omega$, simply choosing the trivial theory $\Delta = L_{\Sigma_{12}}(P)$ would work. Hence, let us assume that this is not the case, and so neither Δ_1 nor Δ_2 are the trivial theory. Further, by Proposition 1, we have that $\Delta_i' = \mathsf{skel}_i(\Delta_i) \in \mathsf{Th}(\langle \Sigma_i, \vdash_i \rangle)$. Since we could always add fresh variables if needed, we can assume, without loss of generality, that $P \setminus \mathsf{var}(\Delta_i')$ is infinite for both $i = 1,2$.

For $i = 1,2$, consider the congruences $\equiv_i \subseteq L_{\Sigma_i}(P)$ defined as $A \equiv_i B$ if and only if $\Delta_i' \vdash_i A \Leftrightarrow B$; let $\mathbb{A}_i$ be the quotient algebra $L_{\Sigma_i}(P)/\equiv_i$. As observed in [30], the Σ_{cpl}-reduct of each $\mathbb{A}_i$ is an infinite countable boolean algebra (ciaB). Letting $v_i : L_{\Sigma_i}(P) \to \mathbb{A}_i$ be the algebra morphism given by $v_i(A) = [A]_{\equiv_i}$ and $\top_i$ be the top element of $\mathbb{A}_i$, we have that $\Delta_i' = v_i^{-1}(\top_i)$.

Consider $\Upsilon = \{C \in \Omega^\sharp : C, \neg C \notin \Delta_1 \cap \Delta_2\}$. Since both theories coincide on $\mathsf{ctx}(\Omega)$, we have that for each $i \in \{1,2\}$, $\Upsilon = \{C \in \Omega^\sharp : v_i(\mathsf{skel}_i(C)) \notin \{\top_i, \bot_i\}\}$. Here, we can split it into two cases:

1. If there is some $C \in \Omega^\sharp$ with $v_i(C) = \top_i$, then $\Upsilon = \emptyset$, as actually we must have $v_i(\mathsf{skel}_i(A)) \in \{\top_i, \bot_i\}$ for all $A \in \Omega^\sharp$. It is known that any two ciaBs are isomorphic. Since the top and bottom elements must be identified, we have $v_1(\mathsf{skel}_1(A))$ and $v_2(\mathsf{skel}_2(A))$ for every $A \in \Omega^\sharp$.
2. Using the boolean-valid equation $x = (x \wedge y) \vee (x \wedge \neg y)$, when $\Upsilon \neq \emptyset$, it is clear that $v_i(\mathsf{skel}_i(\bigvee \Upsilon)) = v_i(\mathsf{skel}_i(\bigvee \Omega^\sharp)) = \top_i$. Furthermore, if $C_1, C_2 \in \Upsilon$ and $C_1 \neq C_2$, then the boolean-valid equation $x \wedge \neg x = \bot$ implies that $v_i(\mathsf{skel}_i(C_1 \wedge C_2)) = \bot_i$. This means that the set of values $v_i(\mathsf{skel}_i(\Upsilon))$ is a partition of $\mathbb{A}_i$. Again, it is known that there is an isomorphism of the two ciaBs that identifies $v_1(\mathsf{skel}_1(A))$ and $v_2(\mathsf{skel}_2(A))$ for every $A \in \Upsilon$ (and, by the same argument as in the previous case, for every $A \in \Omega^\sharp \setminus \Upsilon$.

Hence, in either case, and by identifying the values in the two algebras $\mathbb{A}_1$ and $\mathbb{A}_2$ along the guaranteed suitable isomorphism, we obtain that $v_1(\mathsf{skel}_1(C)) = v_2(\mathsf{skel}_2(C))$ for every $C \in \Omega^\sharp$. Furthermore, we also have that $v_1(\mathsf{skel}_1(A)) = v_2(\mathsf{skel}_2(A))$ for every $A \in \Omega^+$. Namely, letting $\Upsilon_A = \{C_{\underline{\Omega}} \in \Omega^\sharp : A \in \underline{\Omega}\}$, it is clear that $v_i(\mathsf{skel}_i(A)) = v_i(\mathsf{skel}_i(\bigvee \Upsilon_A)) = v_i(\bigvee \mathsf{skel}_i(\Upsilon_A)) = \bigvee v_i(\mathsf{skel}_i(\Upsilon_A)))$. Note, in particular, that $\mathsf{var}(\Omega) \subseteq \Omega^+$.

Let $\mathbb{A}_{12}$ be the Σ_{12}-algebra obtained by merging $\mathbb{A}_1$ and $\mathbb{A}_2$ along the considered isomorphism. We denote the top element of $\mathbb{A}_{12}$ by $\top_{12}$ $(= \top_1 = \top_2)$, and we have that $\mathbb{A}_{12}$'s Σ_{cpl}-reduct is isomorphic to the Σ_{cpl}-reducts of the original algebras, and also that $\Box_{\mathbb{A}_{12}}(x_1, \ldots, x_n) = \Box_{\mathbb{A}_i}(x_1, \ldots, x_n)$ for $\Box \in \Sigma_i^{(n)}$ (modulo the isomorphism). We know that $f : \mathsf{var}(\Omega) \to \mathbb{A}_{12}$ such that $f(p) = v_1(\mathsf{skel}_1(p)) = v_2(\mathsf{skel}_2(p))$ extends to a homomorphism $v : L_{\Sigma_{12}}(P) \to \mathbb{A}_{12}$, which is uniquely determined for formulas with variables in $\mathsf{var}(\Omega)$. Thus, for every $A \in \Omega$, $v(A) = v_1(\mathsf{skel}_1(A)) = v_2(\mathsf{skel}_2(A))$. Consider $\Delta = v^{-1}(\top_{12})$ to be the theory induced by v, and let $g_i : L_{\Sigma_i}(P) \to \mathbb{A}_i$ be $g_i = v \circ \mathsf{unskel}_{\Sigma_i}$, and for $p \in P$, let $\sigma_i(p) = B$ for some formula B with $g_i(p) = [B]_\equiv$; then

$$\Delta_i'' = g_i^{-1}(\top_i) = \sigma_i^{-1}(v_i^{-1}(\top_i)) = \sigma_i^{-1}(\Delta_i') \in \mathsf{Th}(\langle \Sigma_i, \vdash_i \rangle$$

since the set of theories of any logic is closed under inverse images of substitutions. Hence, $\Delta = \mathsf{unskel}_{\Sigma_i}(\Delta_i'') \in \mathsf{Th}(\langle \Sigma_{12}, \vdash_i^{\Sigma_{12}} \rangle)$, and therefore

$$\Delta \in \mathsf{Th}(\langle \Sigma_{12}, \vdash_1^{\Sigma_{12}} \rangle) \cap \mathsf{Th}(\langle \Sigma_{12}, \vdash_2^{\Sigma_{12}} \rangle) = \mathsf{Th}(\langle \Sigma_{12}, \vdash_{12} \rangle).$$

Moreover, as for $i = 1, 2$ and $B \in \Omega$, we have $B \in \Delta$ iff $v(B) = \top_{12}$ iff $v_i(\mathsf{skel}(B)) = \top_i$ iff $B \in \Delta_i$; we know that Δ agrees with Δ_1, Δ_2 in Ω. Thus, we conclude that $\langle \Sigma_1, \vdash_1 \rangle$, $\langle \Sigma_2, \vdash_2 \rangle$ are ctx-extensible. $\Box$

From this fact, and according to Theorem 2, we can conclude that deciding the fusion of logics decidable in **EXPTIME** is in **2EXPTIME**. This complexity upper bound is not too tight, in general, although the fact that it was obtained using a very general, not tailored, result such as Theorem 2 may help to explain why. Indeed, using our result, we can also conclude that combining two modal logics decidable in **PSPACE** yields an **EXPSPACE** upper bound for their fusion, whereas it is well known (see [17]) that the decision problem for the fusion of two copies of the basic normal modal logic **K** is in **PSPACE**, as is also the decision problem for the logic **K**.

It is worth noting, though, that an alternative proof of Proposition 3 using a simpler context function computable in polynomial time is impossible in the general case. For instance, modal logic **S5** is known to be in **coNP**, whereas deciding the fusion of two copies of **S5** is known to be a **PSPACE-complete** problem [17]. According to our Theorem 2, a polynomial time computable ctx function would yield a decision procedure for such a fusion in **coNP**, which is strictly below **PSPACE** unless there is a collapse of the polynomial hierarchy. However, as far as we know, there is no known counterexample eliminating the possibility of finding a suitable ctx computable in polynomial space, even if using exponential time, which would yield the preservation of **PSPACE** by fusion.

It should further be observed that, despite the fact that the context function obtained is exponential, for particular inputs Γ, A, with a logarithmic amount of $\Box$-headed subformulas, deciding if A follows from Γ can still be done with a polynomial time slowdown in the decision time of the algorithms used to decide the component logics using the algorithms in Theorem 2. This is natural, since formulas with head in $\Sigma_i \setminus \Sigma_{\mathsf{cpl}}$ are treated as (new) variables by $\vdash_{3-i}$. This behavior is analogous to the growth in complexity in the SAT-problem for classical logic being strongly dependent on the number of variables of the input rather than on its overall size.

3. Beyond Propositional Logics

We now study the generalization of the previous results beyond propositional logics, in particular in the realm of k-deductive systems [31]. These are consequence relations defined over a (possibly) non-freely generated language, as k-formulas are k-tuples of formulas in an algebraic language. Although, as it will become clear, all the results would be smoothly obtainable for arbitrary k, we focus our attention on the case $k = 2$, and in particular on equational reasoning.

3.1. Syntax

Given a signature Σ, a 2-formula over Σ is a pair (A, B) with $A, B \in L_\Sigma(P)$, which we will simply denote by $A \approx B$. The set of all 2-formulas over Σ is $\mathsf{Eqs}(L_\Sigma(P))$, where $\mathsf{Eqs}(\Gamma) = \{A \approx B : A, B \in \Gamma\}$. Given $\Theta \subseteq \mathsf{Eqs}(L_\Sigma(P))$, it is also useful to define $\mathsf{term}(\Theta) = \{A, B : A \approx B \in \Theta\}$.

Other definitions in Section 2.1 are smoothly adapted to 2-formulas. Substitutions $\sigma : P \to L_\Sigma(P)$ act on 2-formulas and sets thereof in the expected way: $(A \approx B)^\sigma = A^\sigma \approx B^\sigma$, and $\Gamma^\sigma = \{(A \approx B)^\sigma : A \approx B \in \Gamma\}$. Similarly, when $\Sigma_0 \subseteq \Sigma$, we have $\mathsf{skel}_{\Sigma_0}(A \approx B) = \mathsf{skel}_{\Sigma_0}(A) \approx \mathsf{skel}_{\Sigma_0}(B)$ and $\mathsf{unskel}_{\Sigma_0}(A \approx B) = \mathsf{unskel}_{\Sigma_0}(A) \approx \mathsf{unskel}_{\Sigma_0}(B)$.

3.2. 2-*Logics, Equational Logics, and Their Theories*

Let us start by lifting Definition 1 according to [31].

Definition 4. *A* 2-*logic is a pair* $\langle \Sigma, \vdash \rangle$*, where* Σ *is a signature, and* $\vdash \subseteq \wp(\mathsf{Eqs}(L_\Sigma(P))) \times \mathsf{Eqs}(L_\Sigma(P))$ *is a relation satisfying, for* $\Gamma \cup \{A \approx B\} \subseteq \mathsf{Eqs}(L_\Sigma(P))$*:*

($R_\approx$) $\Gamma \vdash A \approx B$ *whenever* $A \approx B \in \Gamma$*;*
($M_\approx$)$\Gamma \vdash A \approx B$ *whenever* $\Gamma' \vdash A \approx B$ *for* $\Gamma' \subseteq \Gamma$*;*
($T_\approx$) $\Gamma \vdash A \approx B$ *whenever* $\Delta \vdash A \approx B$*, and* $\Gamma \vdash C \approx D$ *for every* $C \approx D \in \Delta$*;*
($S_\approx$) $\Gamma \vdash A \approx B$ *implies* $\Gamma^\sigma \vdash A^\sigma \approx B^\sigma$ *for any substitution* $\sigma : P \to L_\Sigma(P)$*.*

We further say that $\langle \Sigma, \vdash \rangle$ *is* compact *whenever it satisfies:*

($F_\approx$) $\Gamma \vdash A \approx B$ *implies there is finite* $\Gamma_0 \subseteq \Gamma$ *such that* $\Gamma_0 \vdash A \approx B$*.*

We also say that a set $\mathsf{R} \subseteq \wp(\mathsf{Eqs}(L_\Sigma(P))) \times \mathsf{Eqs}(L_\Sigma(P))$ axiomatizes $\langle \Sigma, \vdash \rangle$ whenever $\vdash$ is the closure of R by ($\mathrm{R}_\approx$), ($\mathrm{M}_\approx$), ($\mathrm{T}_\approx$), and ($\mathrm{S}_\approx$), and we write $\vdash = \vdash_{\mathsf{R}}$.

For $\Gamma \subseteq \mathsf{Eqs}(L_\Sigma(P))$, we still write $\Gamma^\vdash = \{A \approx B : \Gamma \vdash A \approx B\}$ and $\mathsf{Th}(\langle \Sigma, \vdash \rangle) = \{\Gamma \subseteq \mathsf{Eqs}(L_\Sigma(P)) : \Gamma^\vdash = \Gamma\}$ for the set of theories of $\langle \Sigma, \vdash \rangle$.

The notion of 2-logic covers, in particular, what we will call *equational logics*. Given a set of equations $\mathsf{Eq} = \{A_i \approx B_i : i \in I\} \subseteq \mathsf{Eqs}(L_\Sigma)$, we denote by R_{Eq} the following set of rules.

$$\frac{}{A_i \approx B_i} \text{ for } i \in I$$

$$\frac{}{p \approx p}\,\mathsf{ref} \qquad \frac{p \approx q}{q \approx p}\,\mathsf{symm} \qquad \frac{p \approx q\,,\, q \approx r}{p \approx r}\,\mathsf{trans}$$

$$\frac{p_1 \approx q_1, \ldots, p_k \approx q_k}{\copyright(p_1, \ldots, p_k) \approx \copyright(q_1, \ldots, q_k)}\ \mathsf{cong}_{\copyright} \text{ for each } \copyright \in \Sigma^{(k)}$$

An *equational logic* is a 2-logic $\langle \Sigma, \vdash \rangle$ axiomatized by R_{Eq} for some set of equations $\mathsf{Eq} \subseteq \mathsf{Eqs}(L_\Sigma)$; that is, $\vdash\, = \,\vdash_{\mathsf{R}_{\mathsf{Eq}}}$. Note that since the rules in R_{Eq} are always finitary for any Eq, we have that every equational logic is compact.

This notion of an equational logic corresponds to the quasi-equational theory of the variety axiomatized by Eq, $\mathbb{V}(\mathsf{Eq})$. That is, $A_1 \approx B_1, \ldots, A_n \approx B_n \vdash_{\mathsf{R}_{\mathsf{Eq}}} A \approx B$ if and only if the quasi-equation $A_1 \approx B_1 \wedge \ldots \wedge A_n \approx B_n \to A \approx B$ is valid in all algebras of the variety.

Theories in these logics are sets of equations satisfying $\Delta = \Delta^{\vdash_{\mathsf{Eq}}}$. Clearly, every theory $\Delta \in \mathsf{Th}(\langle \Sigma, \vdash_{\mathsf{Eq}} \rangle)$ defines a congruence on $L_\Sigma(P)$, identifying formulas A and B if and only if $A \approx B \in \Delta$.

Example 3. *The smallest* 2*-logic over a signature* Σ*, corresponding to Example 1, is such that* $\Gamma \vdash_{sml} A \approx B$ *if and only if* $A \approx B \in \Gamma$*, and correspondingly we have that* $Th(\langle \Sigma, \vdash_{sml} \rangle) = \wp(\mathsf{Eqs}(L_\Sigma(P)))$*. This* 2*-logic is axiomatizable by the empty set of rules.*

However, the smallest equational logic is $\langle \Sigma, \vdash_{\mathsf{R}_\emptyset} \rangle$*, that is, the* 2*-logic axiomatized by rules* ref, symm, trans, *and* $\mathsf{cong}_{\copyright}$ *for each* $\copyright \in \Sigma$*.*

We also have that 2-logics are closed for intersections, and it still makes sense to define the extension of a 2-logic $\langle \Sigma_0, \vdash_0 \rangle$ to a larger signature $\Sigma_0 \subseteq \Sigma$ as the least 2-logic with signature Σ such that $\vdash_0 \subseteq \vdash_0^\Sigma$. We can also lift Proposition 1 into an analogue statement characterizing the language extensions of 2-logics.

Proposition 4. *For* $\Gamma \cup \{A \approx B\} \subseteq \mathsf{Eqs}(L_\Sigma(P))$*, we have that* $\Gamma \vdash_0^\Sigma A \approx B$ *if and only if* $\mathsf{skel}(\Gamma) \vdash_0 \mathsf{skel}(A) \approx \mathsf{skel}(B)$*. Hence,* $Th(\langle \Sigma, \vdash_0^\Sigma \rangle) = \mathsf{unskel}(Th(\langle \Sigma_0, \vdash_0 \rangle))$*.*

Proof. The proof is completely analogous to the proof of Proposition 1, using the facts that $\mathsf{unskel} \circ (\mathsf{skel} \circ \sigma) = \sigma$, and skel and unskel are bijections, and the properties $(G_\approx)$ instead of (G) for $G \in \{R, M, T, S\}$. $\square$

Equivalently, $\Gamma \vdash_0^\Sigma A$ if and only if there exist $\Gamma_0 \cup \{A_0 \approx B_0\} \subseteq \mathsf{Eqs}(L_{\Sigma_0}(P))$ and $\sigma : P \to L_\Sigma(P)$ such that $\Gamma_0^\sigma \subseteq \Gamma$, $A_0^\sigma = A$, $B_0^\sigma = B$, and $\Gamma_0 \vdash_0 A_0 \approx B_0$.

3.3. Combining 2-Logics

At this point, it is easy to lift Definition 2.

Definition 5. *The* combination *of* 2*-logics* $\langle \Sigma_1, \vdash_1 \rangle$, $\langle \Sigma_2, \vdash_2 \rangle$*, which is once again denoted by* $\langle \Sigma_1, \vdash_1 \rangle \bullet \langle \Sigma_2, \vdash_2 \rangle$*, is the least* 2*-logic* $\langle \Sigma_{12}, \vdash_{12} \rangle$ *such that* $\vdash_1, \vdash_2 \subseteq \vdash_{12}$*. The combination is said to be* disjoint *if* $\Sigma_1 \cap \Sigma_2 = \emptyset$*.*

Note that it also follows easily that the combination of compact 2-logics is necessarily compact. Namely, if $\langle \Sigma_1, \vdash_1 \rangle$ and $\langle \Sigma_2, \vdash_2 \rangle$ are compact, then so is $\langle \Sigma_{12}, \vdash_{12} \rangle$. Further, if R_1 and R_2 axiomatize each of the given logics, then $\mathsf{R}_1 \cup \mathsf{R}_2$ axiomatizes $\langle \Sigma_{12}, \vdash_{12} \rangle$.

Example 4. *For equational logics* $\langle \Sigma_1, \vdash_{\mathsf{R}_{\mathsf{Eq}_1}} \rangle$ *and* $\langle \Sigma_2, \vdash_{\mathsf{R}_{\mathsf{Eq}_2}} \rangle$*, we have*

$$\langle \Sigma_1, \vdash_{\mathsf{R}_{\mathsf{Eq}_1}} \rangle \bullet \langle \Sigma_2, \vdash_{\mathsf{R}_{\mathsf{Eq}_2}} \rangle = \langle \Sigma_{12}, \vdash_{\mathsf{R}_{\mathsf{Eq}_1 \cup \mathsf{Eq}_2}} \rangle,$$

and $\langle \Sigma_{12}, \vdash_{\mathsf{R}_{\mathsf{Eq}_1}}^{\Sigma_{12}} \rangle = \langle \Sigma_{12}, \vdash_{\mathsf{R}_{\mathsf{Eq}_1} \cup \mathsf{R}_\emptyset} \rangle = \langle \Sigma_1, \vdash_{\mathsf{R}_{\mathsf{Eq}_1}} \rangle \bullet \langle \Sigma_2, \vdash_{\mathsf{R}_\emptyset} \rangle$*.*

As the reader may already suspect, Theorem 1 also adapts to the analogue statement characterizing the combination of 2-logics.

Theorem 3. *Let* $\langle \Sigma_{12}, \vdash_{12} \rangle = \langle \Sigma_1, \vdash_1 \rangle \bullet \langle \Sigma_2, \vdash_2 \rangle$. *For every* $\Gamma \cup \{A \approx B\} \subseteq \mathsf{Eqs}(L_{\Sigma_{12}}(P))$, *we have:*

$$\Gamma \vdash_{12} A \approx B$$

if and only if

$$A \approx B \in \Delta \text{ for every } \Delta \in Th(\langle \Sigma_{12}, \vdash_1^{\Sigma_{12}} \rangle) \cap Th(\langle \Sigma_{12}, \vdash_2^{\Sigma_{12}} \rangle) \text{ with } \Gamma \subseteq \Delta.$$

Hence, $\Gamma^{\vdash_{12}}$ *is the smallest element of both* $Th(\langle \Sigma_{12}, \vdash_1^{\Sigma_{12}} \rangle)$ *and* $Th(\langle \Sigma_{12}, \vdash_1^{\Sigma_{12}} \rangle)$ *that contains* Γ, *and*

$$Th(\langle \Sigma_{12}, \vdash_{12} \rangle) = Th(\langle \Sigma_{12}, \vdash_1^{\Sigma_{12}} \rangle) \cap Th(\langle \Sigma_{12}, \vdash_2^{\Sigma_{12}} \rangle).$$

Proof. The proof is analogous to the proof of Theorem 1 by using properties $(G_\approx)$ instead of (G) for $G \in \{R, M, T, S\}$. □

3.4. Contextual Extensibility and Decidability Preservation Revisited

As before, a 2-logic $\langle \Sigma, \vdash \rangle$ is said to be *decidable* if there exists an algorithm $\mathtt{D}$ that terminates when given any finite set $\Gamma \subseteq \mathsf{Eqs}(L_\Sigma(P))$ and $A \approx B \in \mathsf{Eqs}(L_\Sigma(P))$ as input, and outputs $\mathtt{D}(\Gamma, A \approx B) = \mathtt{yes}$ if $\Gamma \vdash A \approx B$ and $\mathtt{D}(\Gamma, A \approx B) = \mathtt{no}$ if $\Gamma \nvdash A \approx B$.

Again, Theorem 3 is not enough to obtain a decision procedure for the combined logic. As in the propositional case, we consider context functions $\mathsf{ctx} : \mathsf{Eqs}(L_{\Sigma_{12}}(P)) \to \mathsf{Eqs}(L_{\Sigma_{12}}(P))$ with $\Omega \subseteq \mathsf{ctx}(\Omega)$ finite for finite Ω. Furthermore, we can naturally generalize the notion of ctx-extensibility to 2-logics.

Definition 6. *For a fixed context function* ctx, *we say that two* 2-*logics* $\langle \Sigma_1, \vdash_1 \rangle, \langle \Sigma_2, \vdash_2 \rangle$ *are* ctx-extensible *when every* $\Omega \subseteq \mathsf{Eqs}(L_{\Sigma_{12}}(P))$ *and theories* $\Delta_i \in Th(\langle \Sigma_{12}, \vdash_i^{\Sigma_{12}} \rangle)$ *for* $i \in \{1, 2\}$,

if

$$\Delta_1 \cap \mathsf{ctx}(\Omega) = \Delta_2 \cap \mathsf{ctx}(\Omega)$$

then there exists a theory $\Delta \in Th(\langle \Sigma_{12}, \vdash_{12} \rangle)$ *such that*

$$\Delta \cap \Omega = \Delta_1 \cap \Omega = \Delta_2 \cap \Omega.$$

Moreover, Lemma 1 easily lifts as well.

Lemma 2. *Let* $\langle \Sigma_{12}, \vdash_{12} \rangle = \langle \Sigma_1, \vdash_1 \rangle \bullet \langle \Sigma_2, \vdash_2 \rangle$ *be* 2-*logics. For every* $\Gamma \cup \{A \approx B\} \subseteq \mathsf{Eqs}(L_{\Sigma_{12}}(P))$, *we have:*

$$\Gamma \vdash_{12} A \approx B$$

if and only if

$$A \approx B \in \Omega \text{ for every } \Omega = ((\Omega^{\vdash_1^{\Sigma_{12}}} \cup \Omega^{\vdash_2^{\Sigma_{12}}}) \cap \mathsf{ctx}(\Gamma \cup \{A \approx B\})) \text{ with } \Gamma \subseteq \Omega.$$

Proof. Again, the proof is analogous to that of Lemma 1 by invoking Theorem 3 instead of Theorem 1. □

Gathering all these elements, we can also easily adapt the decidability preservation result of Theorem 2 to decide the combination of decidable 2-logics. As before, let ctx be a context function such that ctx is computable in $\mathbf{TIME}(c(n))$ and $\mathbf{SPACE}(d(n))$, obviously with $d(n) \leq c(n)$.

Theorem 4. *Let* $\langle \Sigma_1, \vdash_1 \rangle, \langle \Sigma_2, \vdash_2 \rangle$ *be* ctx-*extensible* 2-*logics. If the decision problems for* $\langle \Sigma_1, \vdash_1 \rangle$, $\langle \Sigma_2, \vdash_2 \rangle$ *are both in complexity class* $\mathbf{C}$, *then the decision problem for* $\langle \Sigma_{12}, \vdash_{12} \rangle$ *is in complexity class* $\mathbf{C}'$, *as given by Table 1.*

Proof. The proof is completely analogous to that of Theorem 2 by adapting the algorithms to work with 2-formulas instead of formulas. □

3.5. Applications

We shall now give two illustrative applications of Theorem 4. In both cases, we shall use the context function $\mathsf{ctx}_\star : \wp(\mathsf{Eqs}(L_\Sigma(P))) \to \wp(\mathsf{Eqs}(L_\Sigma(P)))$ defined by

$$\mathsf{ctx}_\star(\Omega) = \mathsf{Eqs}(\mathsf{sub}(\mathsf{term}(\Omega))),$$

which works for both applications in this section. In a more explicit form:

$$\mathsf{ctx}_\star(\{A_i \approx B_i : i \in I\}) = \{C_i \approx D_i : C_i, D_i \in \mathsf{sub}(\{A_i, B_i : i \in I\})\}.$$

It is clear that $\mathsf{ctx}_\star$ is computable in polynomial time.

3.5.1. Splitting the Smallest Equational Logic

With an eye on the fact that Theorem 4 can be iteratively applied to the combination of any finite number of 2-logics, we use the result to prove the well-known fact that the smallest equation logic $\langle \Sigma, \vdash_{\mathsf{R}_\emptyset} \rangle$ is decidable as a result of combining the decidable 2-logics corresponding, in isolation, to each of the forms ref, symm, trans, and cong.

Let $\vdash_x = \vdash_{\{x\}}$ for $x \in X = \{\mathsf{ref}, \mathsf{symm}, \mathsf{trans}\}$, and set $\vdash_{\mathsf{cong}} = \vdash_{\{\mathsf{cong}_{©} : © \in \Sigma\}}$.

Proposition 5. *Fixed signature Σ and the 2-logics $\langle \Sigma, \vdash_{\mathsf{ref}} \rangle$, $\langle \Sigma, \vdash_{\mathsf{symm}} \rangle$, $\langle \Sigma, \vdash_{\mathsf{trans}} \rangle$, and $\langle \Sigma, \vdash_{\mathsf{cong}} \rangle$ are jointly $\mathsf{ctx}_\star$-extensible.*

Proof. Let $\Delta_x \in \mathsf{Th}(\langle \Sigma, \vdash_x \rangle)$ for $x \in Y = X \cup \{\mathsf{cong}\}$. Assuming that $\Delta_x \cap \mathsf{ctx}_\star(\Omega) = \Gamma$ for every $x \in Y$ for some fixed $\Omega \subseteq \mathsf{Eqs}(L_\Sigma(P))$, we show that there is $\Delta \in \bigcap_{x \in Y} \mathsf{Th}(\langle \Sigma, \vdash_x \rangle)$ such that $\Gamma \cap \Omega = \Delta \cap \Omega$.

Consider $\Delta = \Gamma^{\vdash_{\mathsf{R}_\emptyset}} \in \bigcap_{x \in X} \mathsf{Th}(\langle \Sigma, \vdash_{\mathsf{r}_x} \rangle) = \mathsf{Th}(\langle \Sigma, \vdash_{\mathsf{R}_\emptyset} \rangle)$. It is clear that $\Gamma \subseteq \Delta \cap \mathsf{ctx}_\star(\Omega)$. It is now sufficient to show that the other inclusion also holds.

Knowing that $\Delta_x \in \mathsf{Th}(\langle \Sigma, \vdash_x \rangle)$ and $\Gamma = \Delta_x \cap \mathsf{ctx}_\star(\Omega)$ for $x \in X$, and using the fact that rules ref, symm, and trans are expressed using only variables, it follows that $\Gamma^{\vdash_X} = \Gamma$. Further, if $D \notin \mathsf{term}(\mathsf{ctx}_\star(\Omega))$ and $\Gamma \cup \{C \approx D\} \vdash_X A \approx B$, then either $A \approx B \in \Gamma$, $A \approx C \in \Gamma$ and $B = D$, or $B \approx C \in \Gamma$ and $A = D$.

Furthermore, from $\Delta_{\mathsf{cong}} \in \mathsf{Th}(\langle \Sigma, \vdash_{\mathsf{cong}} \rangle)$ and $\Gamma = \Delta_{\mathsf{cong}} \cap \mathsf{ctx}_\star(\Omega)$, it follows that if $C \approx D \in \Gamma^{\vdash_{\mathsf{cong}}} \setminus \Gamma$, then either C or D is not in $\mathsf{term}(\mathsf{ctx}_\star(\Omega))$. Let it be D, without loss of generality. If $\Gamma \cup \{C \approx D\} \vdash_X A \approx B$, then either $A \approx B \in \Gamma$, $A = D$, or $B = D$. Thus, $(\Gamma \cup \{C \approx D\})^{\vdash_X} \cap \mathsf{ctx}_\star(\Omega) = \Gamma$. This argument can be adapted to an arbitrary finite number of applications of congruence rules. If a derivation of $A \approx B \in \Delta \cap \mathsf{ctx}_\star(\Omega)$ uses exactly k instances of congruence rules, introducing, respectively, $C_i \approx D_i \notin \mathsf{ctx}_\star(\Omega)$ for $1 \leq i \leq k$, then if $\Gamma \cup \{C_j \approx D_j : 1 \leq i \leq k\} \vdash_X A \approx B$ and, assuming without loss of generality, that $D_i \notin \mathsf{term}(\mathsf{ctx}_\star(\Omega))$, we can conclude that either $A \approx B \in \Gamma$, $A \in \{D_i : 1 \leq i \leq k\}$, or $B \in \{D_i : 1 \leq i \leq k\}$. Hence, since $\vdash_{\mathsf{R}_\emptyset}$ is compact, $A \approx B \in \Delta \cap \mathsf{ctx}_\star(\Omega)$ implies $A \approx B \in \Gamma$. Therefore, since $\Omega \subseteq \mathsf{ctx}_\star(\Omega)$, we have that $\Delta \cap \Omega = \Gamma \cap \Omega = \Delta_x \cap \Omega$ for every $x \in Y$. □

Now, the known result that congruences can be computed in polynomial time follows by Theorem 4 just from the fact that calculating $\mathsf{ctx}_\star$ and the closures for each of its requirements (symmetry, reflexivity, transitivity, and congruence) can also separately be done polynomial time.

Corollary 1. *There is a problem deciding if $\langle \Sigma, \vdash_{\mathsf{R}_\emptyset} \rangle$ is in* **P**.

Of course, the same also follows for the 2-logics generated by subsets of the rules in Y, e.g., rules corresponding to equivalence or tolerance relations. Further note that if no congruence rules are involved, it is enough to consider the simpler context function $\mathsf{ctx}_\star(\Omega) = \mathsf{Eqs}(\mathsf{term}(\Omega))$.

3.5.2. Combining Equational Logics with Disjoint Signatures

We now study the combination of equational logics and analyze the preservation of decidability and complexity in the disjoint case along the lines of Theorem 4. This is a particularly interesting case, as it goes in the direction of a myriad of important modular decidability results for reasoning modulo equational theories, which we discuss later.

Proposition 6. *Assuming $\Sigma_1 \cap \Sigma_2 = \emptyset$, we have that equational logics $\langle \Sigma_1, \vdash_{\mathsf{R}_{\mathsf{Eq}_1}} \rangle$ and $\langle \Sigma_2, \vdash_{\mathsf{R}_{\mathsf{Eq}_2}} \rangle$ are $\mathsf{ctx}_\star$-extensible.*

Proof. For simplicity, let $\vdash_i = \vdash_{\mathsf{R}_{\mathsf{Eq}_i}}$ for $i = 1, 2$, $\Sigma = \Sigma_{12}$, and $\vdash = \vdash_{\mathsf{R}_{\mathsf{Eq}_1 \cup \mathsf{Eq}_2}}$. Further, given $\Xi \subseteq L_\Sigma(P)$ and $i = 1, 2$, define

$$\Sigma_i[\Xi] = \bigcup_{j \in \mathbb{N}_0} \{ ©(A_1, \dots, A_j) : A_1, \dots, A_j \in \Xi, © \in \Sigma_i^{(j)} \}$$

and let $\Sigma[\Xi] = \Sigma_1[\Xi] \cup \Sigma_2[\Xi]$.

In order to prove that $\langle \Sigma_1, \vdash_1 \rangle$, $\langle \Sigma_2, \vdash_2 \rangle$ are $\mathsf{ctx}_\star$-extensible, we show that, given $\Omega \subseteq \mathsf{Eqs}(L_\Sigma(P))$, $\Delta_1 \in \mathsf{Th}(\langle \Sigma, \vdash_1^\Sigma \rangle)$ and $\Delta_2 \in \mathsf{Th}(\langle \Sigma, \vdash_2^\Sigma \rangle)$ such that $\Delta_1 \cap \mathsf{ctx}_\star(\Omega) = \Delta_2 \cap \mathsf{ctx}_\star(\Omega)$, there is $\Delta \in \mathsf{Th}(\langle \Sigma, \vdash \rangle)$ such that $\Delta \cap \Omega = \Delta_i \cap \Omega$ for $i = 1, 2$.

If $\Delta_1 \cap \mathsf{ctx}_\star(\Omega) = \Delta_2 \cap \mathsf{ctx}_\star(\Omega) = \mathsf{ctx}_\star(\Omega)$, then picking the trivial theory $\Delta = \mathsf{Eqs}(L_\Sigma(P))$ does the job. We proceed, otherwise, knowing $\Delta_1 \cap \mathsf{ctx}_\star(\Omega) = \Delta_2 \cap \mathsf{ctx}_\star(\Omega) \neq \mathsf{ctx}_\star(\Omega)$.

Let $\Xi_0 = \mathsf{term}(\mathsf{ctx}_\star(\Omega))$, $\Omega_0 = \mathsf{Eqs}(\Xi_0) = \mathsf{ctx}_\star(\Omega)$, and for $k \geq 0$, define

- $\Xi_{k+1} = \Xi_k \cup \Sigma[\Xi_k]$, and
- $\Omega_{k+1} = \mathsf{Eqs}(\Xi_{k+1})$.

Further, let $\Delta_1^0 = \Delta_1$, $\Delta_2^0 = \Delta_2$, and for $k \geq 0$ and $i = 1, 2$, define

- $\Gamma_i^k = \Delta_i^k \cap \Omega_k$;
- $\Theta_i^k = (\Gamma_i^k)^{\vdash_i^\Sigma} \cap \Omega_{k+1}$;
- $\Delta_i^{k+1} = (\Theta_1^k \cup \Theta_2^k)^{\vdash_i^\Sigma}$.

By definition, we have that $\Delta_1^k \in \mathsf{Th}(\langle \Sigma, \vdash_1^\Sigma \rangle)$ and $\Delta_2^k \in \mathsf{Th}(\langle \Sigma, \vdash_1^\Sigma \rangle)$ for every $k \in \mathbb{N}_0$. We show below that for every $k \in \mathbb{N}_0$ and $i = 1, 2$, we have the following two properties.

$$\Delta_1^k \cap \Omega_k = \Delta_2^k \cap \Omega_k \neq \Omega_k \tag{1}$$

$$\Delta_i^{k+1} \cap \Omega_k = \Delta_i^k \cap \Omega_k \tag{2}$$

Thus, using compactness (which holds for any equational logic), we have that (by (1) and (2)) for $i = 1, 2$,

$$\Delta = \bigcup_{k \in \mathbb{N}_0} (\Delta_i^k \cap \Omega_k) \in \mathsf{Th}(\langle \Sigma, \vdash_1^\Sigma \rangle) \cap \mathsf{Th}(\langle \Sigma, \vdash_2^\Sigma \rangle).$$

This finishes the proof, as it immediately follows that $\Delta \cap \Omega = \Delta_i \cap \Omega = \Delta_i^0 \cap \Omega_0$ for $i = 1, 2$ as desired. To prove (1) and (2), we need two technical lemmas.

Lemma 3. *Assuming that* $\Delta_1^k \cap \Omega_k = \Delta_2^k \cap \Omega_k \neq \Omega_k$*, we have, for* $i = 1, 2$*, that the following properties hold.*

$$(\Theta_1^k \cup \Theta_2^k) \cap \Omega_k = \Theta_i^k \cap \Omega_k = \Delta_i^k \cap \Omega_k = \Gamma_i^k \neq \Omega_k \tag{3}$$

$$\Theta_i^k \subseteq \mathsf{Eqs}(\Xi_k \cup \Sigma_i[\Xi_k]) \cup \{A \approx A : A \in \Sigma_{3-i}[\Xi_k]\} \subseteq \Omega_{k+1}\} \tag{4}$$

$$\Theta_1^k \cup \Theta_2^k = (\Theta_1^k \cup \Theta_2^k)^{\vdash_{\mathsf{R}_\emptyset}} \cap \Omega_{k+1} \tag{5}$$

Proof. Note that $\Xi_{k+1} = \Xi_k \cup \Sigma[\Xi_k] = \Xi_k \cup \Sigma_1[\Xi_k] \cup \Sigma_2[\Xi_k]$, but in general, $\mathsf{Eqs}(\Xi_i \cup \Sigma_1(\Xi_k)) \cup \mathsf{Eqs}(\Xi_i \cup \Sigma_2(\Xi_k)) \subsetneq \mathsf{Eqs}(\Xi_{k+1})$. Still, since $\Delta_i^k \in \mathsf{Th}(\langle \Sigma, \vdash_i^\Sigma \rangle)$, then $\Theta_i^k \cap \Omega_k = \Delta_i^k \cap \Omega_k$. Using $(\Theta_1^k \cup \Theta_2^k) \cap \Omega_k = (\Theta_1^k \cap \Omega_k) \cup (\Theta_2^k \cap \Omega_k)$ and the assumption that $\Delta_1^k \cap \Omega_k = \Delta_2^k \cap \Omega_k \neq \Omega_k$, we conclude that (3) holds.

To see that (4) holds, assume by contradiction that we have $A \neq B$ such that $A \approx B \notin \mathsf{Eqs}(\Xi_k \cup \Sigma_i[\Xi_k])$, but $A \approx B \in \Theta_i^k$. Then either A or B (or both) must be in $\Sigma_{3-i}[\Xi_k]$. Without loss of generality, let it be A. Since $\Sigma_1 \cap \Sigma_2 = \emptyset$, $\mathsf{skel}_{\Sigma_i}(A)$ is a variable, and $\mathsf{skel}_{\Sigma_i}(A) \notin \mathsf{var}(\mathsf{skel}_{\Sigma_i}(\Xi_k \cup \Sigma_i[\Xi_k] \cup \{B\}))$. Using the substitution invariance of $\vdash_i$ and Proposition 4, we obtain that $(\Gamma_i^k)^{\vdash_i^\Sigma} = \mathsf{Eqs}(L_\Sigma(P))$, which is absurd, since $(\Gamma_i^k)^{\vdash_i^\Sigma} \cap \Omega_k = \Theta_i^k \cap \Omega_k \neq \Omega_k$ by (3).

In particular, we have that $A \approx B \notin \Theta_i^k$ for any $B \neq A \in \Sigma_{3-i}[\Xi_k]$. Hence, $A \approx B \notin \Theta_1^k \cup \Theta_2^k$ for any $A \in \Sigma_1[\Xi_k]$ and $B \in \Sigma_2[\Xi_k]$. This observation, together with (3) and the fact that $\vdash_{\mathsf{R}_\emptyset} \subseteq \vdash_i$, implies (5). $\square$

Lemma 4. *Still assuming that* $\Delta_1^k \cap \Omega_k = \Delta_2^k \cap \Omega_k \neq \Omega_k$*, for* $i = 1, 2$*, we have that*

$$\Delta_i^{k+1} \cap \Omega_{k+1} = \Theta_1^k \cup \Theta_2^k.$$

Proof. By definition, $\Delta_i^{k+1} = (\Theta_1^k \cup \Theta_2^k)^{\vdash_i^\Sigma}$. Consider the following equivalence relations (for $i = 1, 2$) on $\mathsf{skel}_{\Sigma_i}(\Xi_{k+1})$, where

$$\mathsf{skel}_{\Sigma_i}(A) \equiv_i \mathsf{skel}_{\Sigma_i}(B) \text{ if and only if } \Theta_1^k \cup \Theta_2^k \vdash_{\mathsf{R}_\emptyset} A \approx B.$$

For each $D \in \mathsf{skel}_{\Sigma_i}(\Xi_{k+1})$, let us pick a representative $E_D \in [D]_{\equiv_i} \subseteq \mathsf{skel}_{\Sigma_i}(\Xi_{k+1})$, picking $E_D \in \Xi_k \cup \mathsf{skel}_{\Sigma_i}[\Xi_k]$ whenever possible. Let $Q = \{q \in \mathsf{skel}_{\Sigma_i}(\Sigma_{3-i}[\Xi_k]) : E_q \in \Xi_k\}$ and $Q' = \mathsf{skel}_{\Sigma_i}(\Sigma_{3-i}[\Xi_k]) \setminus Q$. By Lemma 3 (4) and (5), for $q \in Q'$, $E_q \in Q'$.

Consider $\sigma_i : P \to (P \cup \Xi_k \cup Q')$ defined by

$$\sigma_i(p) = \begin{cases} E_p & \text{if } p \in \mathsf{skel}_{\Sigma_i}(\Sigma_{3-i}[\Xi_k]) \\ p & \text{if } p \notin \mathsf{skel}_{\Sigma_i}(\Sigma_{3-i}[\Xi_k]) \end{cases}$$

By construction, $Q^{\sigma_i} \subseteq \Xi_k$ and $Q'' = (Q')^{\sigma_i} \subseteq Q'$. Further, we have that

$$(\mathsf{skel}_{\Sigma_i}(\Xi_{k+1}))^{\sigma_i} = \mathsf{skel}_{\Sigma_i}(\Sigma_1[\Xi_k]) \cup (\mathsf{skel}_{\Sigma_i}(\Sigma_{3-i}[\Xi_k]))^{\sigma_i} = \mathsf{skel}_{\Sigma_i}(\Sigma_i[\Xi_k]) \cup Q'', \text{ and}$$
$$(\mathsf{skel}_{\Sigma_i}(\Theta_1^k \cup \Theta_2^k))^{\sigma_i} = \mathsf{skel}_{\Sigma_i}(\Theta_i^k) \cup \{q \approx q : q \in Q''\}.$$

As $\{q \approx q : q \in Q''\}$ is contained in any theory of every equational logic, we have that $((\mathsf{skel}_{\Sigma_i}(\Theta_1^k \cup \Theta_2^k))^{\sigma_i})^{\vdash_i} = (\mathsf{skel}_{\Sigma_i}(\Theta_i^k))^{\vdash_i}$ and note that

$$(\mathsf{skel}_{\Sigma_i}(\Theta_i^k))^{\vdash_i} \cap \mathsf{skel}_{\Sigma_i}(\Omega_{k+1}) = \mathsf{skel}_{\Sigma_i}(\Theta_i^k) \cup \{q \approx q : q \in Q \cup Q'\}.$$

Since inverse images of theories by substitutions are theories, we have that

$$T_i = \sigma_i^{-1}((\mathsf{skel}_{\Sigma_i}(\Theta_i^k))^{\vdash_i}) \in \mathsf{Th}(\langle \Sigma, \vdash_i \rangle),$$

and from Proposition 4, we obtain that $\mathsf{unskel}_{\Sigma_i}(T_i) \in \mathsf{Th}(\langle \Sigma, \vdash_i^\Sigma \rangle)$.

By definition of σ_i, using Lemma 3 (3) and (5), we obtain that

$$\sigma_i^{-1}(\mathsf{skel}_{\Sigma_i}(\Theta_i^k) \cup \{q \approx q : q \in Q \cup Q'\}) = \mathsf{skel}_{\Sigma_i}(\Theta_1^k \cup \Theta_2^k).$$

We now have that $\Theta_1^k \cup \Theta_2^k \subseteq \Delta_i^{k+1} \subseteq \mathsf{unskel}_{\Sigma_i}(T_i)$, and since skel_{Σ_i} and $\mathsf{unskel}_{\Sigma_i}$ are bijections,

$$\begin{aligned}
\mathsf{unskel}_{\Sigma_i}(T_i) \cap \Omega_{k+1} &= \mathsf{unskel}_{\Sigma_i}((\mathsf{skel}_{\Sigma_i}(\Theta_i^k))^{\vdash_i} \cap \mathsf{skel}_{\Sigma_i}(\Omega_{k+1})) \\
&= \mathsf{unskel}_{\Sigma_i}(\sigma_i^{-1}(\mathsf{skel}_{\Sigma_i}(\Theta_i^k) \cup \{q \approx q : q \in Q \cup Q'\}) \\
&= \mathsf{unskel}_{\Sigma_i}(\mathsf{skel}_{\Sigma_i}(\Theta_1^k \cup \Theta_2^k)) \\
&= \Theta_1^k \cup \Theta_2^k
\end{aligned}$$

Hence, $\Delta_i^{k+1} \cap \Omega_{k+1} = (\Theta_1^k \cup \Theta_2^k)^{\vdash_i^{\Sigma}} \cap \Omega_{k+1} = \Theta_1^k \cup \Theta_2^k$ □

We now prove properties (1) and (2) by induction on $k \in \mathbb{N}_0$.

For the base case $k = 0$, we know that $\Delta_1^0 \cap \Omega_0 = \Delta_2^0 \cap \Omega_0$; thus, (1) holds. Now, we are in position to use Lemmas 3 and 4 with $k = 0$. By Lemma 4, we have that $\Delta_i^1 \cap \Omega_1 = \Theta_1^0 \cup \Theta_2^0$, and from Lemma 3 (3), we know that $(\Theta_1^0 \cup \Theta_2^0) \cap \Omega_0 = \Delta_i^0 \cap \Omega_0$. Since $\Omega_0 \subseteq \Omega_1$, we conclude that

$$\Delta_i^1 \cap \Omega_0 = (\Theta_1^0 \cup \Theta_2^0) \cap \Omega_0 = \Delta_i^0 \cap \Omega_0,$$

and thus (2) holds.

For the step, by induction hypothesis we have that $\Delta_1^k \cap \Omega_k = \Delta_2^k \cap \Omega_k \neq \Omega_k$ holds. Then again, as in the base case, Lemmas 4 and 3 (3) are available. By Lemma 4, we obtain that $\Delta_1^{k+1} \cap \Omega_{k+1} = \Theta_1^k \cup \Theta_2^k = \Delta_2^{k+1} \cap \Omega_{k+1}$ and thus (1) holds. As in the base case, we use the fact that $\Omega_k \subseteq \Omega_{k+1}$ and Lemma 3 (3) to conclude that

$$\Delta_i^{k+1} \cap \Omega_k = (\Theta_1^k \cup \Theta_2^k) \cap \Omega_k = \Delta_i^k \cap \Omega_k.$$

Thus, (2) holds for $i = 1, 2$. □

From Theorem 4 and the fact that $\mathsf{ctx}_\star(\Omega)$ can be calculated in polynomial time on the size of Ω, we conclude that combining equational logics with disjoint signatures thus preserves the upper bound complexity classes for the given logics **P**, **coNP**, **PSPACE**, **EXPTIME**, and **EXPSPACE**.

As far as we know, this exact result has not been stated and proven before, but it is very closely related to many similar and even more ambitious results in the literature. Indeed, in [16], a similar statement is obtained, but in the context of varieties of algebras whose reducts are boolean algebras. Other results focused on deciding the word problem (theoremhood) rather than the associated consequence relations. In [18], it is shown that the Turing degree of the word problem for the variety $\mathbb{V}(\mathsf{Eq}_1 \cup \mathsf{Eq}_2)$, i.e., deciding whether $\varnothing \vdash_{\mathsf{Eq}_1 \cup \mathsf{Eq}_2} A \approx B$, is the join of the Turing degrees for the word problems for $\mathbb{V}(\mathsf{Eq}_1)$ and $\mathbb{V}(\mathsf{Eq}_2)$. In Theorem 4, we assume more and obtain more. Still, our result implies Pigozzi's whenever we depart from from decidable $\vdash_{\mathsf{Eq}_1}$ and $\vdash_{\mathsf{Eq}_2}$. This is so, in particular, when one can reduce the problem of deciding $\vdash_{\mathsf{Eq}}$ to the word problem for $\mathbb{V}(\mathsf{Eq})$, for instance, when both varieties have a strong ternary deductive term [32]. Our result is also reminiscent of Nelson–Oppen-like results, showing preservation of decidability of combined first-order quantifier-free stably-infinite theories with equality over disjoint signatures [19]. Of course, assuming decidability of boolean combinations of equations is more demanding than assuming the decidability of the underlying equational logic. Note, still, that the extra expressivity raises compatibility issues related to the cardinality of the models. These observations also apply to interesting variations and extensions of Nelson and Oppen's seminal result, including some non-disjoint cases such as [18–26].

4. Concluding Remarks

In this paper, we proposed the first generally applicable criterion for the preservation of decidability when combining logics, and analyzed the complexity bounds thus obtained. It is clear from our development that in order to be applied as in Theorem 2 and Theorem 4, the notion of ctx-compatibility can be imposed only for finite Ω since we are considering deciding statements regarding finite sets of hypotheses. Further note that these theorems could be adapted to join any finite set of logics at once by imposing ctx-compatibility as a bunch, the advantage being that the number of logics being combined would enter as a multiplicative factor in the complexity bound obtained, thus improving the bound obtained by joining them iteratively. When joining logics with polynomial time or space, as in Section 3.5.1, this is not so relevant, as it would only affect the degree of the resulting polynomial, but in general it may really yield better upper bounds.

Further, we have shown that our criterion works by providing new proofs for previous results in the area, uniformly using the same abstract idea of contextual extensibility of theories. What is more, due to the generality and abstractness of our notion of extensibility, we have shown that the technique of contextual extensibility can be applied well beyond propositional-based logics, namely in the context of 2-deductive systems and in particular of equational logics. In order to best establish the relationship of our criterion and subsequent decidability preservation proofs, namely with the myriad of important known results for combined equational and first-order theories, it will of course be crucial to adopt other useful extensions of the plain Tarskian notion of logic, namely in order to cover, at least, Horn, clausal, and boolean combinations of atomic formulas, such as equations.

There are several other topics for further research. An obvious one is to pursue specialized decidability preservation results for propositional logics sharing a common base, sufficiently well-behaved but not necessarily classical, thus extending the result for fusions of modal logics covered in Section 2.5.2, for which [21] may be useful. The semantic characterizations of [12], using non-determinism and partiality, may play a crucial role in this setting. Another interesting question is whether there may be a criterion akin to ctx-extensibility that allows us to decide the preservation of decidability of the theoremhood relation of the logics, or of the corresponding satisfiability problem, which in the concrete case of disjoint signatures and by using the ideas in Lemmas 3 and 4, could help us in mimicking Piggozi's proof in [18]. Last but not least, we envisage studying the relationship between our notion of contextual extensibility and model-theoretic techniques involving forms of amalgamation, namely in the lines of [24,33–35].

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