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New Hermite–Hadamard Inequalities for Convex Fuzzy-Number-Valued Mappings via Fuzzy Riemann Integrals

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Abstract: This study uses fuzzy order relations to examine Hermite–Hadamard inequalities (*HH*-inequalities) for convex fuzzy-number-valued mappings (FNVMs). The Kulisch–Miranker order relation, which is based on interval space, is used to define this fuzzy order relation which is defined level-wise. By utilizing this idea, several novel *HH*- and *HH*-Fejér-type inequalities are established in the fuzzy environment via convex FNVMs. Additional novel *HH*-type inequalities for the product of convex FNVMs are also found and proven with the use of practical examples. Additionally, certain unique situations that can be seen as applications of fuzzy *HH*-inequalities are presented. The ideas and methods presented in this work might serve as a springboard for more study in this field.

Keywords: fuzzy-number-valued mapping; fuzzy Riemann integral; convex fuzzy number valued mapping; Hermite–Hadamard inequality; Hermite–Hadamard–Fejér inequality

MSC: 26A33; 26A51; 26D10



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1. Introduction

Convex analysis has contributed significantly and fundamentally to the development of several practical and pure scientific domains. In the recent years, there has been a lot of focus on examining and separating various applications of the traditional concept of convexity. Convex mappings have recently undergone a number of expansions and generalizations. See [1–7] and the references therein for further helpful information. In the classical approach, a real-valued mapping $\mathfrak{U} : K \rightarrow \mathbb{R}$ is called convex if

$$\mathfrak{U}(\mathfrak{s}\mathfrak{x} + (1 - \mathfrak{s})\mathfrak{y}) \leq \mathfrak{s}\mathfrak{U}(\mathfrak{x}) + (1 - \mathfrak{s})\mathfrak{U}(\mathfrak{y}), \quad (1)$$

for all $\mathfrak{x}, \mathfrak{y} \in K$, $\mathfrak{s} \in [0, 1]$, where K is a convex set.

Research on the idea of convexity with integral problems is fascinating. As a result, several inequalities have been applied to convex mappings. A fascinating result of convex analysis is the Hermite–Hadamard inequality (*HH*-inequality, for short). The *HH*-inequality [8,9] for convex mapping $\mathfrak{U} : K \rightarrow \mathbb{R}$ on an interval $K = [\rho, \varsigma]$ is

$$\mathfrak{U}\left(\frac{\rho + \varsigma}{2}\right) \leq \frac{1}{\varsigma - \rho} \int_{\rho}^{\varsigma} \mathfrak{U}(\mathfrak{x}) d\mathfrak{x} \leq \frac{\mathfrak{U}(\rho) + \mathfrak{U}(\varsigma)}{2}, \quad (2)$$

for all $\rho, \varsigma \in K$.

Fejér considered the major generalizations of the *HH*-inequality in [10] which is known as the *HH*-Fejér inequality.

Let $\mathfrak{U} : K \rightarrow \mathbb{R}$ be a convex mapping on a convex set K and, $\varsigma \in K$ with $\rho \leq \varsigma$. Then,

$$\mathfrak{U}\left(\frac{\rho + \varsigma}{2}\right) \leq \frac{1}{\int_{\rho}^{\varsigma} \mathcal{C}(\mathfrak{x}) d\mathfrak{x}} \int_{\rho}^{\varsigma} \mathfrak{U}(\mathfrak{x}) \mathcal{C}(\mathfrak{x}) d\mathfrak{x} \leq \frac{\mathfrak{U}(\rho) + \mathfrak{U}(\varsigma)}{2} \int_{\rho}^{\varsigma} \mathcal{C}(\mathfrak{x}) d\mathfrak{x}, \quad (3)$$

If $\mathcal{C}(\mathfrak{x}) = 1$, then we obtain (2) from (3). Many inequalities may be found using special symmetric mapping $\mathcal{C}(\mathfrak{x})$ for convex mappings with the help of inequality (3).

On the other hand, automated error analysis is performed in order to increase the accuracy of the computation results. Moore [11], Kulish and W. Miranker [12], and others conceived and studied the idea of interval analysis, which substitutes interval operation for real operations. In this field, an interval of real numbers is used to represent an uncertain variable. Based on the aforementioned literature, Zhao et al. [13] proposed h-convex interval-valued mappings in 2018 and demonstrated that the HH-inequality applies specifically to convex i.v.ms as a particular case:

Theorem 1. Let $\mathfrak{U} : [\rho, \varsigma] \subset \mathbb{R} \rightarrow \mathbb{R}_I^+$ be a convex i.v.m given by $\mathfrak{U}(\mathfrak{x}) = [\mathfrak{U}_*(\mathfrak{x}), \mathfrak{U}^*(\mathfrak{x})]$ for all $\mathfrak{x} \in [\rho, \varsigma]$, where $\mathfrak{U}_*(\mathfrak{x})$ is a convex mapping and $\mathfrak{U}^*(\mathfrak{x})$ is a concave mapping. If $\mathfrak{U}$ is Riemann integrable, then

$$\mathfrak{U}\left(\frac{\rho + \varsigma}{2}\right) \supseteq \frac{1}{\varsigma - \rho} (IR) \int_{\rho}^{\varsigma} \mathfrak{U}(\mathfrak{x}) d\mathfrak{x} \supseteq \frac{\mathfrak{U}(\rho) + \mathfrak{U}(\varsigma)}{2}, \quad (4)$$

where $\mathbb{R}_I^+$ is the set of positive real intervals. We refer readers to [14–23] and the references therein for more study of the literature on the uses and characteristics of generalized convex mappings and HH-integral inequalities.

This plays a significant role in the study of a wide range of problems arising in pure mathematics and applied sciences, including operation research, computer science, management sciences, artificial intelligence, control engineering, and decision sciences. In [24], an enormous amount of research on fuzzy sets and systems has been devoted to the development of various fields. Similar to this, the concepts of convexity and non-convexity are crucial in optimization in the fuzzy domain because they allow us to characterize the optimality condition of convexity and produce fuzzy variational inequalities. As a result, the theories of variational inequality and fuzzy complementary problems have powerful mechanisms of mathematical problems and a cordial relationship. This field is fascinating and has produced many writers. Additionally, the concepts of convex fuzzy mapping and finding its optimality condition with the aid of fuzzy variational inequality were studied by Nanda and Kar [25] and Chang [26]. Fuzzy convexity's generalization and extension are crucial to its application in a variety of contexts. Let us remark that preinvex fuzzy mapping is one of the most often discussed kinds of nonconvex fuzzy mapping. This concept was first proposed by Noor [27], who also demonstrated some findings that show how fuzzy variational-like inequality distinguishes the fuzzy optimality condition of differentiable fuzzy preinvex mappings. For a more in-depth review of the literature on the uses and characteristics of generalized convex fuzzy mappings and variational-like inequalities, see [28–42] and the references therein.

The fuzzy mappings are fuzzy mappings with numerical values. There are certain integrals that deal with FNVMs and have FNVMs as their integrands. For instance, Osuna-Gomez et al. [43] and Costa et al. [44] built the Kulisch–Miranker order relation for the Jensen's integral inequality for FNVMs. Costa and Roman-Flores provided Minkowski and Beckenbach's inequalities, where the integrands are FNVMs, by employing the same methodology prompted by [13,43,44] and in particular by Costa et al. [45], who created a connection between the elements of fuzzy number space and interval space and developed level-wise fuzzy order relations on fuzzy number space through Kulisch–Miranker order relations defined on interval space. Using this idea of fuzzy number space, we develop a fuzzy integral inequality for convex FNVM, where the integrands are convex FNVM,

and generalize integral inequality (2) and (3). For more information, see [46–68] and the references therein.

The structure of this study is as follows: Preliminary ideas and findings in interval space, the space of fuzzy numbers, and convex analysis are presented in Section 2. Convex FNVs are used in Section 3 to obtain fuzzy *HH*-inequalities. To support our findings, some compelling instances are also provided. Conclusions and future plans are provided in Section 4.

2. Preliminaries

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_I$ be the collection of all closed and bounded intervals of $\mathbb{R}$, that is, $\mathbb{R}_I = \{[\mathcal{T}_*, \mathcal{T}^*] : \mathcal{T}_*, \mathcal{T}^* \in \mathbb{R} \text{ and } \mathcal{T}_* \leq \mathcal{T}^*\}$. If $\mathcal{T}_* \geq 0$, then $[\mathcal{T}_*, \mathcal{T}^*]$ is called a positive interval. The set of all positive intervals is denoted by $\mathbb{R}_I^+$ and defined as $\mathbb{R}_I^+ = \{[\mathcal{T}_*, \mathcal{T}^*] : \mathcal{T}_*, \mathcal{T}^* \in \mathbb{R}_I \text{ and } \mathcal{T}_* \geq 0\}$.

If $[\mathcal{S}_*, \mathcal{S}^*], [\mathcal{T}_*, \mathcal{T}^*] \in \mathbb{R}_I$ and $\rho \in \mathbb{R}$, then arithmetic operations are defined by

$$[\mathcal{S}_*, \mathcal{S}^*] + [\mathcal{T}_*, \mathcal{T}^*] = [\mathcal{S}_* + \mathcal{T}_*, \mathcal{S}^* + \mathcal{T}^*],$$

$$[\mathcal{S}_*, \mathcal{S}^*] \times [\mathcal{T}_*, \mathcal{T}^*] = [\min\{\mathcal{S}_*\mathcal{T}_*, \mathcal{S}^*\mathcal{T}_*, \mathcal{S}_*\mathcal{T}^*, \mathcal{S}^*\mathcal{T}^*\}, \max\{\mathcal{S}_*\mathcal{T}_*, \mathcal{S}^*\mathcal{T}_*, \mathcal{S}_*\mathcal{T}^*, \mathcal{S}^*\mathcal{T}^*\}],$$

$$\rho \cdot [\mathcal{S}_*, \mathcal{S}^*] = \begin{cases} [\rho\mathcal{S}_*, \rho\mathcal{S}^*] & \text{if } \rho \geq 0, \\ [\rho\mathcal{S}^*, \rho\mathcal{S}_*] & \text{if } \rho < 0. \end{cases}$$

For $[\mathcal{S}_*, \mathcal{S}^*], [\mathcal{T}_*, \mathcal{T}^*] \in \mathbb{R}_I$, the inclusion “ $\subseteq$ ” is defined by

$$[\mathcal{S}_*, \mathcal{S}^*] \supseteq [\mathcal{T}_*, \mathcal{T}^*], \text{ if and only if } \mathcal{T}_* \leq \mathcal{S}_*, \mathcal{S}^* \leq \mathcal{T}^*.$$

Remark 1. The relation “ $\leq_I$ ”, defined on $\mathbb{R}_I$ by

$$[\mathcal{S}_*, \mathcal{S}^*] \leq_I [\mathcal{T}_*, \mathcal{T}^*] \text{ if and only if } \mathcal{S}_* \leq \mathcal{T}_*, \mathcal{S}^* \leq \mathcal{T}^*,$$

for all $[\mathcal{S}_*, \mathcal{S}^*], [\mathcal{T}_*, \mathcal{T}^*] \in \mathbb{R}_I$, is an order relation; see [24]. For given $[\mathcal{S}_*, \mathcal{S}^*], [\mathcal{T}_*, \mathcal{T}^*] \in \mathbb{R}_I$, we say that $[\mathcal{S}_*, \mathcal{S}^*] \leq_I [\mathcal{T}_*, \mathcal{T}^*]$ if and only if $\mathcal{S}_* \leq \mathcal{T}_*, \mathcal{S}^* \leq \mathcal{T}^*$ or $\mathcal{S}_* \leq \mathcal{T}_*, \mathcal{S}^* < \mathcal{T}^*$.

The concept of a Riemann integral for i.v.m, first introduced by Moore [33], is defined as follows:

Theorem 2 ([11]). If $\mathfrak{U} : [c, \varsigma] \subset \mathbb{R} \rightarrow \mathbb{R}_I$ is an i.v.m such that $\mathfrak{U}(\mathfrak{x}) = [\mathfrak{U}_*(\mathfrak{x}), \mathfrak{U}^*(\mathfrak{x})]$. Then, $\mathfrak{U}$ is Riemann integrable over $[c, \varsigma]$ if and only if $\mathfrak{U}_*$ and $\mathfrak{U}^*$ are both Riemann integrable over $[c, \varsigma]$ such that

$$(IR) \int_c^\varsigma \mathfrak{U}(\mathfrak{x}) d\mathfrak{x} = \left[(R) \int_c^\varsigma \mathfrak{U}_*(\mathfrak{x}) d\mathfrak{x}, (R) \int_c^\varsigma \mathfrak{U}^*(\mathfrak{x}) d\mathfrak{x} \right].$$

The collection of all Riemann integrable real-valued mappings and Riemann integrable i.v.m is denoted by $\mathcal{R}_{[c, \varsigma]}$ and $\mathcal{FR}_{[c, \varsigma]}$, respectively.

Let $\mathbb{R}$ be the set of real numbers. A fuzzy subset set $\mathcal{A}$ of $\mathbb{R}$ is distinguished by a mapping $\mathcal{T} : \mathbb{R} \rightarrow [0, 1]$, called the membership mapping. In this study, this depiction is approved. Moreover, the collection of all fuzzy subsets of $\mathbb{R}$ is denoted by $\mathbb{E}$.

A real fuzzy number $\mathcal{T}$ is a fuzzy set in $\mathbb{R}$ with the following properties:

- (1) $\mathcal{T}$ is normal, i.e., there exists $\mathfrak{x} \in \mathbb{R}$ such that $\mathcal{T}(\mathfrak{x}) = 1$;
- (2) $\mathcal{T}$ is upper semi-continuous, i.e., for given $\mathfrak{x} \in \mathbb{R}$, for every $\mathfrak{x} \in \mathbb{R}$ there exist $\varepsilon > 0$ and there exist $\delta > 0$ such that $\mathcal{T}(\mathfrak{x}) - \mathcal{T}(y) < \varepsilon$ for all $y \in \mathbb{R}$ with $|\mathfrak{x} - y| < \delta$.
- (3) $\mathcal{T}$ is fuzzy convex, i.e., $\mathcal{T}((1 - \mathfrak{s})\mathfrak{x} + \mathfrak{s}y) \geq \min(\mathcal{T}(\mathfrak{x}), \mathcal{T}(y))$, $\forall \mathfrak{x}, y \in \mathbb{R}$ and $\mathfrak{s} \in [0, 1]$;
- (4) $\mathcal{T}$ is compactly supported, i.e., $cl\{\mathfrak{x} \in \mathbb{R} | \mathcal{T}(\mathfrak{x}) > 0\}$ is compact.

The collection of all real fuzzy numbers is denoted by $\mathbb{E}_{\mathbb{R}}$.

Since $\mathbb{E}_{\mathbb{F}}$ denotes the set of all real fuzzy numbers, let $\mathcal{T} \in \mathbb{E}_{\mathbb{F}}$ be real fuzzy number if and only if $\mathcal{J}$ -levels $[\mathcal{T}]^{\mathcal{J}}$ is a nonempty compact convex set of $\mathbb{R}$. This is represented by

$$[\mathcal{T}]^{\mathcal{J}} = \{\mathfrak{x} \in \mathbb{R} \mid \mathcal{T}(\mathfrak{x}) \geq \mathcal{J}\}.$$

From these definitions, we have

$$[\mathcal{T}]^{\mathcal{J}} = [\mathcal{T}_*(\mathcal{J}), \mathcal{T}^*(\mathcal{J})],$$

where

$$\mathcal{T}_*(\mathcal{J}) = \inf\{\mathfrak{x} \in \mathbb{R} \mid \mathcal{T}(\mathfrak{x}) \geq \mathcal{J}\}, \mathcal{T}^*(\mathcal{J}) = \sup\{\mathfrak{x} \in \mathbb{R} \mid \mathcal{T}(\mathfrak{x}) \geq \mathcal{J}\}.$$

Theorem 3 ([14,60]). Suppose that $\mathcal{T}_*(\mathcal{J}) : [0, 1] \rightarrow \mathbb{R}$ and $\mathcal{T}^*(\mathcal{J}) : [0, 1] \rightarrow \mathbb{R}$ satisfy the following conditions:

- (1) $\mathcal{T}_*(\mathcal{J})$ is a non-decreasing mapping.
- (2) $\mathcal{T}^*(\mathcal{J})$ is a non-increasing mapping.
- (3) $\mathcal{T}_*(1) \leq \mathcal{T}^*(1)$.
- (4) $\mathcal{T}_*(\mathcal{J})$ and $\mathcal{T}^*(\mathcal{J})$ are bounded, left-continuous on $(0, 1]$, and right-continuous at $\mathcal{J} = 0$.

Moreover, If $\mathcal{T} : \mathbb{R} \rightarrow [0, 1]$ is a real fuzzy number given by $[\mathcal{T}_*(\mathcal{J}), \mathcal{T}^*(\mathcal{J})]$, then mapping $\mathcal{T}_*(\mathcal{J})$ and $\mathcal{T}^*(\mathcal{J})$, we find the conditions (1)–(4).

Proposition 1 ([45]). Let $\mathcal{T}, \mathcal{S} \in \mathbb{E}_{\mathbb{F}}$. Then, the fuzzy order relation " $\preceq$ ", given on $\mathbb{E}_{\mathbb{F}}$ by $\mathcal{T} \preceq \mathcal{S}$ if and only if $[\mathcal{T}]^{\mathcal{J}} \leq_I [\mathcal{S}]^{\mathcal{J}}$ for all $\mathcal{J} \in [0, 1]$, is a partial order relation.

We now discuss some properties of real fuzzy numbers under addition, scalar multiplication, multiplication, and division. If $\mathcal{T}, \mathcal{S} \in \mathbb{E}_{\mathbb{F}}$ and $\rho \in \mathbb{R}$, then arithmetic operations are defined by

$$[\mathcal{T} \tilde{+} \mathcal{S}]^{\mathcal{J}} = [\mathcal{T}]^{\mathcal{J}} + [\mathcal{S}]^{\mathcal{J}}, \quad (5)$$

$$[\mathcal{T} \tilde{\times} \mathcal{S}]^{\mathcal{J}} = [\mathcal{T}]^{\mathcal{J}} \times [\mathcal{S}]^{\mathcal{J}}, \quad (6)$$

$$[\rho \cdot \mathcal{T}]^{\mathcal{J}} = \rho \cdot [\mathcal{T}]^{\mathcal{J}} \quad (7)$$

Remark 2. Obviously, $\mathbb{E}_{\mathbb{F}}$ is closed under addition and nonnegative scalar multiplication and the above-defined properties on $\mathbb{E}_{\mathbb{F}}$ are equivalent to those derived from the usual extension principle. Furthermore, for each scalar number $\rho \in \mathbb{R}$,

$$[\rho \tilde{+} \mathcal{T}]^{\mathcal{J}} = \rho + [\mathcal{T}]^{\mathcal{J}}.$$

Theorem 4 ([17,66]). The space $\mathbb{E}_{\mathbb{F}}$ dealing with a supremum metric, i.e., for $\psi, \mathcal{S} \in \mathbb{E}_{\mathbb{F}}$

$$D(\psi, \mathcal{S}) = \sup_{0 \leq \mathcal{J} \leq 1} H([\mathcal{T}]^{\mathcal{J}}, [\mathcal{S}]^{\mathcal{J}}),$$

is a complete metric space, where H denotes the well-known Hausdorff metric on the space of intervals.

Definition 1 ([45]). A fuzzy-number-valued map $\mathfrak{U} : K \subset \mathbb{R} \rightarrow \mathbb{E}_{\mathbb{F}}$ is called an FNVM. For each $\mathcal{J} \in [0, 1]$, whose $\mathcal{J}$ -levels define the family of i.v.ms, $\mathfrak{U}_{\mathcal{J}} : K \subset \mathbb{R} \rightarrow \mathbb{R}_I$ are given by $\mathfrak{U}_{\mathcal{J}}(\mathfrak{x}) = [\mathfrak{U}_*(\mathfrak{x}, \mathcal{J}), \mathfrak{U}^*(\mathfrak{x}, \mathcal{J})]$ for all $\mathfrak{x} \in K$. Here, for each $\mathcal{J} \in [0, 1]$, the end-point real mappings $\mathfrak{U}_*(\cdot, \mathcal{J}), \mathfrak{U}^*(\cdot, \mathcal{J}) : K \rightarrow \mathbb{R}$ are called the lower and upper mappings.

Remark 3. Let $\mathfrak{U} : K \subset \mathbb{R} \rightarrow \mathbb{E}_{\mathbb{F}}$ be an FNVM. Then, $\mathfrak{U}(\mathfrak{x})$ is said to be continuous at $\mathfrak{x} \in K$, if, for each $\mathcal{J} \in [0, 1]$, both end-point mappings $\mathfrak{U}_*(\mathfrak{x}, \mathcal{J})$ and $\mathfrak{U}^*(\mathfrak{x}, \mathcal{J})$ are continuous at $\mathfrak{x} \in K$.

From the above literature review, the following results can be concluded; see [17,45,66]:

Definition 2. Let $\mathfrak{U} : [c, \varsigma] \subset \mathbb{R} \rightarrow \mathbb{E}_{\mathbb{F}}$ be an FNVM. Then, the fuzzy Riemann integral of $\mathfrak{U}$ over $[c, \varsigma]$, denoted by $(FR) \int_c^{\varsigma} \mathfrak{U}(\mathfrak{x}) d\mathfrak{x}$, it is defined level-wise by

$$\left[(FR) \int_c^{\varsigma} \mathfrak{U}(\mathfrak{x}) d\mathfrak{x} \right]^{\mathcal{J}} = (IR) \int_c^{\varsigma} \mathfrak{U}_{\mathcal{J}}(\mathfrak{x}) d\mathfrak{x} = \left\{ \int_c^{\varsigma} \mathfrak{U}(\mathfrak{x}, \mathcal{J}) d\mathfrak{x} : \mathfrak{U}(\mathfrak{x}, \mathcal{J}) \in \mathcal{R}_{[c, \varsigma]} \right\}, \quad (8)$$

for all $\mathcal{J} \in [0, 1]$, where $\mathcal{R}_{[c, \varsigma]}$ is the collection of end-point mappings of i.v.ms. $\mathfrak{U}$ is FR-integrable over $[c, \varsigma]$ if $(FR) \int_c^{\varsigma} \mathfrak{U}(\mathfrak{x}) d\mathfrak{x} \in \mathbb{E}_{\mathbb{F}}$. Note that, if both end-point mappings are Lebesgue-integrable, then $\mathfrak{U}$ is a fuzzy Aumann-integrable mapping over $[c, \varsigma]$; see [11,65].

Theorem 5. Let $\mathfrak{U} : [c, \varsigma] \subset \mathbb{R} \rightarrow \mathbb{E}_{\mathbb{F}}$ be an FNVM whose $\mathcal{J}$ -levels define the family of i.v.ms $\mathfrak{U}_{\mathcal{J}} : [c, \varsigma] \subset \mathbb{R} \rightarrow \mathbb{R}_I$ that are given by $\mathfrak{U}_{\mathcal{J}}(\mathfrak{x}) = [\mathfrak{U}_*(\mathfrak{x}, \mathcal{J}), \mathfrak{U}^*(\mathfrak{x}, \mathcal{J})]$ for all $\mathfrak{x} \in [c, \varsigma]$ and for all $\mathcal{J} \in [0, 1]$. Then, $\mathfrak{U}$ is FR-integrable over $[c, \varsigma]$ if and only if, $\mathfrak{U}_*(\mathfrak{x}, \mathcal{J})$ and $\mathfrak{U}^*(\mathfrak{x}, \mathcal{J})$ both are R-integrable over $[c, \varsigma]$. Moreover, if $\mathfrak{U}$ is FR-integrable over $[c, \varsigma]$, then

$$\left[(FR) \int_c^{\varsigma} \mathfrak{U}(\mathfrak{x}) d\mathfrak{x} \right]^{\mathcal{J}} = \left[(R) \int_c^{\varsigma} \mathfrak{U}_*(\mathfrak{x}, \mathcal{J}) d\mathfrak{x}, (R) \int_c^{\varsigma} \mathfrak{U}^*(\mathfrak{x}, \mathcal{J}) d\mathfrak{x} \right] = (IR) \int_c^{\varsigma} \mathfrak{U}_{\mathcal{J}}(\mathfrak{x}) d\mathfrak{x}, \quad (9)$$

for all $\mathcal{J} \in [0, 1]$.

The family of all FR-integrable FNVMs and R-integrable mappings over $[c, \varsigma]$ are denoted by $\mathcal{FR}_{([c, \varsigma], \mathcal{J})}$ and $\mathcal{R}_{([c, \varsigma], \mathcal{J})}$, for all $\mathcal{J} \in [0, 1]$.

Definition 3 ([15,25]). Let K be a convex set. Then, FNVM $\mathfrak{U} : K \rightarrow \mathbb{E}_{\mathbb{F}}$ is said to be:

- Convex on K if

$$\mathfrak{U}(\mathfrak{s}\mathfrak{x} + (1 - \mathfrak{s})\mathfrak{y}) \preceq \mathfrak{s}\mathfrak{U}(\mathfrak{x}) \tilde{+} (1 - \mathfrak{s})\mathfrak{U}(\mathfrak{y}), \quad (10)$$

for all $\mathfrak{x}, \mathfrak{y} \in K$, $\mathfrak{s} \in [0, 1]$, where $\mathfrak{U}(\mathfrak{x}) \succcurlyeq \tilde{0}$.

- Concave on K if inequality (10) is reversed; and
- Affine convex on K if

$$\mathfrak{U}(\mathfrak{s}\mathfrak{x} + (1 - \mathfrak{s})\mathfrak{y}) = \mathfrak{s}\mathfrak{U}(\mathfrak{x}) \tilde{+} (1 - \mathfrak{s})\mathfrak{U}(\mathfrak{y}), \text{ for all } \mathfrak{x}, \mathfrak{y} \in K, \mathfrak{s} \in [0, 1]. \quad (11)$$

for all $\mathfrak{x}, \mathfrak{y} \in K$, $\mathfrak{s} \in [0, 1]$, where $\mathfrak{U}(\mathfrak{x}) \succcurlyeq \tilde{0}$.

Theorem 6 ([25]). Let K be a convex set, and let $\mathfrak{U} : K \rightarrow \mathbb{E}_{\mathbb{F}}$ be an FNVM whose $\mathcal{J}$ -levels define the family of i.v.ms $\mathfrak{U}_{\mathcal{J}} : K \subset \mathbb{R} \rightarrow \mathbb{R}_I^+ \subset \mathbb{R}_I$ that are given by

$$\mathfrak{U}_{\mathcal{J}}(\mathfrak{x}) = [\mathfrak{U}_*(\mathfrak{x}, \mathcal{J}), \mathfrak{U}^*(\mathfrak{x}, \mathcal{J})], \quad \forall \mathfrak{x} \in K. \quad (12)$$

for all $\mathfrak{x} \in K$ and for all $\mathcal{J} \in [0, 1]$. Then, $\mathfrak{U}$ is convex on K , if and only if, for all $\mathcal{J} \in [0, 1]$, $\mathfrak{U}_*(\mathfrak{x}, \mathcal{J})$ and $\mathfrak{U}^*(\mathfrak{x}, \mathcal{J})$ are convex.

Example 1. We consider the FNVMs $\mathfrak{U} : [0, 1] \rightarrow \mathbb{E}_{\mathbb{F}}$ defined by,

$$\mathfrak{U}(\mathfrak{x})(\sigma) = \begin{cases} \frac{\sigma}{2\mathfrak{x}^2} & \sigma \in [0, 2\mathfrak{x}^2] \\ \frac{4\mathfrak{x}^2 - \sigma}{2\mathfrak{x}^2} & \sigma \in (2\mathfrak{x}^2, 4\mathfrak{x}^2] \\ 0 & \text{otherwise,} \end{cases} \quad (13)$$

Then, for each $\mathcal{J} \in [0, 1]$, we have $\mathfrak{U}_{\mathcal{J}}(\mathfrak{x}) = [2\mathcal{J}\mathfrak{x}^2, (4 - 2\mathcal{J})\mathfrak{x}^2]$, since end-point mappings $\mathfrak{U}_*(\mathfrak{x}, \mathcal{J})$, $\mathfrak{U}^*(\mathfrak{x}, \mathcal{J})$ are convex mappings for each $\mathcal{J} \in [0, 1]$. Hence, $\mathfrak{U}(\mathfrak{x})$ is a convex FNVM.

3. Fuzzy Hermite–Hadamard Inequalities

In this section, we propose Hermite–Hadamard and Hermite–Hadamard–Fejér inequalities for convex FNVMs and verify them with the help of nontrivial examples.

Theorem 7. Let $\mathfrak{U} : [\rho, \varsigma] \rightarrow \mathbb{E}_{\mathbb{F}}$ be a convex FNVM on $[\rho, \varsigma]$, whose $\mathcal{J}$ -levels define the family of i.v.ms $\mathfrak{U}_{\mathcal{J}} : [\rho, \varsigma] \subset \mathbb{R} \rightarrow \mathbb{R}_I^+$ that are given by $\mathfrak{U}_{\mathcal{J}}(\mathfrak{x}) = [\mathfrak{U}_*(\mathfrak{x}, \mathcal{J}), \mathfrak{U}^*(\mathfrak{x}, \mathcal{J})]$ for all $\mathfrak{x} \in [\rho, \varsigma]$ and for all $\mathcal{J} \in [0, 1]$. If $\mathfrak{U} \in \mathcal{FR}_{([\rho, \varsigma], \mathcal{J})}$, then

$$\mathfrak{U}\left(\frac{\rho + \varsigma}{2}\right) \preccurlyeq \frac{1}{\varsigma - \rho} (FR) \int_{\rho}^{\varsigma} \mathfrak{U}(\mathfrak{x}) d\mathfrak{x} \preccurlyeq \frac{\mathfrak{U}(\rho) \tilde{+} \mathfrak{U}(\varsigma)}{2}. \quad (14)$$

If $\mathfrak{U}(\mathfrak{x})$ is a concave FNVM, then (14) is reversed.

Proof. Let $\mathfrak{U} : [\rho, \varsigma] \rightarrow \mathbb{E}_{\mathbb{F}}$ be a convex FNVM. Then, by hypothesis, we have

$$2\mathfrak{U}\left(\frac{\rho + \varsigma}{2}\right) \preccurlyeq \mathfrak{U}(\mathfrak{s}\rho + (1 - \mathfrak{s})\varsigma) \tilde{+} \mathfrak{U}((1 - \mathfrak{s})\rho + \mathfrak{s}\varsigma).$$

Therefore, for every $\mathcal{J} \in [0, 1]$, we have

$$\begin{aligned} 2\mathfrak{U}_*\left(\frac{\rho + \varsigma}{2}, \mathcal{J}\right) &\leq \mathfrak{U}_*(\mathfrak{s}\rho + (1 - \mathfrak{s})\varsigma, \mathcal{J}) + \mathfrak{U}_*((1 - \mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}), \\ 2\mathfrak{U}^*\left(\frac{\rho + \varsigma}{2}, \mathcal{J}\right) &\leq \mathfrak{U}^*(\mathfrak{s}\rho + (1 - \mathfrak{s})\varsigma, \mathcal{J}) + \mathfrak{U}^*((1 - \mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}). \end{aligned}$$

Then

$$\begin{aligned} 2 \int_0^1 \mathfrak{U}_*\left(\frac{\rho + \varsigma}{2}, \mathcal{J}\right) d\mathfrak{s} &\leq \int_0^1 \mathfrak{U}_*(\mathfrak{s}\rho + (1 - \mathfrak{s})\varsigma, \mathcal{J}) d\mathfrak{s} + \int_0^1 \mathfrak{U}_*((1 - \mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) d\mathfrak{s}, \\ 2 \int_0^1 \mathfrak{U}^*\left(\frac{\rho + \varsigma}{2}, \mathcal{J}\right) d\mathfrak{s} &\leq \int_0^1 \mathfrak{U}^*(\mathfrak{s}\rho + (1 - \mathfrak{s})\varsigma, \mathcal{J}) d\mathfrak{s} + \int_0^1 \mathfrak{U}^*((1 - \mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) d\mathfrak{s}. \end{aligned}$$

It follows that

$$\begin{aligned} \mathfrak{U}_*\left(\frac{\rho + \varsigma}{2}, \mathcal{J}\right) &\leq \frac{1}{\varsigma - \rho} \int_{\rho}^{\varsigma} \mathfrak{U}_*(\mathfrak{x}, \mathcal{J}) d\mathfrak{x}, \\ \mathfrak{U}^*\left(\frac{\rho + \varsigma}{2}, \mathcal{J}\right) &\leq \frac{1}{\varsigma - \rho} \int_{\rho}^{\varsigma} \mathfrak{U}^*(\mathfrak{x}, \mathcal{J}) d\mathfrak{x}. \end{aligned}$$

That is,

$$\left[\mathfrak{U}_*\left(\frac{\rho + \varsigma}{2}, \mathcal{J}\right), \mathfrak{U}^*\left(\frac{\rho + \varsigma}{2}, \mathcal{J}\right) \right] \leq_I \frac{1}{\varsigma - \rho} \left[\int_{\rho}^{\varsigma} \mathfrak{U}_*(\mathfrak{x}, \mathcal{J}) d\mathfrak{x}, \int_{\rho}^{\varsigma} \mathfrak{U}^*(\mathfrak{x}, \mathcal{J}) d\mathfrak{x} \right].$$

Thus,

$$\mathfrak{U}\left(\frac{\rho + \varsigma}{2}\right) \preccurlyeq \frac{1}{\varsigma - \rho} (FR) \int_{\rho}^{\varsigma} \mathfrak{U}(\mathfrak{x}) d\mathfrak{x}. \quad (15)$$

In a similar way as above, we have

$$\frac{1}{\varsigma - \rho} (FR) \int_{\rho}^{\varsigma} \mathfrak{U}(\mathfrak{x}) d\mathfrak{x} \preccurlyeq \frac{\mathfrak{U}(\rho) \tilde{+} \mathfrak{U}(\varsigma)}{2}. \quad (16)$$

Combining (14) and (15), we have

$$\mathfrak{U}\left(\frac{\rho + \varsigma}{2}\right) \preccurlyeq \frac{1}{\varsigma - \rho} (FR) \int_{\rho}^{\varsigma} \mathfrak{U}(\mathfrak{x}) d\mathfrak{x} \preccurlyeq \frac{\mathfrak{U}(\rho) \tilde{+} \mathfrak{U}(\varsigma)}{2}.$$

Hence, the required result. $\square$

Remark 4. If $\mathfrak{U}_*(\mathfrak{x}, \mathcal{J}) = \mathfrak{U}^*(\mathfrak{x}, \mathcal{J})$ with $\mathcal{J} = 1$, then Theorem 7, reduces to the result for convex mapping:

$$\mathfrak{U}\left(\frac{\rho + \varsigma}{2}\right) \leq \frac{1}{\varsigma - \rho} (R) \int_{\rho}^{\varsigma} \mathfrak{U}(\mathfrak{x}) d\mathfrak{x} \leq \frac{\mathfrak{U}(\rho) + \mathfrak{U}(\varsigma)}{2}.$$

We can easily note that, due to the convexity of end-point mappings, $\mathfrak{U}_*(\mathfrak{x}, \mathcal{J})$ and $\mathfrak{U}^*(\mathfrak{x}, \mathcal{J})$ with $\mathcal{J} = 1$ have the following possibilities to satisfy (16): either both are convex or affine convex. However, in the case of the interval HH-integral inequality (2), both end-point mappings should be affine convex because in interval inclusion $\mathfrak{U}_*(\mathfrak{x})$ is convex and $\mathfrak{U}^*(\mathfrak{x})$ is concave.

Example 2. We consider the FNVM $\mathfrak{U} : [\rho, \varsigma] = [0, 2] \rightarrow \mathbb{E}_{\mathbb{F}}$, defined by

$$\mathfrak{U}(\mathfrak{x})(\sigma) = \begin{cases} \frac{\sigma}{2\mathfrak{x}^2} & \sigma \in [0, 2\mathfrak{x}^2] \\ \frac{4\mathfrak{x}^2 - \sigma}{2\mathfrak{x}^2} & \sigma \in (2\mathfrak{x}^2, 4\mathfrak{x}^2] \\ 0 & \text{otherwise.} \end{cases}$$

Then, for each $\mathcal{J} \in [0, 1]$, we have $\mathfrak{U}_{\mathcal{J}}(\mathfrak{x}) = [2\mathcal{J}\mathfrak{x}^2, (4 - 2\mathcal{J})\mathfrak{x}^2]$. Since the end-point mappings $\mathfrak{U}_*(\mathfrak{x}, \mathcal{J}) = 2\mathcal{J}\mathfrak{x}^2$, and $\mathfrak{U}^*(\mathfrak{x}, \mathcal{J}) = (4 - 2\mathcal{J})\mathfrak{x}^2$, are convex mappings for each $\mathcal{J} \in [0, 1]$, then $\mathfrak{U}(\mathfrak{x})$ is a convex FNVM. We now compute the following

$$\mathfrak{U}_*\left(\frac{\rho + \varsigma}{2}, \mathcal{J}\right) \leq \frac{1}{\varsigma - \rho} \int_{\rho}^{\varsigma} \mathfrak{U}_*(\mathfrak{x}, \mathcal{J}) d\mathfrak{x} \leq \frac{\mathfrak{U}_*(\rho) + \mathfrak{U}_*(\varsigma)}{2}.$$

$$\mathfrak{U}_*\left(\frac{\rho + \varsigma}{2}, \mathcal{J}\right) = \mathfrak{U}_*(1, \mathcal{J}) = 2\mathcal{J},$$

$$\frac{1}{\varsigma - \rho} \int_{\rho}^{\varsigma} \mathfrak{U}_*(\mathfrak{x}, \mathcal{J}) d\mathfrak{x} = \frac{1}{2} \int_0^2 2\mathcal{J}\mathfrak{x}^2 d\mathfrak{x} = \frac{8\mathcal{J}}{3},$$

$$\frac{\mathfrak{U}_*(\rho, \mathcal{J}) + \mathfrak{U}_*(\varsigma, \mathcal{J})}{2} = 4\mathcal{J},$$

for all $\mathcal{J} \in [0, 1]$. That means that

$$2\mathcal{J} \leq \frac{8\mathcal{J}}{3} \leq 4\mathcal{J}.$$

Similarly, it can be easily show that

$$\mathfrak{U}^*\left(\frac{\rho + \varsigma}{2}, \mathcal{J}\right) \leq \frac{1}{\varsigma - \rho} \int_{\rho}^{\varsigma} \mathfrak{U}^*(\mathfrak{x}, \mathcal{J}) d\mathfrak{x} \leq \frac{\mathfrak{U}^*(\rho) + \mathfrak{U}^*(\varsigma)}{2}.$$

for all $\mathcal{J} \in [0, 1]$, such that

$$\mathfrak{U}^*\left(\frac{\rho + \varsigma}{2}, \mathcal{J}\right) = \mathfrak{U}^*(1, \mathcal{J}) = 2(2 - \mathcal{J}),$$

$$\frac{1}{\varsigma - \rho} \int_{\rho}^{\varsigma} \mathfrak{U}^*(\mathfrak{x}, \mathcal{J}) d\mathfrak{x} = \frac{1}{2} \int_0^2 (4 - 2\mathcal{J})\mathfrak{x}^2 d\mathfrak{x} = \frac{8(2 - \mathcal{J})}{3},$$

$$\frac{\mathfrak{U}^*(\rho, \mathcal{J}) + \mathfrak{U}^*(\varsigma, \mathcal{J})}{2} = 4(2 - \mathcal{J}),$$

from which it follows that

$$2(2 - \mathcal{J}) \leq \frac{8(2 - \mathcal{J})}{3} \leq 4(2 - \mathcal{J}),$$

that is,

$$[2j, 2(2-j)] \leq_I \left[\frac{8j}{3}, \frac{8(2-j)}{3} \right] \leq_I [4j, 4(2-j)], \text{ for all } j \in [0, 1].$$

Hence,

$$\mathfrak{U}\left(\frac{\rho + \varsigma}{2}\right) \preccurlyeq \frac{1}{\varsigma - \rho} (FR) \int_{\rho}^{\varsigma} \mathfrak{U}(\mathfrak{x}) d\mathfrak{x} \preccurlyeq \frac{\mathfrak{U}(\rho) \tilde{+} \mathfrak{U}(\varsigma)}{2}.$$

Theorem 8. Let $\mathfrak{U} : [\rho, \varsigma] \rightarrow \mathbb{E}_{\mathbb{F}}$ be a convex FNVN on $[\rho, \varsigma]$, whose j -levels define the family of i.v.ms $\mathfrak{U}_j : [\rho, \varsigma] \subset \mathbb{R} \rightarrow \mathbb{R}_I^+$ are given by $\mathfrak{U}_j(\mathfrak{x}) = [\mathfrak{U}_*(\mathfrak{x}, j), \mathfrak{U}^*(\mathfrak{x}, j)]$ for all $\mathfrak{x} \in [\rho, \varsigma]$ and for all $j \in [0, 1]$. If $\mathfrak{U} \in \mathcal{FR}_{([\rho, \varsigma], j)}$, then

$$\mathfrak{U}\left(\frac{\rho + \varsigma}{2}\right) \preccurlyeq \triangleright_2 \preccurlyeq \frac{1}{\varsigma - \rho} (FR) \int_{\rho}^{\varsigma} \mathfrak{U}(\mathfrak{x}) d\mathfrak{x} \preccurlyeq \triangleright_1 \preccurlyeq \frac{\mathfrak{U}(\rho) \tilde{+} \mathfrak{U}(\varsigma)}{2},$$

where

$$\triangleright_1 = \frac{\frac{\mathfrak{U}(\rho) \tilde{+} \mathfrak{U}(\varsigma)}{2} \tilde{+} \mathfrak{U}\left(\frac{\rho + \varsigma}{2}\right)}{2}, \triangleright_2 = \frac{\mathfrak{U}\left(\frac{3\rho + \varsigma}{4}\right) \tilde{+} \mathfrak{U}\left(\frac{\rho + 3\varsigma}{4}\right)}{2}$$

and $\triangleright_1 = [\triangleright_{1*}, \triangleright_{1*}^*], \triangleright_2 = [\triangleright_{2*}, \triangleright_{2*}^*]$.

Proof. Taking $\left[\rho, \frac{\rho + \varsigma}{2}\right]$, we have

$$2\mathfrak{U}\left(\frac{s\rho + (1-s)\frac{\rho + \varsigma}{2} + \frac{(1-s)\rho + s\frac{\rho + \varsigma}{2}}{2}\right) \preccurlyeq \mathfrak{U}\left(s\rho + (1-s)\frac{\rho + \varsigma}{2}\right) \tilde{+} \mathfrak{U}\left((1-s)\rho + s\frac{\rho + \varsigma}{2}\right).$$

Therefore, for every $j \in [0, 1]$, we have

$$\begin{aligned} 2\mathfrak{U}_*\left(\frac{s\rho + (1-s)\frac{\rho + \varsigma}{2} + \frac{(1-s)\rho + s\frac{\rho + \varsigma}{2}}{2}, j\right) &\leq \mathfrak{U}_*\left(s\rho + (1-s)\frac{\rho + \varsigma}{2}, j\right) + \mathfrak{U}_*\left((1-s)\rho + s\frac{\rho + \varsigma}{2}, j\right), \\ 2\mathfrak{U}^*\left(\frac{s\rho + (1-s)\frac{\rho + \varsigma}{2} + \frac{(1-s)\rho + s\frac{\rho + \varsigma}{2}}{2}, j\right) &\leq \mathfrak{U}^*\left(s\rho + (1-s)\frac{\rho + \varsigma}{2}, j\right) + \mathfrak{U}^*\left((1-s)\rho + s\frac{\rho + \varsigma}{2}, j\right). \end{aligned}$$

In consequence, we obtain

$$\begin{aligned} \frac{\mathfrak{U}_*\left(\frac{3\rho + \varsigma}{4}, j\right)}{2} &\leq \frac{1}{\varsigma - \rho} \int_{\rho}^{\frac{\rho + \varsigma}{2}} \mathfrak{U}_*(\mathfrak{x}, j) d\mathfrak{x}, \\ \frac{\mathfrak{U}^*\left(\frac{3\rho + \varsigma}{4}, j\right)}{2} &\leq \frac{1}{\varsigma - \rho} \int_{\rho}^{\frac{\rho + \varsigma}{2}} \mathfrak{U}^*(\mathfrak{x}, j) d\mathfrak{x}. \end{aligned}$$

That is,

$$\left[\frac{\mathfrak{U}_*\left(\frac{3\rho + \varsigma}{4}, j\right), \mathfrak{U}^*\left(\frac{3\rho + \varsigma}{4}, j\right)}{2} \right] \leq_I \frac{1}{\varsigma - \rho} \left[\int_{\rho}^{\frac{\rho + \varsigma}{2}} \mathfrak{U}_*(\mathfrak{x}, j) d\mathfrak{x}, \int_{\rho}^{\frac{\rho + \varsigma}{2}} \mathfrak{U}^*(\mathfrak{x}, j) d\mathfrak{x} \right].$$

It follows that

$$\frac{\mathfrak{U}\left(\frac{3\rho + \varsigma}{4}\right)}{2} \preccurlyeq \frac{1}{\varsigma - \rho} (FR) \int_{\rho}^{\frac{\rho + \varsigma}{2}} \mathfrak{U}(\mathfrak{x}) d\mathfrak{x}. \quad (17)$$

In a similar way as above, we have

$$\frac{\mathfrak{U}\left(\frac{\rho + 3\varsigma}{4}\right)}{2} \preccurlyeq \frac{1}{\varsigma - \rho} (FR) \int_{\frac{\rho + \varsigma}{2}}^{\varsigma} \mathfrak{U}(\mathfrak{x}) d\mathfrak{x}. \quad (18)$$

Combining (17) and (18), we have

$$\frac{\left[\mathfrak{U}\left(\frac{3\rho+\varsigma}{4}\right) \widetilde{+} \mathfrak{U}\left(\frac{\rho+3\varsigma}{4}\right)\right]}{2} \preccurlyeq \frac{1}{\varsigma-\rho} (FR) \int_{\rho}^{\varsigma} \mathfrak{U}(\mathfrak{x}) d\mathfrak{x}.$$

By using Theorem 7, we have

$$\mathfrak{U}\left(\frac{\rho+\varsigma}{2}\right) = \mathfrak{U}\left(\frac{1}{2} \cdot \frac{3\rho+\varsigma}{4} + \frac{1}{2} \cdot \frac{\rho+3\varsigma}{4}\right).$$

Therefore, for every $\mathcal{J} \in [0, 1]$, we have

$$\begin{aligned} \mathfrak{U}_*\left(\frac{\rho+\varsigma}{2}, \mathcal{J}\right) &= \mathfrak{U}_*\left(\frac{1}{2} \cdot \frac{3\rho+\varsigma}{4} + \frac{1}{2} \cdot \frac{\rho+3\varsigma}{4}, \mathcal{J}\right), \\ \mathfrak{U}^*\left(\frac{\rho+\varsigma}{2}, \mathcal{J}\right) &= \mathfrak{U}^*\left(\frac{1}{2} \cdot \frac{3\rho+\varsigma}{4} + \frac{1}{2} \cdot \frac{\rho+3\varsigma}{4}, \mathcal{J}\right), \\ &\leq \left[\frac{1}{2} \mathfrak{U}_*\left(\frac{3\rho+\varsigma}{4}, \mathcal{J}\right) + \frac{1}{2} \mathfrak{U}_*\left(\frac{\rho+3\varsigma}{4}, \mathcal{J}\right)\right], \\ &\leq \left[\frac{1}{2} \mathfrak{U}^*\left(\frac{3\rho+\varsigma}{4}, \mathcal{J}\right) + \frac{1}{2} \mathfrak{U}^*\left(\frac{\rho+3\varsigma}{4}, \mathcal{J}\right)\right], \\ &= \mathfrak{D}_{2*}, \\ &= \mathfrak{D}_{2*}^*, \\ &\leq \frac{1}{\varsigma-\rho} \int_{\rho}^{\varsigma} \mathfrak{U}_*(\mathfrak{x}, \mathcal{J}) d\mathfrak{x}, \\ &\leq \frac{1}{\varsigma-\rho} \int_{\rho}^{\varsigma} \mathfrak{U}^*(\mathfrak{x}, \mathcal{J}) d\mathfrak{x}, \\ &\leq \frac{1}{2} \left[\frac{\mathfrak{U}_*(\rho, \mathcal{J}) + \mathfrak{U}_*(\varsigma, \mathcal{J})}{2} + \mathfrak{U}_*\left(\frac{\rho+\varsigma}{2}, \mathcal{J}\right) \right], \\ &\leq \frac{1}{2} \left[\frac{\mathfrak{U}^*(\rho, \mathcal{J}) + \mathfrak{U}^*(\varsigma, \mathcal{J})}{2} + \mathfrak{U}^*\left(\frac{\rho+\varsigma}{2}, \mathcal{J}\right) \right], \\ &= \mathfrak{D}_{1*}, \\ &= \mathfrak{D}_{1*}^*, \\ &\leq \frac{1}{2} \left[\frac{\mathfrak{U}_*(\rho, \mathcal{J}) + \mathfrak{U}_*(\varsigma, \mathcal{J})}{2} + \frac{\mathfrak{U}_*(\rho, \mathcal{J}) + \mathfrak{U}_*(\varsigma, \mathcal{J})}{2} \right], \\ &\leq \frac{1}{2} \left[\frac{\mathfrak{U}^*(\rho, \mathcal{J}) + \mathfrak{U}^*(\varsigma, \mathcal{J})}{2} + \frac{\mathfrak{U}^*(\rho, \mathcal{J}) + \mathfrak{U}^*(\varsigma, \mathcal{J})}{2} \right], \\ &= \frac{\mathfrak{U}_*(\rho, \mathcal{J}) + \mathfrak{U}_*(\varsigma, \mathcal{J})}{2}, \\ &= \frac{\mathfrak{U}^*(\rho, \mathcal{J}) + \mathfrak{U}^*(\varsigma, \mathcal{J})}{2}, \end{aligned}$$

that is,

$$\mathfrak{U}\left(\frac{\rho+\varsigma}{2}\right) \preccurlyeq \mathfrak{D}_2 \preccurlyeq \frac{1}{\varsigma-\rho} (FR) \int_{\rho}^{\varsigma} \mathfrak{U}(\mathfrak{x}) d\mathfrak{x} \preccurlyeq \mathfrak{D}_1 \preccurlyeq \frac{\mathfrak{U}(\rho) \widetilde{+} \mathfrak{U}(\varsigma)}{2},$$

hence, the result follows. $\square$

Example 3. We consider the FNVM $\mathfrak{U} : [\rho, \varsigma] = [0, 2] \rightarrow \mathbb{E}_{\mathbb{R}}$, defined by $\mathfrak{U}_{\mathcal{J}}(\mathfrak{x}) = [2\mathcal{J}\mathfrak{x}^2, (4-2\mathcal{J})\mathfrak{x}^2]$, as in Example 2, then, $\mathfrak{U}(\mathfrak{x})$ is a convex FNVM and satisfying (10). We have $\mathfrak{U}_*(\mathfrak{x}, \mathcal{J}) = 2\mathcal{J}\mathfrak{x}^2$ and $\mathfrak{U}^*(\mathfrak{x}, \mathcal{J}) = (4-2\mathcal{J})\mathfrak{x}^2$. We now compute the following:

$$\begin{aligned} \frac{\mathfrak{U}_*(\rho, \mathcal{J}) + \mathfrak{U}_*(\varsigma, \mathcal{J})}{2} &= 4\mathcal{J}, \\ \frac{\mathfrak{U}^*(\rho, \mathcal{J}) + \mathfrak{U}^*(\varsigma, \mathcal{J})}{2} &= 4(2-\mathcal{J}), \\ \mathfrak{D}_{1*} &= \frac{\frac{\mathfrak{U}_*(\rho, \mathcal{J}) + \mathfrak{U}_*(\varsigma, \mathcal{J})}{2} + \mathfrak{U}_*\left(\frac{\rho+\varsigma}{2}, \mathcal{J}\right)}{2} = 3\mathcal{J}, \\ \mathfrak{D}_{1*}^* &= \frac{\frac{\mathfrak{U}^*(\rho, \mathcal{J}) + \mathfrak{U}^*(\varsigma, \mathcal{J})}{2} + \mathfrak{U}^*\left(\frac{\rho+\varsigma}{2}, \mathcal{J}\right)}{2} = 3(2-\mathcal{J}), \\ \mathfrak{D}_{2*} &= \frac{\mathfrak{U}_*\left(\frac{3\rho+\varsigma}{4}, \mathcal{J}\right) + \mathfrak{U}_*\left(\frac{\rho+3\varsigma}{4}, \mathcal{J}\right)}{2} = \frac{5}{2}\mathcal{J}, \\ \mathfrak{D}_{2*}^* &= \frac{\mathfrak{U}^*\left(\frac{3\rho+\varsigma}{4}, \mathcal{J}\right) + \mathfrak{U}^*\left(\frac{\rho+3\varsigma}{4}, \mathcal{J}\right)}{2} = \frac{5}{2}(2-\mathcal{J}), \end{aligned}$$

Then we obtain that

$$2j \leq \frac{5}{2}j \leq \frac{8j}{3} \leq 3j \leq 4j, \\ 2(2-j) \leq \frac{5}{2}(2-j) \leq \frac{8(2-j)}{3} \leq 3(2-j) \leq 4(2-j).$$

Hence, Theorem 8 is verified.

We now obtain some *HH*-inequalities for the product of convex FNVMs. These inequalities are refinements of some known inequalities, see [27,37].

Theorem 9. Let $\mathfrak{U}, \mathfrak{S} : [\rho, \varsigma] \rightarrow \mathbb{E}_{\mathbb{F}}$ be two convex FNVMs on $[\rho, \varsigma]$, whose j -levels $\mathfrak{U}_j, \mathfrak{S}_j : [\rho, \varsigma] \subset \mathbb{R} \rightarrow \mathbb{R}_I^+$ are defined by $\mathfrak{U}_j(\mathfrak{x}) = [\mathfrak{U}_*(\mathfrak{x}, j), \mathfrak{U}^*(\mathfrak{x}, j)]$ and $\mathfrak{S}_j(\mathfrak{x}) = [\mathfrak{S}_*(\mathfrak{x}, j), \mathfrak{S}^*(\mathfrak{x}, j)]$ for all $\mathfrak{x} \in [\rho, \varsigma]$ and for all $j \in [0, 1]$. If $\mathfrak{U}, \mathfrak{S}$ and $\mathfrak{U} \tilde{\times} \mathfrak{S} \in \mathcal{FR}_{([\rho, \varsigma], j)}$, then

$$\frac{1}{\varsigma - \rho} (FR) \int_{\rho}^{\varsigma} \mathfrak{U}(\mathfrak{x}) \tilde{\times} \mathfrak{S}(\mathfrak{x}) d\mathfrak{x} \preccurlyeq \frac{\mathcal{M}(\rho, \varsigma)}{3} \tilde{+} \frac{\mathcal{N}(\rho, \varsigma)}{6}.$$

where $\mathcal{M}(\rho, \varsigma) = \mathfrak{U}(\rho) \tilde{\times} \mathfrak{S}(\rho) \tilde{+} \mathfrak{U}(\varsigma) \tilde{\times} \mathfrak{S}(\varsigma)$, $\mathcal{N}(\rho, \varsigma) = \mathfrak{U}(\rho) \tilde{\times} \mathfrak{S}(\varsigma) \tilde{+} \mathfrak{U}(\varsigma) \tilde{\times} \mathfrak{S}(\rho)$, and $\mathcal{M}_j(\rho, \varsigma) = [\mathcal{M}_*(\rho, \varsigma, j), \mathcal{M}^*(\rho, \varsigma, j)]$ and $\mathcal{N}_j(\rho, \varsigma) = [\mathcal{N}_*(\rho, \varsigma, j), \mathcal{N}^*(\rho, \varsigma, j)]$.

Example 4. We consider the FNVMs $\mathfrak{U}, \mathfrak{S} : [\rho, \varsigma] = [0, 1] \rightarrow \mathbb{E}_{\mathbb{F}}$, defined by

$$\mathfrak{U}(\mathfrak{x})(\sigma) = \begin{cases} \frac{\sigma}{2\mathfrak{x}^2}, & \sigma \in [0, 2\mathfrak{x}^2], \\ \frac{4\mathfrak{x}^2 - \sigma}{2\mathfrak{x}^2}, & \sigma \in (2\mathfrak{x}^2, 4\mathfrak{x}^2], \\ 0, & \text{otherwise,} \end{cases} \\ \mathfrak{S}(\mathfrak{x})(\sigma) = \begin{cases} \frac{\sigma}{\mathfrak{x}}, & \sigma \in [0, \mathfrak{x}], \\ \frac{2\mathfrak{x} - \sigma}{\mathfrak{x}}, & \sigma \in (\mathfrak{x}, 2\mathfrak{x}], \\ 0, & \text{otherwise,} \end{cases}$$

Then, for each $j \in [0, 1]$, we have $\mathfrak{U}_j(\mathfrak{x}) = [2j\mathfrak{x}^2, (4-2j)\mathfrak{x}^2]$ and $\mathfrak{S}_j(\mathfrak{x}) = [j\mathfrak{x}, (2-j)\mathfrak{x}]$. Since the end-point mappings $\mathfrak{U}_*(\mathfrak{x}, j) = 2j\mathfrak{x}^2$, $\mathfrak{U}^*(\mathfrak{x}, j) = (4-2j)\mathfrak{x}^2$ and $\mathfrak{S}_*(\mathfrak{x}, j) = j\mathfrak{x}$, $\mathfrak{S}^*(\mathfrak{x}, j) = (2-j)\mathfrak{x}$ are convex mappings for each $j \in [0, 1]$. Hence $\mathfrak{U}, \mathfrak{S}$ both are convex FNVMs. We now compute the following

$$\frac{1}{\varsigma - \rho} \int_{\rho}^{\varsigma} \mathfrak{U}_*(\mathfrak{x}, j) \times \mathfrak{S}_*(\mathfrak{x}, j) d\mathfrak{x} = \frac{j^2}{2}, \\ \frac{1}{\varsigma - \rho} \int_{\rho}^{\varsigma} \mathfrak{U}^*(\mathfrak{x}, j) \times \mathfrak{S}^*(\mathfrak{x}, j) d\mathfrak{x} = \frac{(2-j)^2}{2},$$

$$\frac{\mathcal{M}_*(\rho, \varsigma, j)}{3} = \frac{2j^2}{2}, \\ \frac{\mathcal{M}^*(\rho, \varsigma, j)}{3} = \frac{2(2-j)^2}{3}, \\ \frac{\mathcal{N}_*(\rho, \varsigma, j)}{6} = 0, \\ \frac{\mathcal{N}^*(\rho, \varsigma, j)}{6} = 0,$$

for each $j \in [0, 1]$, that means

$$\frac{j^2}{2} \leq \frac{2j^2}{3}, \\ \frac{(2-j)^2}{2} \leq \frac{2(2-j)^2}{3},$$

Consequently, Theorem 9 is verified.

Theorem 10. Let $\mathfrak{U}, \mathfrak{S} : [\rho, \varsigma] \rightarrow \mathbb{E}_{\mathbb{F}}$ be two convex FNVMS whose $\mathcal{J}$ -levels define the family of i.v.fs $\mathfrak{U}_{\mathcal{J}}, \mathfrak{S}_{\mathcal{J}} : [\rho, \varsigma] \subset \mathbb{R} \rightarrow \mathbb{R}_I^+$ that are given by $\mathfrak{U}_{\mathcal{J}}(\mathfrak{x}) = [\mathfrak{U}_*(\mathfrak{x}, \mathcal{J}), \mathfrak{U}^*(\mathfrak{x}, \mathcal{J})]$ and $\mathfrak{S}_{\mathcal{J}}(\mathfrak{x}) = [\mathfrak{S}_*(\mathfrak{x}, \mathcal{J}), \mathfrak{S}^*(\mathfrak{x}, \mathcal{J})]$ for all $\mathfrak{x} \in [\rho, \varsigma]$ and for all $\mathcal{J} \in [0, 1]$. If $\mathfrak{U} \widetilde{\times} \mathfrak{S} \in \mathcal{FR}_{([\rho, \varsigma], \mathcal{J})}$, then

$$2\mathfrak{U}\left(\frac{\rho+\varsigma}{2}\right) \widetilde{\times} \mathfrak{S}\left(\frac{\rho+\varsigma}{2}\right) \preccurlyeq \frac{1}{\varsigma-\rho} (FR) \int_{\rho}^{\varsigma} \mathfrak{U}(\mathfrak{x}) \widetilde{\times} \mathfrak{S}(\mathfrak{x}) d\mathfrak{x} \widetilde{+} \frac{\mathcal{M}(\rho, \varsigma)}{6} \widetilde{+} \frac{\mathcal{N}(\rho, \varsigma)}{3}.$$

where $\mathcal{M}(\rho, \varsigma) = \mathfrak{U}(\rho) \widetilde{\times} \mathfrak{S}(\rho) \widetilde{+} \mathfrak{U}(\varsigma) \widetilde{\times} \mathfrak{S}(\varsigma)$, $\mathcal{N}(\rho, \varsigma) = \mathfrak{U}(\rho) \widetilde{\times} \mathfrak{S}(\varsigma) \widetilde{+} \mathfrak{U}(\varsigma) \widetilde{\times} \mathfrak{S}(\rho)$, and $\mathcal{M}_{\mathcal{J}}(\rho, \varsigma) = [\mathcal{M}_*((\rho, \varsigma), \mathcal{J}), \mathcal{M}^*((\rho, \varsigma), \mathcal{J})]$ and $\mathcal{N}_{\mathcal{J}}(\rho, \varsigma) = [\mathcal{N}_*((\rho, \varsigma), \mathcal{J}), \mathcal{N}^*((\rho, \varsigma), \mathcal{J})]$.

Proof. By hypothesis, for each $\mathcal{J} \in [0, 1]$, we have

$$\begin{aligned} & \mathfrak{U}_*\left(\frac{\rho+\varsigma}{2}, \mathcal{J}\right) \times \mathfrak{S}_*\left(\frac{\rho+\varsigma}{2}, \mathcal{J}\right) \\ & \mathfrak{U}^*\left(\frac{\rho+\varsigma}{2}, \mathcal{J}\right) \times \mathfrak{S}^*\left(\frac{\rho+\varsigma}{2}, \mathcal{J}\right) \\ & \leq \frac{1}{4} \left[\mathfrak{U}_*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J}) \times \mathfrak{S}_*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J}) \right. \\ & \quad \left. + \mathfrak{U}_*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J}) \times \mathfrak{S}_*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) \right] \\ & \quad + \frac{1}{4} \left[\mathfrak{U}_*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) \times \mathfrak{S}_*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J}) \right. \\ & \quad \left. + \mathfrak{U}_*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) \times \mathfrak{S}_*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) \right], \\ & \leq \frac{1}{4} \left[\mathfrak{U}^*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J}) \times \mathfrak{S}^*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J}) \right. \\ & \quad \left. + \mathfrak{U}^*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J}) \times \mathfrak{S}^*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) \right] \\ & \quad + \frac{1}{4} \left[\mathfrak{U}^*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) \times \mathfrak{S}^*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J}) \right. \\ & \quad \left. + \mathfrak{U}^*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) \times \mathfrak{S}^*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) \right], \\ & \leq \frac{1}{4} \left[\mathfrak{U}_*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J}) \times \mathfrak{S}_*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J}) \right. \\ & \quad \left. + \mathfrak{U}_*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) \times \mathfrak{S}_*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) \right] \\ & \quad + \frac{1}{4} \left[\begin{aligned} & (\mathfrak{s}\mathfrak{U}_*(\rho, \mathcal{J}) + (1-\mathfrak{s})\mathfrak{U}_*(\varsigma, \mathcal{J})) \\ & \times ((1-\mathfrak{s})\mathfrak{S}_*(\rho, \mathcal{J}) + \mathfrak{s}\mathfrak{S}_*(\varsigma, \mathcal{J})) \\ & + ((1-\mathfrak{s})\mathfrak{U}_*(\rho, \mathcal{J}) + \mathfrak{s}\mathfrak{U}_*(\varsigma, \mathcal{J})) \\ & \times (\mathfrak{s}\mathfrak{S}_*(\rho, \mathcal{J}) + (1-\mathfrak{s})\mathfrak{S}_*(\varsigma, \mathcal{J})) \end{aligned} \right], \\ & \leq \frac{1}{4} \left[\mathfrak{U}^*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J}) \times \mathfrak{S}^*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J}) \right. \\ & \quad \left. + \mathfrak{U}^*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) \times \mathfrak{S}^*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) \right] \\ & \quad + \frac{1}{4} \left[\begin{aligned} & (\mathfrak{s}\mathfrak{U}^*(\rho, \mathcal{J}) + (1-\mathfrak{s})\mathfrak{U}^*(\varsigma, \mathcal{J})) \\ & \times ((1-\mathfrak{s})\mathfrak{S}^*(\rho, \mathcal{J}) + \mathfrak{s}\mathfrak{S}^*(\varsigma, \mathcal{J})) \\ & + ((1-\mathfrak{s})\mathfrak{U}^*(\rho, \mathcal{J}) + \mathfrak{s}\mathfrak{U}^*(\varsigma, \mathcal{J})) \\ & \times (\mathfrak{s}\mathfrak{S}^*(\rho, \mathcal{J}) + (1-\mathfrak{s})\mathfrak{S}^*(\varsigma, \mathcal{J})) \end{aligned} \right], \\ & = \frac{1}{4} \left[\mathfrak{U}_*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J}) \times \mathfrak{S}_*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J}) \right. \\ & \quad \left. + \mathfrak{U}_*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) \times \mathfrak{S}_*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) \right] \\ & \quad + \frac{1}{2} \left[\begin{aligned} & \{\mathfrak{s}^2 + (1-\mathfrak{s})^2\} \mathcal{N}_*((\rho, \varsigma), \mathcal{J}) \\ & + \{\mathfrak{s}(1-\mathfrak{s}) + (1-\mathfrak{s})\mathfrak{s}\} \mathcal{M}_*((\rho, \varsigma), \mathcal{J}) \end{aligned} \right] \\ & = \frac{1}{4} \left[\mathfrak{U}^*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J}) \times \mathfrak{S}^*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J}) \right. \\ & \quad \left. + \mathfrak{U}^*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) \times \mathfrak{S}^*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) \right] \\ & \quad + \frac{1}{2} \left[\begin{aligned} & \{\mathfrak{s}^2 + (1-\mathfrak{s})^2\} \mathcal{N}^*((\rho, \varsigma), \mathcal{J}) \\ & + \{\mathfrak{s}(1-\mathfrak{s}) + (1-\mathfrak{s})\mathfrak{s}\} \mathcal{M}^*((\rho, \varsigma), \mathcal{J}) \end{aligned} \right] \end{aligned}$$

R-Integrating over $[0, 1]$, we have

$$\begin{aligned} 2\mathfrak{U}_*\left(\frac{\rho+\varsigma}{2}, \mathcal{J}\right) \times \mathfrak{S}_*\left(\frac{\rho+\varsigma}{2}, \mathcal{J}\right) & \leq \frac{1}{\varsigma-\rho} \int_{\rho}^{\varsigma} \mathfrak{U}_*(\mathfrak{x}, \mathcal{J}) \times \mathfrak{S}_*(\mathfrak{x}, \mathcal{J}) d\mathfrak{x} + \frac{\mathcal{M}_*((\rho, \varsigma), \mathcal{J})}{6} + \frac{\mathcal{N}_*((\rho, \varsigma), \mathcal{J})}{3}, \\ 2\mathfrak{U}^*\left(\frac{\rho+\varsigma}{2}, \mathcal{J}\right) \times \mathfrak{S}^*\left(\frac{\rho+\varsigma}{2}, \mathcal{J}\right) & \leq \frac{1}{\varsigma-\rho} \int_{\rho}^{\varsigma} \mathfrak{U}^*(\mathfrak{x}, \mathcal{J}) \times \mathfrak{S}^*(\mathfrak{x}, \mathcal{J}) d\mathfrak{x} + \frac{\mathcal{M}^*((\rho, \varsigma), \mathcal{J})}{6} + \frac{\mathcal{N}^*((\rho, \varsigma), \mathcal{J})}{3}, \end{aligned}$$

that is,

$$2\mathfrak{U}\left(\frac{\rho+\varsigma}{2}\right) \widetilde{\times} \mathfrak{S}\left(\frac{\rho+\varsigma}{2}\right) \preccurlyeq \frac{1}{\varsigma-\rho} (FR) \int_{\rho}^{\varsigma} \mathfrak{U}(\mathfrak{x}) \widetilde{\times} \mathfrak{S}(\mathfrak{x}) d\mathfrak{x} \widetilde{+} \frac{\mathcal{M}(\rho, \varsigma)}{6} \widetilde{+} \frac{\mathcal{N}(\rho, \varsigma)}{3}.$$

Hence, the required result. $\square$

Example 5. We consider the FNVMs $\mathfrak{U}, \mathfrak{S} : [\rho, \varsigma] = [0, 1] \rightarrow \mathbb{E}_{\mathbb{F}}$. Then, for each $\mathcal{J} \in [0, 1]$, we have $\mathfrak{U}_{\mathcal{J}}(\mathfrak{x}) = [2\mathcal{J}\mathfrak{x}^2, (4 - 2\mathcal{J})\mathfrak{x}^2]$ and $\mathfrak{S}_{\mathcal{J}}(\mathfrak{x}) = [\mathcal{J}\mathfrak{x}, (2 - \mathcal{J})\mathfrak{x}]$, as in Example 4; then, $\mathfrak{U}$ and $\mathfrak{S}$ both are convex mappings. We have $\mathfrak{U}_*(\mathfrak{x}, \mathcal{J}) = 2\mathcal{J}\mathfrak{x}^2$, $\mathfrak{U}^*(\mathfrak{x}, \mathcal{J}) = (4 - 2\mathcal{J})\mathfrak{x}^2$ and $\mathfrak{S}_*(\mathfrak{x}, \mathcal{J}) = \mathcal{J}\mathfrak{x}$, $\mathfrak{S}^*(\mathfrak{x}, \mathcal{J}) = (2 - \mathcal{J})\mathfrak{x}$, then

$$\begin{aligned} 2\mathfrak{U}_*\left(\frac{\rho+\varsigma}{2}, \mathcal{J}\right) \times \mathfrak{S}_*\left(\frac{\rho+\varsigma}{2}, \mathcal{J}\right) &= \frac{\mathcal{J}^2}{2}, \\ 2\mathfrak{U}^*\left(\frac{\rho+\varsigma}{2}, \mathcal{J}\right) \times \mathfrak{S}^*\left(\frac{\rho+\varsigma}{2}, \mathcal{J}\right) &= \frac{(2-\mathcal{J})^2}{2}, \\ \frac{1}{\varsigma-\rho} \int_{\rho}^{\varsigma} \mathfrak{U}_*(\mathfrak{x}, \mathcal{J}) \times \mathfrak{S}_*(\mathfrak{x}, \mathcal{J}) d\mathfrak{x} &= \frac{\mathcal{J}^2}{2}, \\ \frac{1}{\varsigma-\rho} \int_{\rho}^{\varsigma} \mathfrak{U}^*(\mathfrak{x}, \mathcal{J}) \times \mathfrak{S}^*(\mathfrak{x}, \mathcal{J}) d\mathfrak{x} &= \frac{(2-\mathcal{J})^2}{2}, \\ \frac{\mathcal{M}_*((\rho, \varsigma), \mathcal{J})}{6} &= \frac{\mathcal{J}^2}{3}, \\ \frac{\mathcal{M}^*((\rho, \varsigma), \mathcal{J})}{6} &= \frac{(2-\mathcal{J})^2}{3}, \\ \frac{\mathcal{N}_*((\rho, \varsigma), \mathcal{J})}{3} &= 0, \\ \frac{\mathcal{N}^*((\rho, \varsigma), \mathcal{J})}{3} &= 0, \end{aligned}$$

for each $\mathcal{J} \in [0, 1]$, that means

$$\begin{aligned} \frac{\mathcal{J}^2}{2} &\leq \frac{\mathcal{J}^2}{2} + 0 + \frac{\mathcal{J}^2}{3} = \frac{5\mathcal{J}^2}{6}, \\ \frac{(2-\mathcal{J})^2}{2} &\leq \frac{(2-\mathcal{J})^2}{2} + 0 + \frac{5(2-\mathcal{J})^2}{3} = \frac{5(2-\mathcal{J})^2}{6}. \end{aligned}$$

Hence, Theorem 10 is verified.

We now give *HH*-Fejér inequalities for convex FNVMs. Firstly, we obtain the second *HH*-Fejér inequality for a convex FNVM.

Theorem 11. Let $\mathfrak{U} : [\rho, \varsigma] \rightarrow \mathbb{E}_{\mathbb{F}}$ be a convex FNVM with $\rho < \varsigma$, whose $\mathcal{J}$ -levels define the family of i.v.ms $\mathfrak{U}_{\mathcal{J}} : [\rho, \varsigma] \subset \mathbb{R} \rightarrow \mathbb{R}_I^+$ that are given by $\mathfrak{U}_{\mathcal{J}}(\mathfrak{x}) = [\mathfrak{U}_*(\mathfrak{x}, \mathcal{J}), \mathfrak{U}^*(\mathfrak{x}, \mathcal{J})]$ for all $\mathfrak{x} \in [\rho, \varsigma]$ and for all $\mathcal{J} \in [0, 1]$. If $\mathfrak{U} \in \mathcal{FR}_{([\rho, \varsigma], \mathcal{J})}$ and $\mathcal{C} : [\rho, \varsigma] \rightarrow \mathbb{R}$, $\mathcal{C}(\mathfrak{x}) \geq 0$, symmetric with respect to $\frac{\rho+\varsigma}{2}$, then

$$\frac{1}{\varsigma-\rho} (FR) \int_{\rho}^{\varsigma} \mathfrak{U}(\mathfrak{x}) \mathcal{C}(\mathfrak{x}) d\mathfrak{x} \preccurlyeq [\mathfrak{U}(\rho) \tilde{+} \mathfrak{U}(\varsigma)] \int_0^1 \mathfrak{s} \mathcal{C}((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma) d\mathfrak{s}. \quad (19)$$

Proof. Let $\mathfrak{U}$ be a convex FNVM. Then, for each $\mathcal{J} \in [0, 1]$, we have

$$\begin{aligned} &\mathfrak{U}_*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J}) \mathcal{C}(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma) \\ &\leq (\mathfrak{s}\mathfrak{U}_*(\rho, \mathcal{J}) + (1-\mathfrak{s})\mathfrak{U}_*(\varsigma, \mathcal{J})) \mathcal{C}(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma), \\ &\mathfrak{U}^*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J}) \mathcal{C}(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma) \\ &\leq (\mathfrak{s}\mathfrak{U}^*(\rho, \mathcal{J}) + (1-\mathfrak{s})\mathfrak{U}^*(\varsigma, \mathcal{J})) \mathcal{C}(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma). \end{aligned} \quad (20)$$

And

$$\begin{aligned} \mathfrak{U}_*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) \mathcal{C}((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma) &\leq ((1-\mathfrak{s})\mathfrak{U}_*(\rho, \mathcal{J}) + \mathfrak{s}\mathfrak{U}_*(\varsigma, \mathcal{J})) \mathcal{C}((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma), \\ \mathfrak{U}^*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) \mathcal{C}((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma) &\leq ((1-\mathfrak{s})\mathfrak{U}^*(\rho, \mathcal{J}) + \mathfrak{s}\mathfrak{U}^*(\varsigma, \mathcal{J})) \mathcal{C}((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma). \end{aligned} \quad (21)$$

After adding (20) and (21) and integrating over $[0, 1]$, we get

$$\begin{aligned} & \int_0^1 \mathfrak{U}_*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J})\mathcal{C}(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma)d\mathfrak{s} + \int_0^1 \mathfrak{U}_*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J})\mathcal{C}((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma)d\mathfrak{s} \\ & \leq \int_0^1 \left[\mathfrak{U}_*(\rho, \mathcal{J})\{\mathfrak{s}\mathcal{C}(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma) + (1-\mathfrak{s})\mathcal{C}((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma)\} \right. \\ & \quad \left. + \mathfrak{U}_*(\varsigma, \mathcal{J})\{(1-\mathfrak{s})\mathcal{C}(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma) + \mathfrak{s}\mathcal{C}((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma)\} \right] d\mathfrak{s}, \\ & \int_0^1 \mathfrak{U}^*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J})\mathcal{C}((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma)d\mathfrak{s} + \int_0^1 \mathfrak{U}^*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J})\mathcal{C}(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma)d\mathfrak{s} \\ & \leq \int_0^1 \left[\mathfrak{U}^*(\rho, \mathcal{J})\{\mathfrak{s}\mathcal{C}(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma) + (1-\mathfrak{s})\mathcal{C}((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma)\} \right. \\ & \quad \left. + \mathfrak{U}^*(\varsigma, \mathcal{J})\{(1-\mathfrak{s})\mathcal{C}(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma) + \mathfrak{s}\mathcal{C}((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma)\} \right] d\mathfrak{s}, \\ & = 2\mathfrak{U}_*(\rho, \mathcal{J}) \int_0^1 \mathfrak{s}\mathcal{C}(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma) d\mathfrak{s} + 2\mathfrak{U}_*(\varsigma, \mathcal{J}) \int_0^1 \mathfrak{s}\mathcal{C}((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma) d\mathfrak{s}, \\ & = 2\mathfrak{U}^*(\rho, \mathcal{J}) \int_0^1 \mathfrak{s}\mathcal{C}(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma) d\mathfrak{s} + 2\mathfrak{U}^*(\varsigma, \mathcal{J}) \int_0^1 \mathfrak{s}\mathcal{C}((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma) d\mathfrak{s}. \end{aligned}$$

Since $\mathcal{C}$ is symmetric, then

$$\begin{aligned} & = 2[\mathfrak{U}_*(\rho, \mathcal{J}) + \mathfrak{U}_*(\varsigma, \mathcal{J})] \int_0^1 \mathfrak{s}\mathcal{C}((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma) d\mathfrak{s}, \\ & = 2[\mathfrak{U}^*(\rho, \mathcal{J}) + \mathfrak{U}^*(\varsigma, \mathcal{J})] \int_0^1 \mathfrak{s}\mathcal{C}((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma) d\mathfrak{s}. \end{aligned} \quad (22)$$

since

$$\begin{aligned} & \int_0^1 \mathfrak{U}_*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J})\mathcal{C}(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma)d\mathfrak{s} \\ & = \int_0^1 \mathfrak{U}_*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J})\mathcal{C}((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma)d\mathfrak{s} = \frac{1}{\varsigma-\rho} \int_\rho^\varsigma \mathfrak{U}_*(\mathfrak{x}, \mathcal{J})\mathcal{C}(\mathfrak{x})d\mathfrak{x} \\ & \int_0^1 \mathfrak{U}^*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J})\mathcal{C}((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma)d\mathfrak{s} \\ & = \int_0^1 \mathfrak{U}^*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J})\mathcal{C}(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma)d\mathfrak{s} = \frac{1}{\varsigma-\rho} \int_\rho^\varsigma \mathfrak{U}^*(\mathfrak{x}, \mathcal{J})\mathcal{C}(\mathfrak{x})d\mathfrak{x}. \end{aligned} \quad (23)$$

Then, from (22), we have

$$\begin{aligned} & \frac{1}{\varsigma-\rho} \int_\rho^\varsigma \mathfrak{U}_*(\mathfrak{x}, \mathcal{J})\mathcal{C}(\mathfrak{x})d\mathfrak{x} \leq [\mathfrak{U}_*(\rho, \mathcal{J}) + \mathfrak{U}_*(\varsigma, \mathcal{J})] \int_0^1 \mathfrak{s}\mathcal{C}((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma)d\mathfrak{s}, \\ & \frac{1}{\varsigma-\rho} \int_\rho^\varsigma \mathfrak{U}^*(\mathfrak{x}, \mathcal{J})\mathcal{C}(\mathfrak{x})d\mathfrak{x} \leq [\mathfrak{U}^*(\rho, \mathcal{J}) + \mathfrak{U}^*(\varsigma, \mathcal{J})] \int_0^1 \mathfrak{s}\mathcal{C}((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma)d\mathfrak{s}, \end{aligned}$$

that is,

$$\left[\frac{1}{\varsigma-\rho} \int_\rho^\varsigma \mathfrak{U}_*(\mathfrak{x}, \mathcal{J})\mathcal{C}(\mathfrak{x})d\mathfrak{x}, \frac{1}{\varsigma-\rho} \int_\rho^\varsigma \mathfrak{U}^*(\mathfrak{x}, \mathcal{J})\mathcal{C}(\mathfrak{x})d\mathfrak{x} \right] \leq_I [\mathfrak{U}_*(\rho, \mathcal{J}) + \mathfrak{U}_*(\varsigma, \mathcal{J}), \mathfrak{U}^*(\rho, \mathcal{J}) + \mathfrak{U}^*(\varsigma, \mathcal{J})] \int_0^1 \mathfrak{s}\mathcal{C}((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma)d\mathfrak{s}$$

Hence,

$$\frac{1}{\varsigma-\rho} (FR) \int_\rho^\varsigma \mathfrak{U}(\mathfrak{x})\mathcal{C}(\mathfrak{x})d\mathfrak{x} \preceq [\mathfrak{U}(\rho) \tilde{+} \mathfrak{U}(\varsigma)] \int_0^1 \mathfrak{s}\mathcal{C}((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma)d\mathfrak{s}.$$

Next, we construct the first *HH*-Fejér inequality for a convex FNVM, which generalizes the first *HH*-Fejér inequalities for convex mapping, see [10]. $\square$

Theorem 12. Let $\mathfrak{U} : [\rho, \varsigma] \rightarrow \mathbb{E}_{\mathbb{R}}$ be a convex FNVM with $\rho < \varsigma$, whose $\mathcal{J}$ -levels define the family of i.v.ms $\mathfrak{U}_{\mathcal{J}} : [\rho, \varsigma] \subset \mathbb{R} \rightarrow \mathbb{R}_I^+$ that are given by $\mathfrak{U}_{\mathcal{J}}(\mathfrak{x}) = [\mathfrak{U}_*(\mathfrak{x}, \mathcal{J}), \mathfrak{U}^*(\mathfrak{x}, \mathcal{J})]$ for all $\mathfrak{x} \in [\rho, \varsigma]$ and for all $\mathcal{J} \in [0, 1]$. If $\mathfrak{U} \in \mathcal{FR}_{([\rho, \varsigma], \mathcal{J})}$ and $\mathcal{C} : [\rho, \varsigma] \rightarrow \mathbb{R}$, $\mathcal{C}(\mathfrak{x}) \geq 0$, symmetric with respect to $\frac{\rho+\varsigma}{2}$, and $\int_\rho^\varsigma \mathcal{C}(\mathfrak{x})d\mathfrak{x} > 0$; then

$$\mathfrak{U}\left(\frac{\rho+\varsigma}{2}\right) \preceq \frac{1}{\int_\rho^\varsigma \mathcal{C}(\mathfrak{x})d\mathfrak{x}} (FR) \int_\rho^\varsigma \mathfrak{U}(\mathfrak{x})\mathcal{C}(\mathfrak{x})d\mathfrak{x}. \quad (24)$$

Proof. Since $\mathfrak{U}$ is a convex, then for $\mathcal{J} \in [0, 1]$, we have

$$\begin{aligned} & \mathfrak{U}_*\left(\frac{\rho+\varsigma}{2}, \mathcal{J}\right) \leq \frac{1}{2}(\mathfrak{U}_*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J}) + \mathfrak{U}_*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J})), \\ & \mathfrak{U}^*\left(\frac{\rho+\varsigma}{2}, \mathcal{J}\right) \leq \frac{1}{2}(\mathfrak{U}^*(\mathfrak{s}\rho + (1-\mathfrak{s})\varsigma, \mathcal{J}) + \mathfrak{U}^*((1-\mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J})), \end{aligned} \quad (25)$$

Since $\mathcal{C}(\mathfrak{s}\rho + (1 - \mathfrak{s})\varsigma) = \mathcal{C}((1 - \mathfrak{s})\rho + \mathfrak{s}\varsigma)$, then, by multiplying (25) by $\mathcal{C}((1 - \mathfrak{s})\rho + \mathfrak{s}\varsigma)$ and integrating it with respect to $\mathfrak{s}$ over $[0, 1]$, we obtain

$$\begin{aligned} & \mathfrak{U}_*\left(\frac{\rho + \varsigma}{2}, \mathcal{J}\right) \int_0^1 \mathcal{C}((1 - \mathfrak{s})\rho + \mathfrak{s}\varsigma) d\mathfrak{s} \\ & \leq \frac{1}{2} \left(\int_0^1 \mathfrak{U}_*(\mathfrak{s}\rho + (1 - \mathfrak{s})\varsigma, \mathcal{J}) \mathcal{C}(\mathfrak{s}\rho + (1 - \mathfrak{s})\varsigma) d\mathfrak{s} \right. \\ & \quad \left. + \int_0^1 \mathfrak{U}_*((1 - \mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) d\mathfrak{s} \mathcal{C}((1 - \mathfrak{s})\rho + \mathfrak{s}\varsigma) \right), \\ & \mathfrak{U}^*\left(\frac{\rho + \varsigma}{2}, \mathcal{J}\right) \int_0^1 \mathcal{C}((1 - \mathfrak{s})\rho + \mathfrak{s}\varsigma) d\mathfrak{s} \\ & \leq \frac{1}{2} \left(\int_0^1 \mathfrak{U}^*(\mathfrak{s}\rho + (1 - \mathfrak{s})\varsigma, \mathcal{J}) \mathcal{C}(\mathfrak{s}\rho + (1 - \mathfrak{s})\varsigma) d\mathfrak{s} \right. \\ & \quad \left. + \int_0^1 \mathfrak{U}^*((1 - \mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) \mathcal{C}((1 - \mathfrak{s})\rho + \mathfrak{s}\varsigma) d\mathfrak{s} \right). \end{aligned} \quad (26)$$

since

$$\begin{aligned} & \int_0^1 \mathfrak{U}_*(\mathfrak{s}\rho + (1 - \mathfrak{s})\varsigma, \mathcal{J}) \mathcal{C}(\mathfrak{s}\rho + (1 - \mathfrak{s})\varsigma) d\mathfrak{s} \\ & = \int_0^1 \mathfrak{U}^*((1 - \mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) \mathcal{C}((1 - \mathfrak{s})\rho + \mathfrak{s}\varsigma) d\mathfrak{s} = \frac{1}{\varsigma - \rho} \int_\rho^\varsigma \mathfrak{U}^*(\mathfrak{x}, \mathcal{J}) \mathcal{C}(\mathfrak{x}) d\mathfrak{x}, \\ & \int_0^1 \mathfrak{U}^*((1 - \mathfrak{s})\rho + \mathfrak{s}\varsigma, \mathcal{J}) \mathcal{C}((1 - \mathfrak{s})\rho + \mathfrak{s}\varsigma) d\mathfrak{s} \\ & = \int_0^1 \mathfrak{U}^*(\mathfrak{s}\rho + (1 - \mathfrak{s})\varsigma, \mathcal{J}) \mathcal{C}(\mathfrak{s}\rho + (1 - \mathfrak{s})\varsigma) d\mathfrak{s} = \frac{1}{\varsigma - \rho} \int_\rho^\varsigma \mathfrak{U}^*(\mathfrak{x}, \mathcal{J}) \mathcal{C}(\mathfrak{x}) d\mathfrak{x}. \end{aligned} \quad (27)$$

Then, from (27), we have

$$\begin{aligned} \mathfrak{U}_*\left(\frac{\rho + \varsigma}{2}, \mathcal{J}\right) & \leq \frac{1}{\int_\rho^\varsigma \mathcal{C}(\mathfrak{x}) d\mathfrak{x}} \int_\rho^\varsigma \mathfrak{U}_*(\mathfrak{x}, \mathcal{J}) \mathcal{C}(\mathfrak{x}) d\mathfrak{x}, \\ \mathfrak{U}^*\left(\frac{\rho + \varsigma}{2}, \mathcal{J}\right) & \leq \frac{1}{\int_\rho^\varsigma \mathcal{C}(\mathfrak{x}) d\mathfrak{x}} \int_\rho^\varsigma \mathfrak{U}^*(\mathfrak{x}, \mathcal{J}) \mathcal{C}(\mathfrak{x}) d\mathfrak{x}, \end{aligned}$$

from which, we have

$$\begin{aligned} & \left[\mathfrak{U}_*\left(\frac{\rho + \varsigma}{2}, \mathcal{J}\right), \mathfrak{U}^*\left(\frac{\rho + \varsigma}{2}, \mathcal{J}\right) \right] \\ & \leq_I \frac{1}{\int_\rho^\varsigma \mathcal{C}(\mathfrak{x}) d\mathfrak{x}} \left[\int_\rho^\varsigma \mathfrak{U}_*(\mathfrak{x}, \mathcal{J}) \mathcal{C}(\mathfrak{x}) d\mathfrak{x}, \int_\rho^\varsigma \mathfrak{U}^*(\mathfrak{x}, \mathcal{J}) \mathcal{C}(\mathfrak{x}) d\mathfrak{x} \right], \end{aligned}$$

that is,

$$\mathfrak{U}\left(\frac{\rho + \varsigma}{2}\right) \preceq \frac{1}{\int_\rho^\varsigma \mathcal{C}(\mathfrak{x}) d\mathfrak{x}} (FR) \int_\rho^\varsigma \mathfrak{U}(\mathfrak{x}) \mathcal{C}(\mathfrak{x}) d\mathfrak{x}.$$

This completes the proof. $\square$

Remark 5.

1. If $\mathfrak{U}_*(\mathfrak{x}, \mathcal{J}) = \mathfrak{U}^*(\mathfrak{x}, \mathcal{J})$ with $\mathcal{J} = 1$, then Theorems 11 and 12 reduce to classical first and second HH-Fejér inequality for convex mapping, see [10].
2. If $\mathcal{C}(\mathfrak{x}) = 1$, then, combining Theorems 10 and 11, we get Theorem 7.

Example 6. We consider the FNVMS $\mathfrak{U} : [\rho, \varsigma] = [\frac{\pi}{4}, \frac{\pi}{2}] \rightarrow \mathbb{E}_{\mathbb{F}}$, defined by

$$\mathfrak{U}(\mathfrak{x})(\sigma) = \begin{cases} \frac{\sigma}{e^{\sin \mathfrak{x}}}, & \sigma \in [0, e^{\sin \mathfrak{x}}], \\ \frac{2e^{\sin \mathfrak{x}} - \sigma}{e^{\sin \mathfrak{x}}}, & \sigma \in (e^{\sin \mathfrak{x}}, 2e^{\sin \mathfrak{x}}], \\ 0, & \text{otherwise,} \end{cases}$$

Then, for each $\mathcal{J} \in [0, 1]$, we have

$$\mathfrak{U}_{\mathcal{J}}(\mathfrak{x}) = [\mathcal{J}e^{\sin \mathfrak{x}}, (2 - \mathcal{J})e^{\sin \mathfrak{x}}].$$

Since the end-point mappings $\mathfrak{U}_*(\mathfrak{x}, \mathcal{J}) = \mathcal{J}e^{sin\mathfrak{x}}$, $\mathfrak{U}^*(\mathfrak{x}, \mathcal{J}) = (2 - \mathcal{J})e^{sin\mathfrak{x}}$ are convex mappings for each $\mathcal{J} \in [0, 1]$, then, by Theorem 6, $\mathfrak{U}(\mathfrak{x})$ is a convex FNVM. If

$$\mathcal{C}(\mathfrak{x}) = \begin{cases} \mathfrak{x} - \frac{\pi}{4}, & \sigma \in [\frac{\pi}{4}, \frac{3\pi}{8}], \\ \frac{\pi}{2} - \mathfrak{x}, & \sigma \in (\frac{3\pi}{8}, \frac{\pi}{2}]. \end{cases}$$

then, we have

$$\begin{aligned} & \frac{1}{\zeta - \rho} \int_{\rho}^{\zeta} [\mathfrak{U}_*(\mathfrak{x}, \mathcal{J})] \mathcal{C}(\mathfrak{x}) d\mathfrak{x} = \frac{4}{\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\mathfrak{U}_*(\mathfrak{x}, \mathcal{J})] \mathcal{C}(\mathfrak{x}) d\mathfrak{x} \\ &= \frac{4}{\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{8}} [\mathfrak{U}_*(\mathfrak{x}, \mathcal{J})] \mathcal{C}(\mathfrak{x}) d\mathfrak{x} + \frac{4}{\pi} \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \mathfrak{U}_*(\mathfrak{x}, \mathcal{J}) \mathcal{C}(\mathfrak{x}) d\mathfrak{x}, \\ & \frac{1}{\zeta - \rho} \int_{\rho}^{\zeta} [\mathfrak{U}^*(\mathfrak{x}, \mathcal{J})] \mathcal{C}(\mathfrak{x}) d\mathfrak{x} = \frac{4}{\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\mathfrak{U}^*(\mathfrak{x}, \mathcal{J})] \mathcal{C}(\mathfrak{x}) d\mathfrak{x} \\ &= \frac{4}{\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{8}} [\mathfrak{U}^*(\mathfrak{x}, \mathcal{J})] \mathcal{C}(\mathfrak{x}) d\mathfrak{x} + \frac{4}{\pi} \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \mathfrak{U}^*(\mathfrak{x}, \mathcal{J}) \mathcal{C}(\mathfrak{x}) d\mathfrak{x}, \\ &= \frac{4}{\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{8}} [(\mathcal{J}e^{sin\mathfrak{x}})] (\mathfrak{x} - \frac{\pi}{4}) d\mathfrak{x} + \frac{4}{\pi} \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} [(\mathcal{J}e^{sin\mathfrak{x}})] (\frac{\pi}{2} - \mathfrak{x}) d\mathfrak{x} \approx \frac{8\mathcal{J}}{5\pi}, \\ &= \frac{4}{\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{8}} [(2 - \mathcal{J})e^{sin\mathfrak{x}}] (\mathfrak{x} - \frac{\pi}{2}) d\mathfrak{x} + \frac{4}{\pi} \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} [(2 - \mathcal{J})e^{sin\mathfrak{x}}] (\frac{\pi}{2} - \mathfrak{x}) d\mathfrak{x} \approx \frac{8(2 - \mathcal{J})}{5\pi}, \end{aligned} \quad (28)$$

and

$$\begin{aligned} & [\mathfrak{U}_*(\rho, \mathcal{J}) + \mathfrak{U}_*(\zeta, \mathcal{J})] \int_0^1 \mathfrak{s} \mathcal{C}(\rho + \mathfrak{s} \partial(\zeta, \rho)) d\mathfrak{s} \\ & [\mathfrak{U}^*(\rho, \mathcal{J}) + \mathfrak{U}^*(\zeta, \mathcal{J})] \int_0^1 \mathfrak{s} \mathcal{C}(\rho + \mathfrak{s} \partial(\zeta, \rho)) d\mathfrak{s} \\ &= \frac{\pi}{2} \mathcal{J} \left[\int_0^{\frac{1}{2}} \mathfrak{s}^2 d\mathfrak{x} + \int_{\frac{1}{2}}^1 \mathfrak{s}(1 + \mathfrak{s}) d\mathfrak{s} \right] \approx \frac{255\mathcal{J}}{24\pi}, \\ &= \frac{\pi}{2} (2 - \mathcal{J}) \left[\int_0^{\frac{1}{2}} \mathfrak{s}^2 d\mathfrak{x} + \int_{\frac{1}{2}}^1 \mathfrak{s}(1 + \mathfrak{s}) d\mathfrak{s} \right] \approx \frac{255(2 - \mathcal{J})}{24\pi}, \end{aligned} \quad (29)$$

From (28) and (29), we have

$$\left[\frac{8\mathcal{J}}{5\pi}, \frac{8(2 - \mathcal{J})}{5\pi} \right] \leq_I \left[\frac{255\mathcal{J}}{24\pi}, \frac{255(2 - \mathcal{J})}{24\pi} \right], \text{ for all } \mathcal{J} \in [0, 1].$$

Hence, Theorem 11 is verified.

For Theorem 12, we have

$$\begin{aligned} \mathfrak{U}_*\left(\frac{\rho + \zeta}{2}, \mathcal{J}\right) &= \mathfrak{U}_*\left(\frac{3\pi}{8}, \mathcal{J}\right) \approx \frac{5}{2}\mathcal{J}, \\ \mathfrak{U}^*\left(\frac{\rho + \zeta}{2}, \mathcal{J}\right) &= \mathfrak{U}^*\left(\frac{3\pi}{8}, \mathcal{J}\right) \approx \frac{5}{2}(2 - \mathcal{J}), \end{aligned} \quad (30)$$

$$\begin{aligned} \int_{\rho}^{\zeta} \mathcal{C}(\mathfrak{x}) d\mathfrak{x} &= \int_{\frac{\pi}{4}}^{\frac{3\pi}{8}} \left(\mathfrak{x} - \frac{\pi}{4}\right) d\mathfrak{x} + \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \left(\frac{\pi}{2} - \mathfrak{x}\right) d\mathfrak{x} \approx \frac{3}{20}, \\ \frac{1}{\int_{\rho}^{\zeta} \mathcal{C}(\mathfrak{x}) d\mathfrak{x}} \int_{\rho}^{\zeta} \mathfrak{U}_*(\mathfrak{x}, \mathcal{J}) \mathcal{C}(\mathfrak{x}) d\mathfrak{x} &\approx \frac{8}{3}\mathcal{J}, \\ \frac{1}{\int_{\rho}^{\zeta} \mathcal{C}(\mathfrak{x}) d\mathfrak{x}} \int_{\rho}^{\zeta} \mathfrak{U}^*(\mathfrak{x}, \mathcal{J}) \mathcal{C}(\mathfrak{x}) d\mathfrak{x} &\approx \frac{8}{3}(2 - \mathcal{J}). \end{aligned} \quad (31)$$

From (30) and (31), we have

$$\left[\frac{5}{2}\mathcal{J}, \frac{5}{2}(2 - \mathcal{J}) \right] \leq_I \left[\frac{8}{3}\mathcal{J}, \frac{8}{3}(2 - \mathcal{J}) \right].$$

Hence, Theorem 12 is verified.

4. Conclusions

In this paper, we constructed various new *HH*- and *HH*-Fejér-type inequalities for convex FNVMs, and *HH*-inequalities hold for this notion of convex FNVMs. We plan to investigate this idea for non-convex FNVMs and a few applications in fuzzy nonlinear programming in the future. This idea opens up a new area of research for convex analysis

and optimization theory. We think that this idea will be useful to other authors as they play their roles in various scientific domains.

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