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Tightness-Type Properties of the Space of Permutation Degree

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Abstract: In this paper we, prove that if the product X^n of a space X has certain tightness-type properties, then the space of permutation degree $\mathrm{SP}^n X$ has these properties as well. It is proven that the set tightness (T-tightness) of the space of permutation degree $\mathrm{SP}^n X$ is equal to the set tightness (T-tightness) of the product X^n.

Keywords: functor of permutation degree; tightness; set tightness; T-tightness; functional tightness; mini-tightness

MSC: 54A25; 18F60; 54B30

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Citation: Kočinac, L.D.R.; Mukhamadiev, F.G.; Sadullaev, A.K. Tightness-Type Properties of the Space of Permutation Degree. *Mathematics* **2022**, *10*, 3341. https://doi.org/10.3390/math10183341

Academic Editor: Alberto Fiorenza

Received: 19 August 2022
Accepted: 9 September 2022
Published: 15 September 2022

1. Introduction

At the Prague Topological Symposium in 1981, V.V. Fedorchuk [1] posed the following general problem in the theory of covariant functors, which determined a new direction for research in the field of Topology:

- Let $\mathcal{P}$ be some geometric property and F be a covariant functor. If a topological space X has the property $\mathcal{P}$, then whether $F(X)$ has the same property $\mathcal{P}$, or vice versa, whether $F(X)$ has the property $\mathcal{P}$, does it follow that the topological space X has the property $\mathcal{P}$ as well?

In our case, $\mathcal{P}$ is some tightness-type property, X is a topological T_1-space, and F is the functor of the G-permutation degree SP^n_G.

In [1,2] V.V. Fedorchuk and V.V. Filippov investigated the functor of the G-permutation degree and proved that this functor is a normal functor in the category of compact spaces and their continuous mappings.

In recent research, a number of authors have investigated the behaviour of certain cardinal invariants under the influence of various covariant functors. For example, in [3–8] the authors investigated several cardinal invariants under the influence of weakly normal, seminormal, and normal functors.

In [4,5], the authors discussed certain cardinal and geometric properties of the space of the permutation degree $\mathrm{SP}^n X$. They proved that if the product X^n has some Lindelŏf-type properties, then the space $\mathrm{SP}^n X$ has these properties as well. Moreover, they showed that the functor SP^n_G preserves both the homotopy and retraction of topological spaces. In addition, they proved that if the spaces X and Y are homotopically equivalent, then the spaces $\mathrm{SP}^n_G X$ and $\mathrm{SP}^n_G Y$ are homotopically equivalent as well. As a result, it has been proven that the functor SP^n_G is a covariant homotopy functor.

The current paper is devoted to the investigation of cardinal invariants such as the T-tightness, set tightness, functional tightness, mini-tightness (or weak functional tightness), and other topological properties of the space of permutation degree. We mention here that tightness-type properties of function spaces have been studied previously in [9,10].

The concepts of functional tightness and mini-tightness (or weak functional tightness) of a topological space were first introduced and studied by A.V. Arkhangel'skii in [11]. As it turned out, cardinal invariants such as mini-tightness and functional tightness are similar to each other in many ways, and for many natural and classical cases they even coincide. Moreover, there is an example of a topological space with countable mini-tightness and uncountable functional tightness; see [12].

In [13], the action of closed and R-quotient mappings on functional tightness was investigated. The authors proved that R-quotient mappings do not increase functional tightness. Furthermore, in [13] the authors proved that the functional tightness of the product of two locally compact spaces does not exceed the product of the functional tightness of those spaces.

Throughout this paper, all spaces referred to are topological spaces and κ is an infinite cardinal number; furthermore, regular spaces need not be T_1.

2. Definitions and Notations

The following are definitions and notions needed in the rest of this paper.

Definition 1 (see [14])**.** *Let A be a subset of a topological space X; the* tightness of A with respect to X *is the cardinal number*

$$t(A,X) = \min\{\kappa : \forall C \subset X, \text{ such that } A \cap \overline{C} \neq \emptyset\ \exists C_0 \in [C]^{\leq\kappa} \text{ with } A \cap \overline{C}_0 \neq \emptyset\}.$$

If $A = \{x\}$, we briefly write $t(x,X)$ instead of $t(\{x\},X)$. The tightness *of X is defined as $t(X) = \sup\{t(x,X) : x \in X\}$.*

Definition 2 ([15]; see as well [16,17])**.** *Let X be a topological space; then, the* set tightness at a point *$x \in X$, denoted by $t_s(x,X)$, is the smallest cardinal number κ such that whenever $x \in \overline{C} \setminus C$, where $C \subset X$, there exists a family γ of subsets of C such that $|\gamma| \leq \kappa$ and $x \in \overline{\bigcup\gamma} \setminus \bigcup\overline{\gamma}$. The* set tightness *of X is defined as $t_s(X) = \sup\{t_s(x,X) : \ x \in X\}$.*

It is clear that for any topological space X we have $t_s(x,X) \leq t(x,X)$ and $t_s(X) \leq t(X)$.

Definition 3 ([17,18])**.** *For a topological space X, the T-*tightness *of X, denoted by $T(X)$, is the smallest cardinal number κ such that whenever $\{F_\alpha\}_{\alpha\in\Lambda}$ is an increasing sequence of closed subsets of X with $cf(\Lambda) > \kappa$ then $\bigcup_{\alpha\in\Lambda} F_\alpha$ is closed.*

Let κ be an infinite cardinal and let X and Y be topological spaces. A mapping $f : X \to Y$ is said to be *κ-continuous* if for every subspace A of X such that $|A| \leq \kappa$ the restriction $f \restriction A$ is continuous. A mapping $f : X \to Y$ is said to be *strictly κ-continuous* if for every subspace A of X with $|A| \leq \kappa$ there exists a continuous mapping $g : X \to Y$ such that $f \restriction A = g \restriction A$.

Definition 4 ([11]; see as well [13,19,20])**.** *The functional tightness $t_o(X)$ of a space X is the smallest infinite cardinal number κ such that every κ-continuous real-valued function on X is continuous.*

In [13], the following theorem was proven:

Theorem 1. *If X is a locally compact space, then $t_o(X \times Y) \leq t_o(X)t_o(Y)$.*

Note that per Theorem 1, $t_o(X^n) = t_o(X)$ for every compact space X and every $n \in N$.

Definition 5 ([11])**.** *The weak functional tightness (or* minitightness*) $t_m(X)$ of a space X is the smallest infinite cardinal number κ such that every strictly κ-continuous real-valued function on X is continuous.*

Clearly, every strictly κ-continuous mapping is κ-continuous. Therefore, for any topological space X we have

$$t_m(X) \le t_o(X) \le t(X).$$

In [19], the following theorems were provided:

Theorem 2 ([19], Theorem 2.14)**.** *If X is a locally compact space, then, for every space Y,*

$$t_m(X \times Y) \le t_m(X)t_m(Y).$$

Theorem 3 ([19], Theorem 2.7, Corollary 2.8)**.** *For any two spaces X and Y,*

$$t_m(X \times Y) \le t_m(X)\chi(Y).$$

If Y is first countable, $t_m(X \times Y) = t_m(X)$.

The set of all non-empty closed subsets of a topological space X is denoted by $\mathsf{exp}X$. The family of all sets of the form

$$O\langle U_1, U_2, \dots, U_n\rangle = \{F: \ F \in \mathsf{exp}X, \, F \subset \bigcup_{i=1}^{n} U_i, \, F \cap U_i \neq \varnothing, \, i = 1, \dots, n\},$$

where $U_1, U_2, \dots, U_n$ are open subsets of X generates a base of the topology on the set $\mathsf{exp}X$. This topology is called the *Vietoris topology*. The set $\mathsf{exp}X$ with the Vietoris topology is called the *exponential space* or *hyperspace* of a space X. We put [2]

$$\mathsf{exp_n}X = \{F \in \mathsf{exp}X: \ |F| \le n\}.$$

Let S_n denote the permutation group of the set $\{1, 2, \dots, n\}$, and let G be a subgroup of S_n. The group G acts on the n-th power X^n of a space X as permutation of coordinates. Two points $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in X^n$ are considered to be *G-equivalent* if there exists a permutation $\sigma \in G$ such that $y_i = x_{\sigma(i)}$. This relation is called the *symmetric G-equivalence relation* on X. The G-equivalence class of an element $x = (x_1, x_2, \dots, x_n) \in X^n$ is denoted by $[x]_G = [(x_1, x_2, \dots, x_n)]_G$. The sets of all orbits of actions of the group G is denoted by $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. Thus, points of the space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ are finite subsets (equivalence classes) of the product X^n.

Consider the quotient mapping $\pi^s_{n,G} : X^n \to \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ defined by

$$\pi^s_{n,G}((x_1, x_2, \dots, x_n)) = [(x_1, x_2, \dots, x_n)]_G$$

and endow the sets $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ with the quotient topology. This space is called the *space of the n–G-permutation degree*, or simply the *space of the G-permutation degree* of space X.

Let $f : X \to Y$ be a continuous mapping. For an equivalence class $[(x_1, x_2, \dots, x_n)]_G \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$, we can say that

$$\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}f[(x_1, x_2, \dots, x_n)]_G = [(f(x_1), f(x_2), \dots, f(x_n))]_G.$$

In this way, we have the mapping $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}f : \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X \to \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}Y$. It is easy to check that the mapping $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ constructed in this way is a normal functor in the category of compacta. This functor is called the *functor of the G-permutation degree*.

When $G = S_n$, we omit the index or prefix G in all the above definitions. In particular, we speak about the space $\mathsf{SP}^{\mathsf{n}}X$ of the permutation degree, the functor SP^{n}, and the quotient mapping π^s_n.

Equivalence relations by which one obtains spaces $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ and $\mathsf{exp_n}X$ are called the *symmetric* and *hypersymmetric equivalence relations*, respectively.

While any symmetrically equivalent points in X^n are hypersymmetrically equivalent, in general, the converse is not correct. For example, while for $x \neq y$ points (x, x, y), (x, y, y) are hypersymmetrically equivalent, they are not symmetrically equivalent.

The G-symmetric equivalence class $[(x_1, x_2, \ldots, x_n)]_G$ uniquely determines the hypersymmetric equivalence class $[(x_1, x_2, \ldots, x_n)]_G^{hc}$ containing it. Thus, we have the mapping

$$\pi^h_{n,G} : \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X \to \mathsf{exp}_{\mathsf{n}} X,$$

representing the functor $\mathsf{exp}_{\mathsf{n}}$ as the factor functor of the functor $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ [1,2].

3. Results

The functor of the G-permutation degree $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ preserves the κ-continuity of the mappings, i.e., the following holds.

Theorem 4. *If $f : X \to Y$ is a κ-continuous mapping, then the mapping $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f : \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X \to \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} Y$ is κ-continuous as well.*

Proof. Consider an arbitrary subset $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} A$ of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$, such that $|\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} A| \leq \kappa$. We can prove that the restriction of the mapping $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f$ onto the set $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} A$ is continuous.

If we say

$$M = pr_i((\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} A)),$$

where $pr_i : X^n \to X$ is defined as

$$pr_i(z_1, z_2, \ldots, z_n) = z_i,$$

for any $(z_1, z_2, \ldots, z_n) \in X^n, 1 \leq i \leq n$, and $\pi^s_{n,G} : X^n \to \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$, it is clear that $M \subset X$ and $|M| \leq \kappa$. Take an arbitrary element $[x]_G = [(x_1, x_2, \ldots, x_n)]_G$ from $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} A$; then,

$$\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f([x]_G) = [(f(x_1), f(x_2), \ldots, f(x_n))]_G \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} Y.$$

Suppose W is a neighborhood of the orbit $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f([x]_G)$ in $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} Y$. Per the definition of the quotient mapping, there exist neighborhoods $V_1, V_2, \ldots, V_n$ of the points $f(x_1), f(x_2), \ldots, f(x_n)$ such that $[V_1 \times V_2 \times \ldots \times V_n]_G \subset W$. In this case, we have $x_1, x_2, \ldots, x_n \in M$. Because $M \subset X$ and $|M| \leq \kappa$, we find that $f \upharpoonright M : M \to Y$ is continuous. By continuity of f on M, there exist neighborhoods $U_1, U_2, \ldots, U_n$ of the points $x_1, x_2, \ldots, x_n$ satisfying $f(U_j) \subset V_j$ for all $j = 1, 2, \ldots, n$. Then,

$$\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f[U_1 \times U_2 \times \ldots \times U_n]_G = [f(U_1) \times f(U_2) \times, \ldots, \times f(U_n)]_G \subset W.$$

This means that the restriction $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f \upharpoonright \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} A$ is continuous at the point $[x]_G$. As $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} A$ and $[x]_G$ were arbitrary, the theorem is proven. □

Theorem 5. *For every topological space X we have*

$$t_s\big((\pi^s_{n,G})^{\leftarrow}([x]_G),\, X^n\big) \leq t_s([x]_G,\, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X).$$

Proof. Let $t_s\big([x]_G, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X\big) = \kappa$ and $C \subset X^n$ satisfying $\left(\pi^s_{n,G}\right)^{\leftarrow}([x]_G) \subset \overline{C} \setminus C$. Then, we have $[x]_G \in \overline{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} C} \setminus \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} C$. This means that there exists a family $\gamma' \subset \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} C$ such that $|\gamma'| \leq \kappa$ and $[x]_G \in \overline{\bigcup \gamma'} \setminus \bigcup \overline{\gamma'}$. For every $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} S \in \gamma'$, we can choose a set $S \subset C \subset X^n$ such that $\pi^s_{n,G}(S) = \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} S$. Let $\gamma = \{S : \ \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} S \in \gamma'\}$ be a family obtained in this way. It is clear that $|\gamma| \leq \kappa$ and $\left(\pi^s_{n,G}\right)^{\leftarrow}([x]_G) \bigcap (\bigcup \overline{\gamma}) = \emptyset$; thus, by the closedness of the mapping $\pi^s_{n,G}$, we have $\left(\pi^s_{n,G}\right)^{\leftarrow}([x]_G) \bigcap (\overline{\bigcup \gamma}) \neq \emptyset$. This means that $t_s\left(\left(\pi^s_{n,G}\right)^{\leftarrow}([x]_G),\, X^n\right) \leq \kappa$. Theorem 5 is therefore proven. □

Theorem 6. *If X is a regular space, then $t_s(X^n) = t_s(\mathsf{SP}^n_G X)$.*

Proof. Let $\kappa = t_s(X^n)$, $C \subset \mathsf{SP}^n_G X$ and $[y]_G \in \overline{C} \setminus C$. By virtue of the closedness of $\pi^s_{n,G}$, $\left(\pi^s_{n,G}\right)^{\leftarrow}([y]_G) \cap \overline{\left(\pi^s_{n,G}\right)^{\leftarrow}(C)} \neq \emptyset$. Let x be an isolated point of $\left(\pi^s_{n,G}\right)^{\leftarrow}([y]_G) \cap \overline{\left(\pi^s_{n,G}\right)^{\leftarrow}(C)}$. Clearly, $x \in \overline{\left(\pi^s_{n,G}\right)^{\leftarrow}(C)} \setminus \left(\pi^s_{n,G}\right)^{\leftarrow}(C)$. Because $t_s(X^n) = \kappa$, there exists a family $\gamma \subset \left(\pi^s_{n,G}\right)^{\leftarrow}(C)$ such that $|\gamma| = \kappa$ and $x \in \overline{\bigcup \gamma} \setminus \bigcup \overline{\gamma}$. The set $\left\{\left(\pi^s_{n,G}\right)^{\leftarrow}([y]_G) \bigcap (\overline{\bigcup \gamma})\right\} \setminus \{x\}$ is closed and discrete in $\left(\pi^s_{n,G}\right)^{\leftarrow}([y]_G)$; hence, X^n. Due to the regularity of X, there exists a closed neighbourhood U of x such that $U \bigcap \left\{\left\{\left(\pi^s_{n,G}\right)^{\leftarrow}([y]_G) \bigcap (\overline{\bigcup \gamma})\right\} \setminus \{x\}\right\} = \emptyset$. Let $\gamma' = \{B \bigcap U : \ U \in \gamma\}$; then, it is clear that $x \in \overline{\bigcup \gamma'}$ and $\left(\pi^s_{n,G}\right)^{\leftarrow}([y]_G) \bigcap \left(\bigcup \overline{\gamma'}\right) = \emptyset$. Let $\gamma'' = \left\{\pi^s_{n,G}(B) = \mathsf{SP}^n_G B : \ B \in \gamma'\right\}$. By the closedness of $\pi^s_{n,G}$, we have $[y]_G \notin \bigcup \overline{\gamma''}$; however, clearly $[y]_G \in \overline{\bigcup \gamma''}$ and $|\gamma''| = \kappa$. This means that $t_s(\mathsf{SP}^n_G X) = \kappa$. Theorem 6 is therefore proven. □

Proposition 1. *For any topological space X, we have $T(\mathsf{SP}^n_G X) \leq T(X^n)$.*

Proof. Assume that $T(X^n) = \kappa$. This means that for every increasing sequence $\{F_\alpha\}_{\alpha \in \Lambda}$ of closed subsets of X^n with $cf(\Lambda) > \kappa$, we find that $\bigcup_{\alpha \in \Lambda} F_\alpha$ is closed. Because the quotient mapping $\pi^s_{n,G} : X^n \to \mathsf{SP}^n_G X$ is closed onto mapping, it follows immediately that $\{\mathsf{SP}^n_G(F_\alpha)\}_{\alpha \in \Lambda}$ is an increasing sequence of closed subsets of $\mathsf{SP}^n_G X$ and that $\bigcup_{\alpha \in \Lambda} \mathsf{SP}^n_G(F_\alpha)$ is closed. This means that $T(\mathsf{SP}^n_G X) \leq \kappa$. Proposition 1 is therefore proven. □

Theorem 7. *If X is a regular space, then $T(\mathsf{SP}^n_G X) = T(X^n)$.*

Proof. According to Proposition 1, it suffices to show the following equality: $T(\mathsf{SP}^n_G X) \geq T(X^n)$.

Assume that $T(\mathsf{SP}^n_G X) = \kappa$ and $\{F_\alpha\}_{\alpha \in \Lambda}$ is an increasing sequence of closed subsets of X^n such that $cf(\Lambda) > \kappa$. Put $F = \bigcup_{\alpha \in \Lambda} F_\alpha$ and suppose that there exists a point $x \in \overline{F} \setminus F$.

Let $F'_\alpha = (\pi^s_{n,G})^{\leftarrow}([x]_G) \bigcap F_\alpha$ for every $\alpha \in \Lambda$; the family $\{F'_\alpha\}_{\alpha \in \Lambda}$ is an increasing sequence of closed subsets of $(\pi^s_{n,G})^{\leftarrow}([x]_G)$. Because $(\pi^s_{n,G})^{\leftarrow}([x]_G)$ is finite, we find that the set $\bigcup_{\alpha \in \Lambda} F'_\alpha = F \bigcap (\pi^s_{n,G})^{\leftarrow}([x]_G)$ is closed in X^n.

By regularity of X (and hence X^n), there exist two disjoint open sets U and V in X^n such that $x \in U$ and $F \bigcap (\pi^s_{n,G})^{\leftarrow}([x]_G) \subset V$.

Let $F''_\alpha = F_\alpha \setminus V$ for every $\alpha \in \Lambda$. It is clear that $x \in \overline{\bigcup_{\alpha \in \Lambda} F''_\alpha}$ and $(\pi^s_{n,G})^{\leftarrow}([x]_G) \bigcap (\bigcup_{\alpha \in \Lambda} F''_\alpha) = \emptyset$. The family $\{\mathsf{SP}^n_G(F''_\alpha)\}_{\alpha \in \Lambda}$ is an increasing sequence of closed subsets of $\mathsf{SP}^n_G X$. Because $T(\mathsf{SP}^n_G X) = \kappa$, the set $\bigcup_{\alpha \in \Lambda} \mathsf{SP}^n_G(F''_\alpha) = \bigcup_{\alpha \in \Lambda} \pi^s_{n,G}(F''_\alpha) = \pi^s_{n,G}(\bigcup_{\alpha \in \Lambda} F''_\alpha)$ must be closed, and per the continuity of $\pi^s_{n,G}$,

$$\pi^s_{n,G}(x) = [x]_G \in \overline{\pi^s_{n,G}(\bigcup_{\alpha \in \Lambda} F''_\alpha)} = \pi^s_{n,G}(\bigcup_{\alpha \in \Lambda} F''_\alpha).$$

However, this is impossible because

$$(\pi^s_{n,G})^{\leftarrow}([x]_G) \bigcap (\bigcup_{\alpha \in \Lambda} F''_\alpha) = \emptyset.$$

This proves that F is closed, and thus $T(X^n) \leq \kappa$. Theorem 7 is therefore proven. □

If, in the above theorem, the space X is Hausdorff, then the mapping $\pi^s_{n,G}$ is perfect and the assumption about the regularity of X could be weakened.

Corollary 1. *If X is Hausdorff and $T(\mathsf{SP}^n_G X) \leq \kappa$, then $T(X^n) \leq \kappa$.*

Corollary 2. *If X is a locally compact Hausdorff space, then $T(\mathsf{SP}^n_G X) \leq T(X^n) \leq T(X)$.*

Proposition 2. *Let X be any topological space; then,*

(*a*) $t_o(\mathsf{SP}^n_G X) \leq t_o(X^n)$;
(*b*) $t_m(\mathsf{SP}^n_G X) \leq t_m(X^n)$.

Proof. Let f be a κ-continuous (strictly κ-continuous) real-valued function on $\mathsf{SP}^n_G X$ and $t_o(X^n) = \kappa$ (resp. $t_m(X^n) = \kappa$). Then, the composition $g = f \circ \pi^s_{n,G}$ is a κ-continuous (strictly κ-continuous) real-valued function on X^n. In both cases, we find that g is continuous. By continuity of $\pi^s_{n,G}$ and $g = f \circ \pi^s_{n,G}$, it follows that f is continuous; hence, $t_o(\mathsf{SP}^n_G X) \leq \kappa = t_o(X^n)$ ($t_m(\mathsf{SP}^n_G X) \leq \kappa = t_m(X^n)$). Proposition 2 is therefore proven. □

From Proposition 2 and from Theorems 1 and 2, we have the following statement.

Corollary 3. *Let X be a locally compact space; then,*

(*a*) $t_o(\mathsf{SP}^n_G X) \leq t_o(X^n) \leq t_o(X)$;
(*b*) $t_m(\mathsf{SP}^n_G X) \leq t_m(X^n) \leq t_m(X)$.

It follows immediately from Theorem 3 that:

Corollary 4. *For every first countable space X, $t_m(\mathsf{SP}^n_G X) \leq t_m(X^n) \leq t_m(X)$.*

Let us now recall the earlier definitions.

The *weak tightness* $t_c(X)$ of a space X is the smallest (infinite) cardinal κ such that the following condition is fulfilled.

If a set $A \subset X$ is not closed in X, then there is a point $x \in \overline{A} \setminus A$, a set $B \subset A$, and a set $C \subset X$ for which $x \in \overline{B}$, $B \subset \overline{C}$, and $|C| \leq \kappa$.

We can say that $A \subset X$ is a *set of type G_κ* in X if there is a family γ of open sets in X such that $A = \cap\gamma$ and $|\gamma| \leq \kappa$. A set $A \subset X$ is called *κ-placed* in X if for each point $x \in X \setminus A$ there is a set P of type G_κ in X such that $x \in P \subset X \setminus A$.

Put $q(X) = \min\{\kappa \geq \omega : \text{is } \kappa - \text{placed in } \beta X\}$; $q(X)$ is called the *Hewitt–Nachbin number* of X. We can say that X is a *$Q_\kappa-$space* if $q(X) \leq \kappa$.

Proposition 3. *Let X be a compact space; then, $q(\mathsf{SP}^n_G X) \leq d(X)$.*

Proof. It is known (see [14]) that, for any Tychonoff space X, the following relations hold:

$$q(X) = t_m(C_p(X)) = t_o(C_p(X)), \quad t_o(X) \leq t_c(X) \leq d(X).$$

Thus, we have

$$q(\mathsf{SP}^n_G X) = t_m(C_p(\mathsf{SP}^n_G X)) = t_o(C_p(\mathsf{SP}^n_G X)) \leq t_c(C_p(\mathsf{SP}^n_G X))$$
$$\leq d(C_p(\mathsf{SP}^n_G X)) \leq d(\mathsf{SP}^n_G X) = d(X).$$

Proposition 3 is therefore proven. □

Corollary 5. *Let X be a compact and separable space; then, $q(\mathsf{SP}^n_G X) \leq \omega$, i.e., the space $\mathsf{SP}^n_G X$ is a Q_ω-space.*

4. Conclusions

An important question in topology is, if F is a functor and $\mathcal{P}$ is a topological property, whether if a space X has the property $\mathcal{P}$, then $F(X)$ has the same or some other property. This paper is devoted to a study of preservation of tightness-type cardinal invariants (T-tightness, set-tightness, functional tightness, minitightness, weak tightness) of a space X

(and its n-th power X^n) under influence of the functor SP^n of n-permutation degree. It is shown that, for certain classes of spaces, some of these cardinal functions are equal for X^n and $\mathrm{SP}^n X$. We hope that these results may be a first step in the investigation of similar problems for other known functors.

Author Contributions: Supervision, L.D.R.K.; Visualization, F.G.M. and A.K.S. These authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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