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Application of Simplified Homogeneous Balance Method to Multiple Solutions for (2 + l)-Dimensional Burgers' Equations

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Abstract: In this paper, three forms of (2 + l)-dimensional Burgers' equations are investigated. More general multiple solutions of these Burgers' equations are obtained by dependent variable transformation derived using the simplified homogeneous balance method.

Keywords: (2 + l)-dimensional Burgers' equations; multiple solutions; linearization; homogenization; simplified homogeneous balance method

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1. Introduction

In the present paper, we attempt to investigate three forms of (2 + l)-dimensional Burgers' equations as in the following:

The first form is given by

$$u_t - \mu(u_{xx} + u_{yy}) + 2uu_x + 2u_y\partial_x^{-1}u_y = 0, \tag{1}$$

which can be converted into a couple nonlinear PDE by the transformation $u_y = v_x$ as follows [1–3]:

$$u_t - \mu(u_{xx} + u_{yy}) + 2uu_x + 2vu_y = 0, \tag{2a}$$

$$u_y = v_x. \tag{2b}$$

The second form is that [4]

$$u_t - \mu(u_{xx} + u_{yy}) + 2uu_x + 2vu_y = 0, \tag{3a}$$

$$v_t - \mu(v_{xx} + v_{yy}) + 2uv_x + 2vv_y = 0. \tag{3b}$$

Both Equations (2a,b) and (3a,b) have been investigated using the simplified Hirota's direct method in Ref. [1].

The third form of the (2 + l)-dimensional Burgers' equations is taken in the form [5]

$$(u_t + 2uu_x - \mu u_{xx})_x + su_{yy} = 0, \tag{4}$$

which is formally similar to (2 + 1)-dimensional KdV equation, i.e., the KP equation [6]

$$(u_t + 6uu_x - u_{xx})_x + u_{yy} = 0.$$

It should be noted that μ and s appearing in Equations (2)–(4) are constants, and the u_{yy} term represents wave diffraction. Equation (4) describes weakly nonlinear two-dimensional shocks in dissipative media. The shocks described by Equation (4) are weakly two-dimensional in the sense that the scale length of variation in the y direction is much larger than in the x direction.

Burgers' equation is an important nonlinear partial differential equation in fluid mechanics, nonlinear acoustics, gas dynamics and so on, and has been widely studied by scholars. In the present paper, our interest is to apply the simplified homogeneous balance method (SHB) [7–10] to derive a transformation from a solution of a linear equation to a solution of Equations (2a,b) and (3a,b), and a transformation from a solution of a homogeneity equation to a solution of Equation (4). By the transformations obtained here, more general and more types of multiple solutions for the three forms of (2 + 1)-dimensional Burgers' equations can be obtained; since the solutions of the linear equations are known to us, the solutions of the homogeneity equation can be easily obtained.

The paper is organized as follows: in Section 2, both Equations (2a,b) and (3a,b) are linearized by using SHB; in Section 3, more multiple solutions of Equations (2a,b) and (3a,b) are given; in Sections 4 and 5, Equation (4) is homogenized by using SHB, and its multiple solutions are given; in Section 6, some conclusions are made.

2. Linearization of Equations (2a,b) and (3a,b)

2.1. Linearization of Equation (2a,b)

We begin with the (2 + 1)-dimensional Burgers' Equation (2a,b).

Considering the homogeneous balance between highest order term u_{xx} and nonhomogeneous term uu_x (for $x:\ m+2=2m+1 \Rightarrow m=1$), and between u_{yy} and vu_y (for $y:\ m_1+2=n+m_1+1 \Rightarrow n=1$), we can suppose that the solution of Equation (2a,b) is of the form

$$u = A(\ln\varphi)_x = A\frac{\varphi_x}{\varphi},\ v = A(\ln\varphi)_y = A\frac{\varphi_y}{\varphi}, \tag{5}$$

where we use a logarithmic function $A\ln\varphi$ instead of the undetermined function $f = f(\varphi)$ appearing in the original homogeneous balance method (HB) [11] to simplify HB. Constant A and function $\varphi = \varphi(x,y,t)$ are to be determined later. Our goal is to find A and function $\varphi = \varphi(x,y,t)$ such that expression (5) satisfies (2 + 1)-dimensional Burgers' Equation (2a,b) exactly.

Substituting expression (5) into the left side of Equation (2a) yields

$$\begin{aligned} &u_t - \mu(u_{xx}+u_{yy}) + 2uu_x + 2vu_y \\ &= A\frac{\partial}{\partial x}\left\{(\ln\varphi)_t - \mu\left[(\ln\varphi)_{xx} + (\ln\varphi)_{yy}\right] + A(\ln\varphi)_x^2 + A(\ln\varphi)_y^2\right\} \\ &= A\frac{\partial}{\partial x}\left[\frac{\varphi_t-\mu(\varphi_{xx}+\varphi_{yy})}{\varphi} + (A+\mu)\frac{\varphi_x^2+\varphi_y^2}{\varphi^2}\right], \end{aligned} \tag{6}$$

and Equation (2b) becomes an identity: $A(\ln\varphi)_{xy} = A(\ln\varphi)_{yx}$.

Setting the coefficient of $\frac{\varphi_x^2+\varphi_y^2}{\varphi^2}$ in (6) to zero yields $A = -\mu$, and, by using this, expression (5) becomes

$$u(x,y,t) = -\mu\frac{\varphi_x}{\varphi}, \quad v(x,y,t) = -\mu\frac{\varphi_y}{\varphi}, \tag{7}$$

and expression (6) can be simplified as

$$u_t - \mu(u_{xx}+u_{yy}) + 2uu_x + 2vu_y = -\mu\frac{\partial}{\partial x}\left[\frac{\varphi_t - \mu(\varphi_{xx}+\varphi_{yy})}{\varphi}\right], \tag{8}$$

provided that $\varphi = \varphi(x,y,t)$ satisfies the linear equation

$$\varphi_t - \mu(\varphi_{xx}+\varphi_{yy}) = 0. \tag{9}$$

Based upon (7), (8) and (9) we come to the conclusion that (2 + 1)-dimensional Burgers' Equation (2a,b) is transformed into the linear Equation (9) by the transformation of dependent variable (7a). In other words, (2 + 1)-dimensional Burgers' Equation (2a,b) is linearized by the transformation of dependent variable (7), and therefore, by inserting each solution of linear Equation (9) into the transformation of dependent variable (7), we can obtain the solution of (2 + l)-dimensional Burgers' Equation (2a,b).

2.2. Linearization of Equation (3a,b)

To the (2 + l)-dimensional Burgers' Equation (3a,b), by using the SHB, in a similar way, we can also conclude that the transformation

$$u(x,y,t) = -\mu\frac{\varphi_x}{\varphi}, \quad v(x,y,t) = -\mu\frac{\varphi_y}{\varphi}, \tag{10}$$

satisfies (2 + l)-dimensional Burgers' Equation (3a,b):

$$u_t - \mu(u_{xx} + u_{yy}) + 2uu_x + 2vu_y = -\mu\frac{\partial}{\partial x}\left[\frac{\varphi_t - \mu(\varphi_{xx} + \varphi_{yy})}{\varphi}\right], \tag{11a}$$

$$u_t - \mu(v_{xx} + v_{yy}) + 2uv_x + 2vv_y = -\mu\frac{\partial}{\partial y}\left[\frac{\varphi_t - \mu(\varphi_{xx} + \varphi_{yy})}{\varphi}\right], \tag{11b}$$

provided that $\varphi = \varphi(x,y,t)$ is a solution of the linear equation

$$\varphi_t - \mu(\varphi_{xx} + \varphi_{yy}) = 0. \tag{12}$$

We have seen that both Equations (2a,b) and (3a,b) are transformed into the same linear equation by the transformation of the dependent variable, although the forms of them are different. In the next section, we will solve only one of them, as the solutions to them are the same.

3. Multiple Solutions of Equations (2a,b) and (3a,b)

According to the superposition principle for a linear problem, the linear Equation (9) ((12)) can have many solutions, for example, the following functions

$$\varphi_1 = 1 \pm \sum_{i=1}^{N} c_i \mathrm{e}^{\eta_i}, \ \eta_i = m_i x + n_i y + \mu({m_i}^2 + {n_i}^2)t, \tag{13}$$

$$\varphi_2 = 1 \pm \sum_{i=1}^{N} c_i \mathrm{e}^{-\omega_i t}\cos(m_i x + n_i y), \ \omega_i = \mu({m_i}^2 + {n_i}^2)t, \tag{14}$$

$$\varphi_3 = x^3 + y^3 + x^2 + y^2 + (6x + 6y + 4)\mu t + 1, \tag{15}$$

and so on, are the solutions of linear Equation (9)((12)), where $N \geq 1$ is an integer, and $m_i, n_i, c_i (i = 1, 2, \cdots N)$ are constants.

Substituting (13) (with " + " sign) into (7)((10)) we have multiple regular kink solutions of Equations (2a,b) and (3a,b)

$$u_1(x,y,t) = -\mu\frac{\sum\limits_{i=1}^{N} c_i m_i \mathrm{e}^{i\eta_i}}{1 + \sum\limits_{i=1}^{N} c_i \mathrm{e}^{i\eta_i}}, \ \eta_i = m_i x + n_i y + \mu({m_i}^2 + {n_i}^2)t, \tag{16a}$$

$$v_1(x,y,t) = -\mu\frac{\sum\limits_{i=1}^{N} c_i n_i \mathrm{e}^{i\eta_i}}{1 + \sum\limits_{i=1}^{N} c_i \mathrm{e}^{i\eta_i}}, \ \eta_i = m_i x + n_i y + \mu({m_i}^2 + {n_i}^2)t, \tag{16b}$$

In particular, if $N = 1, 2, 3$, respectively, and taking $m_i = n_i = k_i, c_i = 1 (i = 1, 2, 3)$, the results of (16a) and (16b) become single kink solutions, i.e., two kink solutions and three kink solutions for Equations (2a,b) and (3a,b), respectively. These results coincide with those obtained using the simplified form of Hirota's method one by one in Ref. [1]. Consequently, solution (16a) and (16b) of Equations (2a,b) and (3a,b) are more general than those given in Ref. [1].

Substituting (14) (with " + " sign) into (7) ((10)) we have multiple periodic solutions in spacial variable x and y of Equations (2a,b) and (3a,b) as follows

$$u_2(x,y,t) = \mu \frac{\sum_{i=1}^{N} c_i m_i \mathrm{e}^{-\omega_i t} \sin(m_i x + n_i y)}{1 + \sum_{i=1}^{N} c_i \mathrm{e}^{-\omega_i t} \cos(m_i x + n_i y)}, \tag{17a}$$

$$v_2(x,y,t) = \mu \frac{\sum_{i=1}^{N} c_i n_i \mathrm{e}^{-\omega_i t} \sin(m_i x + n_i y)}{1 + \sum_{i=1}^{N} c_i \mathrm{e}^{-\omega_i t} \cos(m_i x + n_i y)}, \tag{17b}$$

where $\omega_i = \mu(m_i{}^2 + n_i{}^2)t, i = 1, 2, \cdots, N$.

If substituting (13) and (14) with "—" sign into (7) ((10)) respectively, we could have multiple singular kink solutions for Equations (2a,b) and (3a,b), respectively, but we omit the results here.

Substituting (15) into (7) ((10)) we have rational solutions of Equations (2a,b) and (3a,b)

$$u_3(x,y,t) = -\mu \frac{3x^2 + 2x + 6\mu t}{x^3 + y^3 + x^2 + y^2 + (6x + 6y + 4)\mu t + 1}, \tag{18a}$$

$$v_3(x,y,t) = -\mu \frac{3y^2 + 2y + 6\mu t}{x^3 + y^3 + x^2 + y^2 + (6x + 6y + 4)\mu t + 1}. \tag{18b}$$

At the end of the section, we point out that the solutions (u_2, v_2) and (u_3, v_3) of Equations (2a,b) and (3a,b) did not appear in Ref. [1].

4. Homogenization for (2 + l)-Dimensional Burgers' Equation (4)

Considering the homogeneous balance between u_{xx} and uu_x in Equation (4), we can suppose that the solution of Equation (4) is of the form

$$u(x,y,t) = A(\ln \varphi)_x = A\frac{\varphi_x}{\varphi}, \tag{19}$$

Substituting (19) into the left-hand side of Equation (4) yields

$$\begin{aligned}
&(u_t + 2uu_x - \mu u_{xx})_x + su_{yy} \\
&= A\frac{\partial}{\partial x}\left[(\ln\varphi)_{xt} + 2A(\ln\varphi)_x(\ln\varphi)_{xx} - \mu(\ln\varphi)_{xxx} + s(\ln\varphi)_{yy}\right] \\
&= A\frac{\partial}{\partial x}\left[\frac{\varphi_{xt}}{\varphi} - \frac{\varphi_x\varphi_t}{\varphi^2} + 2A\left(\frac{\varphi_x\varphi_{xx}}{\varphi^2} - \frac{\varphi_x{}^3}{\varphi^3}\right) - \mu\left(\frac{\varphi_{xxx}}{\varphi} - \frac{3\varphi_x\varphi_{xx}}{\varphi^2} + 2\frac{\varphi_x{}^3}{\varphi^3}\right) + s\left(\frac{\varphi_{yy}}{\varphi} - \frac{\varphi_y{}^2}{\varphi^2}\right)\right] \\
&= A\frac{\partial}{\partial x}\left[\frac{\varphi_{xt} - \mu\varphi_{xxx} + s\varphi_{yy}}{\varphi} + \frac{(2A+3\mu)\varphi_x\varphi_{xx} - \varphi_x\varphi_t - s\varphi_y{}^2}{\varphi^2} - (2A + 2\mu)\frac{\varphi_x{}^3}{\varphi^3}\right].
\end{aligned} \tag{20}$$

Setting the coefficient of $\frac{\varphi_x{}^3}{\varphi^3}$ in (20) to zero yields $A = -\mu$; by using this, expression (19) becomes

$$u(x,y,t) = -\mu\frac{\varphi_x}{\varphi}, \tag{21}$$

and (20) can be simplified as

$$
\begin{aligned}
&(u_t + 2uu_x - \mu u_{xx})_x + su_{yy} \\
&= -\mu \frac{\partial}{\partial x}\left[\frac{\varphi(\varphi_{xt} - \mu\varphi_{xxx} + s\varphi_{yy}) - (\varphi_x\varphi_t - \mu\varphi_x\varphi_{xx} + s{\varphi_y}^2)}{\varphi}\right],
\end{aligned} \tag{22}
$$

provided that $\varphi = \varphi(x, y, t)$ satisfies the homogeneity equation with 2-degree, as follows

$$\varphi(\varphi_{xt} - \mu\varphi_{xxx} + s\varphi_{yy}) - (\varphi_x\varphi_t - \mu\varphi_x\varphi_{xx} + s{\varphi_y}^2) = 0. \tag{23}$$

Based upon (21)–(23) we come to the conclusion that the (2 + l)-dimensional Burgers' Equation (4) is transformed into the homogeneity Equation (23) by the transformation of dependent variable (21). In other words, (2 + 1)-dimensional Burgers' Equation (4) is homogenized by the transformation of dependent variable (21), and inserting each solution of homogeneity Equation (23) into (21) we can obtain the solution of (2 + l)-dimensional Burgers' Equation (4).

5. Multiple Solutions of Equation (4)

To solve (2 + l)-dimensional Burgers' Equation (4), we only need (23). In the following, we use $\varepsilon-$ expansion method to solve Equation (20). Suppose that

$$\varphi = 1 + \varepsilon\varphi^{(1)} + \varepsilon^2\varphi^{(2)} + \varepsilon^3\varphi^{(3)} + \cdots \tag{24}$$

where $\varphi^{(i)}(i = 1, 2, \cdots)$ are to be determined, $\varepsilon-$ small parameter (may take $\varepsilon = 1$, for simplicity). Substituting (24) into the left-hand side of Equation (4), collecting all terms with $\varepsilon^i(i = 1, 2, \cdots)$ together and setting the coefficient of $\varepsilon^i(i = 1, 2, \cdots)$ to zero yields a series of equations for $\varphi^{(i)}(i = 1, 2, \cdots)$ as follows

$$\varepsilon^1 : \ {\varphi_{xt}}^{(1)} - \mu{\varphi_{xxx}}^{(1)} + s{\varphi_{yy}}^{(1)} = 0, \tag{25a}$$

$$\varepsilon^2 : \ {\varphi_{xt}}^{(2)} - \mu{\varphi_{xxx}}^{(2)} + s{\varphi_{yy}}^{(2)} = {\varphi_x}^{(1)}{\varphi_t}^{(1)} - \mu{\varphi_x}^{(1)}{\varphi_{xxx}}^{(1)} + s\left({\varphi_y}^{(1)}\right)^2, \tag{25b}$$

$$
\begin{aligned}
&\varepsilon^3 : \ {\varphi_{xt}}^{(3)} - \mu{\varphi_{xxx}}^{(3)} + s{\varphi_{yy}}^{(3)} = -\varphi^{(1)}\left({\varphi_{xt}}^{(2)} - \mu{\varphi_{xxx}}^{(2)} + s{\varphi_{yy}}^{(2)}\right) \\
&+{\varphi_t}^{(1)}{\varphi_x}^{(2)} + {\varphi_t}^{(2)}{\varphi_x}^{(1)} - \mu{\varphi_x}^{(1)}{\varphi_{xx}}^{(2)} - \mu{\varphi_x}^{(2)}{\varphi_{xx}}^{(1)} + 2s{\varphi_y}^{(1)}{\varphi_y}^{(2)},
\end{aligned} \tag{25c}
$$

and so on, to be solved.

It is noteworthy that the linear Equation (25a) admits an exponential function solution in the form

$$\varphi^{(1)} = \sum_{i=1}^{N} \mathrm{e}^{\xi_i}, \ \ \xi_i = k_i(x + \lambda y) + (\mu{k_i}^2 - s\lambda^2 k_i)t. \tag{26}$$

Substituting (26) into the left-hand side of Equation (25b) yields

$$
\begin{aligned}
{\varphi_{xt}}^{(2)} - \mu{\varphi_{xxx}}^{(2)} + s{\varphi_{yy}}^{(2)} &= \sum_{j=1}^{N}\sum_{i=1}^{N}\left[k_j\left(\mu{k_i}^2 - s\lambda^2 k_i\right)\mathrm{e}^{\xi_j+\xi_i}\right] \\
-\mu\sum_{j=1}^{N}\sum_{i=1}^{N} k_j{k_i}^2\mathrm{e}^{\xi_j+\xi_i} &+ s\lambda^2\sum_{j=1}^{N}\sum_{i=1}^{N} k_jk_i\mathrm{e}^{\xi_j+\xi_i} = 0.
\end{aligned} \tag{27}
$$

So we can take

$$\varphi^{(2)} = 0.$$

Substituting (23) and $\varphi^{(2)} = 0$ into the left-hand side of Equation (22b) yields

$$\varphi^{(3)} = 0,$$

Thus

$$\varphi^{(n)} = 0, \ n = 2, 3, \cdots \tag{28}$$

Substituting $\varphi^{(1)}$ in (23) and $\varphi^{(n)} = 0$ in (28) into (24) and take $\varepsilon = 1$, we have the solution of homogeneity Equation (20) as follows

$$\varphi = 1 + \sum_{i=1}^{N} \mathrm{e}^{\xi_i}, \ \ \xi_i = k_i(x + \lambda y) + (\mu {k_i}^2 - s\lambda^2 k_i)t. \tag{29}$$

Then, substituting (29) into (19), we have the solution of the (2 + l)-dimensional Burgers' Equation (4)

$$u(x, y, t) = -\mu \frac{\sum\limits_{i=1}^{N} k_i \mathrm{e}^{\xi_i}}{1 + \sum\limits_{i=1}^{N} \mathrm{e}^{\xi_i}}, \ \ \xi_i = k_i(x + \lambda y) + (\mu {k_i}^2 - s\lambda^2 k_i)t. \tag{30}$$

6. Conclusions

In this paper, three forms of (2 + l)-dimensional Burgers' equations were studied by using SHB, and we obtained these results:

(1) The transformation of the dependent variable for three forms of (2 + 1)-dimensional Burgers' equations were derived.

(2) Both (2 + 1)-dimensional Burgers' Equations (2a,b) and (3a,b) can be linearized by the same transformation of the dependent variable, and the same multiple regular and singular kink solutions of the two Burgers' equations can be obtained in terms of the solutions of the same linear equation, although forms of the two Burgers equations are different.

(3) The (2 + 1)-dimensional Burgers' Equation (4) is transformed into the homogeneity equation with 2-degree by the transformation of the dependent variable. We have examined in the paper that the homogeneity equation with 2-degree admits multiple solutions. Inserting the multiple solutions of the homogeneity equation into the transformation, we have the multiple regular and multiple singular kink solutions of (2 + 1)-dimensional Burgers' Equation (4).

(4) Compared with the literature, the method used in this paper is more convenient, direct, simple, and general, and more types of multiple solutions for the three forms of (2 + l)-dimensional Burgers' equations can be obtained. The SHB method can solve more nonlinear differential equations in mathematical physics.

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