mathematics

Article

Proposal to Systematize the Reflection and Assessment of the Teacher's Practice on the Teaching of Functions

Yocelyn E. Parra-Urrea [1] and Luis R. Pino-Fan [2,*]

[1] Department of Mathematics and Statistics, Universidad San Sebastián, Santiago 8420524, Chile
[2] Department of Ciencias Exactas, Universidad de Los Lagos, Osorno 5310883, Chile
* Correspondence: luis.pino@ulagos.cl

Abstract: This research proposes theoretical-methodological strategies and tools to guide teaching, reflection, and assessment on the practice of the prospective teacher of mathematics when performing teaching and learning processes on the notion of function. It exemplifies our proposal's use, through the reflection and characterization of the didactic-mathematical knowledge, of a prospective Chilean teacher, underlying his practice of teaching the power function. For the development of our proposal, we used the theoretical-methodological notions of the Didactic-Mathematical Knowledge (DMK) model and micro-teaching contexts, understood as rich and essential spaces for teacher education. As a conclusion, we could observe that the theoretical-methodological proposal here presented serves to systematize the reflection and assessment processes of teaching practice (one's own or that of others) for the teaching of functions, and, thus, determine (or anticipate) actions that improve the teaching processes on this important mathematical notion.

Keywords: teacher education; teacher knowledge; reflection on practice; micro-teaching

MSC: 97B50

Citation: Parra-Urrea, Y.E.; Pino-Fan, L.R. Proposal to Systematize the Reflection and Assessment of the Teacher's Practice on the Teaching of Functions. *Mathematics* **2022**, *10*, 3330. https://doi.org/10.3390/math10183330

Academic Editor: Luis Carlos Contreras-González

Received: 30 July 2022
Accepted: 23 August 2022
Published: 14 September 2022

1. Introduction

Numerous investigations have examined the knowledge that teachers should have in order for their teaching practice to be as effective as possible and to facilitate the learning of their students [1–6]. However, empirical research on the knowledge that mathematics teachers possess is significantly less [7]. There are also few studies on the practical experiences of teacher education, strengthening the didactic-mathematical knowledge of prospective teachers and improving their teaching practice [8].

Currently, teacher education programs (at least in Latin America) aim to develop pedagogical skills from field experiences, a component of professional education over which less control is available. According to Fernández [9], it is essential to seek experiences that allow prospective teachers to interact and explore, in shared contexts (simulated and real), possible pedagogical problems that involve reflection and critical analysis of teaching and learning processes. Dewey [10] proposes that instead of providing extensive practice in typical classrooms, teacher preparation should create opportunities for experimentation (simulations of teaching–learning processes) that allow testing, learning, and the developing of discrete components of complex practice in environments of reduced complexity. Concerning this, Grossman et al. [11] propose incorporating in professional education 'approximations of practice' which would provide prospective teachers with multiple opportunities to get involved in simulated planning (of lessons, units, and classroom management), to explain mathematical notions, and to receive immediate feedback on their performance. The investigation into 'approximations of practice' is known as *micro-teaching*. According to Mergler and Tangen [12], micro-teaching involves planning and conducting an instructional process for a group of prospective teachers who fulfill students' roles. Additionally, it allows future teachers to identify the complexity of teaching by establishing a connection between theory and practice. It admits participating in the feedback and self-reflection of the process. By observing what others do, prospective teachers can reflect on how they will run their lessons. Various studies (e.g., [9,12–16]) have indicated the benefits of micro-teaching in teacher education processes. For example, it allows a safe and controlled teaching environment that reduces real teaching situations' complexity. Besides, prospective teachers can experience a learning process based on collaboration, dialogue, and peer discussion, which, according to Brookfield [17], helps them achieve critical reflection.

This article focuses on one of the most relevant mathematics concepts and its learning, widely discussed in scientific literature: the function. Several studies have investigated the problems related to the teaching–learning processes of functions from different dimensions and approaches. For example, [18] pointed out that the notion of function (given mainly by definition) is cognitively challenging to achieve, and students access the concept through examples in various forms: metaphors, representations, set diagrams, among others. Functions admit various representations (algebraic, graphical, tabular, verbal) that provide relevant information about particular aspects of the mathematical object. However, none of them manages to fully describe the notion of functions. Thus, teachers' lack of competence in coordinating the multiple representations associated with functions could hinder students' understanding [19]. Moreover, from the cognitive dimension, this could lead students to conceive function representations as static and independent entities, i.e., they would have difficulty understanding that different representations describe the same mathematical idea or notion [20]. According to Makonye [21], using different contexts, and interrelating the various representations of the function, promotes learning, procedural fluency and increases students' interest in studying functions.

Dede and Soybaş [22] point out that trainee teachers have difficulties in differentiating and relating mathematical expressions representing a function or equation. That is to say, prospective mathematics teachers' knowledge of these concepts is based only on 'pure' and fragmented definitions, preventing them from moving towards a holistic meaning of function. In this respect, [23] already pointed out that teachers have difficulties in conceptualizing functions, i.e., they approve of informal definitions of the object, considered helpful in determining relationships' functionality, and perceive them as mathematically formal definitions. In this sense, [22] point out that providing students with the definition is not enough to achieve a meaningful understanding of the function object. They also show that trainee teachers define the notion of function without considering some of the essential 'sub-concepts' inherent to the function (e.g., domain, co-domain, preimage, variable, among others). Even [24] refers to two essential features of the notion of function, arbitrariness and univalence. He points out that teachers do not perceive the arbitrariness of functions, and very few can explain the importance and origin of the requirement of univalence.

The complexity underlying the processes of function teaching and learning has led the research community (e.g., [25,26]) to discuss the question: What is the knowledge that the mathematics teacher, or the prospective mathematics teacher, needs to teach functions? However, there is still no consensus on the answer.

Given all of the above, in this article, we have set out to go a step further, and because of the large amount and diversity of information on the subject, we ask ourselves the following questions:

- What dimensions and components might mathematics teachers consider to guide teaching on functions?
- How can we promote and systematize the reflection of future mathematics teachers (or teachers) on the function lessons they design and implement?

To answer these questions, we have combined two research agendas: on the one hand, that of the mathematics teacher's knowledge and, on the other, the spaces of reflection on practice, such as that offered by *micro-teaching*. The results of this research served to build a methodological proposal with criteria and descriptors that allow carrying out processes of reflection on the teaching practice to teach the notion of function. These criteria and descriptors are an essential part of the teacher's didactic-mathematical knowledge.

2. Theoretical Framework

At an international level, various proposals and models attempt to describe the components that make up the knowledge of the teacher of mathematics [27]. This study uses the Didactic-Mathematical Knowledge model (DMK) [6,28]. It allows us to have a more integrated vision of the dimensions of mathematics teaching and learning processes, which a mathematics teacher should, ideally, know. Furthermore, the DMK model allows utilizing the Onto-Semiotic Approach's theoretical-methodological tools [29,30] to operationalize the dimensions and sub-dimensions of the didactic-mathematical knowledge proposed with the model.

The DMK model proposes three major dimensions for interpreting and characterizing the teacher's knowledge: mathematical, didactical, and meta didactic-mathematical. The mathematical dimension refers to the knowledge that allows the teacher to solve a mathematical activity (on a particular notion) intended for the classroom and link it with other mathematical notions found later in the school mathematics curriculum [28]. For its part, the didactical dimension refers to the knowledge and essential aspects of the teacher's work in each of the facets involved in the processes of mathematics instruction. That is, it includes six subcategories: knowledge about the fundamental characteristics of mathematical objects (epistemic facet); knowledge about students' cognitive aspects and prior knowledge (cognitive facet); knowledge about students' interests, needs, attitudes and emotions (affective facet); knowledge about students' capacity for autonomy and the interactions that arise in the teaching and learning processes (interactional facet); knowledge about material resources, classroom conditions and temporality to promote learning (mediational facet); and knowledge about the school curriculum, didactic innovation and sociocultural aspects of students (ecological facet) [6,28].

In this article, the third dimension is of particular relevance. The meta didactic-mathematical [31], which refers to knowledge about institutional conditions, norms and meta-norms that regulate classroom and content management, mechanisms that allow assessing the didactic suitability in the teaching and learning processes to reflect on the teaching practice exercised and determine actions that promote suitable mathematical instruction processes [6].

The DMK model proposes, for each one of its dimensions and facets, various methodological tools that allow its operationalization. Concerning the third dimension, the notion of *didactic suitability* is proposed [29], which articulates, in a systemic and coherent way, the six facets of the didactical dimension mentioned above. The authors in [6] describe the six-partial didactic suitability criteria as follows:

- *Epistemic suitability* refers to the mathematics taught being good mathematics. Besides taking as a reference the prescribed curriculum, it is a matter of taking as a reference the institutional mathematics that has been transposed into the curriculum.
- *Ecological suitability* is the degree of adaptation of the mathematical instruction process to the institutional and curricular guidelines and to the socio-cultural conditions of the students.
- *Cognitive suitability* makes it possible to assess both the relationship between students' prior knowledge and the intended/implemented learning, and the similarity of the acquired learning with respect to the intended/implemented learning.
- *Affective suitability* assesses how involved the student is in the mathematical instructional process.
- *Interactional suitability* is the degree to which interactions (between the teacher and the students and among students) favor the teaching and learning process and allow the resolution of students' concerns and difficulties.
- *Mediational suitability* makes it possible to assess the relevance of the material and temporal resources in the mathematical instruction process.

The notion of didactic suitability, dimensions, and criteria represents a guiding resource for effective classroom management, allowing the assessment of planned or implemented educational actions. For this, in various studies [32,33], tables that describe generic components and indicators that constitute a guide for analysis and reflection on each one of the facets of the DMK model have been presented.

In the next section, didactic suitability criteria are presented to guide teaching, reflection, and assessment on the practice of the prospective teacher of mathematics when performing teaching and learning processes on the notion of function.

3. Systematizing Reflection on the Teaching of Functions

As described in the background section, the notion of function has been the subject of attention in different theoretical approaches, especially on issues regarding students' cognitive difficulties with this mathematical object and on the difficulties of instructional processes for teaching functions. For example, regarding the representations, [34] noted that the cognitive difficulties students encounter are the product of traditional teaching habits, since the vast predominance of the algebraic register (symbolic-formal) prevents students from gaining flexibility in the passage from one register to another. In addition to the works that show the types of representations traditionally used to introduce and study the notion of function, or their advantages and disadvantages (e.g., [35,36]), other studies focus on the resources. For example, the metaphors that teachers use to 'facilitate' students' understanding. They can serve as conceptual bridges (which also allows expanding the meaning of a person's notion of function), but are usually used in an unconscientious way under the belief that the effects on students' understanding are harmless [37].

Sanchez and Llinares [38] point out that prospective mathematics teachers have limitations concerning their students' knowledge, understanding, or skills. Besides, they highlight the need for teacher education programs to emphasize that knowledge of content integrates the different representations of mathematical objects, and that knowledge of pedagogical content allows discerning the problems and difficulties related to concepts. In terms of motivational strategies, teachers should consider it relevant to present contextualized everyday life problems to motivate students and justify the use of functions. Nyikahadzoyi [25] proposes a theoretical framework of the knowledge required by mathematics teachers for teaching functions. The study recognizes the characteristics of arbitrariness and univalence as essential elements of functions. It also describes the importance of approaching the teaching of functions from different representations using an extensive collection of examples to explain the mathematical object's usefulness and link functions with other notions to show their primary and unifying character. Likewise, the study points out the importance of promoting situations presenting functions with explicit domains and ranges in the instructional process. Other aspects described in this theoretical framework refer to the teacher's knowledge of the various cognitive obstacles associated with the use of representations, the school curriculum, and the various standard and advanced technologies that could facilitate learning.

The treatment of teachers' examples in the classroom allows identifying significant aspects of their pedagogical and mathematical knowledge base, supporting or limiting students' learning [39]. In this sense, according to [40], the examples presented by teachers in training in the development of their lessons correspond mostly to typical examples, or exercises that require the use of procedures or the application of theorems with which students learn algorithms, as a resolution tool. On the other hand, more experienced teachers use examples that arise after the exploratory phase of the notion of function, i.e., students face deepening of the concept and its various representations. They are examples that aim to clarify and expand the cognizable characteristics of the notion of function, address students' doubts, and avoid confusion.

Sintema and Marban [41] continue to point out that teachers in training have difficulties understanding the notion of function and have little knowledge about this mathematical object. In their study, they conclude, for example, that teachers are aware of the conditions under which inverse functions exist. However, they confuse the composition of functions with ordinary multiplication, exhibit insufficient knowledge of the meaning of inverse functions, and present inconveniences to promote real-life situations that could favor their students' learning of functions and their motivation.

What is presented so far is an overview of the scientific literature's advances and contributions to the teaching and learning of functions. In this article, we cannot be exhaustive for space reasons. However, Tables 1–6 summarize the in-depth literature review of this work, which constitutes our proposed criteria and descriptors for each one of the six suitability criteria (epistemic, cognitive, affective, mediational, interactional, and ecological). In sum, to elaborate our proposal of components and descriptors, on the criteria of didactic suitability, specific for the teaching of functions, we have based ourselves on the review and contributions of scientific literature on the subject. In [25] a theoretical framework of the teacher's knowledge of functions is presented, and the results of a historical-epistemological and curricular study in [42], which allowed the identification of the following diverse meanings that the notion of function has acquired throughout its origin, evolution, and formalization: (1) as correspondence; (2) relationship between variable magnitudes; (3) as graphic representation; (4) as analytical expression; (5) as arbitrary correspondence; and (6) as function from set theory.

Table 1. Epistemic suitability for teaching functions.

Components	Indicators
Situations- Problems	Problems that mobilize, representatively, the six reference meanings of the function are presented. Problems to reinforce previous knowledge related to the notion of function are presented. Problems to exemplify different definitions (informal, pre-formal, and formal) of the notion of function are presented [40]. Problems in purely mathematical contexts to reinforce learning about functions are presented. Contextualized problems from everyday life or other sciences are presented to reinforce learning about functions [21].
Languages	Representations linked to the function (verbal, symbolic/algebraic, tabular, graphic, and iconic) are mobilized [43,44]. Treatments are promoted in the various representation registers (verbal, algebraic, tabular, and graphic). For example, given the function $f : R \to R$ defined by $f(x) = x^2 + 2x + 1$, a factoring treatment is applied to obtain the function $f : R \to R$ defined by $f(x) = (x+1)^2$. The treatment of the original function must not alter the domain or range of the resulting function. Otherwise, the function is not the same. Conversions are promoted between the various representation registers (verbal, algebraic, tabular, and graphic). For example, to access the idea of continuity, it is convenient to use a graphic register. To enhance the idea of correspondence, it is pertinent to use an algebraic register [35].
Definitions, propositions, procedures, arguments	The definitions and procedures consider arbitrariness and univalence as crucial characteristics of the notion of function [24]. The notion of domain and co-domain are presented as inherent elements of the definition of function [22]. It promotes the meaning of the school curriculum's notion of function to identify and argue functional relationships in its various representations. Introductory statements and procedures relating to the notion of function appropriate to the level of education are presented. Situations where students must justify their guesses and procedures are promoted. The various meanings of the notion of function are identified and articulated, i.e., the function as correspondence, relationship between variable magnitudes, graphical representation, analytical representation, arbitrary correspondence, and from set theory [42].

Table 1. *Cont.*

Components	Indicators
Errors, ambiguities, and beliefs	Working with functions is not limited to the use of algebraic representations to prevent them from being perceived only as formulae and regularities. The belief that a change in the independent variable necessarily implies a change in the dependent variable is avoided; otherwise, a constant function might not be considered a functional relationship [25]. Functional relationships that are not graphable are presented to avoid the belief that any function supports a graphical representation. The error of using continuous curves for discrete functions is avoided [45]. Functional relationships that do not have an algebraic expression, formula, or equation associated with them are presented to avoid the belief that every function supports an algebraic representation. Functions with specific domains and co-domains are presented to avoid the belief that every function has a natural or real domain and co-domain. 'Irregular' graphs are presented to avoid the belief that any graphically represented function has 'good behavior' (symmetrical, regular, smooth, and continuous) [36].

Table 2. Cognitive suitability for teaching functions.

Components	Indicators
Previous knowledge	It is confirmed that students have the necessary prior knowledge to introduce the notion of function: patterns and regularities, proportionality, among others. Previous knowledge is linked to the notion of function. Since the notion of function is introduced, mathematical tasks that allow passing through the various representations associated with that mathematical object are presented, independent of the educational level [44].
Curriculum adaptations to individual differences	Expansion, reinforcement, counter-example, and analogy activities are included. For example, display a graphical representation to illustrate a correspondence that does not verify the definition of function [40].
Learning	Various evaluation tools are used to verify the representativeness of the student's personal meaning regarding the intended or implemented meaning on the notion of function [46].
High cognitive demand	It promotes the study and analysis of the variability of phenomena subject to change, where the notion of function has special significance totally linked to its epistemological origins. Mathematical tasks are proposed where the use of graphical representation is a more efficient strategy of solving the situation problem than tabular or algebraic representation. It differs and relates mathematical expressions that represent a function or equation. For example, a linear function can be represented algebraically as $f : R \to R$ defined by $f(x) = 2x$ and in the Cartesian plane as a line. Students may associate the graphical representation with the equation of the line and not to a functional relationship of real numbers [22].

Table 3. Affective suitability for teaching functions.

Components	Indicators
Interests and needs	The notion of function is presented as a useful tool in solving various mathematical problems from other sciences or everyday life [23]. There are situations in which functions serve as mathematical models for the study of phenomena and for expressing dependence between variables [42]. The study of functions is carried out in a similar way to their historical evolution. That is to say, considering the problems and needs that gave rise to that mathematical object.
Attitudes	Participation in proposed activities, perseverance, responsibility, among others, are promoted [32]. Logical reasoning, argumentation, modeling, analytical thinking, and problem-solving skills are favored.
Emotions	Self-esteem is promoted, avoiding a negative predisposition to the study of functions [32]. The precision and rigorous qualities of mathematics are highlighted.

Table 4. Interactional suitability for teaching functions.

Components	Indicators
Teacher-student interaction	The teacher makes an adequate presentation of the topic (clear and well-organized presentation, does not speak too quickly, emphasizes the fundamental concepts of the notion of function) [22]. The teacher anticipates and determines students' misconceptions, interprets 'incomplete' thoughts, predicts how students solve specific tasks, and estimates those they will find interesting and challenging [25]. The teacher identifies current conceptions of functions that students possess and then uses that conception to deepen or 'reformulate' the teaching of the functions. Various rhetorical and argumentative resources are used to capture students' attention [32]. It facilitates the inclusion of students in the dynamics of the lesson.
Interaction between students	Dialogue and communication between students are encouraged. Consensus is sought from instances of discussion, analysis, and mathematical argumentation. The inclusion and participation of the group are favored, and exclusion is avoided [32].
Autonomy	There are times when students take responsibility for the study (they raise questions and present solutions; they explore examples and counter-examples to investigate and make conjectures; they use a variety of tools to reason, make connections, solve problems, and communicate them).
Formative assessment	Systematic observation of students' cognitive progress [32].

Table 5. Mediational suitability for teaching functions.

Components	Indicators
Material resources (manipulative, calculators, computers)	Different ways of dealing with function teaching are used, various task sequences, representations, procedures, explanations, and arguments linked to the intended meaning are presented [25]. The notion of the function is accessed using set diagrams, value tables, ordered pair sets, charts, algebraic expressions, among others [18]. Computer graphics and graphing calculators are used to model functional relationships that allow visualizing the graph of a function as a static curve and not as the path of a point. Controlled use of metaphors is made, being aware of its disadvantages. The notion of function is introduced as a machine, curves are presented as the trace that leaves a point moving subject to certain conditions, the graph of an f function is the set formed by the $(x, f(x))$ coordinate points, among others [37].
Number of students, schedule and classroom conditions	The distribution of students in the classroom allows carrying out the intended teaching of the notion of functions. An adaptation of the process of mathematical instruction appropriate for the class schedule [32]. The educational space is adequate for the development of the intended instructional process on functions.
Time (of collective learning/mentoring; time of learning)	The teacher identifies frequent errors and anticipates the responses that can emerge from the presented questions on the notion of function. This allows classifying the activities according to their difficulty, designing of assessment instruments, estimating the processing time of a concrete task for students and groups of specific learning.

Table 6. Ecologic suitability for the teaching of functions.

Components	Indicators
Alignment to curriculum	The notion of function is presented as a primary and unifying principle, according to the secondary school curriculum [8]. The mathematical instruction on function is developed from the modeling of daily life situations or other sciences. Solutions of the problems from the arithmetic, algebraic, and geometric points of view are interpreted.
Connections and arguments	The notion of function is linked to other mathematical objects, such as regularities, proportionality, isometric transformations, determinants of matrix, limits, derivatives, among others [25]. Functions are used to respond to simple physical phenomena [40].

Table 6. *Cont.*

Components	Indicators
Socio-Laboral usefulness	Functions are presented as the most suitable tool to respond to situations from mathematics itself, other sciences, or everyday life.
Openness to didactic innovation	It promotes the use of ICT (Information and Communication Technologies), primarily as a support for understanding the notion of function and for manipulating representations linked to that mathematical object [19].

Thus, Tables 1–6 could be considered the didactic-mathematical knowledge of reference for the teaching of functions and also constitute a theoretical-methodological proposal that allows systematization of the reflection and assessment processes of the teaching practice (one's own or that of others) and, thus, determine (or anticipate) actions that improve the teaching processes on the notion of function.

4. Methodology

Our research's methodological paradigm is qualitative [47] since we are interested in determining aspects of the teacher's didactic-mathematical knowledge that could guide and systematize the reflection and evaluation of the instructional processes on the notion of function. For this, in the first stage, specific *didactic suitability* criteria are defined for teaching functions (Tables 1–6). In a second stage, due to the scant research on practical experiences in professional training of teachers [8], we designed a micro-teaching experience, in which a group of Chilean teachers could design, implement and reflect on their mathematics lessons in a simulated context. Specifically, this study shows the class given by one of the teachers participating in the experience, which dealt with the power function.

4.1. Micro-Teaching

In this research, we use a *micro-teaching* space since we recognize, as in various studies (e.g., [9,16]), that they are essential in teacher education. They can foster analysis, self-observation, and self-reflection of the prospective teacher's work in the mathematical instruction processes. Besides, it allows feedback moments to improve teaching performance and achieve more effective classroom and content management. The analysis of video recordings of class simulations represents an element that contributes to the feedback process, since it allows the observing of various aspects of teaching and learning processes: communication-interaction in the classroom, use of didactic-mathematical resources, the relevance of definitions and motivational activities used, among others.

According to Woolfolk Hoy and Burke Spero cited in Mergler and Tangen [12]:

> Watching another demonstrate a required skill (such as teaching) is to engage in a vicarious learning experience. Pre-service teachers need to be exposed to skilled others who can model the teaching 'performance' to a high standard. However, simply viewing teaching is not enough to result in meaningful learning. Being able to then practice the task contributes to mastery of the needed skills. Feedback that pre-service teachers receive about their teaching also plays an important role in bolstering (or lowering) their efficacy.
>
> (p. 200)

In accordance with Mergler and Tangen [12] "this type of social and verbal persuasion encourages pre-service teachers to reflect on their performance and offers an outside perspective that can impact on their efficacy" (p. 200).

4.2. Subjects and Context

In the framework of a mathematics teacher training program at a Chilean university, the first author of this article experimented in the course 'Progressive Practice III' using micro-teaching with prospective teachers. This training program's formative model promotes the early linkage of professional work through progressive internships from the fourth semester of the program. Six prospective teachers participated in this experiment. Following the methodological strategy of micro-teaching, the experimentation dynamics during the course were that prospective teachers taught their lesson while the rest of the future teachers would act as students of the educational level to which the class was directed. The lessons were videotaped and had a maximum duration of 40 min, then the teacher educator (first author of this article) opened the discussion with all the students to reflect on the implemented class, using fragments of the recorded video when required.

At the time of the experimentation, the prospective Chilean teacher who, for this study, we will call *Teacher A*, was in the sixth semester of the aforementioned high school mathematics teacher training program. The total duration of this mathematics teacher education program was eight semesters. Some subjects taken by Teacher A during his training were: elementary algebra, algebra of functions, differential calculus, integral calculus, foundations of the didactics of mathematics, learning and development, diversity learning and metacognition, technological literacy, mathematical software, educational informatics, teaching materials with ICT (Information and Communication Technologies), among others.

It is essential to mention that the six future teachers participating did not receive prior instruction for the design, implementation, and reflection of their lessons. However, the trainer guided the reflection, considering the criteria and descriptors of suitability for teaching the function, presented in Tables 1–6.

Before starting the process of instruction on the power function, teacher A provided, to the teacher educator, the lesson plan where the following was made explicit: the objective of the lesson, the content, the time for the intended teaching, the material resources, and a brief description of the elements to be treated during the beginning, development and closing of the lesson. That is, a lesson plan, describing definitions, examples, the use of representations and sequences of tasks for teaching power function, was not presented. The latter was not a requirement of the progressive practice III course.

The Chilean curriculum approaches mathematical learning in a progressive manner and with increasing conceptual and procedural complexity. It promotes the application of definitions, the use of algorithms, modeling, problem solving and the use of digital tools. The notion of function and its representations is presented for the first time in eighth grade, that is, with students from 13 to 14 years old. In ninth and tenth grade (secondary education), the intention is for students from 14 to 16 years old to expand their knowledge of linear functions by integrating quadratic and inverse functional relationships. In the eleventh and twelfth grades (secondary education), students aged 16 to 18 must apply and construct mathematical models of phenomena involving exponential functions, logarithmic functions, power functions and trigonometric functions [48].

It is worth noting that the study of the notion of the power function is carried out in Chile in twelve grades (secondary school education), that is, with students aged 17–18 years. The Chilean Ministry of Education proposes didactic guidelines for teaching functions. They suggest linking the notion of power function with concepts (e.g., inverse proportionality) from previous educational levels and using contextualized problems. They also propose using power functions to model everyday life situations and interpreting solutions to problems from arithmetic, algebraic and geometric points of view as appropriate [48].

While Teacher A was in charge of the power function instruction process, the other five prospective teacher participants played the role of students aged 17 to 18, whom for this study, we will call Students 1, 2, 3, 4, and 5. After completing the class on power function, developed by Teacher A, an instance of feedback was made between the teacher educator and the six prospective teachers to clarify certain notions and elements involved in the teaching–learning processes on functions. Observing instructional processes allows prospective teachers to reflect on how they will execute their own lessons [12]. Furthermore, strengths were identified, weaknesses explored, and greater reflection of classroom interactions was provided through video recording review. The teacher educator facilitated dialogue for the exchange of ideas and comments on Teacher A's performance. This allowed prospective teachers to reflect on the practice and determine actions to improve the instructional process.

5. Analysis of the Reflection of Teaching Practice

Below is the analysis of a process of mathematical instruction on the notion of the power function, developed in the context of micro-teaching by a Chilean Prospective Teacher.

5.1. Description of the Practice Developed by Teacher A

The teacher begins the teaching process by explaining the lesson's objective: *"Knowing and understanding the concept of the power function and its graphic representation."* At the beginning of the instruction, the notions of power and exponential equation are reinforced. Then he defines the power function: *"It is all that function of the form $f(x) = ax^n$ where a is a non-zero real number and n must be a natural number greater than zero"*.

The teacher then refers to the existence of four power function cases: *"The power function with exponent even is defined as Given $f(x) = ax^n$ if n is an even number, we know that for any x its result will be positive, so both branches (symmetrical to the y axis) grow as quickly in the same direction"*. He then points out: *"An example of this equation … "*. He quickly corrects and states: *"An example of this function is $f(x) = x^6$"*. He then displays the graphical representation of the relationship, using Geogebra. Based on the graphical representation he mentions: *"If the coefficient a is positive, the branches open upwards (case 1) and if the coefficient a is negative, the branches open down"* (case 2). As an example, he proposes the expression $f(x) = 2x^4$, and produces the graphical representation. For this, he uses a tabular representation by assigning values to x and getting the values of y. It is worth noticing that, at the time of graphing, the teacher places the ordered pair in the plane, exchanging the position of the coordinates, which he later corrects, and points out that the representation corresponds to a positive parabola.

He then points out: *"Two other cases of power function are those which exponent is odd with a positive (case 3) or with a negative* (case 4)". He adds: *"This type of function may cause greater difficulty."* Once he presents the $f(x) = 2x^5$ ratio, he asks the students, "*What if x is zero?*" The teacher states: *"if $x = 0$, then $y = 0$ so this type of power function always goes from the point zero, and the positive and negative values will be obtained in branch 1 and 2 respectively, i.e., the numerical values will be equal in each branch, but with the opposite sign*". In the face of this assertion, a student asks, "*What is the zero point?"*. The teacher answers, "*the origin*."

As an activity, the teacher proposes some algebraic representations and requests their graphic representations; for this purpose, he suggests moving from algebraic representation to tabular and then to graphical. He then exposes the case of the expression $f(x) = ax^n$ with $n = 1$ and asks, *"How does the graph of that function behave?"*. After receiving some responses, he points out: *"Effectively, the graph is a line. Therefore, it represents the graph of a linear function. Then, it is not a power function, since its representation does not correspond to a parabola."*

Another question he proposes to the students is "*How does the graph of the $f(x) = x^0$ function behave?"*. The teacher produces the graph based on a tabular representation, assigned the values $\{-1, 0, 1\}$ to the variable x, and mentions that the values of the variable y are $\{1, 1, 1\}$, respectively, and concludes: *"in this case, it is a constant function and not a power function."* One of the students asserts, if $0^0 = 1$. The teacher points out that he cannot explain it but that he will clarify the doubt in the next class. Subsequently, he refers to the domain and the image of the power function. For this, he presents Figure 1. It should be noted that, in subsequent activities, the teacher requests: *"Determine the domain and image of the following functions* [e.g., $f(x) = x^2$; $f(x) = 5x^3$; $f(x) = -7x^4$]." To do this, he suggests using the information in Figure 1.

	Even concave upward function	Even concave downward function	Odd function
Domain	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
Image	$\mathbb{R}^+$	$\mathbb{R}^-$	$\mathbb{R}$

Figure 1. Definition of the domain and image of a power function given by Teacher A.

Before class ends, a student asks, *"What if n is negative?"* The teacher points out: *"Let's see the behavior of the graph of the function $f(x) = x^{-2}$".* To do this, he uses the tabular representation assigning the values $\{-2,\ 0,\ 2\}$ to the variable x and, along with the students, determines that the values of the variable y are $\left\{\frac{1}{4},\ 0,\ \frac{1}{4}\right\}$. It is necessary to indicate that the teacher does not adequately define the domain for the $y = x^{-2}$ relationship, because if we think of the relationship as a function of variable and actual values, it should be noted that the domain is $\mathbb{R} \setminus \{0\}$. That is, the indeterminacy of $\frac{1}{0^2}$ is not considered. The teacher continues to place the ordered pairs $\left(-2,\ \frac{1}{4}\right)$ and $\left(2,\ \frac{1}{4}\right)$ on the Cartesian plane, and calls attention to the fact that he omits the pair $(0,0)$, to then correctly sketch the graph of the function $f : R\backslash\{0\} \to R$ defined by $f(x) = x^{-2}$ and says, *"Here comes the idea of asymptotes."* A student asks, what are asymptotes? The teacher replies, *"Those that will never touch the Y and X axis."*

Finally, the teacher synthetizes the lesson and ends it with the following: *"Let us remember that in the power functions, the exponent must always be a natural number greater than one because if it were equal to one, it would be a linear function and if it were equal to zero it would be a constant function".*

5.2. Reflection on Practice: Teacher A's DMK

From the instructional process developed by Teacher A and the information extracted, it was only possible to describe and analyze in detail the suitability of the epistemic, cognitive, and mediational facets of the didactic dimension of Didactic-Mathematical knowledge (DMK). To do this, we used the built suitability criteria and descriptors (Tables 1–6).

5.2.1. Epistemic Suitability

From the *definitions*, *arguments,* and *justifications* provided by the teacher, we can conclude that he had a partial mastery of the content's knowledge. He failed to give mathematically satisfactory answers to more than one of the tasks he proposed. It should be noted that the proposed activities corresponded to those suggested in the Chilean curriculum of twelfth grade. During the observed practice, connections of the notion of power function with mathematical objects of more advanced educational levels were not observed.

Certain ambiguities were evidenced in the class development. Mainly at the beginning when the teacher provided the *definition* of power function and indicated: "given the function $f(x) = ax^n$, n must be a natural number greater than zero". However, when he summarized the lesson, he indicated that n must be a number greater than one since the expression represented a linear function and not a power function. That is, the teacher did not consider the linear function as a particular case of the power function. Then, it was evident that the teacher had no clarity on the meaning of reference of the power function [49].

On the other hand, the teacher defined domain and image as independent concepts of the notion of function. This was clearly evidenced in the activity he proposed to verify the graphical behavior of the relationship $y = x^{-2}$. In this activity, the teacher did not adequately define the domain so that the relationship $y = x^{-2}$ was a function of variable and real values; that is, the *univalence* condition was not fulfilled for $x = 0$. Furthermore, he did not specify that the relation $y = x^{-2}$ did not meet the definition of a power function given at the beginning of the class. Nor did he make explicit that said relationship could represent a rational function of the form $R(x) = \frac{p(x)}{q(x)}$, where p and q are polynomial functions, and q is not the zero polynomial, which domain is defined as the set of all real numbers except those for which the denominator q is zero, thus, missing the opportunity to show an 'irregular' graphical representation that admits a symmetric, smooth and discontinuous behavior.

Concerning the use of *Languages*, the teacher mobilized various representations (algebraic, tabular, graphical) during the instructional process. However, he only made *conversions* from algebraic to tabular representations and then to graphs. The importance of providing students with a broad spectrum of ways to represent the notion of function to avoid these representations being identified with a particular function is pointed out in [43].

Regarding the *procedures* carried out by the teacher with the activities that he proposed, only the evaluation of algebraic expressions was identified, and then the ordered pairs were located on the Cartesian plane. The *situations-problems* that were developed during the class were *to exemplify definitions presented and non-contextualized problems to reinforce the definitions introduced*. Contextualized tasks that required power functions to model phenomena were not identified, nor situations that represented functional relationships where the *arbitrary* and *univalent* conditions of the function were identified as crucial characteristics of said mathematical object.

Regarding the *arguments*, certain inaccuracies were identified by the teacher. We could observe this when he raised the expression $f(x) = x^0$ and proposed carrying out the graphic representation giving the values $\{-1,\ 0,\ 1\}$ to the variable x and indicated that the values of y were $\{1,\ 1,\ 1\}$. The teacher could not explain what he proposed, and approved that $0^0 = 1$, instead of defining the domain of the 'function' which would answer the student's concern. Furthermore, he presented $f(x) = x^0$ as a power function, when in the initial definition that he proposed, he indicated that n must be a natural number greater than zero. A similar situation occurred when he proposed the relation $y = x^{-2}$ or when he referred to the meaning of the asymptotes. Likewise, during the development of the class, there were no instances in which the Teacher asked the students to *justify* the procedures developed and/or interpret the behavior of the graphs constructed. Although he tried to answer all the students' questions, his answers were sometimes imprecise or even incorrect. This was evidenced by questions such as: "Teacher $0^0 = 1$?", "A linear function is not a power function?", "Teacher, you referred to the zero point. What is that?", "Power function graphs are always parabolas?", "What happens if in the function $f(x) = ax^n$, n is negative?", "What are the asymptotes?", to name a few.

In conclusion, according to the *situations-problems*, *definitions*, *representations*, *properties*, *procedures,* and *arguments* activated by the teacher in his practice, we can say that the meaning of the notion of function implemented was not representative of the meanings of reference. The approximation that he gave to the notion of power function was based fundamentally on the meaning of *graphical representation.* However, as we could observe, the students' problems and questions required using the set theory's function elements and analytic expression.

5.2.2. Cognitive Suitability

At the beginning of the class, the teacher referred to the notions of power and exponential equations as essential *prior concepts* for studying the power function. However, he did not link these concepts with the notion under study. We could observe this when he presented, in the Cartesian plane, the quadratic function $f : R \to R$ defined by $f(x) = x^2$ (a particular case of a power function), obtaining a concave upward parabola, which he did not relate to the parabola equation, wasting the opportunity to reinforce the fact that a quadratic function can be seen as the equation of a parabola. This would have made it possible to collect geometric information on the curve associated with the function. However, it would be necessary to specify that the equation of a parabola does not always represent a functional relationship. Usually, the notion of function is presented as a mathematical object foreign to others previously worked on (proportionality, equations, etc.). In general, the fundamental mathematical notions that allow the introduction of a particular function are not precisely identified [8]. Thus, it was found that Teacher A did not corroborate the current conceptions that his students had about the notion of function. According to [50], when teachers in training examine students' procedures and inquire about the learner's mathematical understanding, they can identify and analyze what aspects are required in the instructional process to maximize their students' learning. Makonye [21] suggests that using different contexts and interrelating diverse representations of the notion of function favors students' learning. In this sense, Teacher A's intention to incorporate graphic, tabular, and algebraic representations in his instructional process was appreciated. Although all the representations that he used contain the same information, they took different cognitive processes. That is, graphic representation favors visualization and is related to geometry, tabular representation highlights the idea of correspondence and numerical aspects, and algebraic representation emphasizes symbolic capacity and is linked to algebra [44].

As for curriculum adaptations, examples were only used in purely mathematical contexts, which were presented after introducing a definition to show general aspects of the power function. While typical activities of implementing procedures that promote student participation to verify learning were presented, no situations involving power functions or examples of deepening or counter-examples that allowed students to anticipate difficulties and reduce confusion situations were proposed. This practice is consistent with what was proposed by [40], who state that novice teachers show particular difficulties in the choice and sequencing of examples, i.e., they mainly present typical examples or exercises that require the use of algorithmic procedures, seen as a resolution tool.

Given that the instructional process developed by Teacher A focused on algorithmic techniques, there were no activities that promoted the study and analysis of the variability of phenomena subject to change, where the notion of function has special significance. Although Teacher A proposed a definition and presented various representations of the power function, the cognitive structure associated with a mathematical object, in addition to formal definitions and representations, involves mental images, models, examples, counter-examples, and relations to other mathematical notions [45]. In this sense, Teacher A had the challenge to include these elements to achieve high cognitive demand.

In this way, we can observe that the teacher's practice conformed to traditional teaching habits. That is, there was a significant predominance of mechanical and algorithmic processes that prevented a significant appropriation of the notion of power function by students. Sierpinska [43] explains that the usual or traditional ways of coping with the teaching of functions are not sufficient for students to build the object's holistic meaning or understand its entire range of applications. Although teacher A used elements from set theory to define the notion of power function, function-formula, and function-curve associations predominated in the development of his class. These criteria could lead to reject functional relationships and admit non-functional objects [34].

5.2.3. Mediational Suitability

During the class, the teacher used *technological resources*; mainly, he used the Geogebra software as a digital tool to show students the graphic behavior of the relationships he presented. However, in the instructional process, the Geogebra software was only manipulated by the teacher; that is, the students did not have access to the software, which, accompanied by a well-planned activity, could strengthen one of the partial meanings of the notion of function, as a *graphic representation*.

The teacher's intention to incorporate in the development of his class tables of values and graphical representations was valued, as this allowed students to visualize the covariation of the variables involved [35]. However, the usual forms of representation were not sufficient for students to construct the holistic meaning of the object or understand the full range of its applications [43]. Focusing exclusively on algorithmic processes leads to considering the mathematical function as a mechanical formula, where the construction of tables is a simple requirement for graphing.

Although the teacher's use of *metaphorical resources* was not explicit, the existence of the metaphor described in [37] was perceived in his speech, "Curves are the trace that leaves a point that moves subject to certain conditions, and the analysis of these conditions allows finding an equation that meets the points of the curve" (p. 405) when he stated: "*given the function* $f(x) = 2x^5$, *if* $x = 0$, *then* $y = 0$, *therefore, this type of power function always goes from zero, and the positive and negative values in branches 1 and 2 respectively will be obtained, i.e., the numeric values will be equal in each branch, but with an opposite sign.*"

It is important to note that "from the work of Fourier, Cauchy, and Dirichlet, among others, it was accepted as graphs of functions, curves that could not be trajectories" [37] (p. 408). In this case, the teacher explained something static in dynamic terms, which is a metaphor that he did not control and was unaware of its consequences.

Finally, given the micro-teaching context in which the power function class occurred, both the physical space and students' distribution in the classroom were appropriate. Perhaps, for this reason, the teacher did not see the need to reorganize the room's distribution and the students. This aspect could be explored in studies conducted in authentic contexts.

5.3. Some Aspects of Interest in Teacher A's Reflection: Meta Didactic-Mathematical Dimension of the DMK

In the process of instruction on the power function developed by teacher A, a deep reflection on affective, interactional and ecological suitability was not achieved. A possible explanation could lie in characteristics of the micro-teaching contexts. However, at the end of the instructional process, a feedback activity was carried out between the teacher educator and prospective teachers. This was a reflective space in which we also used our criteria and suitability descriptors proposal for teaching functions. In this space, questions, comments, guidance, and suggestions were raised that led to the clarification of certain notions and elements involved in the teaching–learning processes on functions. Likewise, the video recording was analyzed to generate spaces of self-observation and self-reflection of Teacher A's work and the questions of other prospective teachers (that acted as students). This allowed visualizing the suitability of the use of resources, and the precision of definitions and given answers, among others. One of the aspects reflected upon was the meanings of the function. In this regard, the following discussion was held between the prospective teachers and the teacher educator:

> **Teacher educator:** *While in your class you defined the power function, but you did not reinforce the meaning of functional relationship. How would you explain or elaborate on the meaning of function?*
>
> **Teacher A:** *I would say that a function is any element of a set, starting, that corresponds to an element in the set of arrival. Mmm [...] I think I would explain it with a diagram [Teacher A draws on the whiteboard a diagram like the one presented in Figure 2].*

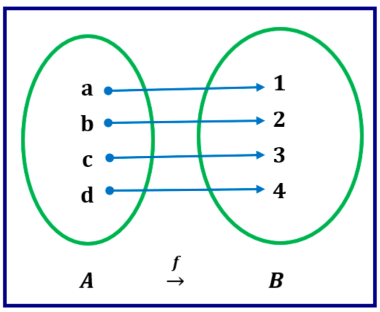

Figure 2. Image similar to the one made by Teacher A.

Student 1: *I would start by pointing out that the function is a relationship between two sets and use the machine to explain the relationship, but I think the machine is relevant when first teaching functions, then it could generate confusion and students might be left with only the idea of the machine.*

Teacher A: *Hmm, I also think I would use the machine to explain that we take one number, transform it and get another.*

Teacher educator: *In the definition you provide on the power function, you do not refer to the domain and co-domain of the function, this is mentioned in another part of the class. Do you consider that it would be convenient to incorporate in your definition of function the notions of domain and co-domain?*

Teacher A: *I thought about it, but I considered that giving the definition of the power function and then referring to the domain and range would be easier to understand. Now I question it, and I do not know why I left it aside if the domain and range are part of the definition.*

Teacher educator: *If we consider the notion of domain and co-domain as inherent elements of the definition of a function, can we answer the problem that arose when you proposed the expression* $f(x) = 0^0$*?*

Teacher A: *Maybe yes [...] I have to think about it more.*

Student 2: *In that case the domain of the function* $f(x) = 0^0$ *would have to exclude zero because*0^0 *is not 1.*

From the above dialogue, it was found that the explanations given by the prospective teachers considered only some aspects of the reference meaning of functions (e.g. the idea of correspondence, relation, elements of set theory). This led the teacher educator to present the historical and epistemological journey of the notion of function to strengthen prospective teachers' personal meaning and bring that significance closer to the holistic meaning of reference [42,43,51]. Another issue discussed was how to teach functions to encourage student learning. While Teacher A referred to the importance of going through different representations and explained the difficulties this creates for twelve grade students, he only emphasized the relevance of transiting from algebraic to tabular representations and from there to graphs, as exemplified in the following excerpt:

Teacher A: *I have taught lessons on power functions, and when I presented the expression* $f(x) = 2x^4$ *and asked students to determine the graph, the students would say, "Teacher, what do we have to do?". They did not automatically resort to the table. They did not understand that in order to produce the graph, it was necessary to give values to the variable* x *and then to evaluate to obtain the values of the variable* y*.*

Teacher educator: *Would you employ other representations to enhance the learning of the functions?*

Teacher A: *I would explain to them that they should give values to the variable* x *to then obtain the values of* y*. Then, they could determine several points and locate them in the Cartesian plane.*

This instance discussed the importance of identifying the mathematical object's different representations and transiting through all of them. It also reflected on the previous knowledge needed to address power function teaching, making it possible to conclude that it is essential to clarify a mathematical function's general definition.

Another of the elements discussed was that of conflict interactions. Interactions between teachers and students are often regulated by *rules*, *habits*, *obligations*, or non-explicit norms, such as *social norms* and *socio-mathematical norms* (e.g., [52–54]. Social norms regulate the interaction between teachers and students, while socio-mathematical norms regulate the specific aspects of students' mathematical discussions and influence learning opportunities. Concerning the developed class, we could identify some social norms in which students (prospective teachers) questioned the explanations described by Teacher A. For example, when they raised concerns about proposition $0^0 = 1$, given by Teacher A, or when he indicated that the graph of power functions was always a parabola. However, there were other situations where students did not adopt a critical attitude based on knowledge already 'acquired,' i.e., the case of the activity in which they jointly approved (teacher and students) that $\frac{1}{0^2} = 0$, or when they recognized "functions" that had not been well defined. The above is in line with [22] on the importance of considering functional sub-concepts as part of the function's definition. This was reinforced by the teacher educator, who emphasized the importance of presenting functions with explicit domains and co-domains to avoid the belief that every function has a natural or real domain and co-domain. Another element that was reflected upon referred to when Teacher A presented the case of the expression $f(x) = ax^n$ with $n = 1$ and asked "How does the graph of that function behave?", and immediately indicated that it was a linear function and not a power function:

> **Teacher educator:** *Among your examples, you posed verifying the graph of the expression* $f(x) = ax^n$ with $n = 1$ What was the purpose of the task?
>
> **Teacher A:** *Show that the exponent cannot be equal to 1 because it would become a linear function.*
>
> **Teacher educator:** *But in the definition provided at the beginning of the class it was made explicit:* $f(x) = ax^n$ *where a is a non-zero real number and n must be a natural number greater than zero.*
>
> **Teacher A:** *Hmm [...] then the definition should be for n a real number other than zero and one. Because when it is raised to zero, the function is constant, and when it is raised to one, the function is linear.*
>
> **Teacher educator:** *In this case, would we have to exclude* $n = 2$ *because with* $n = 2$ *we would obtain a quadratic function?*
>
> **Teacher A:** *Hmm [...] so I couldn't discard zero nor one?*
>
> **Student 3:** *It is that in that case, the constant, linear, quadratic functions etc., would be particular cases of a power function.*

It is worth noticing that Teacher A also proposed in the development of his class, analyzing the graphical behavior of expressions such as $f(x) = x^{-2}$ which did not satisfy the definition he proposed. This forced Teacher A's reflection on the power function's definition and implied the teacher educator's reinforcement of this mathematical object's conceptualization.

On the other hand, on the relevance of using resources and means that could facilitate understanding functions, Teacher A stated, *"It is essential that students participate during class, when they resolve an exercise on the board, they can reinforce the contents."* He also noted, *"Using Geogebra allows me to display the behavior of graphs more accurately."* In this regard, the teacher educator referred to what is indicated in various investigations (e.g., [55,56]). That is, to achieve an adequate understanding of the notion of function, it is necessary to transit from algebraic representation to graph and vice versa. Therefore, it is essential to visualize the graph of a function with its algebraic and tabular expression, and this is where Geogebra can have a significant impact. That is to say, the difficulties that may arise in modeling activities with functions can be supported with the use of ICT. However, it is clear that dynamic graphics facilitate, even if unconsciously, the structuring of graphs as traces of points. Furthermore, that the explicit or implicit use of dynamic metaphors (like "the graph is a path" type) has its advantages, but also its disadvantages when the graph is perceived as a trajectory and not as a static representation [37].

Finally, the teacher's responsibility for creating environments conducive to learning was discussed, proposing mathematically rich activities that belong to the students' field of interest. In this sense, the teacher educator and future teachers reflected on how to motivate students to study functions.

Teacher educator: *What kind of mathematical tasks should be proposed?*

Teacher A: *I consider that the class I just taught is very mechanical and does not motivate students. I would look for a new way to teach the power function, perhaps using problem-solving.*

Student 5: *I believe that contextualized examples should be shown with situations of interest to students. For example, the relationship between shirts and player. A shirt belongs only to one player, but a player could have more than one shirt. Hmm [...] we would have to think it through.*

Given this statement, the idea of developing mathematical instruction processes promoting diverse problem situations (some contextualized within mathematics and others based on functional models, it is possible to study phenomena from everyday life or from other sciences) was discussed. Collaborative work and dialogical and argumentative interaction must prevail since they potentially have greater interactional suitability than those of a magisterial type, since students would have more significant interaction with the mathematical objects that are intended to institutionalize and formalize, in our case, the notion of function.

To conclude this section, it is worth saying that the micro-teaching processes allowed prospective teachers to broadly perceive their performance and that of other teachers in training through discussion, analysis, feedback, and reflection of the experiences. For example, in another micro-teaching process developed within the framework of the subject "Progressive Practice III," a class on inverse function was implemented. In this instance, the instruction process was in the charge of a trainee teacher who participated in the role of student in the class on the power function of Teacher A. We could observe that this future teacher (Student 2 in this article) incorporated into his inverse function class elements discussed in Teacher A's class reflection. This is exemplified in the following excerpt:

Teacher educator: *When planning your class on inverse function, what aspects did you consider relevant to mention?*

Student 2: *Well, the first thing was to mention what are the requirements for the function to admit inverse. That's why I started the class reinforcing what is an injective, surjective and bijective function. Also, I think contextualized examples are useful for modeling life situations, and physics. Hmm [...] and well students always learn more when examples are shown.*

Teacher educator: *To exemplify the notion of injective function, you used the following situation: "If we define a first set, Chilean persons and the arrival set the RUN (Chilean ID number). This means that each person has a designated RUN number that is unique for that person, which does not mean that there are more RUN numbers that are not yet used, but every person has a RUN and two Chilean persons cannot have the same RUN". What is the intention of this type of examples?*

Student 2: *First, show a contextualized example where the relationship between the elements of the two sets is not with numbers and reinforce the idea that each element of the arrival set has at most one element or a preimage in the departure set.*

Teacher educator: *That is, was your intention to show a problem that mobilizes one of the meanings of the notion of function?*

Student 2: *Yes, the function as an arbitrary correspondence. And present a problem contextualized to everyday life to exemplify the definition of injective function.*

In general, the experimentation showed that through micro-teaching processes and the use of proposed criteria that guide reflection in such contexts (such as those included in Tables 1–6), future teachers were able to identify aspects through which they could improve their teaching practice and move towards more suitable teaching.

6. Final Reflections

In this article, we report the results of a study carried out with six prospective teachers during one semester to promote their meta didactic-mathematical knowledge required for reflection on their practice (or that of others) when teaching lessons on functions. In doing so, we seek to answer the questions: *What dimensions and components could mathematics teachers consider to guide teaching on functions? How can the reflection of future mathematics teachers on function classes be promoted and systematized?*

The first challenge that arose regarding the first question was a large amount of information and recommendations from the scientific literature on the processes of teaching and learning of functions and on the knowledge required by the teacher to teach this notion. We carried out a thorough review of the scientific literature and, based on the proposal of the DMK model (its dimensions, sub-dimensions, and, specifically, the notion of didactic suitability), it was possible to organize the information into six facets (epistemic, cognitive, affective, interactional, mediational and ecological), each with particular components and descriptors for the notion of function. Thus, the six partial suitability criteria, presented in Tables 1–6, for the teaching and learning of functions constitute a proposed theoretical-methodological tool to guide teaching and reflection of classes on functions. Such components and their indicators could be considered a first approximation to the didactic-mathematical knowledge of reference of teachers on the function.

In this research, our proposal of didactic-mathematical knowledge of reference on functions was used during a semester to guide six future teachers' reflections on their classes regarding this notion, which were developed in micro-teaching contexts. We exemplified the above with a specific case, that of the lesson developed by teacher A on the power function. The prospective teacher participants in this research could experience spaces for reflection based on dialogue, discussion, and analysis among peers, on the epistemic (mathematical richness of objects and processes), cognitive (features of students thinking and learning, and how to enhance their learning), and mediational (resources and material means, time, etc.) suitability of the practice developed by Teacher A. This also generated opportunities to promote, in the six participants, their DMK on functions. However, we obtained insufficient information to perform an in-depth analysis of affective, interactional, and ecological suitability. A possible explanation for the latter could lie in the very nature of micro-teaching contexts, for, as we saw, these are controlled environments and, in our case, had only a small number of prospective teachers. For this reason, it is necessary to continue researching spaces of reflection to analyze the suitability criteria that could not be analyzed in this article.

Thus, regarding the second question, we can assert that another essential element of our proposal is micro-teaching spaces [12], which provide prospective teachers with a safe and controlled environment to develop their practice and reflect on their performance. In no way does this methodology replace the teaching practice in a real context, but it allows gaining experience, self-assessment, and receiving of immediate feedback on the elements that present more significant strengths and/or deficiencies. In other words, it is recommended to incorporate in the initial training of teachers experiences in simulated and real contexts to explore possible pedagogical problems that lead to the reflection and critical analysis of teaching. Furthermore, the micro-teaching spaces allow working on the development of the diverse dimensions of the didactic-mathematical knowledge of the participating teachers. For example, the prospective teacher presenting the class has to use DMK on functions to plan and implement his lessons, and the prospective teachers that impersonate students have to "put on the shoes" of the students from the corresponding level. That is to say, acting and thinking as real students would (cognitive and affective facets of the DMK).

Thus, conjunctively considering micro-teaching spaces and the *suitability criteria on functions* that we propose herein (to guide reflection) can be a useful strategy for initial teacher education. Nevertheless, our proposal is not restricted to initial education. Tables 1–6 could be used in diverse moments of the didactic design (preliminary study, design, implementation, and evaluation) by researchers, teachers, and teacher educators. In the case that we used as an example, the authors of this article (teacher educators) used the criteria and suitability descriptors on functions to analyze and evaluate Teacher A's didactic-mathematical knowledge at the implementation moment, and, subsequently, to develop a space for reflection with future teachers around the analysis and assessment of Teacher A's practice. Teachers (or future teachers) can use these criteria to reflect on their planning and implementation of classes on functions, seeking potential improvements to rework their teaching designs, and, thus, begin to "make the practice of reflection habitual" [4]. Finally, it is necessary to point out that teaching practice is complex by nature and that, in the process, the six facets we have presented intervene simultaneously and dynamically. Therefore, the DMK model used has been fundamental for our study since few models provide theoretical-methodological notions that allow the identification and individualization of such facets and the analysis of their components separately for research and training purposes.

Author Contributions: Conceptualization, Y.E.P.-U. and L.R.P.-F.; Methodology, Y.E.P.-U. and L.R.P.-F.; Theoretical Framework, Y.E.P.-U. and L.R.P.-F.; Investigation (criteria and descriptors for the teaching of functions), Y.E.P.-U. and L.R.P.-F.; Analysis, Y.E.P.-U. and L.R.P.-F.; Writing-original draft preparation, Y.E.P.-U. and L.R.P.-F.; Writing-review and editing, Y.E.P.-U. and L.R.P.-F. All authors have read and agreed to the published version of the manuscript.

Funding: This work has been developed within the framework of the research project Fondecyt 1200005 funded by de Angencia Nacional de Investigación y Desarrollo (ANID) of Chile.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Shulman, L.S. Knowledge, and teaching: Foundations of the new reform. *Harv. Educ. Rev.* **1987**, *57*, 1–23. [CrossRef]
2. Ball, D.L. Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *J. Teach. Educ.* **2000**, *51*, 241–247. [CrossRef]
3. Hill, H.C.; Ball, D.L.; Schilling, S.G. Unpacking pedagogical content knowledge of students. *J. Res. Math. Educ.* **2008**, *39*, 372–400. [CrossRef]
4. Schoenfeld, A.; Kilpatrick, J. Towards a theory of profiency in teaching mathematics. In *Tools and Processes in Mathematics Teacher Education*; Tirosh, D., Wood, T.L., Eds.; Sense Publishers: Rotterdam, The Netherlands, 2008; pp. 321–354.
5. Mishra, P.; Koehler, M. Technological pedagogical content knowledge: A framework for teacher knowledge. *Teach. Coll. Rec.* **2006**, *108*, 1017–1054. [CrossRef]
6. Pino-Fan, L.; Godino, J.D.; Font, V. Assessing key epistemic features of didactic-mathematical knowledge of prospective teachers: The case of the derivate. *J. Math. Teach. Educ.* **2018**, *21*, 63–94. [CrossRef]
7. Wilkie, K.J. Upper primary school teachers' mathematical knowledge for teaching functional thinking in algebra. *J. Math. Teach. Educ.* **2013**, *17*, 397–428. [CrossRef]
8. Steele, M.D.; Hillen, A.F.; Smith, M.S. Developing mathematical knowledge for teaching in a methods course: The case of function. *J. Math. Teach. Educ.* **2013**, *16*, 451–482. [CrossRef]
9. Fernández, M.L. Investigating how and what prospective teachers learn through microteaching lesson study. *Teach. Teach. Educ.* **2010**, *26*, 351–362. [CrossRef]
10. Dewey, J. The relation of theory to practice in education. In *Teacher Education in America: A Documentary History*; Borrowman, M., Ed.; Teachers College Press: New York, NY, USA, 1965; pp. 140–171.
11. Grossman, P.; Compton, C.; Igra, D.; Ronfeldt, M.; Shahan, E.; Williamson, P. Teaching practice: A cross-professional perspective. *Teach. Coll. Rec.* **2009**, *111*, 2055–2100. [CrossRef]
12. Mergler, A.G.; Tangen, D. Using microteaching to enhance teacher efficacy in preservice teachers. *Teach. Educ.* **2010**, *21*, 199–210. [CrossRef]
13. Benton-Kupper, J. The Microteaching Experience: Student Perspectives. *Education* **2001**, *121*, 830–835.
14. Donnelly, R.; Fitzmaurice, M. Towards productive reflective practice in microteaching. *Innov. Educ. Teach. Int.* **2011**, *48*, 335–346. [CrossRef]

15. Altuk, Y.G.; Kaya, V.H.; Bahceci, D. A Study on Developing "Microteaching Scale" for Student Teachers. *Procedia-Soc. Behav. Sci.* **2012**, *46*, 2964–2969. [CrossRef]
16. Arsal, Z. Microteaching and preservice teachers' sense of self-efficacy in teaching. *Eur. J. Teach. Educ.* **2014**, *37*, 453–464. [CrossRef]
17. Brookfield, S.D. *Becoming a Critically Reflective Teacher*, 1st ed.; Josse-Bass: Hoboken, NJ, USA, 1995; pp. 1–320.
18. Akkoç, H.; Tall, D.O. The simplicity, complexity, and complication of the function concept. In Proceedings of the 26th Conference of PME, Norwich, UK, 21–26 July 2002; Cockburn, A.D., Nardi, E., Eds.; Volume 2, pp. 25–32.
19. Panaoura, A.; Michael-Chrysanthou, P.; Gagatsis, A.; Elia, I.; Philippou, A. A Structural Model Related to the Understanding of the Concept of Function: Definition and Problem Solving. *Int. J. Sci. Math. Educ.* **2017**, *15*, 723–740. [CrossRef]
20. Wang, Y.; Barmby, P.; Bolden, D. Understanding Linear Function: A Comparison of Selected Textbooks from England and Shanghai. *Int. J. Sci. Math. Educ.* **2015**, *15*, 131–153. [CrossRef]
21. Makonye, J.P. Teaching functions using a realistic mathematics education approach: A theoretical perspective. *Int. J. Educ. Sci.* **2014**, *7*, 653–662. [CrossRef]
22. Dede, Y.; Soybaş, D. Preservice Mathematics Teachers' Experiences about Function and Equation Concepts. *Eurasia J. Math. Sci. Technol. Educ.* **2011**, *7*, 89–102. [CrossRef]
23. Norman, A. Teachers mathematical knowledge of the concept of function. In *The Concept of Function. Aspects of Epistemology and Pedagogy*; Harel, G., Dubinsky, E., Eds.; Mathematical Association of America: Washington, DC, USA, 1992; Volume 25, pp. 215–232.
24. Even, R. Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *J. Res. Math. Educ.* **1993**, *24*, 94–116. [CrossRef]
25. Nyikahadzoyi, M.R. Teachers' knowledge of the concept of a function: A theoretical framework. *Int. J. Sci. Math. Educ.* **2013**, *13*, 261–283. [CrossRef]
26. Hatisaru, V.; Erbas, A. Mathematical Knowledge for Teaching the Function Concept and Student Learning Outcomes. *Int. J. Sci. Math. Educ.* **2017**, *15*, 703–722. [CrossRef]
27. Neubrand, M. Conceptualizations of professional knowledge for teachers of mathematics. *ZDM Int. J. Math. Educ.* **2018**, *50*, 601–612. [CrossRef]
28. Pino-Fan, L.; Assis, A.; Castro, W.F. Towards a methodology for the characterization of teachers' didactic-mathematical knowledge. *EURASIA J. Math. Sci. Techonol. Educ.* **2015**, *11*, 1429–1456. [CrossRef]
29. Godino, J.D.; Batanero, C.; Font, V. The onto-semiotic approach to research in mathematics education. *ZDM Int. J. Math. Educ.* **2007**, *39*, 127–135. [CrossRef]
30. Godino, J.D.; Batanero, C.; Font, V. The onto-semiotic approach: Implications for the prescriptive character of didactics. *Learn. Math.* **2019**, *39*, 37–42.
31. Breda, A.; Pino-Fan, L.; Font, V. Meta didactic-mathematical knowledge of teachers: Criteria for the reflection and assessment on teaching practice. *Eurasia J. Math. Sci. Technol. Educ.* **2017**, *13*, 1893–1918. [CrossRef]
32. Godino, J.D.; Batanero, C.; Rivas, H.; Arteaga, P. Componentes e indicadores de idoneidad de programas de formación de profesores en didáctica de las matemáticas [Components and indicators of suitability of teacher training programmes in didactics of mathematics]. *REVEMAT* **2013**, *8*, 46–74. [CrossRef]
33. Breda, A.; Font, V.; Pino-Fan, L. Criterios valorativos y normativos en la Didáctica de las Matemáticas: El caso del constructo idoneidad didáctica [Valuative and normative criteria in the Didactics of Mathematics: The case of the didactic suitability construct]. *Bolema* **2018**, *32*, 255–278. [CrossRef]
34. Artigue, M. La enseñanza de los principios del cálculo: Problemas epistemológicos, cognitivos y didácticos [Teaching the principles of calculus: Epistemological, cognitive and didactic problems]. In *Ingeniería Didáctica en Educación Matemática*; Gómez, P., Ed.; Grupo Editorial Iberoamericano: México, Mexico, 1995; pp. 97–140.
35. Nitsch, R.; Fredebohm, A.; Bruder, R.; Kelava, A.; Naccarella, D.; Leuders, T.; Wirtz, M. Students competencies in working with functions in secondary mathematics education—empirical examination of a competence structure model. *Int. J. Sci. Math. Educ.* **2015**, *13*, 657–682. [CrossRef]
36. Adu-Gyamfi, K.; Bossé, M.J. Processes and reasoning in representations of linear functions. *Int. J. Sci. Math. Educ.* **2013**, *12*, 167–192. [CrossRef]
37. Font, V.; Acevedo, J. Fenómenos relacionados con el uso de metáforas en el discurso del profesor. El caso de las gráficas de funciones [Phenomena related to the use of metaphors in the teacher's speech. The case of function graphs]. *Enseñanza De Las Cienc.* **2003**, *21*, 405–418.
38. Sánchez, V.; Llinares, S. Four student teachers' pedagogical reasoning on functions. *J. Math. Teach. Educ.* **2003**, *6*, 5–25. [CrossRef]
39. Zaslavsky, O.; Harel, G.; Manaster, A. A teacher's treatment of examples as reflection of her knowledge-base. In Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education (PME), Prague, Czech Republic, 16–21 July 2006; Novotná, J., Moraová, H., Krátká, M., Stehlíková, N., Eds.; Volume 5, pp. 457–464.
40. Figueiredo, C.; Contreras, L.; Blanco, L. La ejemplificación del concepto función: Diferencias entre profesores noveles y profesores expertos [The exemplification of the function concept: Differences between novice and expert teachers]. *Educ. Matemática* **2012**, *24*, 73–105.
41. Sintema, E.J.; Marban, J.M. Preservice Teachers' Knowledge of Identifying and Clearing Pupils' Misconceptions about Inverse and Composite Functions via Vignettes. *Eurasia J. Math. Sci. Technol. Educ.* **2021**, *17*, em1930. [CrossRef]

42. Pino-Fan, L.; Parra-Urrea, Y.; Castro, W.F. Significados de la función pretendidos por el currículo de matemáticas chileno [Meanings of function intended by the Chilean mathematics curriculum]. *Magis Rev. Int. De Investig. En Educ.* **2019**, *11*, 201–220. [CrossRef]
43. Sierpinska, A. Understanding the Notion of Function. In *The Concept of Function Aspects of Epistemology and Pedagogy*; Harel, A., Dubinsky, E., Eds.; Mathematical Association of America: Washington, DC, USA, 1992; pp. 25–58.
44. Janvier, C. *Problems of Representation in the Teaching and Learning of Mathematics*; Lawrence Erlbaum: Hillsdale, NJ, USA, 1987; p. 247.
45. Vinner, S.Y.; Dreyfus, T. Images and definitions for the concept of function. *J. Res. Math. Educ.* **1989**, *20*, 356–366. [CrossRef]
46. Ramírez, R.; Ibarra, S.; Pino-Fan, L. Prácticas evaluativas y significados evaluados por profesores del bachillerato mexicano sobre la noción de ecuación lineal [Evaluative practices and meanings evaluated by Mexican high school teachers on the notion of linear equation]. *Educ. Matemática* **2020**, *32*, 69–98. [CrossRef]
47. Creswell, J.W. *Research Design. Qualitative, Quantitative, and Mixed Method Approaches*; Sage Publications: London, UK, 2009; pp. 1–342.
48. Mineduc. Bases Curriculares tercero y cuarto año medio. Ministerio de Educación Gobierno de Chile. Bases Curricula for Eleventh and Twelfth Grade (Secondary Education); Ministry of Education Government of Chile. 2019. Available online: https://www.curriculumnacional.cl/614/articles-91414_bases.pdf (accessed on 20 August 2022).
49. Sullivan, M. *Algebra y Trigonometry Eighth Edition*; Pearson Education: Upper Saddle River, NJ, USA, 2008; pp. 1–1088.
50. Llinares, S.; Valls, J.; Roig, A.I. Aprendizaje y diseño de entornos de aprendizaje basado en videos en los programas de formación de profesores de matemáticas [Video-based learning and design of learning environments in mathematics teacher training programs]. *Educ. Matemática* **2008**, *20*, 31–54.
51. Biehler, R. Reconstruction of Meaning as a Didactical Task: The Concept of Function as an Example. In *Meaning in Mathematics Education*; Kilpatrick, J., Hoyles, C., Skovsmose, O., Eds.; Kluwer: Dordrecht, The Netherlands, 2005; pp. 61–81.
52. Godino, J.D.; Font, V.; Wilhelmi, M.R.; Castro, C. Aproximación a la dimensión normativa en didáctica de las matemáticas desde un enfoque ontosemiótico [Approximation to the normative dimension in didactics of mathematics from an ontosemiotic approach]. *Enseñanza Cienc.* **2009**, *27*, 59–76.
53. Yackel, E.; Cobb, P. Sociomathematical Norms, Argumentation, and Autonomy in Mathematics. *J. Res. Math. Educ.* **1996**, *27*, 458–477. [CrossRef]
54. McClain, K.; Cobb, P. An analysis of development of sociomathematical norms in one first-grade classroom. *J. Res. Math. Educ.* **2001**, *32*, 236–266. [CrossRef]
55. Elia, I.; Panaoura, A.; Eracleous, A.; Gagatsis, A. Relations between Secondary Pupils' Conceptions about Functions and Problem Solving in Different Representations. *Int. J. Sci. Math. Educ.* **2007**, *5*, 533–556. [CrossRef]
56. Gómez Chacón, I.; Joglar, N. Developing competencies to teach exponential and logarithmic functions using Geogebra from a holistic approach. *Educ. Matemática Pesqui.* **2010**, *12*, 485–513.