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A Mathematical Model for Ovine Brucellosis during Dynamic Transportation of Sheep, and Its Applications in Jalaid Banner and Ulanhot City

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Abstract: Brucellosis a the serious infectious disease in Hinggan League. Research has demonstrated that a large amount of transportation is one of the main reasons for so many cases. However, the specific transmission mechanism of brucellosis is not clear. In this paper, we utilize a multi-patch model to study the effect of the transportation of sheep on the spread of brucellosis in Hinggan League. Theoretically, we prove the global stability of the disease-free equilibrium and the uniform persistence of the endemic equilibrium. In a practical application, we apply the model to investigate the spread of brucellosis in Ulanhot city and Jalaid Banner, which are geographically adjacent in Hinggan League. The strains carried by humans are B.melitensis bv.1 and B.melitensis bv.3. We use the two-patch model to fit reported brucellosis cases data of two places by Markov Chain Monte Carlo (MCMC) simulations. It is found that the global basic reproduction number R_0 is larger than 1, but the isolated basic reproduction numbers in Ulanhot city and Jalaid Banner are both less than 1. This indicates that the prevalence of brucellosis may be caused by the transportation of sheep. Sensitivity analysis of parameters on R_0 shows that it is the most effective means to control the transportation of sheep from Jalaid to Ulanhot on preventing brucellosis. Moreover, we also discover that improving vaccine efficiency is an effective method compared with strengthening the vaccination coverage rate and improving the detection rate of sheep with brucellosis. Our dynamic behavior analysis of the two-patch model can provide a reference for the dynamic behavior analysis of the n-patch model, and our results provide a guide for how to control brucellosis based on transportation.

Keywords: brucellosis; basic reproduction number; transportation; vaccine efficiency; sensitivity analysis

MSC: 37N25; 34D23; 37M05

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1. Introduction

Brucellosis, as a type of zoonosis disease [1] that is extremely prevalent in pastoral areas. When human beings are exposed to infected animals or contaminated environments [2,3], they may be infected by brucellosis. They usually have symptoms such as fever, hyper-hidrosis, fatigue, pain in bone joints and muscles, lymphadenopathy, and hepatosplenomegaly [4,5]. Animals are infected via contacting with infected animals or contaminated environment. They will present miscarriage, infertility, decreased production and lameness, and the fetus will die [6–8].

Dynamical models have become the important tool to characterize the space distributed laws of animals [9,10] and plants [11–13], and the spread laws of infectious diseases [14]. The amount of research that uses dynamical model to analyze the prevention and control assessment of the prevalence of brucellosis is increasing. Considering the bacteria in the environment, Zhang et al., based on SEIV model, showed that the external

input of dairy cows may be the main reason for the high fluctuation of the number of dairy cows with brucellosis in Zhejiang Province [15]. Hou et al. constructed the coupling dynamic model of humans and sheep, and found that disinfection of the environment and immunization of sheep are effective means to control the transmission of brucellosis in Inner Mongolia [16]. Sun et al. proposed the SEIVW model to study the global dynamics of the model, and concluded that reducing immigration and self-sufficiency of the flock are effective to controlling sheep brucellosis [17]. Zhang et al. utilized a two-patch SEIV model to reveal the dispersal of the susceptible population in each patch and found that concentrating the infected cattle to large breeding scale is effective to control the brucellosis [18]. Li et al. proposed a deterministic model to investigate the transmission dynamics of brucellosis in Hinggan League. They demonstrated that a combination of prohibiting mixed feeding between basic ewes and other sheep, vaccination, detection and elimination is useful in controlling human brucellosis in Hinggan League [19]. It has been demonstrated that the multi-patch model is the most appropriate mathematical model for studying the transmission dynamics between regions [18,20].

Brucellosis is an important public health problem in China, particularly in pastoral areas of northern China [21–23], although many developed countries control brucellosis well. It accounts for more than 40% of Chinese human brucellosis in Inner Mongolia [24], where the incidence rate of brucellosis cases is the highest in northern China. In Hinggan League, as one of the largest pastoral areas in Inner Mongolia, a total of 121,151 new human cases were reported from 2001 to 2010 [25]. Figure 1 shows the time series of human cases from 2010 to 2020 in Hinggan League. It could be said that the curve of Figure 1 decreases first and then increases. An important reason for the rise in the number of brucellosis cases is the growing sheep breeding and trafficking, which enhances the transportation of sheep among these places and increases the infected risk of brucellosis. According to the investigation [3], most of the cases occurred in slaughterhouses, fur and hair processing factories, sheep breeding, transportation and food factories processing milk and meat in rural areas. Jalaid and Ulanhot, located in Horqin prairie, are a pair of representative areas. The two places are geographically adjacent, and the strains carried by human are B.melitensis bv.1 and B.melitensis bv.3 [3]. It could be inferred that there is mutual transportation between the two places, but the mechanism of transportation of sheep between the two places is still unclear. Few researchers apply the multi-patch model of human-sheep coupling to actual circumstances to study the impact of transportation on the prevalence of brucellosis. In this paper, we proposed a deterministic two-patch model to study the impact of transportation on the spread of brucellosis between Jalaid and Ulanhot, and put forward some practical prevention and control measures.

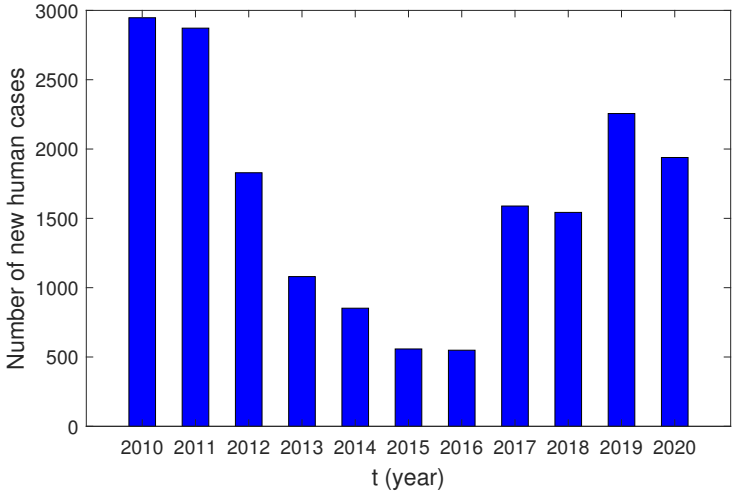


Figure 1. Number of human brucellosis cases in Hinggan League from January 2010 to January 2020 (data from NNDRS [26]).

The paper is organised in the following way. In Section 2, we present the dynamical model. In Sections 3 and 4, we analyze the dynamical behavior of the model. We fit the number of human brucellosis cases in Ulanhot and Jalaid from 2010 to 2020 and give the sensitivity analysis on R_0 in Section 5. In Section 6, we verify the theorem of the Section 3 through numerical simulation. Sections 7 and 8 gives a brief discussion of the main results, shortcomings and future work.

2. Dynamical Model

In order to explore the transportation relationship of sheep, we construct a human-sheep coupling transportation model based on the characteristics of brucellosis transmission. According to the transmission mechanism of brucellosis, the following assumptions are made.

(H_1) The susceptible sheep could be infected through touching the exposed and infected sheep, or brucella in the environment.

(H_2) Susceptible people could be infected by exposure to contaminated environments and exposed and infected sheep, but they will not be infected by people with brucellosis.

(H_3) Exposed sheep usually have no symptoms, and they will also be vaccinated. It is assumed that susceptible and exposed sheep have immunity within the validity period after vaccination, that is, they will not be infected with brucellosis.

(H_4) Since the migration of sheep is mainly directional transportation through trade, we consider the directed migration of sheep between patches, but do not consider the spatial dispersal caused by the free movement of individuals.

The follow chart of transmission dynamic of brucellosis from sheep to sheep and from sheep to human is in Figure 2.

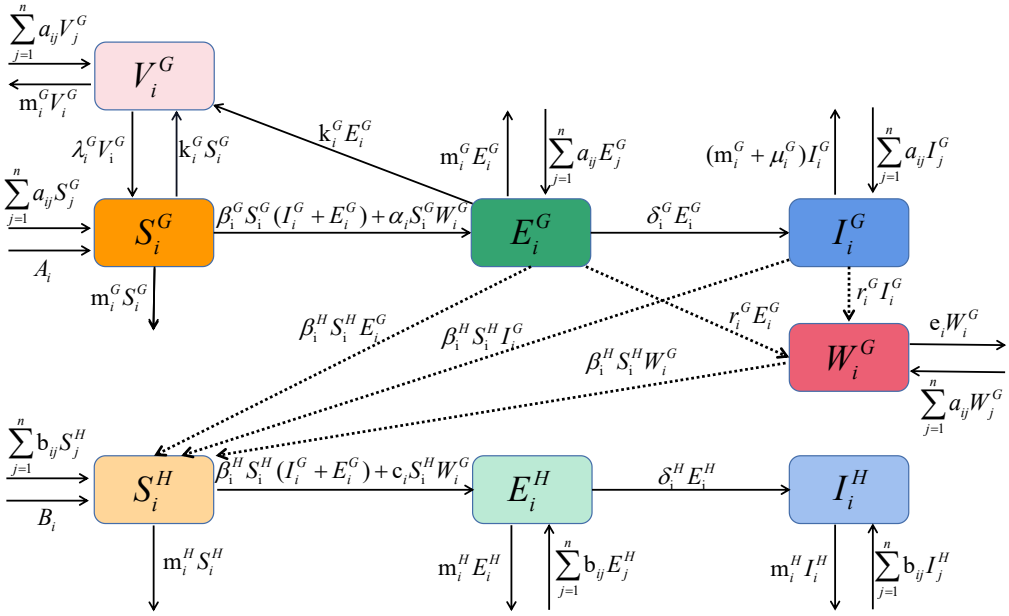


Figure 2. The transportation of brucella between sheep and from sheep with brucellosis to humans and from patch to patch due to the transportation of sheep in patch i.

In Figure 2, the superscript G and H represent sheep and human, the subscript i and j represent patch i and patch j, respectively. The number of susceptible, exposed, infected and vaccinated sheep are indicated by S_i^G, E_i^G, I_i^G, V_i^G. Meanwhile, the numbers of susceptible, exposed, and infected humans are represented by S_i^H, E_i^H, I_i^H. The brucella amount in the environment is W_i^G. The meaning of other parameters is shown in Table 1, in which parameters A_i, B_i, λ_i^G, β_i^G, ε, α_i, m_i^G, k_i^G, δ_i^G, μ_i,r_i^G,e_i, c_i, δ_i^H, $a_{ij}^K (K = S^G, E^G, I^G, V^G)$,

$b_{ij}^K (K = S^H, E^H, I^H)$ are nonnegative constants. Based on the above assumption, a dynamical model of brucellosis transmission between sheep and human is established as follows.

$$\begin{cases} \frac{dS_i^G}{dt} = A_i + \lambda_i^G V_i^G - \beta_i^G S_i^G (I_i^G + E_i^G) - \alpha_i S_i^G W_i^G - (m_i^G + k_i^G) S_i^G + \Sigma_{j=1}^n a_{ij} S_j^G, \\ \frac{dE_i^G}{dt} = \beta_i^G S_i^G (I_i^G + E_i^G) + \alpha_i S_i^G W_i^G - (m_i^G + \delta_i^G + k_i^G) E_i^G + \Sigma_{j=1}^n a_{ij} E_j^G, \\ \frac{dI_i^G}{dt} = \delta_i^G E_i^G - (\mu_i + m_i^G) I_i^G + \Sigma_{j=1}^n a_{ij} I_j^G, \\ \frac{dV_i^G}{dt} = k_i^G (S_i^G + E_i^G) - (\lambda_i^G + m_i^G) V_i^G + \Sigma_{j=1}^n a_{ij} V_j^G, \\ \frac{dW_i^G}{dt} = r_i^G (E_i^G + I_i^G) - e_i W_i^G + \Sigma_{j=1}^n a_{ij} W_j^G, \\ \frac{dS_i^H}{dt} = B_i - m_i^H S_i^H - \beta_i^H S_i^H (I_i^G + E_i^G) - c_i S_i^H W_i^G + \Sigma_{j=1}^n b_{ij} S_j^H, \\ \frac{dE_i^H}{dt} = \beta_i^H S_i^H (I_i^G + E_i^G) + c_i S_i^H W_i^G - m_i^H E_i^H - \delta_i^H E_i^H + \Sigma_{j=1}^n b_{ij} E_j^H, \\ \frac{dI_i^H}{dt} = \delta_i^H E_i^H - m_i^H I_i^H + \Sigma_{j=1}^n b_{ij} I_j^H. \end{cases} \tag{1}$$

Table 1. Definitions of variables and parameters.

Parameters	Description
A_i	the annual birth rate of sheep in patch i
B_i	the annual birth rate of humans in patch i
λ_i^G	the loss rate of vaccination immunity for sheep in patch i
β_i^G	the transmission coefficient of infectious sheep to susceptible sheep in patch i
β_i^H	the transmission coefficient of infectious sheep to susceptible humans in patch i
α_i	the transmission coefficient of polluted environment to susceptible sheep in patch i
m_i^G	natural death and slaughter elimination rate coefficient of sheep in patch i
k_i^G	vaccination rate of sheep
m_i^H	the natural death rate of people in patch i
δ_i^G	the rate of clinical outcome of exposed sheep in patch i
μ_i	disease-related culling rate of infectious sheep in patch i
r_i^G	the amount of brucella per unit time emitted by exposed and infected sheep
e_i	brucella decay rate in patch i
c_i	the transmission rate coefficient of brucella in environment-to-susceptible human in patch i
δ_i^H	the rate of clinical outcome of exposed human in patch i
a_{ij}	the migration rate of sheep from patch j to patch i ($i \neq j$)
b_{ij}	the migration rate of people from patch j to patch i ($i \neq j$)

All solutions of (1) satisfy $S_i^G(t) > 0$, $E_i^G(t) \geq 0$, $I_i^G(t) \geq 0$, $V_i^G(t) \geq 0$, $W_i^G(t) \geq 0$, $S_i^H(t) \geq 0$, $E_i^H(t) \geq 0$ and $I_i^H(t) \geq 0$, if initial condition satisfy $S_i^G(0) \geq 0$, $E_i^G(0) \geq 0$, $I_i^G(0) \geq 0$, $V_i^G(0) \geq 0$, $W_i^G(0) \geq 0$, $S_i^H(0) \geq 0$, $E_i^H(0) \geq 0$ and $I_i^H(0) \geq 0$ for $i = n$.

Proof. For the first equation of system (1), there is

$$\frac{dS_i^G}{S_i^G dt} = \frac{A_i + \lambda_i^G V_i^G - \beta_i^G S_i^G (I_i^G + E_i^G) - \alpha_i S_i^G W_i^G - (m_i^G + k_i^G) S_i^G + \Sigma_{j=1}^n a_{ij} S_j^G}{S_i^G}.$$

By calculating,

$$S_i^G(t) = S_i^G(0)e^{\int_0^t A_i+\lambda_i^G V_i^G(t)-\beta_i^G S_i^G(t)(I_i^G(t)+E_i^G(t))-\alpha_i S_i^G(t)W_i^G(t)-(m_i^G+k_i^G)S_i^G(t)+\Sigma_{j=1}^n a_{ij}S_j^G(t)dt} \geq 0.$$

Similarly, $E_i^G(t) \geq 0$, $I_i^G(t) \geq 0$, $V_i^G(t) \geq 0$, $W_i^G(t) \geq 0$, $S_i^H(t) \geq 0$, $E_i^H(t) \geq 0$ and $I_i^H(t) \geq 0$, when $S_i^G(0) \geq 0$, $E_i^G(0) \geq 0$, $I_i^G(0) \geq 0$, $V_i^G(0) \geq 0$, $W_i^G(0) \geq 0$, $S_i^H(0) \geq 0$, $E_i^H(0) \geq 0$ and $I_i^H(0) \geq 0$ for $i = n$. □

Ignoring births and deaths in transportation, we get the relationship related to the transportation rate, as follows,

$$\Sigma_{i=1}^n a_{ij} = 0, \ \ \Sigma_{i=1}^n b_{ij} = 0 \quad 1 \leq i \leq n.$$

System (1) is a system of n patches. In next section, we will discuss the case when $n = 2$. Brucellosis generally does not spread from person to person and the equations for sheep (first five equation of system (1)) are independent of those for humans (last three equation of system (1)). Thus, dynamical analysis of the first five equations of system (1) is presented in Section 3. In Section 4, we analyze the dynamic behavior of the last three equations of system (1). In order two simplify the presentation, we consider the following notation,

$$g_i = (S_i^G, E_i^G, I_i^G, V_i^G, W_i^G), \quad h_i = (S_i^H, E_i^H, I_i^H) \quad i = 1, 2.$$

3. Dynamic Analysis of First Five Equation of System (1) for n = 2

When transportation is not considered, the first five equations of system (1) could be transformed into system (2); when we consider transportation, system (1) is converted to system (3). In the following, we analyze the dynamical behaviours according to theory [27–33].

3.1. The Single Patch Model without Sheep Transportation

Ignoring the transportation of the sheep, $a_{ij} = 0$ $(i = 1, 2)$. Then system (1) reduces to the following model.

$$\begin{cases} \frac{dS_i^G}{dt} = A_i + \lambda_i^G V_i^G - \beta_i^G S_i^G (I_i^G + E_i^G) - \alpha_i S_i^G W_i^G - (m_i^G + k_i^G) S_i^G, \\ \frac{dE_i^G}{dt} = \beta_i^G S_i^G (I_i^G + E_i^G) + \alpha_i S_i^G W_i^G - (m_i^G + \delta_i^G + k_i^G) E_i^G, \\ \frac{dI_i^G}{dt} = \delta_i^G E_i^G - (\mu_i + m_i^G) I_i^G, \\ \frac{dV_i^G}{dt} = k_i^G (S_i^G + E_i^G) - (\lambda_i^G + m_i^G) V_i^G, \\ \frac{dW_i^G}{dt} = r_i^G (E_i^G + I_i^G) - e_i W_i^G. \end{cases} \tag{2}$$

$\limsup_{t\to\infty}(S_i^G + E_i^G + I_i^G + V_i^G) = \frac{A_i}{m_i^G}$, $\limsup_{t\to\infty} W_i^G = \frac{r_i^G A_i}{m_i^G e_i}$ $(i = 1, 2)$. So, the positive invariant set of system (2) is expressed as

$$D = \left\{ (g_1, g_2) \in \mathbb{R}_+^{10} \ | S_i^G + E_i^G + I_i^G + V_i^G \leq \frac{A_i}{m_i^G}, \ W_i^G \leq \frac{r_i^G A_i}{m_i^G e_i} \right\}.$$

By calculating, the disease free equilibrium are

$$P^* = (g_1^*, g_2^*) = (S_1^*, 0, 0, V_1^*, 0, S_2^*, 0, 0, V_2^*, 0),$$

where

$$S_i^* = \frac{(\lambda_i^G + m_i^G) A_i}{(m_i^G (k_i^G + \lambda_i + m_i^G)}, \ V_i^* = \frac{k_i^G A_i}{m_i^G (k_i^G + \lambda_i^G + m_i^G)} \ (i = 1, 2).$$

Then, using the next generation method [27], we could obtain the basic reproduction number

$$R_0^i = \frac{S_i^*(\alpha_i\delta_i^G r_i^G + \alpha_i m_i^G r_i^G + \alpha_i r_i^G \mu_i^G + \delta_i^G \beta_i^G e_i + \beta_i^G e_i m_i^G + \beta_i^G e_i u_i)}{e_i(\delta_i^G m_i^G + \delta_i^G \mu_i^G + k_i^G m_i^G + k_i^G \mu_i + {m_i^G}^2 + m_i^G \mu_i)} \ (i = 1, 2).$$

Since the exposed and infected people are not infectious, the basic reproduction number of the system (2) is the basic reproduction number of the system (1) without transmission of humans and sheep.

3.2. The Two Patch Model with the Transportation of Sheep between Two Patches

We consider the transmission of sheep between two patches, and system (1) could be rewritten as

$$\begin{cases} \frac{dS_i^G}{dt} = A_i + \lambda_i^G V_i^G - \beta_i^G S_i^G (I_i^G + E_i^G) - \alpha_i S_i^G W_i^G - (m_i^G + k_i^G) S_i^G - a_{21} S_1^G + a_{12} S_1^G, \\ \frac{dE_i^G}{dt} = \beta_i^G S_i^G (I_i^G + E_i^G) + \alpha_i S_i^G W_i^G - (m_i^G + \delta_i^G + k_i^G) E_i^G - a_{21} E_1^G + a_{12} E_1^G, \\ \frac{dI_i^G}{dt} = \delta_i^G E_i^G - (\mu_i + m_i^G) I_i^G - a_{21} I_1^G + a_{12} I_1^G, \\ \frac{dV_i^G}{dt} = k_i^G (S_i^G + E_i^G) - (\lambda_i^G + m_i^G) V_i^G - a_{21} V_1^G + a_{12} V_1^G, \\ \frac{dW_i^G}{dt} = r_i^G (E_i^G + I_i^G) - e_i W_i^G - a_{21} W_1^G + a_{12} W_1^G. \end{cases} \tag{3}$$

The disease-free equilibrium of system (3) is

$$P^0 = (g_1^0, g_2^0) = (S_1^0, 0, 0, V_1^0, 0, S_2^0, 0, 0, V_2^0, 0),$$

where

$$S_1^0 = \frac{a_{12}(A_2 + \lambda_2^G Z_2) + (A_1 + \lambda_1^G Z_1)(\lambda_2^G + c_6 + k_2^G)}{-a_{12}a_{21} + (\lambda_1^G + c_5 + k_1^G)(\lambda_2^G + c_6 + k_2^G)},$$

$$S_2^0 = \frac{a_{21}(A_1 + \lambda_1^G Z_1) + (A_2 + \lambda_2^G Z_2)(\lambda_1^G + c_5 + k_1^G)}{-a_{12}a_{21} + (\lambda_1^G + c_5 + k_1^G)(\lambda_2^G + c_6 + k_2^G)},$$

$$V_1^0 = \frac{a_{12}k_2^G(A_1a_{21} + A_2c_5) + k_1^G(A_2a_{12} + A_1c_6)(\lambda_2^G + c_6 + k_2^G)}{[-a_{12}a_{21} + (\lambda_1^G + c_5 + k_1^G)(\lambda_2^G + c_6 + k_2^G)](c_5c_6 - a_{12}a_{21})},$$

$$V_2^0 = \frac{a_{21}k_1^G(A_2a_{12} + A_1c_6) + k_2^G(A_1a_{21} + A_2c_5)(\lambda_1^G + c_5 + k_1^G)}{[-a_{12}a_{21} + (\lambda_1^G + c_5 + k_1^G)(\lambda_2^G + c_6 + k_2^G)](c_5c_6 - a_{12}a_{21})},$$

with

$$c_5 = m_1^G + a_{21}, \ \ c_6 = m_2^G + a_{12},$$

$$Z_1 = \frac{A_1c_6 + A_2a_{12}}{c_5c_6 - a_{12}a_{21}}, \ \ Z_2 = \frac{A_1a_{21} + A_2c_5}{c_5c_6 - a_{12}a_{21}}.$$

Assuming that $N_i = S_i^G + E_i^G + I_i^G + V_i^G$ $(i = 1, 2)$, we sum the equations in system (3) and find that,

$$\begin{cases} \frac{dN_1(t)}{dt} = A_1 - c_5 N_1(t) + a_{12} N_2(t) - \mu_1^G I_1^G(t), \\ \frac{dN_2(t)}{dt} = A_2 - c_6 N_2(t) + a_{21} N_1(t) - \mu_2^G I_2^G(t). \end{cases} \tag{4}$$

Here, N_i represents the total population of patch i. The dynamical behavior of the linearized system corresponding to system (4) near the equilibrium point is similar to that of system (4). Therefore, we consider an auxiliary linear system,

$$\begin{cases} \dfrac{dx_1(t)}{dt} = A_1 - c_5 x_1(t) + a_{12} x_2(t), \\ \dfrac{dx_2(t)}{dt} = A_2 - c_6 x_2(t) + a_{21} x_1(t). \end{cases} \tag{5}$$

It could be seen that system (5) has a unique equilibrium (Z_1, Z_2). The corresponding characteristic equation of system (5) at (Z_1, Z_2) is,

$$\lambda^2 + (c_5 + c_6)\lambda + c_5 c_6 - a_{12} a_{21} = 0. \tag{6}$$

As $c_5 = m_1^G + a_{21} > 0$, $c_6 = m_2^G + a_{12} > 0$, so $c_5 + c_6 > 0$ and $c_5 c_6 - a_{12} a_{21} = (m_1^G + a_{21})(m_2^G + a_{12}) - a_{12} a_{21} = m_1^G m_2^G + a_{12} m_1^G + a_{21} m_2^G > 0$. It follows that all roots of system (6) have negative real parts, and (Z_1, Z_2) is locally asymptotically stable. It follows that,

$$\lim_{t \to \infty} (x_1(t), x_2(t)) = (Z_1, Z_2).$$

As system (5) is a cooperative and irreducible system, and the local stability of linear system is global stability. It is known by comparison principle [28] that $N_i(t) \leq Z_i + \epsilon$, $(i = 1, 2)$ for all $\epsilon \geq 0$ and all large enough t through the comparison principle. It implies that, as $t \to \infty$, all solutions of system (3) with nonnegative conditions ultimately turn into positively invariant set

$$\Omega = \Bigg\{ (g_1, g_2) \in \mathbb{R}_+^{10} | \ (S_i^G + E_i^G + I_i^G + V_i^G) \leq Z_i + \epsilon, \\ W_1^G \leq \frac{Z_1 r_1^G (a_{12} + e_2) + a_{12} Z_2 r_2^G}{(a_{12} + e_2)(a_{21} + e_1) - a_{21} a_{12}}, \ W_2^G \leq \frac{Z_2 r_2^G (a_{21} + e_1) + a_{21} Z_1 r_1^G}{(a_{12} + e_2)(a_{21} + e_1) - a_{21} a_{12}} \Bigg\}. \tag{7}$$

According to the next generation method [27], we define

$$\mathcal{F} = \begin{pmatrix} \beta_1^G S_1^0 (I_1^0 + E_1^0) + \alpha_1 S_1^0 W_1^0 \\ \beta_2^0 S_2^0 (I_2^0 + E_2^0) + \alpha_2 S_2^0 W_2^0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \mathcal{V} = \begin{pmatrix} (m_1^G + \delta_1^G + k_1^G + a_{21}) E_1^0 - a_{12} E_2^0 \\ (m_2^G + \delta_2^G + k_2^G + a_{12}) E_2^0 - a_{21} E_1^0 \\ -\delta_1^G E_1^0 + (\mu_1 + m_1^G + a_{21}) I_1^0 - a_{12} I_2^0 \\ -\delta_2^G E_2^0 + (\mu_2 + m_2^G + a_{12}) I_2^0 - a_{21} I_1^0 \\ -r_1^G (E_1^0 + I_1^0) + (e_1 + a_{21}) W_1^0 - a_{12} W_2^0 \\ -r_2^G (E_2^0 + I_2^0) + (e_2 + a_{12}) W_2^0 - a_{21} W_1^0 \end{pmatrix},$$

$$F = \begin{pmatrix} \beta_1^G S_1^0 & 0 & \beta_1^G S_1^0 & 0 & \alpha_1 S_1^0 & 0 \\ 0 & \beta_2^G S_2^0 & 0 & \beta_2^G S_2^0 & 0 & \alpha_2 S_2^0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} m_1^G + \delta_1^G + k_1^G + a_{21} & -a12 & 0 & 0 & 0 & 0 \\ -a_{21} & m_2^G + \delta_2^G + k_2^G + a_{12} & 0 & 0 & 0 & 0 \\ -\delta_1^G & 0 & \mu_1 + m_1^G + a_{21} & -a12 & 0 & 0 \\ 0 & -\delta_2^G & -a21 & \mu_2 + m_2^G + a_{12} & 0 & 0 \\ -r_1^G & 0 & -r_1^G & 0 & e_1 + a_{21} & -a12 \\ 0 & -r_2^G & 0 & -r_2^G & -a21 & e_2 + a_{12} \end{pmatrix}.$$

We also define,

$$J = F - V,$$

$$y_1 = m_1^G + \delta_1^G + k_1^G + a_{21}, \ y_2 = m_2^G + \delta_2^G + k_2^G + a_{12}, \ y_3 = m_1^G + \mu_1 + a_{21},$$

$$y_4 = m_2^G + \mu_2 + a_{12},\ \ y_5 = e_1 + a_{21},\ \ y_6 = e_2 + a_{12},$$

$$b_{11} = \frac{y_2\delta_1^G y_4 + a_{12}\delta_2^G a_{21}}{(a_{12}a_{21} - y_1y_2)(a_{12}a_{21} - y_3y_4)},\ \ b_{12} = \frac{y_2\delta_1^G a_{12} + a_{12}\delta_1^G y_3}{(a_{12}a_{21} - y_1y_2)(a_{12}a_{21} - y_3y_4)},$$

$$b_{13} = \frac{y_2 r_1^G y_6 + a_{12} r_1^G a_{21}}{(a_{12}a_{21} - y_1y_2)(a_{12}a_{21} - y_5y_6)},\ \ b_{14} = \frac{y_2 r_1^G a_{12} + a_{12} r_2^G y_5}{(a_{12}a_{21} - y_1y_2)(a_{12}a_{21} - y_5y_6)},$$

$$b_{21} = \frac{a_{21}\delta_1^G y_4 + y_1\delta_2^G a_{21}}{(a_{12}a_{21} - y_1y_2)(a_{12}a_{21} - y_3y_4)},\ \ b_{22} = \frac{a_{21}\delta_1^G a_{12} + y_1\delta_1^G y_3}{(a_{12}a_{21} - y_1y_2)(a_{12}a_{21} - y_3y_4)},$$

$$b_{23} = \frac{a_{21} r_1^G y_6 + y_1 r_2^G a_{21}}{(a_{12}a_{21} - y_1y_2)(a_{12}a_{21} - y_5y_6)},\ \ b_{14} = \frac{a_{21} r_1^G a_{12} + y_1 r_1^G y_5}{(a_{12}a_{21} - y_1y_2)(a_{12}a_{21} - y_5y_6)}.$$

Through a series of simplification, the basic reproduction number is

$$R_0 = \rho(FV^{-1}) = \frac{A + D + \sqrt{(A+D)^2 - 4(AD - BC)}}{2}. \tag{8}$$

with

$$A = -\frac{\beta_1 S_1^0 y_2}{(a_{12}a_{21} - y_1y_2)} + \beta_1 S_1^0 b_{11} + \alpha_1 S_1^0(-\frac{b_{11}r_1y_6 + b_{12}r_2^G a_{21}}{a_{12}a_{21} - y_5y_6} + b_{13}),$$

$$B = -\frac{\beta_1 S_1^0 a_{12}}{(a_{12}a_{21} - y_1y_2)} + \beta_1 S_1^0 b_{12} + \alpha_1 S_1^0(-\frac{b_{11}r_1^G a_{12} + b_{12}r_2^G y_5}{a_{12}a_{21} - y_5y_6} + b_{14}),$$

$$C = -\frac{\beta_2 S_2^0 a_{21}}{(a_{12}a_{21} - y_1y_2)} + \beta_2 S_2^0 b_{21} + \alpha_2 S_2^0(-\frac{b_{21}r_1^G y_6 + b_{22}r_2^G a_{21}}{a_{12}a_{21} - y_5y_6} + b_{23}),$$

$$D = -\frac{\beta_2 S_2^0 y_1}{(a_{12}a_{21} - y_1y_2)} + \beta_2 S_2^0 b_{22} + \alpha_2 S_2^0(-\frac{b_{21}r_1^G a_{12} + b_{22}r_2^G y_5}{a_{12}a_{21} - y_5y_6} + b_{24}).$$

Since the exposed and infected people are not infectious, the basic reproduction number of the system (3) is the basic reproduction number of the system (1).

Lemma 1. *Let us consider the system* (3), R_0 *defined on* (8) *and the notation* $s(J) := \max\{Re\ Z:$ *Z is an eigenvalue of* $J\}$. *Then, the following assertions are satisfied*

(i) $R_0 < 1$ *is equivalent to* $S(J) < 0$,
(ii) $R_0 > 1$ *is equivalent to* $S(J) > 0$.

Proof. We can prove the Lemma by follow [27]. □

By Theorem 2 in [27], it is easy to conclude that the disease-free equilibrium P^0 is locally asymptotically stable when $R_0 < 1$ and P^0 is unstable when $R_0 > 1$. Then we further investigate the global dynamical behavior of P^0.

Theorem 1. *When* $R_0 < 1$, *the disease-free equilibrium* P^0 *is global asymptotically stable.*

Proof. In this section, we only need to prove the global attraction of P^0. From system (3), for sufficient large t, we have

$$\begin{cases} \dfrac{dV_1^G}{dt} = k_1^G(S_1^G + E_1^G) - (\lambda_1^G + m_1^G + a_{21})V_1^G + a_{12}V_2^G \\ \qquad = k_1^G(N_1^G - I_1^G - V_1^G) - (\lambda_1^G + m_1^G + a_{21})V_1^G + a_{12}V_2^G \\ \qquad \le k_1^G(Z_1 + \epsilon) - c_1V_1^G + a_{12}V_2^G, \\ \dfrac{dV_2^G}{dt} = k_2^G(S_2^G + E_2^G) - (\lambda_2^G + m_2^G + a_{12})V_2^G + a_{21}V_1^G \\ \qquad = k_2^G(N_2^G - I_2^G - V_2^G) - (\lambda_2^G + m_2^G + a_{12})V_2^G + a_{21}V_1^G \\ \qquad \le k_2^G(Z_2 + \epsilon) - c_2V_2^G + a_{21}V_1^G, \end{cases} \tag{9}$$

with $c_1 = \lambda_1^G + m_1^G + a_{21} + k_1^G$, $c_2 = \lambda_2^G + m_2^G + a_{12} + k_2^G$. Next considering an auxiliary linear system

$$\begin{cases} \dfrac{du_1(t)}{dt} = k_1^G(Z_1 + \epsilon) - c_1u_1 + a_{12}u_2, \\ \dfrac{du_2(t)}{dt} = k_2^G(Z_2 + \epsilon) - c_2u_2 + a_{21}u_1. \end{cases} \tag{10}$$

The endemic equilibrium of system (10) is,

$$\begin{aligned} (\tilde{Z}_1, \tilde{Z}_2) &= \frac{1}{c_1c_2 - a_{12}a_{21}}(c_2k_1^G(Z_1 + \epsilon) + a_{12}k_2^G(Z_2 + \epsilon), c_1k_2^G(Z_2 + \epsilon) + a_{21}k_1^G(Z_1 + \epsilon)) \\ &= (V_1^0 + \epsilon_{11}, V_2^0 + \epsilon_{12}), \end{aligned}$$

where $\epsilon_{11} = \frac{\epsilon(c_2k_1^G + a_{12}k_2^G)}{c_1c_2 - a_{12}a_{21}}$, $\epsilon_{12} = \frac{\epsilon(c_1k_2^G + a_{21}k_1^G)}{c_1c_2 - a_{12}a_{21}}$. Since system (10) is similar to (5), $(\tilde{Z}_1, \tilde{Z}_2)$ is globally asymptotically stable. Using comparison theorem, we know that for a small enough $\epsilon > 0$, there is a small enough $\epsilon_1 > 0$ such that $V_1(t) \le V_1^0 + \epsilon_1, V_2(t) \le V_2^0 + \epsilon_1$, when $t > t_1 (t_1 > 0)$. Consequently, we have

$$\begin{cases} \dfrac{d(S_1^G(t) + E_1^G(t))}{dt} = A_1 + \lambda_1^GV_1^G - c_3S_1^G - (c_3 + \delta_1)E_1^G + a_{12}S_2^G + a_{12}E_2^G \\ \qquad \le A_1 + \lambda_1^G(V_1^0 + \epsilon_{11}) - c_3(S_1^G + E_1^G) + a_{12}(S_2^G + E_2^G), \\ \dfrac{d(S_2^G(t) + E_2^G(t))}{dt} = A_2 + \lambda_2^GV_2^G - c_4S_2^G - (c_4 + \delta_2)E_2^G + a_{21}S_2^G + a_{21}E_1^G \\ \qquad \le A_2 + \lambda_2^G(V_2^0 + \epsilon_{12}) - c_4(S_2^G + E_2^G) + a_{21}(S_1^G + E_1^G), \end{cases} \tag{11}$$

with $c_3 = m_1^G + a_{21} + k_1^G$, $c_4 = m_2^G + a_{12} + k_2^G$. Now considering the following system,

$$\begin{cases} \dfrac{du_3}{dt} = A_1 + \lambda_1^G(V_1^0 + \epsilon_{11}) - c_3u_3 + a_{12}u_4, \\ \dfrac{du_4}{dt} = A_2 + \lambda_2^G(V_2^0 + \epsilon_{12}) - c_4u_4 + a_{21}u_3, \end{cases} \tag{12}$$

the endemic equilibrium of the system (12) is

$$\begin{aligned} (\overline{Z_1}, \overline{Z_2}) = \frac{1}{c_3c_4 - a_{12}a_{21}}&(c_4(A_1 + \lambda_1^G(V_1^0 + \epsilon_1)) + a_{12}(A_2 + \lambda_2^G(V_2^0 \\ &+ \epsilon_1)), c_3(A_2 + \lambda_2^G(V_2^0 + \epsilon_1)) + a_{21}(A_1 + \lambda_1^G(V_1^0 + \epsilon_1))) = (S_1^0 + \epsilon_{13}, S_2^0 + \epsilon_{14}), \end{aligned}$$

which is globally asymptotically stable, here $\epsilon_{13} = \frac{\epsilon_1(\lambda_1^Gc_4 + \lambda_2^Ga_{12})}{c_3c_4 - a_{12}a_{21}}$, $\epsilon_{14} = \frac{\epsilon_1(\lambda_2^Gc_3 + \lambda_1^Ga_{21})}{c_1c_2 - a_{12}a_{21}}$. Therefore, there exist $\epsilon_2 > 0$, $t_2 > t_1 > 0$, $S_1^G(t) \le S_1^G(t) + E_1^G(t) \le S_1^0 + \epsilon_2$ and $S_2(t) \le S_2^G(t) + E_2^G(t) \le S_2^0 + \epsilon_2$, when $t > t_2$. So, we have

$$\begin{cases} \frac{dE_1^G}{dt} \leq \beta_1^G(S_1^0+\epsilon_2)I_1^G + \alpha_1(S_1^0+\epsilon_2)W_1^G - (m_1^G+\delta_1^G+k_1^G+a_{21}-\beta_1^G(S_1^0+\epsilon_2))E_1^G \\ \quad + a_{12}E_2^G, \\ \frac{dI_1^G}{dt} = \delta_1^G E_1^G - (\mu_1+m_1^G+a_{21})I_1^G + a_{12}I_2^G, \\ \frac{dW_1^G}{dt} = r_1^G(E_1^G+I_1^G) - (e_1+a_{21})W_1^G + a_{12}W_2^G, \\ \frac{dE_2^G}{dt} \leq \beta_2^G(S_2^0+\epsilon_2)I_2^G + \alpha_2(S_2^0+\epsilon_2)W_2^G - (m_2^G+\delta_2^G+k_1^G+a_{12}-\beta_2^G(S_2^0+\epsilon_2))E_2^G \\ \quad + a_{21}E_1^G, \\ \frac{dI_2^G}{dt} = \delta_2^G E_2^G - (\mu_2+m_2^G+a_{12})I_2^G + a_{21}I_2^G, \\ \frac{dW_2^G}{dt} = r_2^G(E_2^G+I_2^G) - (e_2+a_{12})W_2^G + a_{21}W_2^G. \end{cases} \tag{13}$$

Similarly consider the following auxiliary system,

$$\begin{cases} \frac{dx_1}{dt} = \beta_1^G(S_1^0+\epsilon_2)x_2 + \alpha_1(S_1^0+\epsilon_2)x_3 - (m_1^G+\delta_1^G+k_1^G+a_{21}-\beta_1^G(S_1^0+\epsilon_2))x_1 + a_{12}x_4, \\ \frac{dx_2}{dt} = \delta_1^G x_1 - (\mu_1+m_1^G+a_{21})x_1^G + a_{12}x_2, \\ \frac{dx_3}{dt} = r_1^G x_1 + r_1^G x_2 - (e_1+a_{21})x_3 + a_{12}x_6, \\ \frac{dx_4}{dt} = \beta_2^G(S_2^0+\epsilon_2)x_5 + \alpha_2(S_2^0+\epsilon_2)x_6 - (m_2^G+\delta_2^G+k_1^G+a_{12}-\beta_2^G(S_2^0+\epsilon_2))x_4 + a_{21}x_1, \\ \frac{dx_5}{dt} = \delta_2^G x_4 - (\mu_2+m_2^G+a_{12})x_5 + a_{21}x_2, \\ \frac{dx_6}{dt} = r_2^G x_4 + r_2^G x_6 - (e_2+a_{12})x_6 + a_{21}x_3. \end{cases} \tag{14}$$

We conclude that (14) has an equilibrium $P^1 = (0,0,0,0,0,0)$. Then we rewrite the system (14) as the following form,

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \\ \frac{dx_4}{dt} \\ \frac{dx_5}{dt} \\ \frac{dx_6}{dt} \end{pmatrix} = J_{\epsilon_2} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix},$$

where

$$J^* = \begin{pmatrix} \beta_1^G & 0 & \beta_1^G & 0 & \alpha_1 & 0 \\ 0 & \beta_2^G & 0 & \beta_2^G & 0 & \alpha_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$J_{\epsilon_2} = J + \epsilon_2 J^*.$$

The root of the characteristic equation corresponding to system (14) at the equilibrium point P^1 is the eigenvalue of J_{ϵ_2}. From the previous analysis, we know that when $R_0 < 1$, $s(J) < 0$. Hence, there exists a small enough number $\epsilon_2 > 0$ such that $s(J_{\epsilon_2}) < 0$, and all the eigenvalues of matrix J_{ϵ_2} have negative real parts. Therefore, the solution of system (14) satisfies $x_i(t) \to 0$ $(i = 1, 2, 3, 4, 5, 6)$ as $t \to \infty$. According to the comparison principle, we have $E_i^G(t) \to 0$, $I_i^G(t) \to 0$, $W_i^G(t) \to 0$ $(i = 1, 2)$. Furthermore, we consider the limiting system of system (3),

$$\begin{cases} \frac{dS_1^G}{dt} = A_1 + \lambda_1^G V_1^G - c_3 S_1^G + a_{12} S_2^G, \\ \frac{dV_1^G}{dt} = k_1^G S_1^G - c_1 V_1^G + a_{12} V_2^G, \\ \frac{dS_2^G}{dt} = A_2 + \lambda_2^G V_2^G - c_4 S_2^G + a_{21} S_1^G, \\ \frac{dV_2^G}{dt} = k_2^G S_2^G - c_2 c_2^G + a_{21} V_1^G. \end{cases} \tag{15}$$

It has an equilibrium $P^2 = (S_1^0, V_1^0, S_2^0, V_2^0)$. And the characteristic equation at P^2 is

$$f(x) = x^4 + b_1 x^3 + b_2 x^2 + b_3 x + b_4 = 0,$$

where

$$\begin{aligned} b_1 &= c_1 + c_2 + c_5 + c_6 > 0, \\ b_2 &= c_1 c_5 + c_2 c_6 + (c_1 + c_5)(c_2 + c_6) - 2a_{12} a_{21}, \\ b_3 &= (c_1 + c_5) c_2 c_6 + (c_2 + c_6) c_1 c_5 - a_{12} a_{21} (c_1 + c_2 + c_5 + c_6), \\ b_4 &= (c_1 c_2 - a_{12} a_{21})(c_3 c_4 - a_{12} a_{21}) > 0. \end{aligned}$$

Based on Routh Hurwitz criterion, all roots of $f(x)$ have negative real parts. Therefore, P^2 is locally asymptotically stable. Considering (15) is a linearized system, P^2 is global asymptotically stable. By the theory of asymptotic autonomous systems [29], P^0 is globally attractive when $R_0 < 1$. Then, we obtain that disease free equilibrium P^0 is globally asymptotically stable if $R_0 < 1$. This completes the proof of Theorem 1. □

We define

$$\begin{aligned} X &= \{(S_i^G, E_i^G, I_i^G, V_i^G, W_i^G) : S_i^G \geq 0, E_i^G \geq 0, I_i^G \geq 0, V_i^G \geq 0, W_i^G \geq 0, i = 1, 2\} \\ X_0 &= \{(S_i^G, E_i^G, I_i^G, V_i^G, W_i^G) \in X : E_i^G > 0, I_i^G > 0, W_i^G > 0, i = 1, 2\} \\ \partial X_0 &= X \backslash X_0. \end{aligned}$$

Then, we have following theorem.

Theorem 2. *If $R_0 > 1$, there admits a positive constant $\epsilon_3 > 0$ such that when $(S_1^G(0), E_1^G(0), I_1^G(0), V_1^G(0), W_1^G(0), S_2^G(0), E_2^G(0), I_2^G(0), V_2^G(0), W_2^G(0)) \in X_0$ and $\|E_1^G(0), I_1^G(0), W_1^G(0), E_2^G(0), I_2^G(0), W_2^G(0)\| < \epsilon_3$, we have*

$$\limsup_{t \to \infty} \max \|E_i^G(t), I_i^G(t), W_i^G(t)\| \geq \epsilon_3. \quad (i = 1, 2)$$

Proof. According to the previous proof, system (15) has a positive equilibrium $P^2 = (S_1^0, V_1^0, S_2^0, V_2^0)$ which is globally asymptotically stable. $R_0 > 1$ is equivalent to $s(J) > 0$. Therefore, we could choose a small enough $\varepsilon > 0$ such that $s(J_\varepsilon) > 0$ $(J_\varepsilon = J - \varepsilon J^*)$. Consider the following perturbed system,

$$\begin{cases} \dfrac{d\overline{S_1^G}}{dt} = A_1 + \lambda_1^G \overline{V_1^G} - (c_3 + \beta_1\delta)\overline{S_1^G} + a_{12}\overline{S_2^G}, \\ \dfrac{d\overline{V_1^G}}{dt} = k_1^G \overline{S_1^G} - c_1 \overline{V_1^G} + a_{12}\overline{V_2^G}, \\ \dfrac{d\overline{S_2^G}}{dt} = A_2 + \lambda_2^G \overline{V_2^G} - (c_4 + \beta_2\delta)\overline{S_2^G} + a_{21}\overline{S_1^G}, \\ \dfrac{d\overline{V_2^G}}{dt} = k_2^G \overline{S_2^G} - c_2 \overline{V_2^G} + a_{21}\overline{V_1^G}. \end{cases} \tag{16}$$

System (16) has an unique positive equilibrium $P^3 = (\overline{S_1^*}, \overline{V_1^*}, \overline{S_2^*}, \overline{V_2^*})$, where

$$\begin{aligned} \overline{S_1^*} =& \frac{c_6[A_1(c_1c_2 - a_{12}a_{21}) + \lambda_1^G(A_2a_{12} + A_1c_2)] + a_{12}[A_2(c_1c_2 - a_{12}a_{21}) + \lambda_2^G(A_1a_{21} + A_2c_1)]}{\Delta} \\ &+ \frac{A_1\beta_2^G(\lambda_2^G c_1 + c_1c_2 - a_{12}a_{21})\delta}{\Delta}, \\ \overline{V_1^*} =& \frac{a_{12}k_2^G(A_1a_{21} + A_2c_1) + c_6k_1^G(A_2a_{12} + A_1c_2) + [k_1^G\beta_2^G(\lambda_2^G + c_2)A_1 + k_2^G A_2 a_{12}\beta_1^G]\delta}{\Delta}, \\ \overline{S_2^*} =& \frac{c_5[A_2(c_1c_2 - a_{12}a_{21}) + \lambda_2^G(A_1a_{21} + A_2c_1)] + a_{21}[A_1(c_1c_2 - a_{12}a_{21}) + \lambda_1^G(A_2a_{12} + A_1c_2)]}{\Delta} \\ &+ \frac{A_2\beta_1^G[(\lambda_1^G + c_1)(\lambda_2^G + c_2) - a_{12}a_{21}]\delta}{\Delta}, \\ \overline{V_2^*} =& \frac{a_{21}k_1^G(A_2a_{12} + A_2c_2) + c_6k_2^G(A_1a_{21} + A_2c_1) + [k_2^G\beta_1^G(\lambda_1^G + c_1)A_2 + k_1^G A_1 a_{21}\beta_2^G]\delta}{\Delta}, \end{aligned}$$

here $\Delta = (c_5c_6 - a_{12}a_{21})(c_1c_2 - a_{12}a_{21}) + \{[(\lambda_1^G + c_1)(\lambda_2^G + c_2) - a_{12}a_{21}][\beta_1(\beta_2\delta + c_4) + \beta_2c_3] - k_2\lambda_2^G\beta_1(\lambda_1^G + c_1) - \lambda_1^G k_1^G\beta_2(\lambda_2^G + c_2)\}\delta$. Since system (15) has a globally asymptotically stable equilibrium P^2, we could choose a small enough $\delta > 0$ such that P^3 is globally asymptotically stable. Moreover we find $\lim\limits_{\delta\to 0}(\overline{S_1^*}, \overline{V_1^*}, \overline{S_2^*}, \overline{V_2^*}) = (S_1^0, V_1^0, S_2^0, V_2^0)$. So, there exists $\delta > 0$ small enough such that $\overline{S_1^*}(t) \geq S_1^0 - \varepsilon$, $\overline{S_2^*}(t) \geq S_2^0 - \varepsilon$. Nextly we prove theorem 2 by means of contradiction. Supposed that $T > 0$ and $E_i^G(t) < \delta$, $I_i^G(t) < \delta$, $W_i^G(t) < \delta$, $i = 1, 2$. When $t \geq T$, we have

$$\begin{cases} \dfrac{dS_1^G}{dt} \geq A_1 + \lambda_1^G V_1^G - (c_3 + \beta_1\delta)S_1^G + a_{12}S_2^G, \\ \dfrac{dV_1^G}{dt} = k_1^G S_1^G - c_1 V_1^G + a_{12}V_2^G, \\ \dfrac{dS_2^G}{dt} \geq A_2 + \lambda_2^G V_2^G - (c_4 + \beta_2\delta)S_2^G + a_{21}S_1^G, \\ \dfrac{dV_2^G}{dt} = k_2^G S_2^G - c_2 V_2^G + a_{21}V_1^G. \end{cases} \tag{17}$$

System (16) has a globally asymptotically stable positive equilibrium and $\lim\limits_{\delta\to 0}(\overline{S_1^*}, \overline{V_1^*}, \overline{S_2^*}, \overline{V_2^*})$ $= (S_1^0, V_1^0, S_2^0, V_2^0)$, $\overline{S_1^*}(t) \geq S_1^0 - \varepsilon_1$, $\overline{S_2^*}(t) \geq S_2^0 - \varepsilon_1$. When $T_1 > T > 0$, there is $S_i^G(t) > S_i^0 - \varepsilon_1 (\, i = 1, 2;\ t > T_1)$. Therefore, for any $t > T_1$, we have

$$\begin{cases} \frac{dE_1^G}{dt} \geq \beta_1^G(S_1^0-\varepsilon_1)(E_1^G+I_1^G)+\alpha_1(S_1^0-\varepsilon_1)W_1^G-(m_1^G+\delta_1^G+k_1^G+a_{21})E_1^G+a_{12}E_2^G, \\ \frac{dI_1^G}{dt} = \delta_1^G E_1^G-(\mu_1+m_1^G+a_{21})I_1^G+a_{12}I_2^G, \\ \frac{dW_1^G}{dt} = r_1^G(E_1^G+I_1^G)-(e_1+a_{21})W_1^G+a_{12}W_2^G, \\ \frac{dE_2^G}{dt} \geq \beta_2^G(S_2^0-\varepsilon_1)(E_2^G+I_2^G)+\alpha_2(S_2^0-\varepsilon_1)W_2^G-(m_2^G+\delta_2^G+k_2^G+a_{12})E_2^G+a_{12}E_2^G, \\ \frac{dI_2^G}{dt} = \delta_2^G E_2^G-(\mu_2+m_2^G+a_{12})I_2^G+a_{21}I_2^G, \\ \frac{dW_2^G}{dt} = r_2^G(E_2^G+I_2^G)-(e_2+a_{12})W_2^G+a_{21}W_2^G. \end{cases} \tag{18}$$

Considering the following auxiliary system,

$$\begin{cases} \frac{d\overline{E_1^G}}{dt} = \beta_1^G(S_1^0-\varepsilon_1)(\overline{E_1^G}+\overline{I_1^G})+\alpha_1(S_1^0-\varepsilon_1)\overline{W_1^G}-(m_1^G+\delta_1^G+k_1^G+a_{21})\overline{E_1^G}+a_{12}\overline{E_2^G}, \\ \frac{d\overline{I_1^G}}{dt} = \delta_1^G\overline{E_1^G}-(\mu_1+m_1^G+a_{21})\overline{I_1^G}+a_{12}\overline{I_2^G}, \\ \frac{d\overline{W_1^G}}{dt} = r_1^G(\overline{E_1^G}+\overline{I_1^G})-(e_1+a_{21})\overline{W_1^G}+a_{12}\overline{W_2^G}, \\ \frac{d\overline{E_2^G}}{dt} = \beta_2^G(S_2^0-\varepsilon_1)(\overline{E_2^G}+\overline{I_2^G})+\alpha_2(S_2^0-\varepsilon_1)\overline{W_2^G}-(m_2^G+\delta_2^G+k_2^G+a_{12})\overline{E_2^G}+a_{12}\overline{E_2^G}, \\ \frac{d\overline{I_2^G}}{dt} = \delta_2^G\overline{E_2^G}-(\mu_2+m_2^G+a_{12})\overline{I_2^G}+a_{21}\overline{I_2^G}, \\ \frac{d\overline{W_2^G}}{dt} = r_2^G(\overline{E_2^G}+\overline{I_2^G})-(e_2+a_{12})W_2^G+a_{21}\overline{W_2^G}, \end{cases} \tag{19}$$

and $R_0 > 1$ is equivalent to $s(J) > 0$, it has

$$(\overline{E_1^G(t)}, \overline{I_1^G(t)}, \overline{W_1^G(t)}, \overline{E_2^G(t)}, \overline{I_2^G(t)}, \overline{W_2^G(t)}) \to (\infty,\infty,\infty,\infty,\infty,\infty) \quad (t\to\infty).$$

Using the principle of comparison [28], there is

$$(E_1^G(t), I_1^G(t), W_1^G(t), E_2^G(t), I_2^G(t), W_2^G(t)) \to (\infty,\infty,\infty,\infty,\infty,\infty) \quad (t\to\infty),$$

which is contradictory with our assumptions, it follows that

$$\limsup_{t\to\infty} \max \|(E_i^G(t), I_i^G(t), W_i^G(t))\| \geq \epsilon_3 \ \ (i=1,2). \quad \square$$

Theorem 3. *When $R_0 > 1$, system (3) admits at least one positive equilibrium, and there is a positive constant ϵ_3 such that every solution of system (3) with $(S_i^G(0), E_i^G(0), I_i^G(0), V_i^G(0), W_i^G(0)) \in X_0$ $(i = 1, 2)$ satisfies*

$$\min\{\lim_{t\to\infty} inf\ E_i^G(t),\ \lim_{t\to\infty} inf\ I_i^G(t),\ \lim_{t\to\infty} inf\ W_i^G(t)\} \geq \epsilon_3 \ \ (i=1,2).$$

Proof. First of all, we prove that system (3) is uniformly persistent with respect to $(X_0, \partial X_0)$. Both X and X_0 are positively invariant and ∂X_0 is relatively closed in X. System (3) is point dissipative because of (7). According to [30] and for convenience of proof, we set

$$M_\partial = \{(S_i^G(0), E_i^G(0), I_i^G(0), V_i^G(0), W_i^G(0)) : (S_i^G(t), E_i^G(t), I_i^G(t), V_i^G(t), W_i^G(t)) \in \partial X_0, \ \forall t \geq 0 \ \ i = 1, 2\}.$$

Then we show

$$M_\partial = \{(S_i^G(t), 0, 0, V_i^G(t), 0) : S_i^G(t) \geq 0, V_i^G(t) \geq 0, \ \ i = 1, 2\}. \tag{20}$$

As $\{(S_i^G(t), 0, 0, V_i^G(t), 0) : S_i^G(t) \geq 0, V_i^G(t) \geq 0\} \subseteq M_\partial$, we mainly prove

$$M_\partial \subseteq \{(S_i^G(t), 0, 0, V_i^G(t), 0) : S_i^G(t) \geq 0, V_i^G(t) \geq 0, \ \ i = 1, 2\}.$$

Assume that $(S_i^G(0), E_i^G(0), I_i^G(0), V_i^G(0), W_i^G(0)) \in M_\partial$ $(i = 1, 2)$. It is sufficient to show that $E_i^G(t) = 0, I_i^G(t) = 0$ and $W_i^G(t) = 0$ $(\forall t \geq 0, \ i = 1, 2)$. Reductio ad absurdum, there is a $t_0 > 0$ such that $(E_1^G(t_0), I_1^G(t_0), W_1^G(t_0))^T > 0$ or $(E_2^G(t_0), I_2^G(t_0), W_2^G(t_0))^T > 0$. In order not to lose generality, assume that $E_1^G(t_0) > 0, \ I_1^G(t_0) = 0, \ W_1^G(t_0) = 0, \ E_2^G(t_0) = 0, \ I_2^G(t_0) = 0, \ W_2^G(t_0) = 0$. Then the following inequality holds, $\frac{dI_1^G(t_0)}{dt} = \delta_1 E_1^G > 0$ and $\frac{dW_1^G(t_0)}{dt} = r_1 E_1^G > 0$. These mean that there is a small enough $\eta_1 > 0$ such that $E_1^G(t) > 0, I_1(t) > 0$ and $W_1^G(t) > 0$ for $t_0 < t < t_0 + \eta_1$. For $\frac{dE_2^G(t_0)}{dt} \geq a_{21} E_1^G(t_0) > 0$, there is a $\eta_2 > 0$ such that $E_2^G(t) > 0$ for $t_0 < t < t_0 + \eta_2$. Just as the above proof method, there is a $\eta_3 > 0$ such that $I_2^G(t) > 0 \ \ W_2^G(t) > 0$ for $t_0 < t < t_0 + \eta_3$. It means that $(S_i^G(t), E_i^G(t), I_i^G(t), V_i^G(t), W_i^G(t)) \notin \partial X_0$ $(i = 1, 2)$ for $t_0 < t < t_0 + \eta_4, \ \ \eta_4 = \min(\eta_1, \eta_2, \eta_3)$. This contradicts to the hypothesis, so (20) holds. Considering that the disease-free equilibrium P^0 of system (3) is globally asymptotically stable, there is only one equilibrium point P^0 in set M_∂. According to Theorem 2, P^0 is an isolated invariant set in X and $W^s(P^0) \bigcap X_0 = \varnothing$. The disease-free equilibrium P^0 which has been calculated is globally asymptotically stable. Based on the above argument P^0 is the unique fixed point and acyclic in ∂X_0. By Theorem 4.6 [31], it follows that system (3) is uniformly persistent with respect to $(X, \partial X_0)$.

Theorem 2.4 of [32] implies that system (3) has one equilibrium $P^* = (S_1^*, E_1^*, I_1^*, V_1^*, W_1^*, S_2^*, E_2^*, I_2^*, V_2^*, W_2^*) \in X_0$, which implies that $E_1^* > 0, I_1^* > 0, W_1^* > 0, E_2^* > 0, I_2^* > 0, W_2^* > 0$. Nextly, we prove $S_1^* > 0, V_1^* > 0, S_2^* > 0, V_2^* > 0$ by contradiction. Assume that $S_1^* = 0, V_1^* = 0, S_2^* = 0, V_2^* = 0$, we have $E_1^* = 0, I_1^* = 0, W_1^* = 0, E_2^* = 0, I_2^* = 0, W_2^* = 0$, because of $A_i + \lambda_i^G V_i^G - \beta_i^G S_i^G (I_i^G + E_i^G) - \alpha_i S_i^G W_i^G - (m_i^G + k_i^G) S_i^G + \Sigma_{j=1}^n a_{ij} S_j^G = 0$ with $A_i > 0$, and $k_i^G (S_i^G + E_i^G) - (\lambda_i^G + m_i^G) V_i^G + \Sigma_{j=1}^n a_{ij} V_j^G = 0$. It is contradict to Theorem 2. Therefore $(S_1^*, E_1^*, I_1^*, V_1^*, W_1^*, S_2^*, E_2^*, I_2^*, V_2^*, W_2^*)$ is a positive equilibrium of system (3). □

4. Dynamic Analysis of Last Three Equations of System (1) for n = 2

When transportation is not considered, the last three equations of system (1) could be transformed into system (21); when considering transportation, system (1) is converted to system (22).

4.1. The Single Patch Model without Transmission of the Humans

Ignoring the transmission of the humans, $b_{ij} = 0$ $(i = 1, 2)$, then system (1) reduces to the following model.

$$\begin{cases} \frac{dS_i^H}{dt} = B_i - m_i^H S_i^H - \beta_i^H S_i^H (I_i^G + E_i^G) - c_i S_i^H W_i^G, \\ \frac{dE_i^H}{dt} = \beta_i^H S_i^H (I_i^G + E_i^G) + c_i S_i^H W_i^G - m_i^H E_i^H - \delta_i^H E_i^H, \\ \frac{dI_i^H}{dt} = \delta_i^H E_i^H - m_i^H I_i^H. \end{cases} \tag{21}$$

$\limsup_{t\to\infty}(S_i^H + E_i^H + I_i^H) = \frac{A_i}{m_i^H}$ $(i = 1, 2)$. So, the positive invariant set of system (21) is expressed as

$$D = \left\{ (h_1, h_2) \in \mathbb{R}_+^{10} \ | S_i^H + E_i^H + I_i^H \leq \frac{B_i}{m_i^H} \right\}.$$

By calculating, the disease free equilibrium is

$$P_1^* = (h_1^*, h_2^*) = (\frac{B_1}{m_1^H}, 0, 0, \frac{B_2}{m_2^H}, 0, 0),$$

4.2. The Two Patch Model with the Transmission of the Humans between Two Patches

We consider the transmission of the humans between two patches, and system (1) could be rewritten as

$$\begin{cases} \frac{dS_i^H}{dt} = B_i - m_i^H S_i^H - \beta_i^H S_i^H (I_i^G + E_i^G) - c_i S_i^H W_i^G + \Sigma_{j=1}^n b_{ij} S_j^H, \\ \frac{dE_i^H}{dt} = \beta_i^H S_i^H (I_i^G + E_i^G) + c_i S_i^H W_i^G - m_i^H E_i^H - \delta_i^H E_i^H + \Sigma_{j=1}^n b_{ij} E_j^H, \\ \frac{dI_i^H}{dt} = \delta_i^H E_i^H - m_i^H I_i^H + \Sigma_{j=1}^n b_{ij} I_j^H. \end{cases} \tag{22}$$

By calculating, the disease free equilibrium of system (22) is

$$P_2^* = (h_1^{**}, h_2^{**}) = (S_1^{**}, 0, 0, S_2^{**}, 0, 0).$$

Here, $S_1^{**} = \frac{(m_2^H + b_{12})B_1 + B_2 b_{12}}{(m_1^H + b_{21})(m_2^H + b_{12}) - b_{12}b_{21}}$, $S_2^{**} = \frac{(m_1^H + b_{21})B_2 + B_1 b_{21}}{(m_1^H + b_{21})(m_2^H + b_{12}) - b_{12}b_{21}}$.
According to Theorem 1, when $t \to \infty$, $I_i^G(t) \to 0$ and $E_i^G(t) \to 0$. Similar to the previous proof method, when $t \to \infty$, $I_i^H(t) \to 0$ and $E_i^H(t) \to 0$. The limiting system of system (22) is

$$\begin{cases} \frac{dS_i^H}{dt} = B_1 - m_1^H S_1^H - b_{21} S_1^H + b_{12} S_2^H, \\ \frac{dS_i^H}{dt} = B_2 - m_2^H S_2^H - b_{12} S_2^H + b_{21} S_1^H. \end{cases} \tag{23}$$

Its equilibrium (S_1^{**}, S_2^{**}) is locally asymptotically stable based on Routh Hurwitz criterion. As system (23) is a linear system, (S_1^{**}, S_2^{**}) is globally asymptotically stable. According to theory of asymptotic autonomous system [31], the disease free equilibrium P_2^* of system (22) is global asymptotically stable when $R_0 < 1$.

The endemic equilibrium point of system (22) is

$$P_3^* = (h_1^{***}, h_2^{***}) = (S_1^{***}, E_1^{***}, I_1^{***}, S_2^{***}, E_1^{***}, I_1^{***}).$$

Here, $S_1^{***} = \frac{A_2 B_1 + B_2 b_{12}}{A_1 A_2 - b_{12} b_{21}} > 0$, $S_2^{***} = \frac{A_1 B_2 + B_1 b_{21}}{A_1 A_2 - b_{12} b_{21}} > 0$, $E_1^{***} = \frac{A_4 b_{12} + A_3 A_6}{A_5 A_6 - b_{12} b_{21}} > 0$, $E_2^{***} = \frac{A_3 b_{21} + A_4 A_5}{A_5 A_6 - b_{12} b_{21}} > 0$, $I_1^{***} = \frac{A_8 E_1^{***} \delta_1 + E_2 b_{12} \delta_2}{A_7 A_8 - b_{12} * b_{21}} > 0$, $I_2^{***} = \frac{A_7 E_2^{***} \delta_2 + E_1 b_{21} \delta_1}{A_7 A_8 - b_{12} * b_{21}} > 0$, with $A_1 = m_1^H + \beta_1^H (I_1^* + E_1^*) + c_1 W_1^* + b_{21}$, $A_2 = m_2^H + \beta_1^H (I_2^* + E_2^*) + c_2 W_2^* + b_{12}$, $A_3 = \beta_1^H S_1^{***} (I_1^* + E_1^*) + c_1 S_1^{***} W_1^*$, $A_4 = \beta_2^H S_2^{***} (I_2^* + E_2^*) + c_2 S_2^{***} W_2^*$, $A_5 = m_1^H + \delta_1^H + b_{21}$, $A_6 = m_2^H + \delta_2^H + b_{12}$, $A_7 = m_1^H + b_{21}$, $A_8 = m_2^H + b_{12}$.

5. Case Study of Brucellosis in Ulanhot and Jalaid

Hinggan League, located in Horqin grassland in Inner Mongolia, is an important pastoral area in China. In recent years, the living standard of people has improved significantly, which leads to the huge demand for mutton and the rise of mutton price. This makes more and more ares in Hinggan League, such as Jalaid banner (patch 1) and Ulanhot (patch 2), start to breed sheep in large quantities. Both Ulanhot and Jalaid are in Horqin prairie; they are geographically adjacent. The strains carried by human are B.melitensis bv.1 and B.melitensis bv.3 in these places. Therefore, the transportation of sheep between them has had a significant effect on brucellosis transmission, especially in the past 10 years (see Figure 1).

5.1. Parameter Estimation

In this section, we mainly estimate the parameters by fitting the model (3) with the reported data of human with brucellosis in Ulanhot and Jalaid. Firstly, all parameters are with units of year, and the values of parameters are listed in Table 2. We explain the parameter values as follows.

[A] According to the Inner Mongolia Bureau of Statistics 2010–2020 [34] and Hinggan League Bureau of Statistics 2015–2020 [35], the average birth rate of human in Jalaid banner and Ulanhot are $B_1^H = 3,840$, $B_2^H = 214$, and the average mortality are $m_1^H = 7.23‰$ and $m_2^H = 6‰$ in these places. According to [19], the survival time of sick sheep with positive serum results is generally one month. Almost all sick sheep related to the disease die within one year, so we set the mortality rate $\mu_1 = \mu_2 = 1$.

[B] Here we use the data from 2015 to 2019 in [35] to calculate the average mortality and average birth rate of sheep, since the number of births of lambs, slaughters of sheep, deaths of sheep, sheep eaten by sheep breeders, and the transportation of humans and sheep from 2010 to 2014 are not recorded. In 2015, 348,584 lambs were born in Jalaid. Thus, the average birth rate of sheep in Jalaid and Ulanhot from 2015 to 2019 are $A_1 = (348,584 + 484,702 + 247,380 + 879,302)/4 = 489,992$ and $A_2 = 135,776$. The rate of natural deaths, self consumption and slaughter of local sheep in Jalaid and Ulanhot are $m_1^G = 0.3385$ and $m_2^G = 0.4873$.

[C] According to [36], the effective protection period of the vaccine is three years, so we set $\lambda_1^G = \lambda_2^G = 1/3 = 0.3$. According to [37], the effective vaccinated rate is 0.316, we set $k_1^G = k_2^G = 0.316$. According to [16], we set $\delta_1^G = \delta_2^G = 1$, $\beta_1^G = 1 \times 10^{-7}$, $\beta_2^G = 3.5 \times 10^{-6}$, $\alpha_1 = \alpha_2 = 6 \times 10^{-8}$, $c_1 = 1.7453 \times 10^{-12}$, and $c_2 = 5.0825 \times 10^{-11}$. The average survival time of brucella in the environment is 3.3 months [16], so $e_1^G = e_2^G = 3.6$. We set the averaged Brucella shedding rate of incubation sheep $r_1^G = r_2^G = 15$ based on [16]. Since the exposed period of human with brucellosis is generally two weeks, people usually have no symptoms during the exposed period. Therefore, most people could not get timely treatment to enter the infected period, we set $\delta_1^H = \delta_2^H = 1$. Next, we calculate the transportation coefficient of sheep and people. For sheep, using the data in Hinggan League of Statistics Yearbook from 2015 to 2018 [35], we divide the number of sheep sold in Jalaid in this year by the total number of sheep in this year in Jalaid. The values for four years are averaged to estimate the transportation coefficient of sheep transferred from Jalaid to Ulanhot as $a_{21} = 40.92\%$ and the transportation coefficient of sheep from Ulanhot to Jalaid $a_{12} = 50.71\%$. For humans, we take the transportation from Ulanhot to Jalaid banner as an example. We take the outgoing population of Ulanhot as the molecule, and the outgoing population of five counties except Jalaid as the denominator to calculate a result. Then we multiply this result by the immigration population of Jalaid. That is, the number of people from Ulanhot to Jalaid banner. Finally, we divide the number of people from Ulanhot to Jalaid banner by the total population of Ulanhot to get b_{12} in current year. Then we average the values from 2015 to 2018 to get the mean value $b_{12} = 27.89\%$ and $b_{21} = 5.36\%$.

Table 2. Definitions and values of variables and parameters.

Parameter	Mean Value	Unit	Source
A_1	489,992	year^{-1}	[B]
A_2	135,776	year^{-1}	[B]
B_1	3840	year^{-1}	[A]
B_2	214	year^{-1}	[A]
λ_1^G	1/3	year^{-1}	[C]
λ_2^G	1/3	year^{-1}	[C]
β_1^G	1×10^{-7}	year^{-1}	[C]
β_2^G	3.5×10^{-6}	year^{-1}	[C]
β_1^H	3.1647×10^{-7}	year^{-1}	MCMC
β_2^H	1.2219×10^{-6}	year^{-1}	MCMC
α_1	6×10^{-8}	year^{-1}	[C]
α_2	6×10^{-8}	year^{-1}	[C]
m_1^G	0.3385	year^{-1}	[B]
m_2^G	0.4873	year^{-1}	[B]
k_1^G	0.316	year^{-1}	[C]
k_2^G	0.316	year^{-1}	[C]
m_1^H	7.23‰	year^{-1}	[A]
m_2^H	6‰	year^{-1}	[A]
δ_1^G	1	year^{-1}	[C]
δ_2^G	1	year^{-1}	[C]
μ_1	1	year^{-1}	[A]
μ_2	1	year^{-1}	[A]
r_1^G	15	year^{-1}	[C]
r_2^G	15	year^{-1}	[C]
e_1	3.6	year^{-1}	[C]
e_2	3.6	year^{-1}	[C]
c_1	1.7453×10^{-12}	year^{-1}	[C]
c_2	5.0825×10^{-11}	year^{-1}	[C]
δ_1^H	1	year^{-1}	[C]
δ_2^H	1	year^{-1}	[C]
$a_{12}^K(K = S^G, E^G, I^G, V^G, W^G)$	50.71%	year^{-1}	[C]
$a_{21}^K(K = S^G, E^G, I^G, V^G, W^G)$	40.92%	year^{-1}	[C]
$b_{12}^K(K = S^H, E^H, I^H)$	27.89%	year^{-1}	[C]
$b_{21}^K(K = S^H, E^H, I^H)$	5.36%	year^{-1}	[C]

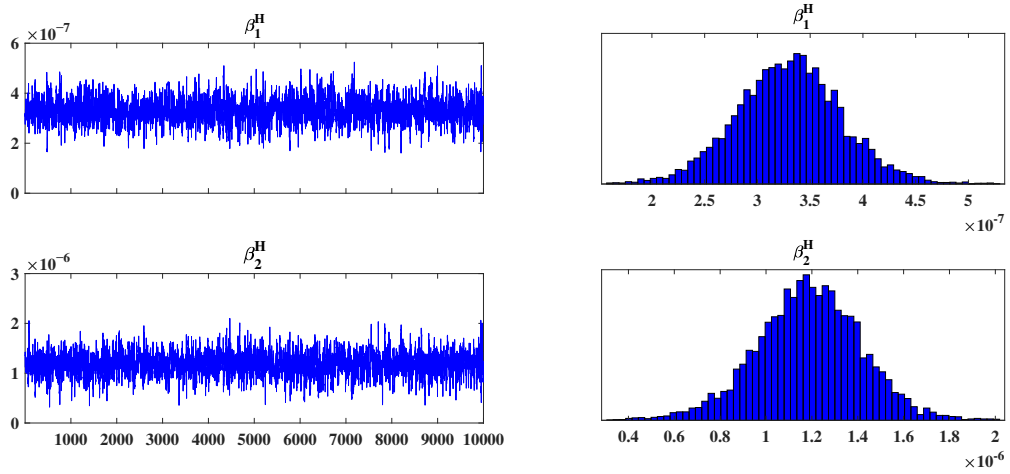
In this paragraph, we set the initial conditions. We regard the population as residents in rural areas, since residents in agricultural and pastoral areas are more likely to contact infected sheep. For humans, the total number of newly reported cases in Jalaid in 2010 is 343 [34]. People with brucellosis are usually cured after two months, so $I_1^H(0) = 369$. Considering the average exposed period of humans with brucellosis is about two weeks, we set $E_1^H(0) = 343 \div 2 = 172$. At the beginning of 2010, the population in rural areas is about 318,063 [34], so we assume $S_1^H(0) = 318{,}063 - 369 - 172 = 317{,}522$. For sheep, there are 633,500 sheep in the end of the 2009 [34]. According to the national brucellosis

control plan (2016–2020) [38], the individual positive rate of sheep in key areas of brucellosis was 3.3%. As the population positive rate was 34% in 2015, we estimate $I_1^G(0) = 3.3\% \times 34\% \times 633{,}500 = 7107$, $V_1^G(0) = 633{,}500 \times 31.6\% \times \frac{11}{12} = 200{,}186$. $S_1^G(0) = 633{,}500 - 7107 - 200{,}186 = 426{,}207$ with $E_1^G(0) = 0$, $W_1(0) = 1$. Similarly, we set $S_2^H(0) = 81{,}401$, $I_2^H(0) = 273$, $E_2^H(0) = 122$, $E_2^G(0) = 0$, $I_2^G(0) = 1996$, $V_2^G(0) = 56{,}216$, $W_2(0) = 0.01$. Since the data is the cumulative number of new cases, we use $X_i(t)$ to correspond to the solution of the equation, which is

$$\frac{dX_1(t)}{dt} = \delta_1^H E_1^H + b_{12} I_2^H, \quad \frac{dX_2(t)}{dt} = \delta_1^H E_2^H + b_{21} I_1^H.$$

According to the initial condition above, we set $X_1(0) = 369$, $X_2(0) = 273$.

Based on parameters and initial conditions giving above, we use the Latin Hypercube Sampling and Markov Chain Monte Carlo(MCMC) simulations (the algorithm similar to research in [39–42]) to estimate β_1^H, β_2^H. We use 10,000 times simulation, and the parameter values of β_1^H and β_2^H with MCMC chain are in Figure 3. The mean value, the standard deviation, MCMC error and Geweke of β_1^H and β_2^H are in Table 3. It could be seen from Figure 3 that the Markov-chains of parameters β_1^H and β_2^H are converged. The fitting results, the 95% percent interval, and the median of these simulation outputs are seen in Figure 4. Then we were able to estimate the basic reproduction number $R_0 = 1.0134$, $R_0^1 = 0.325$, $R_0^2 = 0.6915$. It could be seen that the basic reproduction number $R_0 > 1$ and the isolated basic reproduction number $R_0^1 < 1$, $R_0^2 < 1$ according to dynamical analysis. When there is no transportation, the disease die out in two ares. In contrast, when we consider the transportation, the disease is persistent. This indicates that the transportation of sheep between Ulanhot and Jalaid is one of the reasons for disease persistence. Next, we will further analyze the impact of transportation on the prevalence of brucellosis.



(**a**) (**b**)

Figure 3. (**a**) Simulation results for parameter β_1^H and β_2^H of Markov chain with 10,000 sample realizations. (**b**) The histogram of parameter β_1^H and β_2^H.

Table 3. Parameter estimation for β_1^H and β_2^H with the method of MCMC.

Parameter	Mean Value	Standard	MC Error	Geweke
β_1^H	3.3433×10^{-7}	4.8114×10^{-8}	1.1761×10^{-9}	0.99557
β_2^H	1.1681×10^{-6}	2.2907×10^{-7}	5.6932×10^{-9}	0.99545

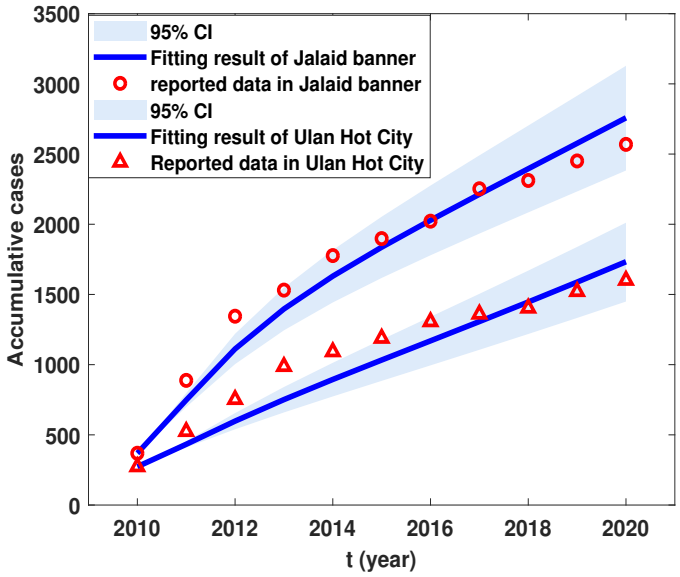


Figure 4. Fitting results of cumulative new cases from 2010 to 2020 in Jalaid banner and Ulanhot City. The circles and triangles are the number of cumulative newly reported human cases, and the curve represents the solution of the system.

5.2. Influence of Transportation Restriction on Brucellosis Transmission Dynamic

This paragraph analyzes the influence of transportation coefficient on the basic reproduction number R_0. Firstly, when the transportation rates of sheep in both patches are the same ($a_{12} = a_{21} = a$) and the other parameters are the same as Figure 3, we found that with the increase of a, the basic reproduction number R_0 increased (see Figure 5a). This means that, with the increase of transportation of sheep, the disease will breakout. Secondly, when the transportation from Jalaid banner to Ulanhot is zero, the basic reproduction number $R_0 < 1$ no matter how much a is (see Figure 5b). This reveals that the transportation of sheep from Ulanhot to Jalaid could not make the disease epidemic between the two places. We also find that the basic reproduction number R_0 first increases and then decreases with the increase of a from Figure 5b. This is because the number of sick sheep in Ulanhot is much smaller than that in Jalaid banner. When there is only transportation from Ulanhot to Jalaid banner, the rate of exposed and infectious sheep in Jalaid banner could be reduced, and then the basic reproduction number R_0 could be reduced. However, when the basic reproduction number R_0 decreases to a certain extent, the rate of exposed and infectious sheep in Jalaid banner and Ulanhot tends to be the same. At this time, further transportation will slightly increase the basic reproduction number R_0. Finally, when the transportation from Ulanhot to Jalaid banner is zero, the basic reproduction number R_0 increases rapidly with the increase of a (see Figure 5c), and the increase amplitude is higher than Figure 5a. This shows that the transportation of sheep from Jalaid banner to Ulanhot is a critical factor for the prevalence of brucellosis in the two places.

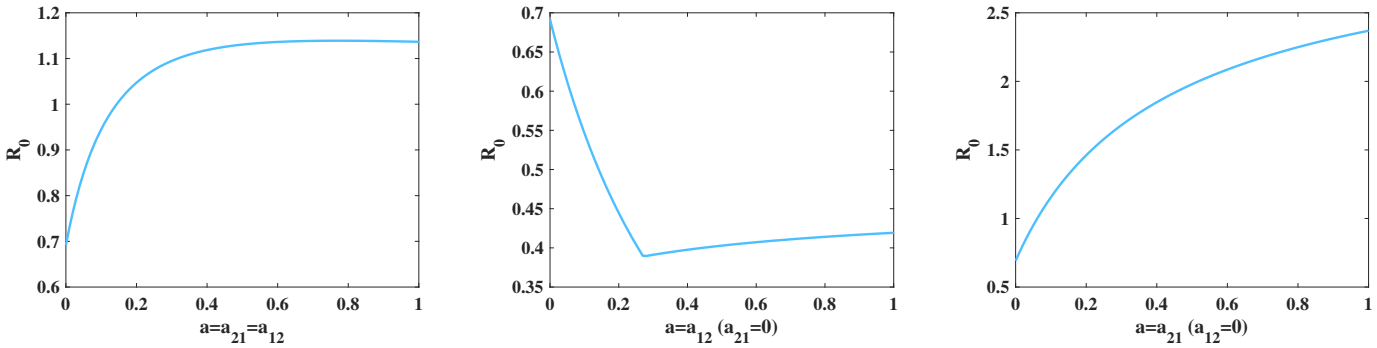


Figure 5. (**a**) R_0 versus a (The transportation rate between the two places is the same). (**b**) R_0 in terms of a (There is only transportation from Ulanhot to Jalaid banner). (**c**) R_0 in terms of a (There is only transportation from Jalaid banner to Ulanhot).

In this paragraph, we study the influence of the transportation coefficient on the cumulative number of new infections. From a practical point of view, people with brucellosis could move everywhere. However, we often slaughter sheep with brucellosis. This leads to the fact that few sheep with brucellosis could be transported between patches. Considering inhibiting the transportation of infected sheep from Ulanhot to Jalaid in each patch, we set $a_{12} = 0$ for the third and the 11th equations of system (1). Figure 6a shows that when this transportation ban policy appears, the cumulative number of human with brucellosis will increase in the next years. Then we set $a_{21} = 0$ for the third and the 11th equations of (1) when the transportation of infected sheep from Jalaid to Ulanhot is prohibited in each patch. For Figure 6b, when this transportation ban policy is executed, the cumulative number of humans with brucellosis will decrease in the next years. Thirdly, we set $a_{12} = a_{21} = 0$ for the third and the 11th equations of (1) when there is no transportation of infected sheep between the two places. We could see from Figure 6c that when this transportation ban policy appears, the cumulative number of human with brucellosis will increase. We consider a more ideal situation, which is that the exposed and infected sheep and the bacteria in the environment will not be transported. We set $a_{12} = 0$ for the second, the third, fifth, 10th, 11th and 13th equations of (1). We could see from Figure 6d that when this transportation ban policy appears, the cumulative number of humans with brucellosis in Ulanhot and Jalaid will increase. Similarly, we set $a_{21} = 0$ for the second, third, fifth, 10th, 11th and 13th equations of (1). Figure 6e shows that when this transportation ban policy appears, the cumulative number of humans with brucellosis in Ulanhot and in Jalaid banner will decrease. Finally, we set $a_{21} = a_{12} = 0$ for the second, third, fifth, 10th, 11th and 13th equations of (1). According to Figure 6f, we find that, when this transportation ban policy appears, the cumulative number of human with brucellosis in Ulanhot and Jalaid banner will increase.

To sum up, border control does not always have a positive influence on the epidemic of brucellosis. Brucellosis cases will be well controlled if we suitably control unidirectional transmission of sheep (especially exposed and infected sheep) and brucella in the environment from Jalaid to Ulanhot, and if appropriate release in the unidirectional transmission of sheep (especially exposed and infected sheep) and brucella in the environment from Ulanhot to Jalaid is observed. This means that expanding the breeding scale of sheep in these places is also effective in preventing the spread of brucellosis.

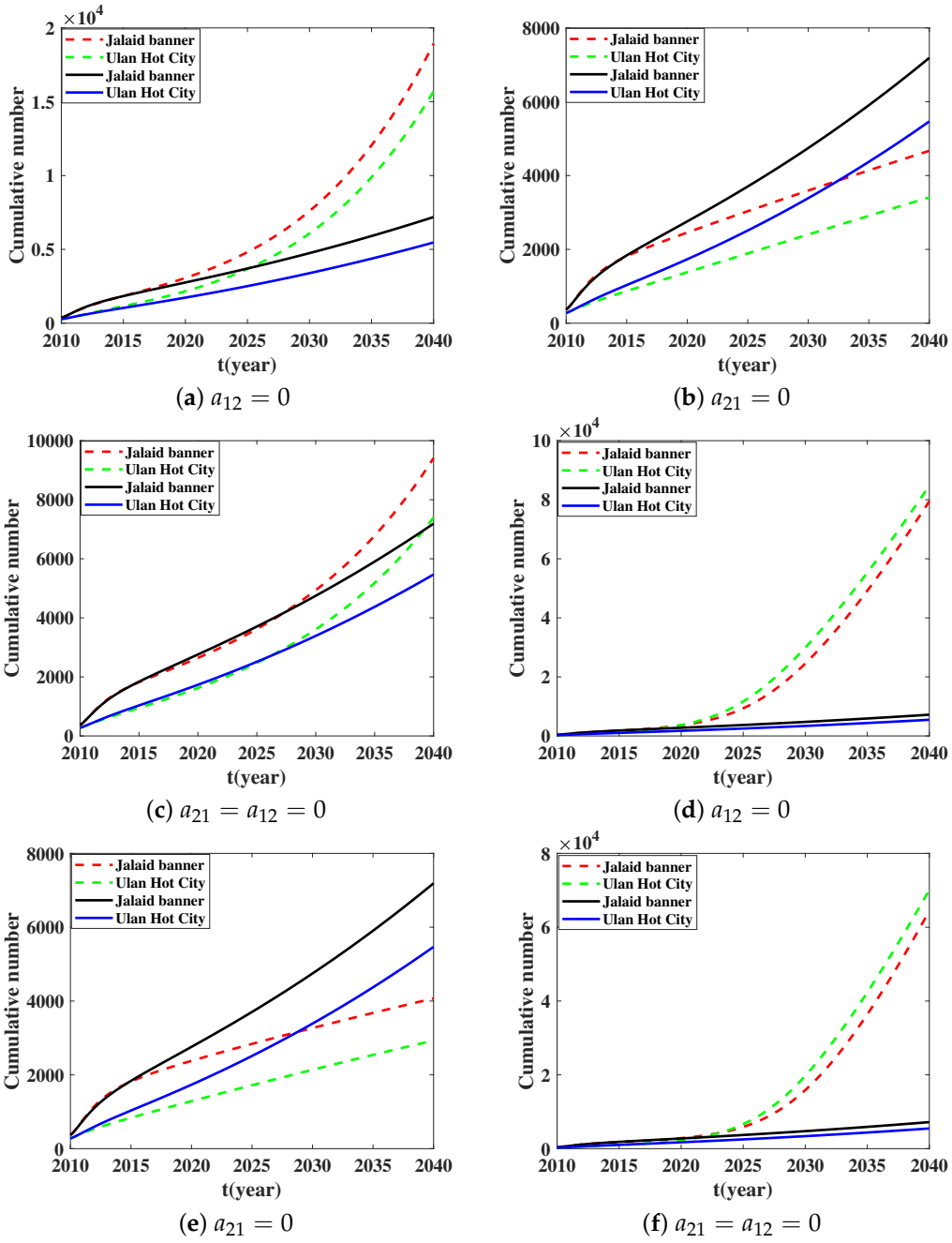


Figure 6. Cumulative number versus time t. Solid lines represent the solution of System (1). In (**a**–**c**), the dotted line represents three cases: zero transportation rate of infected sheep in one direction ($a_{12} = 0$ or $a_{21} = 0$) and zero transportation rate of infected sheep in two directions ($a_{21} = a_{12} = 0$). In (**d**–**f**), the dotted line represents three cases: zero transportation rate of exposed, infected sheep and brucella in the environment in one direction ($a_{12} = 0$ or $a_{21} = 0$) and zero transportation rate of exposed, infected sheep and brucella in the environment in two directions ($a_{21} = a_{12} = 0$).

5.3. Sensitivity Analysis

The PRCC (Partial Rank Correlation Coefficient)-based sensitivity analysis evaluates the influence of parameters on the basic reproduction number R_0. Here, PRCC values of some parameters are given based on Latin Hypercube Sampling [41]. We take the sample size $N = 100{,}000$. The λ_1^G, λ_2^G, k_1^G, k_2^G, μ_1, μ_2, β_1^G, β_2^G, α_1 and α_2 are considered to be input variables, which mean value are shown in Table 2. Furthermore, the values of R_0 are the output variables. We assume that all parameters are uniformly distributed and the respective standard deviations of λ_1^G and λ_2^G are 0.01, and $k_1^G = 0.001$, $k_2^G = 0.001$,

$\mu_1 = 0.01$, $\mu_2 = 0.01$, $\beta_1^G = 1 \times 10^{-9}$, $\beta_2^G = 1 \times 10^{-9}$, $\alpha_1 = 1 \times 10^{-9}$, $\alpha_2 = 1 \times 10^{-9}$. The magnitude of the partial rank correlation coefficient value of each input parameter with respect to the basic reproduction number R_0 is proportional to the correlation of this parameter to R_0. That is, the larger the bias correlation coefficient of this parameter is, the greater the influence of this parameter on R_0 is. We can see from Figure 7 that λ_1,λ_2, β_1^G, β_2^G, α_1 and α_2 are positively correlated with R_0, and k_1^G, k_2^G, μ_1 and μ_2 are negatively correlated with R_0. This further shows that vaccination and capture of the sheep with brucellosis are effective means to control the disease. We could further find that R_0 is the most sensitive to the values of λ_1^G and λ_2^G from Figure 7. This shows that improving vaccine efficiency is the best way to prevent brucellosis. At the same time, strengthening the vaccination rate, killing sick sheep and disinfecting the environment are effective means to prevent and control brucellosis. Considering the effect of μ_1 and μ_2 on R_0 is slightly greater than that of k_1^G and k_2^G, we believe that timely killing of sheep with brucellosis is more effective than vaccination.

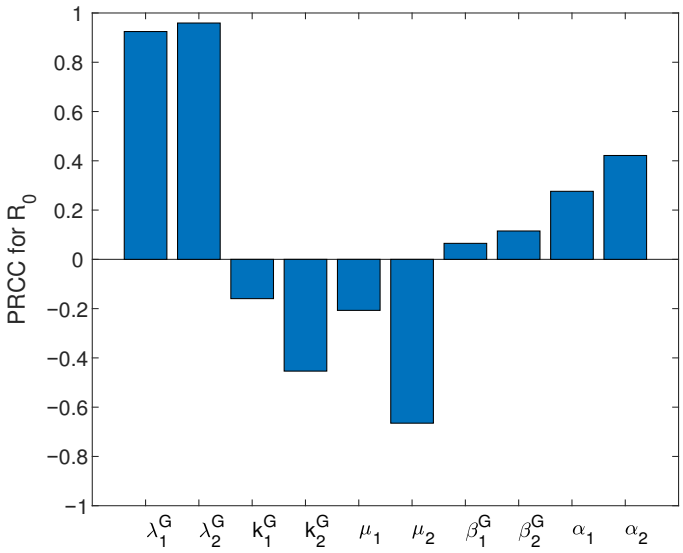


Figure 7. PRCC value of R_0 with λ_1^G,λ_2^G, k_1^G, k_2^G, μ_1, μ_2, β_1^G, β_2^G, α_1 and α_2.

6. Validation of Theories

In this section, firstly we verify Theorem 1 through numerical simulation. Then we use numerical simulation to illustrate that the endemic equilibrium point is globally asymptotically stable. First, we set $\beta_2^G = 3.5 \times 10^{-8}$ and the values of other parameters to be the same as those in Figure 5, then we get R_0 = 0.2146. Through Theorem 1, the disease-free equilibrium point $P^0 = (72659, 0, 0, 31090, 0, 484330, 0, 0, 37390, 0, 0, 18957, 0, 9189, 0, 0)$ is globally asymptotically stable. In Figure 8a, it could be observed that the sheep in the susceptible stage and the immunized stage are stable at a fixed value. It can be seen from the results in Figure 8b that the sheep in the exposed and infectious period and the brucella in the environment will eventually be extinct. Figure 8c shows that the number of susceptible humans will tend to a fixed value. For Figure 8d, we found that exposed and infected humans will tend to be 0. Next, assuming that $\beta_2^G = 3.5 \times 10^{-5}$, we get R_0 = 9.3435 when the values of other parameters are the same as those in Figure 5. It could be seen from Figure 8e–h that the endemic equilibrium point $P^* = (52{,}495, 4780, 5282, 23{,}086, 49{,}414, 5793, 219, 42{,}419, 3943, 15{,}342, 8778, 12{,}007, 93{,}031, 585, 137, 8463)$ may be globally asymptotically stable.

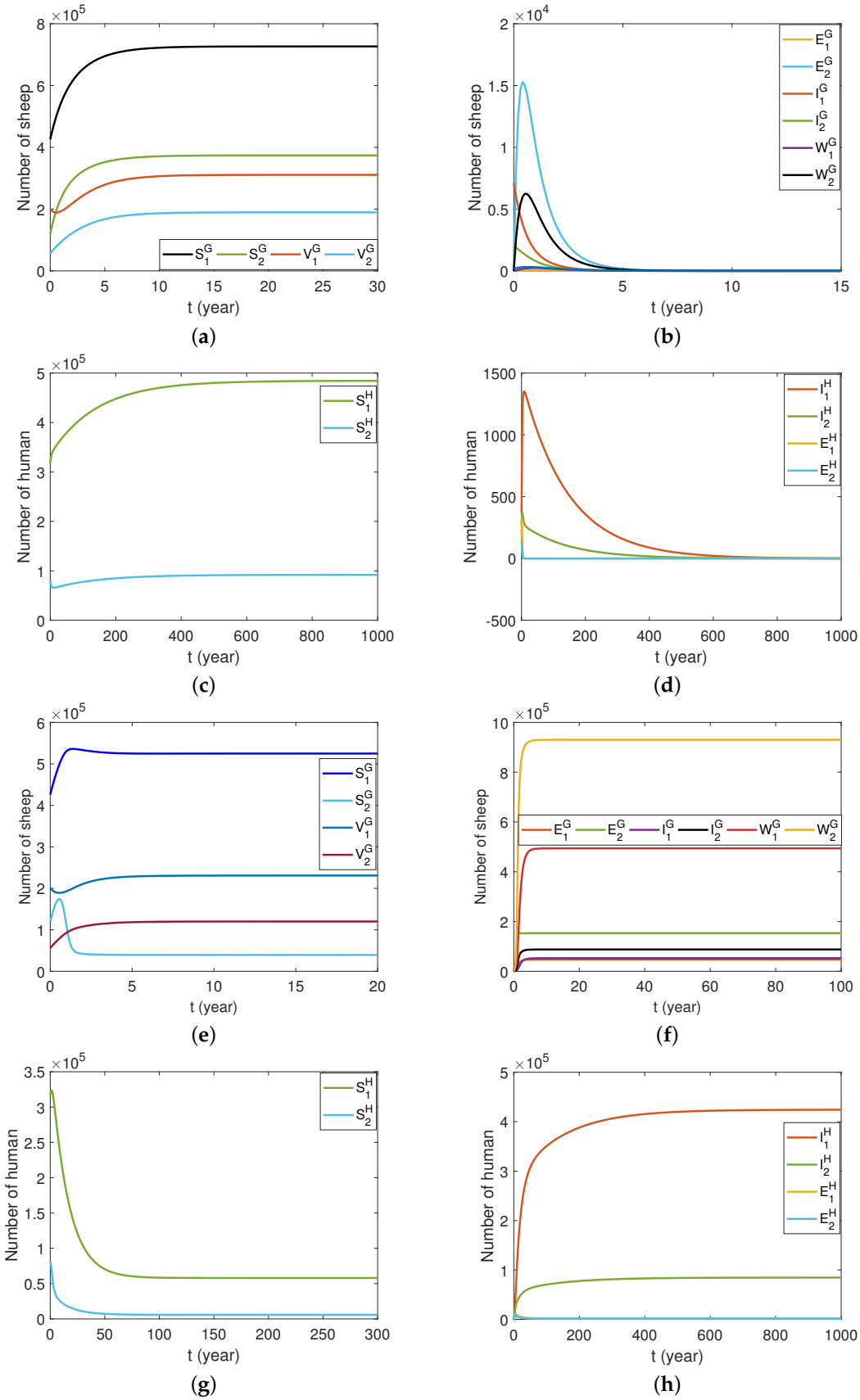


Figure 8. (**a**–**d**) represent the global stability of equilibrium P^0 with $R_0 = 0.2146$; (**e**–**h**) represent the global stability of equilibrium P^* with $R_0 = 9.3435$.

7. Discussion

Comparing with Hou et al. [16], we considered the transportation of sheep and human between different patches. Following Zhang et al. [18] and Liu et al. [20], we considered the sheep-human coupling model, and applied the model in practice. From the perspective of transportation, we only study the transportation of sheep between Jalaid and Ulanhot because of the frequent transportation between the two regions, ignoring the transportation between other regions in Hinggan League. This may lead to a gap between our simulation and the actual situation, which suggests that the model needs to add more patches to study the problem.

8. Conclusions

Brucellosis, as a national epidemic disease, has brought great disaster to people's health and socioeconomic development. There have been many studies on prevention and control of brucellosis [43–45], but the disease is still prevalent. As an important means of epidemic spreading, transportation of sheep is more and more frequent in China because of the development of the social economy. However, the effect of transportation of sheep is still poorly understood. In this paper, firstly we constructed a two-patch model and analyzed its dynamical behavior. In particular, this provided a reference for the dynamical behavior analysis of the n-patch model. Then we considered Jalaid and Ulanhot in Hinggan League of Inner Mongolia as an example to study how the transportation of sheep affects the spread of brucellosis. The results show that controlling the transportation of sheep from Jalaid to Ulanhot is the most effective means.

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