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Theorems of Common Fixed Points for Some Mappings in b_2 Metric Spaces

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Abstract: In this research paper, we concentrate on the existence and uniqueness of common fixed points of various mappings in b_2 metric space under generalized $(\phi, f)_\lambda$-expansive conditions and implicit contractive conditions. Additionally, we derive a progression of noteworthy approaches and ideas extending the conclusions on the real metric space to the b_2 metric space, which advances the investigation of the b_2 metric space will be able to advance.

Keywords: common fixed point; expansive mappings; implicit relation; b_2 metric spaces; generalized contraction

MSC: 47H10; 54H10

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1. Introduction

Since Alber and Guerre-Delabrere [1] introduced a class of strongly compressed maps called weakly contracted maps on closed convex sets of Hilbert spaces and proved that it was an iterative operator on Hilbert spaces that extended the Banach principle, which was formerly applied to strongly contracted maps alone, Rhoades [2] extended these works to arbitrary Banach spaces and proved that their conclusions still held. On the basis of the conclusions proposed by the previous authors, Chuanxi Zhu [3] succeeded in gaining some results, which were related to the common fixed point of four mappings under a generalized weak contraction of a partially ordered metric space. Inspired by this research, we demonstrate that the results still hold when the space is replaced by a b_2 metric space [4] consisting of a b-metric space [5–8] and a 2-metric space [9–15].

Simultaneously, we found that the authors [4,16–18] discussed and obtained the common fixed point theorem for a limited mapping family in the b_2 metric space, but they largely studied it under explicit or semiexplicit contraction conditions. Notwithstanding, by introducing implicit contraction conditions, the authors [19] discussed the common fixed point problem in the metric space and gained better results. Consequently, if one continues to introduce a new class of functions in the b_2 metric space and establish implicit contraction conditions, is it feasible to obtain the presence and uniqueness theorems for common fixed points of many mappings.

Through the analysis presented above, this paper attempts to establish a new generalized weak contraction condition in the b_2 metric space to demonstrate that when the metric space is replaced by the b_2 metric space, there are still common points between various mappings, and uniqueness can still be proven. In addition, we establish an implicit contraction condition in the b_2 metric space and obtain that there still exist unique fixed points between various mappings in the b_2 metric space when the explicit or semiexplicit contraction condition is changed to an implicit contraction condition.

The following core ideas are necessary for comprehending and validating our major findings.

Definition 1 ([5]). *Assume that X is a nonempty set, R_+ denotes the set of all non-negative numbers, and $s \geq 0$ is a specified real number, then $b : X^2 \to R_+$ is a b-metric on X, if the following requirements hold true for any $x, y, z \in X$:*

(1) $b(x, y) = 0$ *if and only if* $x = y$*;*
(2) $b(x, y) = b(y, x)$*;*
(3) $b(x, z) \leq s[b(x, y) + b(y, z)]$.

In this scenario, a pair of (X, b) is referred to as a b-metric space with the parameter s.

Definition 2 ([14]). *Assume that X is a nonempty set. A function $d : X^3 \to [0, +\infty)$ is a 2-metric if and only if the following conditions are satisfied for all $x, y, z \in X$:*

(1) *If $x \neq y$, then there is a point $z \in X$ such that $d(x, y, z) \neq 0$;*
(2) *$d(x, y, z) = 0$ if at least two of the three points are equal;*
(3) *$d(x, y, z) = d(x, z, y) = d(y, z, x)$ (symmetry about three variables);*
(4) *$d(x, y, z) \leq d(x, y, u) + d(y, z, u) + d(z, x, u)$, for all $u \in X$.*

Then, (X, d) is referred to as a 2-metric space.

Example 1 ([15]). *Let $X = \{1, 2, 3\}$ and $d(x, y, z) = min\{|x - y|, |y - z|, |z - x|\}$ for all $x, y, z \in X$. Then, (X, d) is a 2-metric space.*

b_2 metric spaces are introduced as an extension of 2-metric spaces and b-metric spaces, and are described in detail below.

Definition 3 ([4]). *Assume that X is a nonempty set and $\lambda > 1$ is a given real number. Suppose that the following criteria are met for the function $d : X^3 \to R$, for all $a, b, c \in X$:*

(b_1) *If $a \neq b$, then there is a point $c \in X$ such that $d(a, b, c) \neq 0$;*
(b_2) *If at least two of three points a, b, c are the same, then $d(a, b, c) = 0$;*
(b_3) *$d(a, b, c) = d(b, a, c) = d(c, a, b)$ (symmetry about three variables);*
(b_4) *$d(a, b, c) \leq \lambda[d(a, b, u) + d(b, c, u) + d(c, a, u)]$, $u \in X$.*

The d in X is thus termed b_2 metric, the pair (X, d) is referred to as a b_2 metric space with the parameter λ in this case. Obviously, for $\lambda = 1$, a b_2 metric reduces to a 2-metric.

In [4], Zead Mustafa also gave some basic properties about b_2 metric spaces after giving the definition of b_2 metric spaces in 2014.

Definition 4 ([4]). *Assume a sequence $\{x_n\}$ in b_2-metric space (X, d).*

(1) *If $\lim\limits_{n\to\infty} d(x_n, x, \alpha) = 0$ for all $\alpha \in X$, then the sequence $\{x_n\}$ is b_2-convergent to $x \in X$, denoted by $\lim\limits_{n\to\infty} x_n = x$.*
(2) *If $\lim\limits_{n,m\to\infty} d(x_n, x_m, \alpha) = 0$ for all $\alpha \in X$, then the sequence $\{x_n\}$ is b_2-Cauchy sequence.*
(3) *If all b_2-Cauchy sequences are b_2-convergent, then the pair (X, d) is b_2-complete.*

Definition 5 ([20]). *Let (X, ω_λ) be a modular metric space and let C be a nonempty subset of X. If A, S: $C \to C$ are two mappings, then A and S are said to be:*

(1) *Commuting if $ASx = SAx$, for all $x \in C$;*
(2) *Weakly commuting if $\omega_\lambda(ASx, SAx) \leq \omega_\lambda(Ax, Sx)$, for all $x \in C$;*
(3) *Compatible if $\lim\limits_{n\to\infty} \omega_\lambda(ASx_n, SAx_n) = 0$ for each sequence $\{x_n\}$ in C such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Sx_n$;*
(4) *Noncompatible if there exists a sequence $\{x_n\}$ in C such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Sx_n$ but $\lim\limits_{n\to\infty} \omega_\lambda(ASx_n, SAx_n)$ is either nonzero or nonexistent;*
(5) *Weakly compatible if they commute at their coincidence points, that is, $ASx = SAx$ whenever $Ax = Sx$, for some $x \in C$.*

We write the following definition in the b_2 metric space, following Definition 5.

Definition 6. *Let X be a nonempty set and (X, d) be a b_2 metric space. If A, B: $X \to X$ are two mappings, then A and B are said to be:*

(1) Commuting if $ABx = BAx$, for all $x \in X$;
(2) Compatible if $\lim_{n\to\infty} d(ABx_n, BAx_n, \alpha) = 0$ for each sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n$;
(3) Noncompatible if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n$ but $\lim_{n\to\infty} d(ABx_n, BAx_n, \alpha)$ is either nonzero or nonexistent;
(4) Weakly compatible if they commute at their coincidence points, that is, $ABx = BAx$ whenever $Ax = Bx$, for some $x \in X$.

Example 2 ([4])**.** *Let $X = [0, \infty)$ and $d(x, y, z) = [xy + yz + zx]^p$ if $x \neq y \neq z \neq x$, and otherwise $d(x, y, z) = 0$, where $p \geq 1$ is a real number. Evidently, from the convexity of function $f(x) = x^p$ for $x \geq 0$, then by Jensen's inequality, we have*

$$(a + b + c)^p \leq 3^{p-1}(a^p + b^p + c^p).$$

Therefore, one can obtain the result that (X, d) is a b_2-metric space with $s = 3^{p-1}$.

Example 3 ([21])**.** *Let (X, d) be a 2-metric space and $\rho(x, y, w) = (d(x, y, w))^p$, where $p \geq 1$ is a real number. We see that ρ is a b_2-metric with $s = 3^{p-1}$. In view of the convexity of $f(x) = x^p$ on $[0, \infty)$ for $p \geq 1$ and Jensen's inequality, we have*

$$(a + b + c)^p \leq 3^{p-1}(a^p + b^p + c^p).$$

Therefore, condition (b_4) of Definition 3 is satisfied and ρ is a b_2-metric on X.

Example 4 ([4])**.** *Let a mapping $d : R^3 \to [0, \infty)$ be defined by*

$$d(x, y, z) = \min\{|x - y|, |y - z|, |z - x|\}.$$

Then, d is a 2-metric on R, i.e., the following inequality holds:

$$d(x, y, z) \leq d(x, y, t) + d(y, z, t) + d(z, x, t),$$

for arbitrary real numbers x, y, z, t. Using the convexity of the function $f(x) = x^p$ on $[0, \infty)$ for $p \geq 1$, we obtain that

$$d_p(x, y, z) = [\min\{|x - y|, |y - z|, |z - x|\}]^p$$

is a b_2-metric on R with $s < 3^{p-1}$.

2. Expansive Mappings

It is vital to highlight that the majority of past scholarly study results concern contracted fixed point results in b-metric spaces and 2-metric spaces, whereas relatively few results concern expansive fixed-point results in these two types of spaces. Furthermore, the research on expansive mappings is a highly intriguing research topic in the theory of fixed points, so influenced by the literature [3], we propose to introduce generalized $(\phi, f)_\lambda$-expansive mappings into a b_2 metric space.

Definition 7. *We define Φ to be the set of functions ϕ, $\phi : {R_+}^6 \to R_+$ satisfying the below situations:*

(1) ϕ is a continuous and nondecreasing function;

(2) *For* $h_i \in [0, +\infty)$, $i = 1, 2, \ldots, 6$, $\phi(h_1, h_2, h_3, h_4, h_5, h_6) > \min\{h_1, \frac{h_2+h_3}{2}\}$, *where* $\min\{h_1, \frac{h_2+h_3}{2}\} > 0$;

(3) $\phi(0, 0, 0, 0, 0, 0) = 0$ *and* $\phi(h_1, h_2, h_3, h_4, h_5, h_6) > \min\{h_1, h_5\}$, *where* $\min\{h_1, h_5\} > 0$.

Example 5. *Some simple examples of the* ϕ *function are given next:*

$$\phi(h_1, h_2, h_3, h_4, h_5, h_6) = h_1 + h_5;$$
$$\phi(h_1, h_2, h_3, h_4, h_5, h_6) = \max\{\min\{h_1, \frac{h_2 + h_3}{2}\}, \min\{h_1, h_5\}\} + \psi(\min\{h_1, \frac{h_2 + h_3}{2}\}, \min\{h_1, h_5\}),$$

where the continuous function ψ *is nondecreasing on the range of real numbers, if and only if* $t = 0$ *yields* $\psi(t) = 0$.

Definition 8. *Consider the* b_2 *metric space* (X, d) *is* b_2*-complete, and* A*,* B*,* S*, and* T *are four self-mappings of* X *meeting the generalized* $(\phi, f)_\lambda$*-expansive condition:*

$$f(\frac{d(Sx, Ty, \alpha)}{\lambda^2}) \geq \phi\Big(d(Ax, By, \alpha), d(Ax, Sx, \alpha), d(By, Ty, \alpha), \frac{d(Ax, Sx, \alpha) + d(By, Ty, \alpha)}{2}, \frac{d(By, Sx, \alpha)}{\lambda}, \max\{d(Sx, Ty, \alpha), \frac{d(Sx, Ax, \alpha)d(By, Ty, \alpha)}{d(Ax, By, \alpha)}\}\Big), \tag{1}$$

for all $\alpha, x, y \in X$ *and* $\lambda \geq 1$*, where* $\phi \in \Phi$*, continuous function* $f : [0, \infty) \to [0, \infty)$ *is nondecreasing,* $f(0) = 0$*, and for all* $h > 0$*,* $\phi(h_1, h_2, h_3, h_4, h_5, h_6) > f(h)$*, where* $\min\{h_1, \frac{h_2+h_3}{2}\} = h$ *or* $\min\{h_1, h_5\} = h$.

The following are the key theorems that we have developed regarding expansive maps.

Theorem 1. *Assume that* (X, d) *is a* b_2*-complete* b_2 *metric space and that* A*,* B*,* S*, and* T *are four self-mappings of* X *that meet the condition (1). Assume, moreover, that the mappings also meet the below requirements:*

(1) $A(X) \subseteq T(X)$*,*$B(X) \subseteq S(X)$*;*

(2) *A(or B) and S(or T) are continuous, A(or B) and S(or T) are compatible, and B(or A) and T(or S) are weakly compatible.*

Then, the four mappings have a unique common fixed point in X.

Proof. Given that $x_0 \in X$. Since $A(X) \subset T(X)$, there is an $x_1 \in X$ that makes $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, there is an $x_2 \in X$ that makes $Bx_1 = Sx_2$.
Constructing the sequences of number $\{x_n\}$ and $\{u_n\}$ such that

$$u_{2n-1} = Ax_{2n-2} = Tx_{2n-1}, u_{2n} = Bx_{2n-1} = Sx_{2n}, n = 0, 1, 2, \ldots \tag{2}$$

First, we prove

$$\lim_{n\to\infty} d(u_n, u_{n+1}, \alpha) = 0. \tag{3}$$

Case 1: Suppose $d(u_n, u_{n+1}, \alpha) = 0$ for some $n = n_0$; when $n_0 = 2p$, we have $d(u_{2p}, u_{2p+1}, \alpha) = 0$ and by (1), one has

$$\begin{aligned}
0 =& f(\frac{d(u_{2p}, u_{2p+1}, \alpha)}{\lambda^2}) \\
\geq& \phi\Big(d(Ax_{2p}, Bx_{2p+1}, \alpha), d(Ax_{2p}, Sx_{2p}, \alpha), d(Bx_{2p+1}, Tx_{2p+1}, \alpha), \\
& \frac{d(Ax_{2p}, Sx_{2p}, \alpha) + d(Bx_{2p+1}, Tx_{2p+1}, \alpha)}{2}, \frac{d(Bx_{2p+1}, Sx_{2p}, \alpha)}{\lambda}, \\
& \max\{d(Sx_{2p}, Tx_{2p+1}, \alpha), \frac{d(Sx_{2p}, Ax_{2p}, \alpha)d(Bx_{2p+1}, Tx_{2p+1}, \alpha)}{d(Ax_{2p}, Bx_{2p+1}, \alpha)}\}\Big) \\
=& \phi\Big(d(u_{2p+1}, u_{2p+2}, \alpha), d(u_{2p+1}, u_{2p}, \alpha), d(u_{2p+2}, u_{2p+1}, \alpha), \\
& \frac{d(u_{2p+1}, u_{2p}, \alpha) + d(u_{2p+2}, u_{2p+1}, \alpha)}{2}, \frac{d(u_{2p+2}, u_{2p}, \alpha)}{\lambda}, \\
& \max\{d(u_{2p}, u_{2p+1}, \alpha), \frac{d(u_{2p}, u_{2p+1}, \alpha)d(u_{2p+2}, u_{2p+1}, \alpha)}{d(u_{2p+1}, u_{2p+2}, \alpha)}\}\Big) \\
=& \phi\Big(d(u_{2p+1}, u_{2p+2}, \alpha), 0, d(u_{2p+1}, u_{2p+2}, \alpha), \frac{d(u_{2p+2}, u_{2p+1}, \alpha)}{2}, \frac{d(u_{2p+2}, u_{2p}, \alpha)}{\lambda}, 0\Big),
\end{aligned}$$

which implies

$$\min\{h_1, \frac{h_2 + h_3}{2}\} = \frac{d(u_{2p+1}, u_{2p+2}, \alpha)}{2} = 0,$$

hence

$$d(u_{2p+1}, u_{2p+2}, \alpha) = 0.$$

Similarly, when $n_0 = 2p + 1$, we have $d(u_{2p+1}, u_{2p+2}, \alpha) = 0$ and by (1), one has

$$\begin{aligned}
0 =& f(\frac{d(u_{2p+1}, u_{2p+2}, \alpha)}{\lambda^2}) \\
\geq& \phi\Big(d(Ax_{2p+2}, Bx_{2p+1}, \alpha), d(Ax_{2p+2}, Sx_{2p+2}, \alpha), d(Bx_{2p+1}, Tx_{2p+1}, \alpha), \\
& \frac{d(Ax_{2p+2}, Sx_{2p+2}, \alpha) + d(Bx_{2p+1}, Tx_{2p+1}, \alpha)}{2}, \frac{d(Bx_{2p+1}, Sx_{2p+2}, \alpha)}{\lambda}, \\
& \max\{d(Sx_{2p+2}, Tx_{2p+1}, \alpha), \frac{d(Sx_{2p+2}, Ax_{2p+2}, \alpha)d(Bx_{2p+1}, Tx_{2p+1}, \alpha)}{d(Ax_{2p+2}, Bx_{2p+1}, \alpha)}\}\Big) \\
=& \phi\Big(d(u_{2p+3}, u_{2p+2}, \alpha), d(u_{2p+3}, u_{2p+2}, \alpha), d(u_{2p+2}, u_{2p+1}, \alpha), \\
& \frac{d(u_{2p+3}, u_{2p+2}, \alpha) + d(u_{2p+2}, u_{2p+1}, \alpha)}{2}, \frac{d(u_{2p+2}, u_{2p+2}, \alpha)}{\lambda}, \\
& \max\{d(u_{2p+1}, u_{2p+2}, \alpha), \frac{d(u_{2p+2}, u_{2p+3}, \alpha)d(u_{2p+2}, u_{2p+1}, \alpha)}{d(u_{2p+3}, u_{2p+2}, \alpha)}\}\Big) \\
=& \phi\Big(d(u_{2p+2}, u_{2p+3}, \alpha), d(u_{2p+2}, u_{2p+3}, \alpha), 0, \frac{d(u_{2p+2}, u_{2p+3}, \alpha)}{2}, 0, 0\Big),
\end{aligned}$$

which implies

$$\min\{h_1, \frac{h_2 + h_3}{2}\} = \frac{d(u_{2p+2}, u_{2p+3}, \alpha)}{2} = 0,$$

hence

$$d(u_{2p+2}, u_{2p+3}, \alpha) = 0.$$

Thus, for $n \geq n_0$, we can get $d(u_n, u_{n+1}, \alpha) = 0$. Hence, we have $\lim_{n \to \infty} d(u_n, u_{n+1}, \alpha) = 0$.
Case 2: Suppose $d(u_n, u_{n+1}, \alpha) > 0$, for every n, by (1), when $n_0 = 2p$, we obtain

$$\begin{aligned}
f(d(u_{2p},u_{2p+1},\alpha)) \geq & f(\frac{d(Sx_{2p},Tx_{2p+1},\alpha)}{\lambda^2}) \\
\geq & \phi\Big(d(Ax_{2p},Bx_{2p+1},\alpha),d(Ax_{2p},Sx_{2p},\alpha),d(Bx_{2p+1},Tx_{2p+1},\alpha), \\
& \frac{d(Ax_{2p},Sx_{2p},\alpha)+d(Bx_{2p+1},Tx_{2p+1},\alpha)}{2},\frac{d(Bx_{2p+1},Sx_{2p},\alpha)}{\lambda}, \\
& \max\{d(Sx_{2p},Tx_{2p+1},\alpha),\frac{d(Sx_{2p},Ax_{2p},\alpha)d(Bx_{2p+1},Tx_{2p+1},\alpha)}{d(Ax_{2p},Bx_{2p+1},\alpha)}\}\Big) \\
= & \phi\Big(d(u_{2p+1},u_{2p+2},\alpha),d(u_{2p+1},u_{2p},\alpha),d(u_{2p+2},u_{2p+1},\alpha), \\
& \frac{d(u_{2p+1},u_{2p},\alpha)+d(u_{2p+2},u_{2p+1},\alpha)}{2},\frac{d(u_{2p+2},u_{2p},\alpha)}{\lambda}, \\
& \max\{d(u_{2p},u_{2p+1},\alpha),\frac{d(u_{2p},u_{2p+1},\alpha)d(u_{2p+2},u_{2p+1},\alpha)}{d(u_{2p+1},u_{2p+2},\alpha)}\}\Big).
\end{aligned} \tag{4}$$

If

$$\min\{h_1,\frac{h_2+h_3}{2}\}=\frac{d(u_{2p+1},u_{2p},\alpha)+d(u_{2p+2},u_{2p+1},\alpha)}{2},$$

then $d(u_{2p+1},u_{2p},\alpha)\leq d(u_{2p+2},u_{2p+1},\alpha)$, by (4) and the characteristics of ϕ and f, we obtain

$$\begin{aligned}
f(d(u_{2p+1},u_{2p},\alpha)) \geq & \phi\Big(d(u_{2p+1},u_{2p+2},\alpha),d(u_{2p+1},u_{2p},\alpha),d(u_{2p+2},u_{2p+1},\alpha), \\
& \frac{d(u_{2p+1},u_{2p},\alpha)+d(u_{2p+2},u_{2p+1},\alpha)}{2},\frac{d(u_{2p+2},u_{2p},\alpha)}{\lambda}, \\
& \max\{d(u_{2p},u_{2p+1},\alpha),\frac{d(u_{2p},u_{2p+1},\alpha)d(u_{2p+2},u_{2p+1},\alpha)}{d(u_{2p+1},u_{2p+2},\alpha)}\}\Big) \\
> & f(\min\{h_1,\frac{h_2+h_3}{2}\}) \\
= & f(\frac{d(u_{2p},u_{2p+1},\alpha)+d(u_{2p+2},u_{2p+1},\alpha)}{2}).
\end{aligned}$$

Since f is a nondecreasing function, $d(u_{2p},u_{2p+1},\alpha)\geq d(u_{2p+1},u_{2p+2},\alpha)$, and the result is contradictory.

Therefore, we can infer that $d(u_{2p+1},u_{2p+2},\alpha)\leq d(u_{2p},u_{2p+1},\alpha)$.

When $n_0=2p+1$, one has

$$\begin{aligned}
f(d(u_{2p+1},u_{2p+2},\alpha)) \geq & f(\frac{d(Sx_{2p+2},Tx_{2p+1},\alpha)}{\lambda^2}) \\
\geq & \phi\Big(d(Ax_{2p+2},Bx_{2p+1},\alpha), \\
& d(Ax_{2p+2},Sx_{2p+2},\alpha),d(Bx_{2p+1},Tx_{2p+1},\alpha), \\
& \frac{d(Ax_{2p+2},Sx_{2p+2},\alpha)+d(Bx_{2p+1},Tx_{2p+1},\alpha)}{2}, \\
& \frac{d(Bx_{2p+1},Sx_{2p+2},\alpha)}{\lambda},\max\{d(Sx_{2p+2},Tx_{2p+1},\alpha), \\
& \frac{d(Sx_{2p+2},Ax_{2p+2},\alpha)d(Bx_{2p+1},Tx_{2p+1},\alpha)}{d(Ax_{2p+2},Bx_{2p+1},\alpha)}\}\Big) \\
\geq & \phi\Big(d(u_{2p+3},u_{2p+2},\alpha),d(u_{2p+3},u_{2p+2},\alpha),d(u_{2p+2},u_{2p+1},\alpha), \\
& \frac{d(u_{2p+3},u_{2p+2},\alpha)+d(u_{2p+2},u_{2p+1},\alpha)}{2},\frac{d(u_{2p+2},u_{2p+2},\alpha)}{\lambda}, \\
& \max\{d(u_{2p+1},u_{2p+2},\alpha),\frac{d(u_{2p+2},u_{2p+3},\alpha)d(u_{2p+2},u_{2p+1},\alpha)}{d(u_{2p+3},u_{2p+2},\alpha)}\}\Big).
\end{aligned} \tag{5}$$

If

$$\min\{h_1, \frac{h_2+h_3}{2}\} = \frac{d(u_{2p+2}, u_{2p+3}, \alpha) + d(u_{2p+1}, u_{2p+2}, \alpha)}{2},$$

then $d(u_{2p+1}, u_{2p+2}, \alpha) \leq d(u_{2p+2}, u_{2p+3}, \alpha)$, by (5) and the characteristics of ϕ and f, we obtain

$$\begin{aligned} f(d(u_{2p+1}, u_{2p+2}, \alpha)) \geq &\phi\Big(d(u_{2p+3}, u_{2p+2}, \alpha), d(u_{2p+3}, u_{2p+2}, \alpha), d(u_{2p+2}, u_{2p+1}, \alpha), \\ &\frac{d(u_{2p+3}, u_{2p+2}, \alpha) + d(u_{2p+2}, u_{2p+1}, \alpha)}{2}, \frac{d(u_{2p+2}, u_{2p+2}, \alpha)}{\lambda}, \\ &\max\{d(u_{2p+1}, u_{2p+2}, \alpha), \frac{d(u_{2p+2}, u_{2p+3}, \alpha)d(u_{2p+2}, u_{2p+1}, \alpha)}{d(u_{2p+3}, u_{2p+2}, \alpha)}\}\Big) \\ > &f(\min\{h_1, \frac{h_2+h_3}{2}\}) \\ = &f(\frac{d(u_{2p+3}, u_{2p+2}, \alpha)d(u_{2p+2}, u_{2p+1}, \alpha)}{2}). \end{aligned}$$

Since f is a nondecreasing function, $d(u_{2p+1}, u_{2p+2}, \alpha) \geq d(u_{2p+2}, u_{2p+3}, \alpha)$. Thus, the result is contradictory.

On the basis of the above results, we can deduce $d(u_{2p+2}, u_{2p+3}, \alpha) \leq d(u_{2p+1}, u_{2p+2}, \alpha)$. In addition, since $\{d(u_n, u_{n+1}, \alpha)\}$ is a decreasing sequence of nonnegative real numbers, there is a $r \geq 0$ that yields $\lim_{n\to\infty}\{d(u_n, u_{n+1}, \alpha)\} = r$. (b_4) in Definition 3 gives us

$$\begin{aligned} d(u_n, u_{n+2}, \alpha) = &d(u_n, u_{n+2}, \alpha) + d(u_{n+1}, u_{n+1}, \alpha) \\ \leq &\lambda d(u_n, u_{n+2}, u_{n+1}) + \lambda d(u_{n+2}, \alpha, u_{n+1}) + \lambda d(\alpha, u_n, u_{n+1}). \end{aligned} \tag{6}$$

Obviously, $\{d(u_n, u_{n+2}, \alpha)\}$ and $\{d(u_{n+1}, u_{n+1}, \alpha)\}$ are two bounded sequences. Therefore, the sequence $\{d(u_n, u_{n+2}, \alpha)\}$ has subsequences $\{d(u_{n_p}, u_{n_p+2}, \alpha)\}$ that converge to $a \leq 2\lambda r$, and the sequence $\{d(u_{n_p+1}, u_{n_p+1}, \alpha)\}$ also has subsequences $\{d(u_{n_{p_l}+1}, u_{n_{p_l}+1}, \alpha)\}$ that converge to $b \leq 2\lambda r$.

By (4), we get

$$\begin{aligned} f(d(u_{2n_{p(l)}}, u_{2n_{p(l)}+1}, \alpha)) \geq &\phi\Big(d(u_{2n_{p(l)}+1}, u_{2n_{p(l)}+2}, \alpha), \\ &d(u_{2n_{p(l)}+1}, u_{2n_{p(l)}}, \alpha), d(u_{2n_{p(l)}+2}, u_{2n_{p(l)}+1}, \alpha), \\ &\frac{d(u_{2n_{p(l)}+1}, u_{2n_{p(l)}}, \alpha) + d(u_{2n_{p(l)}+1}, u_{2n_{p(l)}+1}, \alpha)}{2}, \\ &\frac{d(u_{2n_{p(l)}+2}, u_{2n_{p(l)}}, \alpha)}{\lambda}, \max\{d(u_{2n_{p(l)}}, u_{2n_{p(l)}+1}, \alpha), \\ &\frac{d(u_{2n_{p(l)}}, u_{2n_{p(l)}+1}, \alpha)d(u_{2n_{p(l)}+2}, u_{2n_{p(l)}+1}, \alpha)}{d(u_{2n_{p(l)}+1}, u_{2n_{p(l)}+2}, \alpha)}\}\Big). \end{aligned}$$

In the above inequality, let $n_{p_l} \to \infty$, and then from the properties of f and ϕ, we can get

$$f(r) \geq \phi(r, r, r, \frac{r+b}{2}, \frac{a}{\lambda}, r),$$

with $r = 0$, hence, $\lim_{n\to\infty} d(u_n, u_{n+1}, \alpha) = 0$. Furthermore, because $\{d(u_n, u_{n+1}, \alpha)\}$ is a monotonically decreasing sequence, if $d(u_n, u_{n+1}, \alpha) = 0$, then $d(u_{n-1}, u_n, \alpha) = 0$. Then, it can be known that when $d(u_0, u_1, u_0) = 0$, $d(u_n, u_{n+1}, u_0) = 0$, for $\forall n \in N$.
Furthermore, because $d(u_{m-1}, u_m, u_m) = 0$, we obtain

$$d(u_n, u_{n+1}, u_m) = 0, \tag{7}$$

for $n + 1 \geq m$. By (7), we can easily acquire

$$d(u_{m-1}, u_m, u_{n+1}) = d(u_{m-1}, u_m, u_n) = 0. \tag{8}$$

$$\begin{aligned} d(u_n, u_{n+1}, u_m) \leq &\lambda d(u_n, u_{n+1}, u_{m-1}) + \lambda d(u_{n+1}, u_m, u_{m-1}) + \lambda d(u_m, u_n, u_{m-1}) \\ = &\lambda d(u_n, u_{n+1}, u_{m-1}). \end{aligned}$$

Since $d(u_n, u_{n+1}, u_{n+1}) = 0$, from the above inequality, we can get

$$d(u_n, u_{n+1}, u_m) \leq \lambda^{m-n-1} d(u_n, u_{n+1}, u_{n+1}) = 0, 0 \leq n < m-1, \tag{9}$$

combining (7) and (9), we get

$$d(u_n, u_{n+1}, u_m) = 0, \tag{10}$$

for all $l, q, p \in N, l < q$, and one obtains

$$d(u_{q-1}, u_q, u_l) = d(u_{q-1}, u_q, u_p) = 0, \tag{11}$$

as a result, by (11) and Definition 3, we acquire

$$\begin{aligned} d(u_l, u_q, u_p) \leq &\lambda[d(u_l, u_q, u_{q-1}) + d(u_q, u_p, u_{q-1}) + d(u_p, u_l, u_{q-1})] \\ = &\lambda d(u_l, u_{q-1}, u_p) \leq \ldots \leq \lambda^{q-l} d(u_l, u_q, u_p) = 0, \end{aligned}$$

which proves that for all $l, q, p \in N$, we have $d(u_l, u_q, u_p) = 0$.

Second, $\{u_n\}$ must be demonstrated to be a b_2-Cauchy sequence.

As a matter of fact, we have demonstrated that $\lim\limits_{m,n\to\infty} d(u_n, u_m, \alpha) = 0$, and there is $\lim\limits_{m,n\to\infty} d(u_n, u_m, \alpha) = 0$ which can make $\lim\limits_{m,n\to\infty} d(u_{2n}, u_{2m}, \alpha) = 0$ true.

Using counter-evidence, we assume the opposite, $\varepsilon > 0$, so that we can find two subsequences of $\{u_{2n}\}$, $\{u_{2n_t}\}$, and $\{u_{2m_t}\}$, such that $m(t)$ is the minimum value that satisfies this condition,

$$m(t) > n(t) > t, d(u_{2m_{(t)}}, u_{2n_{(t)}}, \alpha) \geq \varepsilon, \forall t \in N, \tag{12}$$

which means

$$d(u_{2m_{(t)}-2}, u_{2n_{(t)}}, \alpha) < \varepsilon. \tag{13}$$

Using the triangle inequality for (13), we gain

$$\begin{aligned} 0 \leq \varepsilon \leq &d(u_{2m_{(t)}}, u_{2n_{(t)}}, \alpha) \\ \leq &\lambda d(u_{2m_{(t)}}, u_{2n_{(t)}}, u_{2m_{(t)}-1}) + \lambda d(u_{2n_{(t)}}, \alpha, u_{2m_{(t)}-1}) + \lambda d(\alpha, u_{2m_{(t)}}, u_{2m_{(t)}-1}) \\ \leq &\lambda d(u_{2m_{(t)}}, u_{2n_{(t)}}, u_{2m_{(t)}-1}) + \lambda^2 [d(u_{2n_{(t)}}, u_{2n_{(t)}+1}, \alpha) + d(u_{2m_{(t)}-1}, u_{2n_{(t)}+1}, \alpha) \\ &+ d(u_{2n_{(t)}}, u_{2n_{(t)}+1}, u_{2m_{(t)}-1})] + \lambda d(u_{2m_{(t)}}, u_{2m_{(t)}-1}, \alpha), \end{aligned}$$

and taking $t \to \infty$ in the above formula, we get

$$\frac{\varepsilon}{\lambda} \leq \lim_{t\to\infty} \inf d(u_{2n_{(t)}}, u_{2m_{(t)}-1}, \alpha) \leq \lim_{t\to\infty} \sup d(u_{2n_{(t)}}, u_{2m_{(t)}-1}, \alpha),$$

$$\varepsilon \leq \lim_{k\to\infty} \inf d(u_{2m_{(t)}}, u_{2n_{(t)}}, \alpha) \leq \lim_{k\to\infty} \sup d(u_{2m_{(t)}}, u_{2n_{(t)}}, \alpha).$$

It is easy to get

$$\begin{aligned}
f(d(u_{2n_{(t)}},u_{2m_{(t)}-1},\alpha)) \geq & f(\frac{d(Sx_{2n_{(t)}},Tx_{2m_{(t)}-1},\alpha)}{\lambda^2}) \\
> & \phi\Big(d(Ax_{2n_{(t)}},Bx_{2m_{(t)}-1},\alpha), \\
& d(Ax_{2n_{(t)}},Sx_{2n_{(t)}},\alpha), d(Bx_{2m_{(t)-1}},Tx_{2m_{(t)}-1},\alpha), \\
& \frac{d(Ax_{2n_{(t)}},Sx_{2n_{(t)}},\alpha)+d(Bx_{2m_{(t)}-1},Tx_{2m_{(t)}-1},\alpha)}{2}, \\
& \frac{d(Bx_{2m_{(t)}-1},Sx_{2n_{(t)}},\alpha)}{\lambda}, \max\{d(Sx_{2n_{(t)}},Tx_{2m_{(t)}-1},\alpha), \\
& \frac{d(Sx_{2n_{(t)}},Ax_{2n_{(t)}},\alpha)d(Bx_{2m_{(t)}-1},Tx_{2m_{(t)}-1},\alpha)}{d(Ax_{2n_{(t)}},Bx_{2m_{(t)}-1},\alpha)}\}\Big) \\
= & \phi\Big(d(u_{2n_{(t)}+1},u_{2m_{(t)}},\alpha), \\
& d(u_{2n_{(t)}+1},u_{2n_{(t)}},\alpha), d(u_{2m_{(t)}},u_{2m_{(t)}-1},\alpha), \\
& \frac{d(u_{2n_{(t)}+1},u_{2n_{(t)}},\alpha)+d(u_{2m_{(t)}},u_{2m_{(t)}-1},\alpha)}{2}, \\
& \frac{d(u_{2m_{(t)}},u_{2n_{(t)}},\alpha)}{\lambda}, \max\{d(u_{2n_{(t)}},u_{2m_{(t)}-1},\alpha), \\
& \frac{d(u_{2n_{(t)}},u_{2n_{(t)+1}},\alpha)d(u_{2m_{(t)}},u_{2m_{(t)}-1},\alpha)}{d(u_{2n_{(t)}+1},u_{2m_{(t)}},\alpha)}\}\Big).
\end{aligned}$$

Now, when $t \to \infty$, taking the upper limit in the preceding inequality, according to the characteristics of ϕ and f, we get

$$f(\frac{\varepsilon}{\lambda}) \geq \phi(\frac{\varepsilon}{\lambda},0,0,0,\frac{\varepsilon}{\lambda},\frac{\varepsilon}{\lambda}),$$

therefore, $\varepsilon = 0$ and $\lim\limits_{m,n\to\infty} d(u_n,u_m,\alpha) = 0$, which means $\{u_n\}$ is a b_2-Cauchy sequence on X.

Last but not least, we establish that A, B, S, and T have a unique common fixed point.

Since $\{u_n\}$ is a b_2-Cauchy sequence on X, and (X,d) is b_2-complete, there is a point θ in X where $\{u_n\}$ is b_2- converges to θ, so we have

$$\lim_{n\to\infty} u_{2n-1} = \lim_{n\to\infty} Ax_{2n-2} = \lim_{n\to\infty} Tx_{2n-1} = \theta,$$
$$\lim_{n\to\infty} u_{2n} = \lim_{n\to\infty} Bx_{2n-1} = \lim_{n\to\infty} Sx_{2n} = \theta.$$

Suppose that A and S are continuous. Furthermore, since $\{A,S\}$ is compatible, we can easily obtain

$$\lim_{n\to\infty} ASx_{2n} = \lim_{n\to\infty} SAx_{2n} = S\theta = A\theta,$$

$$\begin{aligned}
f(d(SAx_{2n+2},Tx_{2n+3},\alpha)) \geq & \phi\Big(d(AAx_{2n+2},Bx_{2n+3},\alpha), \\
& d(AAx_{2n+2},SAx_{2n+2},\alpha), d(Bx_{2n+3},Tx_{2n+3},\alpha), \\
& \frac{d(AAx_{2n+2},SAx_{2n+2},\alpha)+d(Bx_{2n+3},Tx_{2n+3},\alpha)}{2}, \\
& \frac{d(Bx_{2n+3},SAx_{2n+2},\alpha)}{\lambda}, \max\{d(SAx_{2n+2},Tx_{2n+3},\alpha), \\
& \frac{d(SAx_{2n+2},AAx_{2n+2},\alpha)d(Bx_{2n+3},Tx_{2n+3},\alpha)}{d(AAx_{2n+2},Bx_{2n+3},\alpha)}\}\Big),
\end{aligned}$$

and taking $n \to \infty$ for the above formula, we get

$$\begin{aligned} f(d(A\theta,\theta,\alpha)) \geq &\phi\Big(d(A\theta,\theta,\alpha), d(A\theta,A\theta,\alpha), d(\theta,\theta,\alpha), \frac{d(A\theta,A\theta,\alpha)+d(\theta,\theta,\alpha)}{2}, \\ &\frac{d(\theta,A\theta,\alpha)}{\lambda}, \max\{d(A\theta,\theta,\alpha), \frac{d(A\theta,A\theta,\alpha)d(\theta,\theta,\alpha)}{d(A\theta,\theta,\alpha)}\}\Big) \\ =&\phi(d(A\theta,\theta,\alpha), 0, 0, 0, \frac{d(\theta,A\theta,\alpha)}{\lambda}, d(A\theta,\theta,\alpha)), \end{aligned}$$

which can illustrate $A\theta = \theta$. Therefore, we continue to get

$$\begin{aligned} f(d(S\theta,Tx_{2n+3},\alpha)) \geq &\phi\Big(d(A\theta,Bx_{2n+3},\alpha), d(A\theta,S\theta,\alpha), d(Bx_{2n+3},Tx_{2n+3},\alpha), \\ &\frac{d(A\theta,S\theta,\alpha)+d(Bx_{2n+3},Tx_{2n+3},\alpha)}{2}, \frac{d(Bx_{2n+3},S\theta,\alpha)}{\lambda}, \\ &\max\{d(S\theta,Tx_{2n+3},\alpha), \frac{d(S\theta,A\theta,\alpha)d(Bx_{2n+3},Tx_{2n+3},\alpha)}{d(A\theta,Bx_{2n+3},\alpha)}\}\Big), \end{aligned}$$

and taking $n \to \infty$ for the above formula again, we obtain

$$\begin{aligned} f(d(S\theta,\theta,\alpha)) \geq &\phi\Big(d(\theta,\theta,\alpha), d(\theta,S\theta,\alpha), d(\theta,\theta,\alpha), \frac{d(\theta,S\theta,\alpha)+d(\theta,\theta,\alpha)}{2}, \frac{d(\theta,S\theta,\alpha)}{\lambda}, \\ &\max\{d(S\theta,\theta,\alpha), \frac{d(\theta,S\theta,\alpha)d(\theta,\theta,\alpha)}{d(\theta,\theta,\alpha)}\}\Big) \\ =&\phi(0, d(S\theta,\theta,\alpha), 0, \frac{d(S\theta,\theta,\alpha)}{2}, \frac{d(S\theta,\theta,\alpha)}{\lambda}, d(S\theta,\theta,\alpha)). \end{aligned}$$

From the above formula, we deduce that $S\theta = \theta$. Furthermore, because $A(x) \subset T(x)$, there must be a point $\omega \in X$ making $A\theta = T\omega$ hold. Then, assuming $B\omega \neq T\omega$, by (1),

$$\begin{aligned} f(d(S\theta,T\omega,\alpha)) \geq &\phi\Big(d(A\theta,B\omega,\alpha), d(A\theta,S\theta,\alpha), d(B\omega,T\omega,\alpha), \\ &\frac{d(A\theta,S\theta,\alpha)+d(B\omega,T\omega,\alpha)}{2}, \frac{d(B\omega,S\theta,\alpha)}{\lambda}, \\ &\max\{d(S\theta,T\omega,\alpha), \frac{d(S\theta,A\theta,\alpha)d(B\omega,T\omega,\alpha)}{d(A\theta,B\omega,\alpha)}\}\Big) \\ =&\phi(d(\theta,B\omega,\alpha), d(\theta,\theta,\alpha), d(B\omega,T\omega,\alpha), \\ &\frac{d(B\omega,T\omega,\alpha)}{2}, \frac{d(B\omega,\theta,\alpha)}{\lambda}, d(\theta,T\omega,\alpha)), \end{aligned}$$

this shows that $0 \geq \frac{d(B\omega,\theta,\alpha)}{\lambda}$, where $B\omega = \theta$ can be inferred, so $B\omega = T\omega$.

As $\{B, T\}$ is weakly compatible, then $B\theta = BA\theta = BT\omega = TB\omega = TA\theta = T\theta$, where θ is the common point of B and T.

When $Ax_{2n} \to \theta$, if $n \to \infty$, then $x_{2n} \to \theta$.

$$
\begin{aligned}
f(d(Sx_{2n}, T\theta, \alpha)) \geq & \phi\Big(d(Ax_{2n}, B\theta, \alpha), d(Ax_{2n}, Sx_{2n}, \alpha), d(B\theta, T\theta, \alpha), \\
& \frac{d(Ax_{2n}, Sx_{2n}, \alpha) + d(B\theta, T\theta, \alpha)}{2}, \frac{d(B\theta, Sx_{2n}, \alpha)}{\lambda}, \\
& \max\{d(Sx_{2n}, T\theta, \alpha), \frac{d(Sx_{2n}, Ax_{2n}, \alpha) d(B\theta, T\theta, \alpha)}{d(Ax_{2n}, B\theta, \alpha)}\}\Big) \\
= & \phi(d(\theta, B\theta, \alpha), d(\theta, \theta, \alpha), d(B\theta, B\theta, \alpha), \frac{d(\theta, \theta, \alpha) + d(B\theta, B\theta, \alpha)}{2}, \\
& \frac{d(B\theta, \theta, \alpha)}{\lambda}, \max\{d(\theta, B\theta, \alpha), \frac{d(\theta, \theta, \alpha) d(B\theta, B\theta, \alpha)}{d(\theta, B\theta, \alpha)}\}) \\
= & \phi(d(\theta, B\theta, \alpha), 0, 0, 0, \frac{d(B\theta, \theta, \alpha)}{\lambda}, d(\theta, B\theta, \alpha)),
\end{aligned}
$$

which means $f(d(\theta, B\theta, \alpha)) \geq 0$. Furthermore, because it contradicts the condition $\phi(h_1, h_2, h_3, h_4, h_5, h_6) > f(h)$ in the theorem, $\theta = B\theta$.

Following similar arguments, we can obtain $\theta = T\theta$, and it is easy to obtain $A\theta = B\theta = S\theta = T\theta = \theta$.

Assuming that fixed points are not unique, which means that $Az_1 = Bz_1 = Sz_1 = Tz_1 = z_1$, $Az_2 = Bz_2 = Sz_2 = Tz_2 = z_2$, where $z_1 \neq z_2$, In (2.1), substituting z_1 for x and z_2 for y, one has

$$
\begin{aligned}
f(d(Sz_1, Tz_2, \alpha)) \geq & \phi\Big(d(Az_1, Bz_2, \alpha), d(Az_1, Sz_1, \alpha), d(Bz_2, Tz_2, \alpha), \\
& \frac{d(Az_1, Sz_1, \alpha) + d(Bz_2, Tz_2, \alpha)}{2}, \frac{d(Bz_2, Sz_1, \alpha)}{\lambda}, \\
& \max\{d(Sz_1, Tz_2, \alpha), \frac{d(Sz_1, Az_1, \alpha) d(Bz_2, Tz_2, \alpha)}{d(Az_1, Bz_2, \alpha)}\}\Big),
\end{aligned}
$$

$$
\begin{aligned}
f(d(z_1, z_2, \alpha)) \geq & \phi\Big(d(z_1, z_2, \alpha), d(z_1, z_1, \alpha), d(z_2, z_2, \alpha), \\
& \frac{d(z_1, z_1, \alpha) + d(z_2, z_2, \alpha)}{2}, \frac{d(z_2, z_1, \alpha)}{\lambda}, \\
& \max\{d(z_1, z_2, \alpha), \frac{d(z_1, z_1, \alpha) d(z_2, z_2, \alpha)}{d(z_1, z_2, \alpha)}\}\Big) \\
= & \phi(d(z_1, z_2, \alpha), 0, 0, 0, \frac{d(z_2, z_1, \alpha)}{\lambda}, d(z_1, z_2, \alpha)),
\end{aligned}
$$

and since $\phi(h_1, h_2, h_3, h_4, h_5, h_6) > f(h)$, where $h = \min\{h_1, \frac{h_2+h_3}{2}\} = 0$, we have $z_1 = z_2$. Consequently, we get that the fixed point is unique, and the theorem is proved. □

Example 6. *Let* $X = \{(\alpha, 0) : \alpha \in [0,1] \cup \{2,3,4,\ldots\}\} \bigcup (0,2)$ *and let* $d(x,y,z)$ *denote the square of the area of a triangle with vertices* $x, y, z \in X$, *e.g.,* $d((\alpha,0), (\beta,0), (0,2)) = (\alpha-\beta)^2$. *It is straightforward to verify that* d *is a* b_2*-metric with parameter* $\lambda = 2$. *Consider the mapping* $f : [0, +\infty) \to [0, +\infty)$ *given by*

$$
f((\alpha, 0)) = \begin{cases} 3\alpha + 1 & 0 \leq \alpha \leq 1, \\ 4\alpha^2 & \alpha > 1, \end{cases}
$$

for $\alpha \in [0,1] \cup \{2,3,4,\ldots\}$ *and* $f(0,2) = (0,2)$.

In addition, $\phi(h_1, h_2, h_3, h_4, h_5, h_6) = \max\{\min\{h_1, \frac{h_2+h_3}{2}\}, \min\{h_1, h_5\}\}$*, and four mappings of X are defined by*

$$Ax = \begin{cases} 0 & x = 0 \\ \frac{1}{4} & x \in (0, \frac{1}{2}] \\ \frac{1}{2} & x \in (\frac{1}{2}, 1] \\ x & x \in \{2, 3, 4, \ldots\}, \end{cases} \quad Bx = \begin{cases} 0 & x = 0 \\ \frac{1}{4} & x \in (0, \frac{1}{2}] \\ \frac{x}{2} & x \in (\frac{1}{2}, 1] \\ x+1 & x \in \{2, 3, 4, \ldots\}, \end{cases}$$

$$Sx = \begin{cases} 0 & x = 0 \\ \frac{1}{4} & x \in (0, \frac{1}{2}] \\ \frac{1}{2} & x \in (\frac{1}{2}, 1] \\ x+1 & x \in \{2, 3, 4, \ldots\}, \end{cases} \quad Tx = \begin{cases} 0 & x = 0 \\ \frac{1}{3} & x \in (0, \frac{1}{2}] \\ \frac{x}{2} & x \in (\frac{1}{2}, 1] \\ x^2 & x \in \{2, 3, 4, \ldots\}, \end{cases}$$

Finally, in order to check the contractive condition (1), only the case when $x = (\alpha, 0)$*,* $y = (\beta, 0)$*, and* $a = (0, 2)$ *is nontrivial, but then* $d(x, y, a) = (\alpha - \beta)^2$*. Note that, all of the requirements given in Theorem 2.1 are met by A, B, S, and T. Moreover,* 0 *is the only common fixed point shared by A, B, S, and T.*

Corollary 1. *Let us consider* (X, d) *as a* b_2*-complete* b_2 *metric space; M, N, P, Q, H, and R are six self-mappings of X meeting the generalized* $(\phi, f)_\lambda$*-expansive condition:*

$$\begin{aligned} f(\frac{d(HQx, RPy, \alpha)}{\lambda^2}) \geq \phi\Big(& d(Mx, Ny, \alpha), d(Mx, HQx, \alpha), d(Ny, RPy, \alpha), \\ & \frac{d(Mx, HQx, \alpha) + d(Ny, RPy, \alpha)}{2}, \frac{d(Ny, HQx, \alpha)}{\lambda}, \\ & \max\{d(HQx, RPy, \alpha), \frac{d(HQx, Mx, \alpha) d(Ny, RPy, \alpha)}{d(Mx, Ny, \alpha)}\}\Big), \end{aligned} \tag{14}$$

for all $\alpha, x, y \in X$ *and* $\lambda \geq 1$*, where* $\phi \in \Phi$*, continuous function* $f : [0, \infty) \to [0, \infty)$ *is non-decreasing,* $f(0) = 0$*, and for all* $h > 0$*,* $\phi(h_1, h_2, h_3, h_4, h_5, h_6) > f(h)$*, where* $\min\{h_1, \frac{h_2+h_3}{2}\} = h$ *or* $\min\{h_1, h_5\} = h$*. The following requirements are considered to be met by these mappings:*

(1) $M(X) \subset RP(X)$*,* $N(X) \subset HQ(X)$*,*

(2) $\{M, HQ\}$ *are compatible, M and HQ are continuous, and* $\{N, RP\}$ *are weakly compatible or* $\{N, RP\}$ *are compatible, N and RP are continuous, and* $\{M, HQ\}$ *are weakly compatible.*

(3) (M, HQ) *and* (N, RP) *are interchangeable in pairs, that is*

$$MH = HM, MQ = QM, HQ = QH, NR = RN, NP = PN, RP = PR,$$

then M, N, P, Q, H and R have a unique common fixed point in X.

Proof. By Theorem 1, it is not difficult to see that M, N, HQ, and RP have a unique common fixed point $\theta \in X$, and it follows that it is also the only common fixed point of M, N, P, Q, H, and R.
Since M, N, HQ, and RP have a unique common fixed point, it is obvious that $M\theta = N\theta = HQ\theta = RP\theta = \theta$, taking $x = Q\theta$ and $y = \theta$. Therefore, we can easily acquire

$$\begin{aligned}f(d(HQQ\theta,RP\theta,\alpha))\geq& f(\frac{d(HQQ\theta,RP\theta,\alpha)}{\lambda^2})\\ \geq&\phi\Big(d(MQ\theta,N\theta,\alpha),d(MQ\theta,HQQ\theta,\alpha),d(N\theta,RP\theta,\alpha),\\ &\frac{d(MQ\theta,HQQ\theta,\alpha)+d(N\theta,RP\theta,\alpha)}{2},\frac{d(N\theta,HQQ\theta,\alpha)}{\lambda},\\ &\max\{d(HQQ\theta,RP\theta,\alpha),\frac{d(HQQ\theta,MQ\theta,\alpha)d(Nz,RPz,\alpha)}{d(MQ\theta,N\theta,\alpha)}\}\Big),\end{aligned}$$

$$\begin{aligned}f(d(Q\theta,\theta,\alpha))\geq& f(\frac{d(Q\theta,\theta,\alpha)}{\lambda^2})\\ \geq&\phi\Big(d(Q\theta,\theta,\alpha),d(Q\theta,Q\theta,\alpha),d(\theta,\theta,\alpha),\\ &\frac{d(Q\theta,Q\theta,\alpha)+d(\theta,\theta,\alpha)}{2},\frac{d(\theta,Q\theta,\alpha)}{\lambda},\\ &\max\{d(Q\theta,\theta,\alpha),\frac{d(Q\theta,Q\theta,\alpha)d(\theta,\theta,\alpha)}{d(Q\theta,\theta,\alpha)}\}\Big),\end{aligned}$$

and from the above equation, we obtain

$$f(d(Q\theta,\theta,\alpha))\geq\phi(d(Q\theta,\theta,\alpha),0,0,0,\frac{d(\theta,Q\theta,\alpha)}{\lambda},d(Q\theta,\theta,\alpha)),$$

so

$$f(d(Q\theta,\theta,\alpha))\geq\min\{h_1,\frac{h_2+h_3}{2}\}=0.$$

Furthermore, since $\phi(h_1,h_2,h_3,h_4,h_5,h_6)>f(h)$, we get $Q\theta=\theta$, which means that $H(Q\theta)=H\theta=\theta$ is true. In the same way, we can obtain $P\theta=\theta$, that is $R(P\theta)=R\theta=\theta$. As a result,

$$\theta=M\theta=N\theta=H\theta=Q\theta=R\theta=P\theta,$$

and θ is the unique common fixed point of M, N, P, Q, H, and R; the corollary is proven. □

Corollary 2. *Let (X,d) be a b_2-complete b_2 metric space. Then, A, B, S, and T are four self-mappings of X meeting the generalized $(\phi,f)_\lambda$-expansive condition:*

$$\begin{aligned}f(\frac{d(S^px,T^qy,\alpha)}{\lambda^2})\geq&\phi\Big(d(A^mx,B^ny,\alpha),d(A^mx,S^px,\alpha),d(B^ny,T^qy,\alpha),\\ &\frac{d(A^mx,S^px,\alpha)+d(B^ny,T^qy,\alpha)}{2},\frac{d(B^ny,S^px,\alpha)}{\lambda},\\ &\max\{d(S^px,T^qy,\alpha),\frac{d(S^px,A^mx,\alpha)d(B^ny,T^qy,\alpha)}{d(A^mx,B^ny,\alpha)}\}\Big),\end{aligned}\tag{15}$$

for all $\alpha,x,y\in X$, $\lambda\geq 1$ and $m,n,p,q\in N$, where $\phi\in\Phi$, the continuous function $f:[0,\infty)\to[0,\infty)$ is nondecreasing, $f(0)=0$, and for all $h>0$, $\phi(h_1,h_2,h_3,h_4,h_5,h_6)>f(h)$, where $\min\{h_1,\frac{h_2+h_3}{2}\}=h$ or $\min\{h_1,h_5\}=h$. It is assumed that these mappings also satisfy the below circumstances:

(1) $A(X)\subset T(X)$, $B(X)\subset S(X)$,
(2) *A(or B) and S(or T) are continuous, A(or B) and S(or T) are compatible, and B(or A) and T(or S) are weakly compatible.*

If $AS=SA$, $BT=TB$, then A, B, S and T have a unique common fixed point in X.

3. Implicit Relations

As it is well accepted, previous literature mostly studied and generalized fixed points under explicit or semiexplicit contraction conditions. Now, by introducing a new class of functions in a b_2 metric space and providing implicit contraction requirements, we can likewise achieve satisfactory results. Notwithstanding, currently, we can also achieve satisfactory results by introducing a new class of functions in the b_2 metric space and establishing implicit contraction conditions.

Definition 9. *Let Γ be the set of real function $F(t_1, t_2, t_3, \ldots, t_6) : {R_+}^6 \to R_+$ fulfilling the below circumstances, where $\lambda \geq 1$:*

(F_1) F is monotonically decreasing with respect to the fourth and fifth variables and there exists $h_1, h_2 \in [0, 1)$ such that

(F_{11}) if $F(t, z, t, \frac{t+z}{2}, t+z, z) \leq 0$ exists, then $t \leq h_1 z$;

(F_{12}) if $F(t, t, z, \frac{t+z}{2}, 0, z) \leq 0$ exists, then $t \leq h_2 z$;

(F_{21}): For all $t > 0$, $F(t, 0, t, \frac{t}{2}, 0, 0) > 0$;

(F_{22}): For all $t > 0$, $F(t, 0, 0, 0, \frac{t}{\lambda}, t) > 0$;

(F_{31}): For all $t > 0$, $F(t, t, 0, \frac{t}{2}, 0, 0) > 0$;

(F_{32}): For all $t > 0$, $F(t, 0, t, \frac{t}{2}, \frac{t}{\lambda}, 0) > 0$.

Example 7. *$F(t_1, t_2, t_3, \ldots, t_6) = t_1^2 - t_1(at_2 + bt_4 + ct_6) - dt_3t_5$, where $a, b, c, d \geq 0, a + b + d \leq 1, a - \frac{d}{\lambda} < 0$.*

F_1 is satisfied since the fourth and fifth variables, namely t_4 and t_5, are both in the subtrahend position in the expression of F, meaning that as they grow, the value of F decreases.

F_{11}: Let $t > 0$, $F(t, z, t, \frac{t+z}{2}, t+z, z) = t^2 - t(az + \frac{b}{2}(t+z) + cz) - dt(t+z) = (1 - \frac{b}{2} - d)t^2 - (a + \frac{b}{2} - c + d)tz \leq 0$, then $t \leq \frac{a+\frac{b}{2}-c+d}{1-\frac{b}{2}-d} z = h_1 z$, where $h_1 = \frac{a+\frac{b}{2}-c+d}{1-\frac{b}{2}-d} < 1$.

F_{12}: Let $t > 0$, $F(t, t, z, \frac{t+z}{2}, 0, z) = t^2 - t(at + \frac{b}{2}(t+z) + cz) = (1 - a - \frac{b}{2})t^2 - (\frac{b}{2} + c)tz \leq 0$, then $t \leq \frac{\frac{b}{2}+c}{1-a-\frac{b}{2}} z = h_2 z$, where $h_2 = \frac{\frac{b}{2}+c}{1-a-\frac{b}{2}}$.

Therefore, $t \leq hz$, where $h = \max\{h_1, h_2\}$, if $t = 0$, then $t \leq hz$.

$F_{21} = (1 - \frac{b}{2})t^2 > 0$;

$F_{22} = t^2 > 0$;

$F_{31} = (1 - a - \frac{b}{2}) > 0$;

$F_{32} = (1 - \frac{b}{2} - \frac{d}{\lambda})t^2 > 0$.

So all the conditions of the F function are satisfied.

Example 8. *$F(t_1, t_2, t_3, \ldots, t_6) = t_1^3 - at_1t_2t_3 - bt_1^2t_4 - ct_1t_3t_5 - dt_1^2t_6$, where $a > 0, c \geq 0, 0 \leq b < 2, 0 \leq d < 1, 2a + b \leq 2, 1 - \frac{b}{2} - \frac{\lambda}{c} > 0$.*

F_1 is satisfied since the fourth and fifth variables, namely t_4 and t_5, are both in the subtrahend position in the expression of F, meaning that as they grow, the value of F decreases.

F_{11}: Let $t > 0$, $F(t, z, t, \frac{t+z}{2}, t+z, z) = t^3 - at^2z - \frac{b}{2}t^2(t+z) - ct^2(t+z) - dt^2z = (1 - \frac{b}{2} - c)t^3 - (a - \frac{b}{2} - c - d)t^2z \leq 0$, then $t \leq \frac{a-\frac{b}{2}-c-d}{1-\frac{b}{2}-c} z = h_1 z$, where $h_1 = \frac{a-\frac{b}{2}-c-d}{1-\frac{b}{2}-c} < 1$.

F_{12}: Let $t > 0$, $F(t, t, z, \frac{t+z}{2}, 0, z) = t^3 - at^2z - \frac{b}{2}t^2(t+z) - dt^2z = (1 - \frac{b}{2})t^3 - (a + \frac{b}{2} + d)t^2z \ \leq 0$, then $t \leq \frac{a+\frac{b}{2}+d}{1-\frac{b}{2}} z = h_2 z$, where $h_2 = \frac{a+\frac{b}{2}+d}{1-\frac{b}{2}}$.

Therefore, $t \leq hz$, where $h = \max\{h_1, h_2\}$, if $t = 0$, then $t \leq hz$.

$F_{21} = (1 - \frac{b}{2})t^3 > 0$;

$F_{22} = (1 - d)t^3 > 0$;

$F_{31} = (1 - \frac{b}{2})t^3 > 0$;

$F_{32} = (1 - \frac{b}{2} - \frac{c}{\lambda})t^3 > 0$.

So all the conditions of the F function are satisfied.

According to the literature [14], a sequence of 2-metric spaces is a 2-Cauchy sequence and we provide the following lemma for use in the subsequent proof.

Lemma 1 ([17]). *Assume that a sequence $\{x_n\}_{n \in N}$ exists in b_2 metric space (X, d); if there exists $k \in [0, 1)$, for any $\alpha, n \in X$ with $d(x_{n+2}, x_{n+1}, \alpha) \leq kd(x_{n+1}, x_n, \alpha)$, then $\{x_n\}_{n \in N}$ is a b_2-Cauchy sequence.*

Definition 10. *Let (X, d) be a b_2-complete b_2 metric space, S, T, I, and J are four self-mappings of X fulfilling $S(X) \subset J(X)$, $T(X) \subset I(X)$; if for any $x, y, \alpha \in X$ and $\lambda \geq 1$, one has*

$$F\Big(d(Sx,Ty,\alpha), d(Sx,Ix,\alpha), d(Ty,Jy,\alpha), \frac{d(Sx,Ix,\alpha)+d(Ty,Jy,\alpha)}{2}, \\ \frac{d(Ty,Ix,\alpha)}{\lambda}, \max\{d(Ix,Jy,\alpha), \frac{d(Ix,Sx,\alpha)d(Ty,Jy,\alpha)}{d(Sx,Ty,\alpha)}\}\Big) \leq 0, \tag{16}$$

where $F \in \Gamma$.

Theorem 2. *Let (X, d) be a b_2-complete b_2 metric space, and S, T, I and J are four self-mappings of X meeting implicit relations (16) and $S(X) \subset J(X)$, $T(X) \subset I(X)$. If one of the mappings in S, T, I, and J is continuous, and $\{S, I\}$ and $\{T, J\}$ are, respectively, compatible, then S, T, I, and J have a unique common fixed point.*

Proof. Since $S(X) \subset J(X)$, there exists $x_1 \in X$ making $Sx_0 = Jx_1$, and since $T(X) \subset I(X)$, there exists $x_2 \in X$ making $Tx_1 = Ix_2$.

Constructing two sequences $\{x_n\}$ and $\{u_n\}$, respectively, such that

$$u_{2n+1} = Sx_{2n} = Jx_{2n+1}, u_{2n+2} = Tx_{2n+1} = Ix_{2n+2}. \forall n = 0, 1, 2, \ldots \tag{17}$$

Substituting $x = x_{2n}, u = x_{2n+1}$ into (16), we obtain

$$F\Big(d(Sx_{2n},Tx_{2n+1},\alpha), d(Sx_{2n},Ix_{2n},\alpha), d(Tx_{2n+1},Jx_{2n+1},\alpha), \\ \frac{d(Sx_{2n},Ix_{2n},\alpha)+d(Tx_{2n+1},Jx_{2n+1},\alpha)}{2}, \frac{d(Tx_{2n+1},Ix_{2n},\alpha)}{\lambda}, \\ \max\{d(Ix_{2n},Jx_{2n+1},\alpha), \frac{d(Ix_{2n},Sx_{2n},\alpha)d(Tx_{2n+1},Jx_{2n+1},\alpha)}{d(Sx_{2n},Ty_{2n+1},\alpha)}\}\Big) \leq 0,$$

and formula (18) can be obtained by arranging

$$F\Big(d(u_{2n+1},u_{2n+2},\alpha), d(u_{2n},u_{2n+1},\alpha), d(u_{2n+1},u_{2n+2},\alpha), \\ \frac{d(u_{2n},u_{2n+1},\alpha)+d(u_{2n+1},u_{2n+2},\alpha)}{2}, \frac{d(u_{2n},u_{2n+2},\alpha)}{\lambda}, \\ \max\{d(u_{2n},u_{2n+1},\alpha), \frac{d(u_{2n},u_{2n+1},\alpha)d(u_{2n+1},u_{2n+2},\alpha)}{d(u_{2n+1},u_{2n+2},\alpha)}\}\Big) \leq 0. \tag{18}$$

Since X is a b_2 metric space, according to the fourth property in the definition of a b_2 metric space, we get the below formula

$$d(u_{2n},u_{2n+2},\alpha) \leq \lambda[d(u_{2n},u_{2n+1},u_{2n+2}) + d(u_{2n},u_{2n+1},\alpha) + d(\alpha,u_{2n+1},u_{2n+2})],$$

then recalling $t = d(u_{2n+1},u_{2n+2},\alpha)$, $z = d(u_{2n},u_{2n+1},\alpha)$, the above formula can be simplified to

$$d(u_{2n},u_{2n+2},\alpha) \leq \lambda[t + z + d(u_{2n},u_{2n+1},u_{2n+2})],$$

and substituting it in (18), we get

$$F(t, z, t, \frac{t+z}{2}, t + z + d(u_{2n},u_{2n+1},u_{2n+2}), z) \leq 0.$$

Faced with this situation, we take $\alpha = u_{2n}$ in (18), and then arranging according to the definition of the b_2 metric space yields

$$F\Big(d(u_{2n+1},u_{2n+2},u_{2n}), 0, d(u_{2n+1},u_{2n+2},u_{2n}), \frac{d(u_{2n+1},u_{2n+2},u_{2n})}{2}, 0, 0\Big) \leq 0.$$

Obviously, we can see that this contradicts (F_{21}) in Definition 9, so there is

$$d(u_{2n+1}, u_{2n+2}, u_{2n}) = 0, \forall n = 0, 1, 2, \ldots$$

and

$$F(t, z, t, \frac{t+z}{2}, t+z, z) \leq 0.$$

Then, according to (F_{11}) in Definition 9, one has

$$d(u_{2n+1}, u_{2n+2}, \alpha) \leq h_1 d(u_{2n}, u_{2n+1}, \alpha), \forall n = 0, 1, 2, \ldots, \alpha \in X.$$

When $x = x_{2n+2}, u = x_{2n+1}$, we substitute in (16) to get

$$F\Big(d(Sx_{2n+2}, Tx_{2n+1}, \alpha), d(Sx_{2n+2}, Ix_{2n+2}, \alpha), d(Tx_{2n+1}, Jx_{2n+1}, \alpha), \frac{d(Sx_{2n+2}, Ix_{2n+2}, \alpha) + d(Tx_{2n+1}, Jx_{2n+1}, \alpha)}{2}, \frac{d(Tx_{2n+1}, Ix_{2n+2}, \alpha)}{\lambda}, \max\{d(Ix_{2n+2}, Jx_{2n+1}, \alpha), \frac{d(Ix_{2n+2}, Sx_{2n+2}, \alpha) d(Tx_{2n+1}, Jx_{2n+1}, \alpha)}{d(Sx_{2n+2}, Tx_{2n+1}, \alpha)}\}\Big) \leq 0,$$

and using (17) in the above formula, we obtain

$$F\Big(d(u_{2n+3}, u_{2n+2}, \alpha), d(u_{2n+3}, u_{2n+2}, \alpha), d(u_{2n+1}, u_{2n+2}, \alpha), \frac{d(u_{2n+3}, u_{2n+2}, \alpha) + d(u_{2n+1}, u_{2n+2}, \alpha)}{2}, \frac{d(u_{2n+2}, u_{2n+2}, \alpha)}{\lambda}, \max\{d(u_{2n+2}, u_{2n+1}, \alpha), \frac{d(u_{2n+2}, u_{2n+3}, \alpha) d(u_{2n+1}, u_{2n+2}, \alpha)}{d(u_{2n+3}, u_{2n+2}, \alpha)}\}\Big) \leq 0.$$

Continuing to simplify

$$F(t, t, z, \frac{t+z}{2}, 0, z) \leq 0,$$

from (F_{12}) in Definition 9 yields

$$d(u_{2n+2}, u_{2n+3}, \alpha) \leq h_2 d(u_{2n+1}, u_{2n+2}, \alpha), \forall n = 0, 1, 2, \ldots, \alpha \in X.$$

Let $h = \max\{h_1, h_2\}, h \in [0, 1)$, so

$$d(u_{n+2}, u_{n+1}, \alpha) \leq h d(u_{n+1}, u_n, \alpha), \forall n = 0, 1, 2, \ldots, \alpha \in X,$$

therefore, from the lemma, we can prove that $\{u_n\}$ is a b_2-Cauchy sequence.

Next, since X is b_2-complete, we know that $\{u_n\}$ b_2-converges to $r \in X$, which is to say that r is the limit of $\{Sx_{2n}\} = \{Jx_{2n+1}\}$ and $\{Tx_{2n+1}\} = \{Ix_{2n+2}\}$.

Assuming I is continuous, then $\{ISx_{2n}\}$ b_2-converges to $\{Ir\}$. According to the triangular inequality of the b_2 metric space, we obtain

$$d(SIx_{2n}, Ir, \alpha) \leq \lambda[d(SIx_{2n}, Ir, ISx_{2n}) + d(ISx_{2n}, Ir, \alpha) + d(SIx_{2n}, ISx_{2n}, \alpha)],$$

I is continuous, and S and I are compatible, so let $n \to \infty$, and we know that $\{SIx_{2n}\}$ also b_2-converges to $\{Ir\}$.

Replacing x and y in (16), with Ix_{2n} and x_{2n+1}, respectively, yields

$$F\Big(d(SIx_{2n}, Tx_{2n+1}, \alpha), d(SIx_{2n}, I^2x_{2n}, \alpha), d(Tx_{2n+1}, Jx_{2n+1}, \alpha), \frac{d(SIx_{2n}, I^2x_{2n}, \alpha) + d(Tx_{2n+1}, Jx_{2n+1}, \alpha)}{2}, \frac{d(Tx_{2n+1}, I^2x_{2n}, \alpha)}{\lambda}, \max\{d(I^2x_{2n}, Jx_{2n+1}, \alpha), \frac{d(I^2x_{2n}, SIx_{2n}, \alpha) d(Tx_{2n+1}, Jx_{2n+1}, \alpha)}{d(STx_{2n}, Tx_{2n+1}, \alpha)}\}\Big) \leq 0,$$

then taking n to infinity and using the continuity of F, we obtain

$$F\Big(d(Ir,r,\alpha),d(Ir,Ir,\alpha),d(r,r,\alpha),\frac{d(Ir,Ir,\alpha)+d(r,r,\alpha)}{2},\frac{d(r,Ir,\alpha)}{\lambda},d(Ir,r,\alpha)\Big)\leq 0,$$

which can be simplified to get

$$F\Big(d(Ir,r,\alpha),0,0,0,\frac{d(r,Ir,\alpha)}{\lambda},d(Ir,r,\alpha)\Big)\leq 0.$$

Since it contradicts (F_{22}) in Definition 9, it is easy to know $d(Ir,r,\alpha)=0$, so $r=Ir$. Then, replacing x in (16) with r and y with x_{2n+1} yields

$$\begin{aligned}F\Big(&d(Sr,Tx_{2n+1},\alpha),d(Sr,Ir,\alpha),d(Tx_{2n+1},Jx_{2n+1},\alpha),\\ &\frac{d(Sr,Ir,\alpha)+d(Tx_{2n+1},Jx_{2n+1},\alpha)}{2},\frac{d(Tx_{2n+1},Ir,\alpha)}{\lambda},\\ &\max\{d(Ir,Jx_{2n+1},\alpha),\frac{d(Ir,Sr,\alpha)d(Tx_{2n+1},Jx_{2n+1},\alpha)}{d(Sr,Tx_{2n+1},\alpha)}\}\Big)\leq 0.\end{aligned}$$

Let $n\to\infty$, combined with the continuity of F, we can get the formula that contradicts (F_{31}) in Definition 9, that is,

$$F\Big(d(Sr,r,\alpha),d(Sr,r,\alpha),0,\frac{d(Sr,r,\alpha)}{2},0,0\Big)\leq 0,$$

so we can get $d(Sr,r,\alpha)=0$ and $r=Sr$. Therefore, r is the common fixed point of mapping S and T.

Furthermore, because $S(X)\subset J(X)$, there exists $w\in X$, such that $Jw=z$. Substitute $x=r,y=w$ into (3.1) to get

$$\begin{aligned}F\Big(&d(Sr,Tw,\alpha),d(Sr,Ir,\alpha),d(Tw,Jw,\alpha),\frac{d(Sr,Ir,\alpha)+d(Tw,Jw,\alpha)}{2},\\ &\frac{d(Tw,Ir,\alpha)}{\lambda},\max\{d(Ir,Jw,\alpha),\frac{d(Ir,Sr,\alpha)d(Tw,Jw,\alpha)}{d(Sr,Tw,\alpha)}\}\Big)\leq 0,\end{aligned}$$

and simplify the formula to get

$$F\Big(d(r,Tw,\alpha),0,d(r,Tw,\alpha),\frac{d(r,Tw,\alpha)}{2},\frac{d(r,Tw,\alpha)}{\lambda},0\Big)\leq 0,$$

then, from (F_{32}) in Definition 9, we get $d(r,Tw,\alpha)=0$ and $r=Tw$.

Because $Tw=Jw=r$ and T and J are compatible, $TJw=JTw$, and combined with the content that has been deduced above, we can get $Tr=TJw=JTw=Jr$.

When $x=r,y=r$ in (16), we have

$$\begin{aligned}F\Big(&d(Sr,Tr,\alpha),d(Sr,Ir,\alpha),d(Tr,Jr,\alpha),\frac{d(Sr,Ir,\alpha)+d(Tr,Jr,\alpha)}{2},\\ &\frac{d(Tr,Ir,\alpha)}{\lambda},\max\{d(Ir,Jr,\alpha),\frac{d(Ir,Sr,\alpha)d(Tr,Jr,\alpha)}{d(Sr,Tr,\alpha)}\}\Big)\leq 0,\end{aligned}$$

and continuing to simplify, we can get

$$F(d(r,Tr,\alpha),0,0,0,\frac{d(r,Tr,\alpha)}{\lambda},d(r,Tr,\alpha))\leq 0.$$

Obviously, this contradicts (F_{22}) in Definition 9, so $d(r,Tr,\alpha)=0$ and $r=Tr$, and r is the common fixed point of S, T, I, and J.

Finally, we show that the common fixed points are unique.

Let r' be a common fixed point of S, T, I, and J different from r. Let $x = r$ and $y = r'$ in (16), one has

$$F\Big(d(Sr,Tr',\alpha),d(Sr,Ir,\alpha),d(Tr',Jr',\alpha),\frac{d(Sr,Ir,\alpha)+d(Tr',Jr',\alpha)}{2},$$
$$\frac{d(Tr',Ir,\alpha)}{\lambda},\max\{d(Ir,Jr',\alpha),\frac{d(Ir,Sr,\alpha)d(Tr',Jr',\alpha)}{d(Sr,Tr',\alpha)}\}\Big)\leq 0,$$

and

$$F(d(r,r',\alpha),0,0,0,\frac{d(r,r',\alpha)}{\lambda},d(r,r',\alpha))\leq 0,$$

then it can be known from (F_{22}) in Definition 9 that $d(r,r',\alpha) = 0$, so $r = r'$.

To sum up, S, T, I, and J have a unique common fixed point: r.

The method of proof is almost identical when S, T and J are, respectively, continuous. □

Corollary 3. *Let (X,d) be a b_2-complete b_2 metric space; I, J, and $\{T_i\}_i \in N^*$ are some self-mappings of X fulfilling $T(i) \subset J(x) \bigcap I(X)$, if for any $x, y, \alpha \in X$ and $\lambda \geq 1$, one has*

$$F\Big(d(T_ix,T_{i+1}y,\alpha),d(T_ix,Ix,\alpha),d(T_{i+1}y,Jy,\alpha),\frac{d(T_ix,Ix,\alpha)+d(T_{i+1}y,Jy,\alpha)}{2},$$
$$\frac{d(T_{i+1}y,Ix,\alpha)}{\lambda},\max\{d(Ix,Jy,\alpha),\frac{d(Ix,T_ix,\alpha)d(T_{i+1}y,Jy,\alpha)}{d(Sx,T_{i+1}y,\alpha)}\}\Big)\leq 0, \tag{19}$$

where $F \in \Gamma$. If one of the mappings in I, J, and $\{T_i\}_i \in N^*$ is continuous, and $\{T_i, I\}$ and $\{T_{i+1}, J\}$ are, respectively, compatible, then I, J, and $\{T_i\}_i \in N^*$ have a unique common fixed point.

Now, introduce two continuous functions $\delta : [0,\infty)^3 \to (-\infty,+\infty)$, $\rho : [0,\infty)^3 \to (-\infty,+\infty)$.

Corollary 4. *Let (X,d) be a b_2-complete b_2 metric space; S, T, I, and J are four self-mappings of X meeting $S(X) \subset J(X)$, $T(X) \subset I(X)$, if for any $x, y, \alpha \in X$ and $\lambda \geq 1$, one has*

$$\delta(d(Sx,Ty,\alpha),d(Sx,Ix,\alpha),d(Ty,Jy,\alpha))$$
$$\leq \rho(\frac{d(Sx,Ix,\alpha)+d(Ty,Jy,\alpha)}{2},\frac{d(Ty,Ix,\alpha)}{\lambda},$$
$$\max\{d(Ix,Jy,\alpha),\frac{d(Ix,Sx,\alpha)d(Ty,Jy,\alpha)}{d(Sx,Ty,\alpha)}\}), \tag{20}$$

where δ and ρ satisfy the following conditions:

(i) ρ is monotonically increasing with respect to the first and second variables;

(ii) There exist $h_1, h_2 \in [0,1)$, such that

if $\delta(t,z,t) \leq \rho(\frac{t+z}{2}, t+z, z)$, then $t \leq h_1 z$;

if $\delta(t,t,z) \leq \rho(\frac{t+z}{2}, 0, z)$, then $t \leq h_2 z$;

(iii) For any $t > 0$, $\delta(t,0,t) > \rho(\frac{t}{2},0,0)$; $\delta(t,0,0) > \rho(0,\frac{t}{\lambda},t)$; $\delta(t,t,0) > \rho(\frac{t}{2},0,0)$; and $\delta(t,0,t) > \rho(\frac{t}{2},\frac{t}{\lambda},0)$.

If there is a continuous mapping in S, T, I, and J, and $\{S, I\}$ and $\{T, J\}$ are, respectively, compatible, then S, T, I, and J have a unique common fixed point.

Proof. Let $F(t_1,t_2,t_3,t_4,t_5,t_6) = \delta(t_1,t_2,t_3) - \rho(t_4,t_5,t_6)$, so F is monotonically decreasing with respect to the fourth and fifth variables. Additionally, and because other conditions of Γ can be easily satisfied on the function F, we can determine $F \in \Gamma$.

Finally, according to Theorem 2, the corollary is proved. □

4. Conclusions

In this paper, we predominantly focused on some questions about common fixed points in a b_2 metric space, obtaining the above theorems. The above theorems can be adapted to prove the existence and uniqueness of fixed points under generalized extended mappings and implicit functions. We can alternatively design the spaces that meet the criteria and then use the theory to achieve the desired results under the theorem's given condition. Future research should continue in this approach. The application of this essay to numerous realms of reality is also a direction for future investigation.

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