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Analysis of a Class of Predation-Predation Model Dynamics with Random Perturbations

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Abstract: In this paper, we study a class of predation–prey biological models with random perturbation. Firstly, the existence and uniqueness of systematic solutions can be proven according to Lipschitz conditions, and then we prove that the systematic solution exists globally. Moreover, the article discusses the long-term dynamical behavior of the model, which studies the stationary distribution and gradual properties of the system. Next, we use two different methods to give the conditions of population extinction. From what has been discussed above, we can safely draw the conclusion that our results are reasonable by using numerical simulation.

Keywords: random perturbation; Lipschitz conditions; stationary distribution; gradual properties; population extinction; numerical simulation

MSC: 34C23; 34D05

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Citation: Tan, X.; Liu, P.; Luo, W.; Chen, H. Analysis of a Class of Predation-Predation Model Dynamics with Random Perturbations. *Mathematics* **2022**, *10*, 3238. https://doi.org/10.3390/math10183238

Academic Editor: Ivo Siekmann

Received: 2 July 2022
Accepted: 31 August 2022
Published: 6 September 2022

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1. Introduction

Along with the remarkable development of science and technology, mathematics is applied everywhere around us. In this way, it is possible for us to use mathematics to solve various problems in our daily lives. The application of mathematics in biology is more significant, in 2015, J. Banasiak [1] solved systems biology problems by establishing a mathematical framework. He studied multi-scale problems in the complex domains in biology, which can help us to use mathematical methods to solve biological problems. By studying population models, we can also effectively predict and control biological populations. Grunert K [2], Luoyi Wu [3] and Àngel Calsina [4] have conducted research on this.

Research on biological populations has acquired many good results, which can be referred to [5–10], the study of biological populations is of great significance to us. By analyzing the rules of the biological population change and studying the factors affecting the population change, we can take appropriate measures to control the population number. In addition, the study of population models can make reasonable predictions for population numbers, which can provide correct suggestions for industrial and animal protection. Due to the fact that the number of populations is of great importance to ecosystems, the study of population models is reasonable.

Predators initially grow when they enter the ecosystem, but they are unlikely to grow indefinitely due to limiting factors, such as food and habitat space. When the population reaches a certain degree, the population reaches saturation, so the introduction of the saturation factor $-y(t)^2 + Dy(t)$ is crucial. The non-monotonic functional predator-bait model is mainly explored in this paper, the specific mathematical expression is given below:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = rx(t)(1-\dfrac{x(t)}{k}) - \dfrac{x(t)y(t)}{a+x(t)+y(t)} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{ux(t)y(t)}{a+x(t)+y(t)} - y(t)^2 + Dy(t) \end{cases} \tag{1}$$

where $x(t)$ and $y(t)$ are the populations of prey and predators at the time t, the parameter r represents the intrinsic growth rate of the prey, k represents the environmental accommodation, $\dfrac{x(t)y(t)}{a+x(t)+y(t)}$ is the functional response function for both predators and prey, u represents the maximum growth rate generated by the predation, the parameter D represents the natural growth rate of the predators, and all the parameters are positive numbers.

The model has been discussed a lot by plenty of scholars before. However, in spite of that, the population is influenced not only by the above parameters but also by environmental factors such as climate, pollution and disease, as well as various human factors, the model is determinstic, so the model (1) may have certain limitations, it is very necessary to add a certain random perturbation to the original model, this paper mainly analyzes the influence of white noise [11–13] on the model, so the model (1) becomes:

$$\begin{cases} \mathrm{d}x(t) = (rx(t)(1-\dfrac{x(t)}{k}) - \dfrac{x(t)y(t)}{a+x(t)+y(t)})\mathrm{d}t + \sigma_1 x(t)\mathrm{d}B_1(t) \\ \mathrm{d}y(t) = (\dfrac{ux(t)y(t)}{a+x(t)+y(t)} - y(t)^2 + Dy(t))\mathrm{d}t + \sigma_2 y(t)\mathrm{d}B_2(t) \end{cases} \tag{2}$$

where $B_1(t)$ and $B_2(t)$ are independent wiener processes, σ_1 and σ_2 are the strength of white noise, and they are all non-negative and bounded.

This model can be regarded as a stochastic process, so this system can be studied through the relevant laws of geometric Brownian motion. According to the modeling method of reference [14], this model can also be seen as the application of two stochastic resetting scenarios [15] in the heterogeneous diffusion process, one is a stochastic resetting with a diffusion coefficient $\frac{\sigma_1^2 x^2}{2}$, and the other is a stochastic resetting with a diffusion coefficient $\frac{\sigma_2^2 y^2}{2}$, so this model can also be considered with the resetting idea, by establishing the relationship between geometric Brownian motion and stochastic resetting, it facilitates us to study stochastic processes in different minds.

With further research, there are many models that focus on the role of resetting, especially in the geometric Brownian movement, related articles can be referred to [16]. Resetting dynamics plays an important role in quantum physics and biophysics, and specific information can refer to the literature [17–20], resetting also has important applications in computer fields [21] and economics [22].

White noise plays a crucial role in biological population research, and the introduction of white noise makes the model more reasonable in the actual situation. Moreover, the study of white noise has profound significance in population control and pest elimination. Through the control of noise intensity, it can achieve the purpose of population control, which is of great significance in ecological balance. Moreover, the purpose of eliminating pests can be achieved by reasonable control of noise intensity, correlational research is discussed in this paper.

In Section 2, firstly, the article explains that the solution of the system 2 exists locally and is unique according to the Lipshiz conditions [23–25], secondly, we show using proof by contradiction to prove that the solution exists globally.

In Section 3, the thesis mainly explores the properties of the solution. Firstly, the paper studies the long-term dynamical behavior of the system and gives a sufficient condition for the existence of a stationary distribution [26,27]. Secondly, the gradual stabilization of the solution is proved in the article, it proves that the solution of the system oscillates at the

internal equilibrium state [28] of the corresponding deterministic system. Finally, the paper gives numerical simulations of the stationary distribution.

In Section 4, the paper explores the extinction of the populations with two methods. The first way is to achieve pest control purposes by controlling the intensity and size of the random perturbations, and another is to control the pests by natural predation.

In Section 5, we have summarized the full text. First, the advantages and disadvantages of the article are analyzed, and I put forward the relevant requirements and expectations for myself, hoping that I could make progress in future scientific research and study.

2. Global Existence and Uniqueness of the Solution

Before discussing the various properties of the system, first judges whether the system solution exists, if so, whether the system solution exists is global. In this section, the article focuses on the global existence and uniqueness of the solution by constructing an auxiliary function.

Theorem 1. *For any initial condition*$(x(0), y(0))$*, system* (2) *must exists a unique solution* $(x(t), y(t))$*, and the solution exists globally.*

Proof of Theorem 1. We observe that parameters of the system do not satisfy the Lipschitz condition, first, we discuss the following system of equations:

$$\begin{cases} \mathrm{d}p(t) = [r - \dfrac{\sigma_1^2}{2} - \dfrac{re^{p(t)}}{k} - \dfrac{e^{q(t)}}{a + e^{p(t)} + e^{q(t)}}]\mathrm{d}t + \sigma_1 \mathrm{d}B_1 t \\ \mathrm{d}q(t) = [D - \dfrac{\sigma_2^2}{2} - e^{q(t)} + \dfrac{ue^{p(t)}}{a + e^{p(t)} + e^{q(t)}}]\mathrm{d}t + \sigma_2 \mathrm{d}B_2 t \end{cases} \tag{3}$$

For any initial value $p(0) = \ln x(0)), q(0) = \ln y(0)$, system (3) has a unique solution $(p(t), q(t))$ at $t \in [0, \tau_e)$, τ_e is blasting time, according to the Ito formula, system 2 has a unique solution$(x(t), y(t)) = (e^{p(t)}, e^{q(t)})$. The next step is to prove that the solution exists globally, that is $\tau_e \to \infty$ a.s. Let m_0 be big enough, the initial condition $(x(0), y(0))$ belongs to the interval $[\frac{1}{m_0}, m_0]$, for any positive number $m \geq m_0$, define the stopping time [29]:

$$\tau_m = inf\{t \in [0, \tau_e) : x(t) \notin [\frac{1}{m_0}, m_0] \quad or \quad y(t) \notin [\frac{1}{m_0}, m_0]\} \tag{4}$$

It is easy to see that τ_m is rising monotonically, stipulate $\tau_\infty = \lim\limits_{t \to +\infty} \tau_m$, in the following proof, we prove that $\tau_\infty = +\infty$ a.s.

Proof by contradiction: Suppose $\tau_\infty = +\infty$ does not hold, there must be a finite positive value T and $0 < \epsilon < 1$, the following results hold true

$$P(\tau_m \leq T) \geq \epsilon \tag{5}$$

Construct a Lyapunov function [30–32] $V_1(x, y) = (x + 1 - \ln x) + c(y + 1 - \ln y)$, and c is a positive parameter that is determined in the proof, apparently V_1 is a map from R_2^+ to R^+, according to the Ito formula:

$$\mathrm{d}V_1(x, y) = LV_1(x, y)\mathrm{d}t + (1 - \frac{1}{x})\sigma_1 x \mathrm{d}B_1(t) + (c - \frac{c}{y})\sigma_1 y \mathrm{d}B_2(t) \tag{6}$$

where

$$\begin{aligned} LV_1(x,y) &= (1-\frac{1}{x})(rx-\tfrac{rx^2}{k}-\tfrac{xy}{a+x+y})+\tfrac{\sigma_1^2x^2}{2x^2}+(c-\tfrac{c}{y})(\tfrac{uxy}{a+x+y}+Dy-y^2)+\frac{c\sigma_2^2y^2}{2y^2} \\ &= rx-\tfrac{r}{k}x^2-\tfrac{xy}{a+x+y}-r+\tfrac{r}{k}x+\tfrac{y}{a+x+y}+\tfrac{\sigma_1^2}{2} \\ &\quad +\tfrac{cuxy}{a+x+y}+cDy-cy^2-\tfrac{cux}{a+x+y}-cD+cy+\tfrac{c\sigma_2^2}{2} \\ &\le (r+\tfrac{r}{k})x+(\tfrac{1}{a}+cD+c)y+\frac{cu-1}{a+x+y}xy+\lambda \end{aligned}$$

where $\lambda=\frac{\sigma_1^2}{2}+\frac{c\sigma_2^2}{2}$, take $c=\dfrac{1}{u}$, obviously, $z<2(z+1-\ln z)$ is established by the relation of derivatives and extremvalues in the interval $(0,+\infty)$, so above formula becomes

$$\begin{aligned} LV_1(x,y) &\le (r+\frac{r}{k})x+(\frac{1}{a}+cD+c)y+\lambda \\ &\le \beta(x+cy)+\lambda \\ &\le 2\beta[x+1-\ln x+c(y+1-\ln y)]+\lambda \\ &= 2\beta V_1(x,y)+\lambda \end{aligned} \tag{7}$$

where $\beta=max\{r+\frac{r}{k},\frac{1}{ac}+D+1\}$, substitute (7) to Formula (6), next both sides of the upper formula simultaneously integrate in the interval $[0,\tau_m]$, and take the expectation.

$$E\int_0^{\tau_m}\mathrm{d}V_1(x,y)\le E\int_0^{\tau_m}[\lambda+2\beta V(x,y)]\mathrm{d}t$$

that is

$$EV_1(x(\tau_m),y(\tau_m))\le V_1(x_0,y_0)+\lambda T+2\beta\int_0^{\tau_m}EV_1(x,y)\mathrm{d}t \tag{8}$$

according to Gronwall's inequation

$$\begin{aligned} EV_1(x(\tau_m),y(\tau_m)) &\le [V_1(x_0,y_0)+\lambda T]e^{\int_0^{\tau_m}2\beta\mathrm{d}t} \\ &\le [V_1(x_0,y_0)+\lambda T]e^{2\beta T} \\ &= \gamma \end{aligned}$$

When $m>m_0$, take event Ω_m, according to the hypothesis (5), then $P(\Omega_m)\ge\epsilon>0$, while $\omega\in\Omega_m$, according to the expectation definition has the following results:

$$\gamma\ge E[I_{\Omega_m}V_1(x(\tau_m),y(\tau_m))]\ge\epsilon[m+1-\ln m+\frac{1}{u}(m+1-\ln m)]$$

then take the limit on both sides

$$\gamma\ge\infty$$

there is a contradiction, so $\gamma=+\infty$ a.s., so $T=+\infty$, so solutions exist globally. □

3. The Nature of the Solution

3.1. Existence and Ergodicity of the Stationary Distribution

In this section, we mainly discuss the long-term kinetic behavior of predators and prey, we study the weak stability of the system, when the parameters of the system meet certain conditions, the populations are long-term lasting, and the population number will be stable around a determined value, the existence of the stationary distribution of the model (2) is discussed below.

Theorem 2. *Any initial value condition* (x_0,y_0)*, if* $r-1-\sigma_1^2>0$ *and* $D-\sigma_2^2>0$ *hold true, a stationary distribution* $\mu(\cdot)$ *exists in system (2), and the stationary distribution* $\mu(\cdot)$ *is ergodic.*

Proof of Theorem 2. Define a Lyapunov function $V_2(x,y) = x + x^\delta + y + y^\delta$, the δ is a constant to be determined later, obviously, $V_3(x,y)$ is a positive definite function, according to the Ito formula

$$\mathrm{d}V_2(x,y) = LV_2(x,y)\mathrm{d}t + (1+\delta x^{\delta-1})\sigma_1 x \mathrm{d}B_1(t) + (1+\delta y^{\delta-1})\sigma_2 y \mathrm{d}B_2(t) \tag{9}$$

where

$$\begin{aligned} LV_2(x,y) =& (x+\delta x^\delta)(r - \frac{rx}{k} - \frac{y}{a+x+y}) + \frac{1}{2}\delta(\delta-1)\sigma_1^2 x^\delta \\ &+ (y+\delta y^\delta)(\frac{ux}{a+x+y} - y + D) + \frac{1}{2}\delta(\delta-1)\sigma_2^2 y^\delta \\ =& rx - \frac{r}{k}x^2 - \frac{xy}{a+x+y} + r\delta x^\delta - \frac{r\delta}{k}x^{\delta+1} - \frac{\delta x^\delta y}{a+x+y} + \frac{1}{2}\delta(\delta-1)\sigma_1^2 x^\delta \\ &+ \frac{uxy}{a+x+y} - y^2 + Dy + \frac{\delta u x y^\delta}{a+x+y} - \delta y^{\delta+1} + D\delta y^\delta + \frac{1}{2}\delta(\delta-1)\sigma_2^2 y^\delta \end{aligned}$$

So just make $\delta < 0$, let $\delta = -1$, the above formula sub changes to

$$\begin{aligned} LV_2(x,y) \leq& -\frac{r}{k}x^2 + rx - \frac{r}{x} + \frac{r}{k} + \frac{1}{x} + \frac{\sigma_1^2}{x} - y^2 + (u+D)y - \frac{D}{y} + 1 + \frac{\sigma_2^2}{y} \\ =& -\frac{r}{k}x^2 + rx - \frac{r-1-\sigma_1^2}{x} - y^2 + (u+D)y - \frac{D-\sigma_2^2}{y} + (\frac{r}{k}+1) \end{aligned}$$

For any small amount ϵ_1 and ϵ_2, consider the following set

$$U = \{(x,y) \in R_+^2 | \epsilon_1 < x < \frac{1}{\epsilon_1}, \epsilon_2 < y < \frac{1}{\epsilon_2}\}$$

then for the set $E = R_+^2 \backslash U$, LV equals to negative infinite constant holds, so there is a positive constant M, make the $LV < -M$ established at set E, so system (2) possesses a stationary distribution $\mu(\cdot)$. In addition, there is a constant N

$$N = min\{\alpha_1^2 x^2, \alpha_2^2 y^2, (x,y) \in E\} > 0$$

for any η, there is the following result

$$\sum_{i,j=1}^{2} a_{ij}\eta_i\eta_j = \alpha_1^2 x^2 \eta_1^2 + \alpha_2^2 y^2 \eta_2^2 \geq N||\eta||^2$$

so the stationary distribution $\mu(\cdot)$ is ergodic. □

Brief summary: According to the proof of Theorem 2, when the system parameters meet the above condition, the system (2) has a stationary distribution, so both of the populations have always been positive, it indicates that all populations of the system (2) will persist.

3.2. The Progressive Nature of the System (2) at the Internal Equilibrium Point (x_1, y_1) in the System (1)

If $(x_1, y_1)$is the internal equilibrium state of the system (1), we can clearly see that (x_1, y_1) is not the solution of a system (2), so what is the relationship between (x_1, y_1) and the solution of a stochastic system(2)? Next, the solution of the system (2) proceeds near the internal equilibrium state (x_1, y_1) of the deterministic system (1) is studied.

Theorem 3. *For any initial value condition*(x_0, y_0)*, if* $\frac{r}{k} - \frac{y_1}{a(a+x_1+y_1)} > 0$*, then the solution of the stochastic system* (2) *has the following properties*

$$\lim_{t\to\infty} sup \frac{1}{t} \cdot E \int_0^t [(x-x_1)^2 + (y-y_1)^2] \mathrm{d}s \leq \frac{w}{h}$$

both w and h are positive constants, they are given specifically later.

Proof of Theorem 3. Define a function $V_3(x,y) = (x - x_1 - x_1 \ln \frac{x}{x_1}) + \lambda_1 (y - y_1 - y_1 \ln \frac{y}{y_1})$ = $V_{31}(x,y) + \lambda_1 V_{32}(x,y)$, where λ_1 is a positive constant to be determined, obviously $R_2^+ \to R^+$.

Paramount consideration $V_{31}(x,y)$, according to the Ito formula.

$$\begin{aligned} LV_{31}(x,y) &= (x-x_1)(r - \tfrac{r}{k}x - \tfrac{y}{a+x+y}) + \tfrac{x_1\sigma_1^2}{2} \\ &= (x-x_1)[\tfrac{r}{k}x_1 + \tfrac{y_1}{a+x_1+y_1} - \tfrac{r}{k}x - \tfrac{y}{a+x+y}] + \tfrac{x_1\sigma_1^2}{2} \\ &= (x-x_1)[-\tfrac{r}{k}(x-x_1) + \tfrac{ay_1+y_1x-ay-x_1y}{(a+x_1+y_1)(a+x+y)}] + \tfrac{x_1\sigma_1^2}{2} \\ &= \tfrac{-a(y-y_1)(x-x_1)+y_1(x-x_1)^2-x_1(y-y_1)(x-x_1)}{(a+x_1+y_1)(a+x+y)} - \tfrac{r}{k}(x-x_1)^2 + \tfrac{x_1\sigma_1^2}{2} \end{aligned} \tag{10}$$

Then discuss $V_{32}(x,y)$, according to the Ito formula

$$\begin{aligned} LV_{32}(x,y) &= (y-y_1)(\frac{ux}{a+x+y} - y - \frac{ux_1}{a+x+y} - y_1) + \frac{y_1\sigma_2^2}{2} \\ &= (y-y_1)(\frac{ux(a+x_1+y_1) - ux_1(a+x+y)}{a+x+y} - (y-y_1)) + \frac{y_1\sigma_2^2}{2} \\ &= (y-y_1)(\frac{au(x-x_1) - u(x_1y - y_1x)}{(a+x+y)(a+x_1+y_1)} - (y-y_1)) + \frac{y_1\sigma_2^2}{2} \\ &= (y-y_1)(\frac{au(x-x_1) - u(x_1(y-y_1) - y_1(x-x_1))}{(a+x+y)(a+x_1+y_1)} - (y-y_1)) + \frac{y_1\sigma_2^2}{2} \\ &= \frac{(au+uy_1)(x-x_1)(y-y_1) - ux_1(y-y_1)^2}{(a+x+y)(a+x_1+y_1)} - (y-y_1)^2 + \frac{y_1\sigma_2^2}{2} \end{aligned} \tag{11}$$

Thus

$$\begin{aligned} LV_3(x,y) \leq &-(\frac{r}{k} - \frac{y_1}{a(a+x_1+y_1)})(x-x_1)^2 - \lambda_1(y-y_1)^2 \\ &+ \frac{-a-x_1+\lambda_1 au+\lambda_1 uy_1}{(a+x+y)(a+x_1+y_1)}(x-x_1)(y-y_1) + \frac{x_1\sigma_1^2}{2} + \frac{y_1\sigma_2^2}{2} \end{aligned} \tag{12}$$

so just take $\lambda_1 = \frac{a+x_1}{au+uy_1}$, $h = min\{\frac{r}{k} - \frac{y_1}{a(a+x_1+y_1)}, \lambda_1\} > 0$, bring them into the Formula (12), both sides are integrated in the interval $t \in (0,t)$, and then take the expectation.

$$E[V(x,y)] - V(0,0) \leq -hE \int_0^t [(x-x_1)^2 + (y-y_1)^2] \mathrm{d}s + wt$$

where $w = \frac{x_1\sigma_1^2}{2} + \frac{y_1\sigma_2^2}{2}$, so

$$\lim_{t\to\infty} sup \frac{1}{t} \cdot E \int_0^t [(x-x_1)^2 + (y-y_1)^2] \mathrm{d}s \leq \frac{w}{h} \tag{13}$$

so the Theorem 3 is proved. □

Brief summary: If the system (2) parameters satisfy the above condition, both predators and prey are persistent, and the solution of the system (2) will fluctuate up and down near the internal equilibrium state of its corresponding deterministic system. We can also get a conclusion, the system (2) is deterministic when the perturbation coefficient σ_i are zero $i = 1, 2$, and the system is randomly stable.

3.3. Numerical Simulation

According to the milsteins discretization method [33–35], the following discretization equations are obtained

$$\begin{cases} x_{i+1} = x_i + (rx_i(1 - \frac{x_i}{k}) - \frac{x_i y_i}{a + x_i + y_1})\Delta t + \sigma_1 x_i \sqrt{\Delta t}\xi_1 + \frac{1}{2}\sigma_1^2 x_i(\xi_1^2 - 1)\Delta t \\ y_{i+1} = y_i + (\frac{u x_i y_i}{a + x_i + y_i} - y_i^2 + Dy)\Delta t + \sigma_2 x_i \sqrt{\Delta t}\xi_2 + \frac{1}{2}\sigma_2^2 x_i(\xi_2^2 - 1)\Delta t \end{cases} \tag{14}$$

Let initial value $x_0 = 1, y_0 = 12$, and take the parameter $r = 10, a = 10, \sigma_1 = 1, \sigma_2 = 1$, $u = 0.3, D = 5, \Delta = \frac{200}{2^{16}}, k = 1$, then $10 - 1 - 1^2 = 8 > 0, 5 - 1^2 = 4 > 0$, thus the values satisfy the conditions of Theorem 2, the numerical simulation are as follows, see Figure 1.

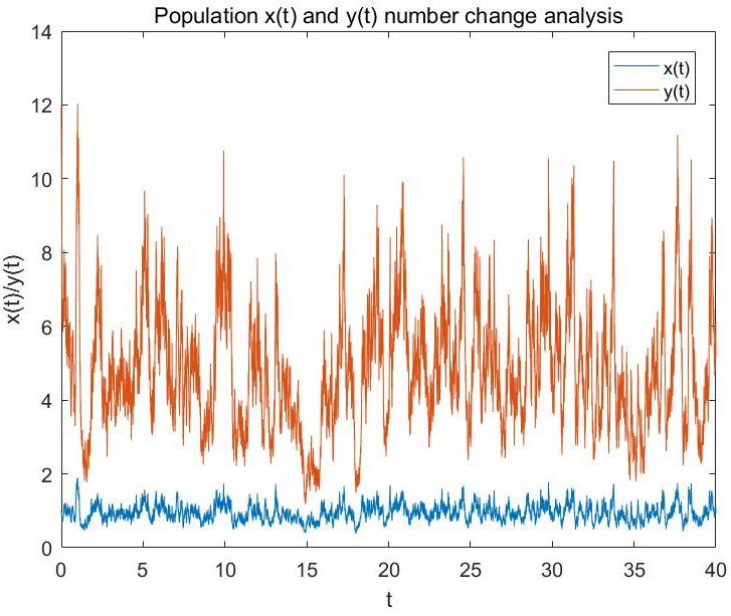


Figure 1. Stationary distribution of the model (2).

By observing Figure 1, it shows that the system does have a stationary distribution, populations $x(t)$ and $y(t)$ are persistent, and they fluctuate up and down around the determined values, so the conclusion of the Theorem 2 is correct.

4. The Final Behavior of the Population

4.1. Environmental Forces to Eliminate Insect Pests

In nature, the number of pests will be affected by various factors such as climate, infectious diseases, so environmental factors can affect the number of pests, the next subsection discusses adjusting the stochastic disturbance coefficient to eliminate pests.

Theorem 4. *For any initial value condition $x_0 > 0$, if $r - \frac{\sigma_1^2}{2} < 0$, then $\lim_{t\to\infty} x(t) = 0$, the pests can be eliminated.*

Proof of Theorem 4. Observe the first formula of the system (2), we have following results

$$\mathrm{d}x(t) \leq (rx(t)(1-\frac{x(t)}{k}))\mathrm{d}t + \sigma_1 x(t)\mathrm{d}B_1(t)$$

consider the following formula

$$\begin{cases} \mathrm{d}\psi(t) = (r\psi(t)(1-\frac{\psi(t)}{k})\mathrm{d}t + \sigma_1\psi(t)\mathrm{d}B_1(t) \\ \psi(0) = x(0) \end{cases} \tag{15}$$

according to the principle of comparison

$$x(t) \leq \psi(t) \quad a.s.$$

the solution of Equation (15) is that

$$\psi(t) = \frac{e^{(r-\frac{\sigma_1^2}{2})t+\sigma_1 B_1(t)}}{\frac{1}{\psi(0)} + \frac{r}{k}\int_0^t e^{(r-\frac{\sigma_1^2}{2})s+\sigma_1 B_1(s)}\mathrm{d}s} \tag{16}$$

when $r - \frac{\sigma_1^2}{2} < 0$, both sides are simultaneously divided t, and take the limit

$$\lim_{t\to\infty} x(t) \leq \lim_{t\to\infty} \psi(0)e^{(r-\frac{\sigma_1^2}{2})t+\sigma_1 B_1(t)} = 0$$

so $\lim_{t\to\infty} x(t) = 0$. □

Brief summary: According to the conclusion $x(t) \leq \psi(t) = \psi(0)e^{(r-\frac{\sigma_1^2}{2})t+\sigma_1 B_1(t)}$, it shows that the number of pests can be controlled by reasonably controlling the strength of the stochastic disturbance, and if σ_1 is large enough, the purpose of pest eradication will be achieved.

4.2. The Extinction of the Populations

Theorem 2 shows that both predators and bait can survive in the system (2) when the parameters meet certain conditions. Theorem 4 shows that it can control and even eliminate pests by controlling the size of the σ_1. We should also focus on the conditions in which both populations become extinct, and specific proof is given in this section.

Theorem 5. *For any initial value condition* (x_0, y_0), *if* $r - \frac{\sigma_1^2}{2} < 0$ *and* $(r - \frac{\sigma_1^2}{2})(u + D - \frac{\sigma_2^2}{2}) - \frac{(r+u+D)^2}{4} > 0$, *then both predators and prey become extinct, that is* $\lim_{t\to\infty} x(t) = 0, \lim_{t\to\infty} y(t) = 0$.

Proof Theorem 5. Define a Lyapunov function $V_4(x, y) = \ln(x + y)$, according to the Ito formula

$$\mathrm{d}V_4(x, y) = LV_4(x, y)\mathrm{d}t + \frac{1}{x+y}\sigma_1 x\mathrm{d}B_1(t) + \frac{1}{x+y}\sigma_2 y\mathrm{d}B_2(t) \tag{17}$$

where

$$\begin{aligned} LV_4(x,y) &= \tfrac{x}{x+y}(r-\tfrac{rx}{k}-\tfrac{y}{a+x+y})-\tfrac{\sigma_1^2x^2}{2(x+y)^2}+\tfrac{y}{x+y}(\tfrac{ux}{a+x+y}+D-y)-\tfrac{\sigma_2^2y^2}{2(x+y)^2} \\ &= \tfrac{x(x+y)(r-\frac{rx}{k}-\frac{y}{a+x+y})+y(x+y)(\frac{ux}{a+x+y}+D-y)-\frac{\sigma_1^2x^2}{2}-\frac{\sigma_2^2y^2}{2}}{(x+y)^2} \\ &= \tfrac{rx^2-\frac{rx^3}{k}-\frac{x^2y}{a+x+y}+rxy-\frac{rx^2y}{k}-\frac{xy^2}{a+x+y}+\frac{ux^2y}{a+x+y}+Dxy-xy^2+\frac{uxy^2}{a+x+y}+Dy^2-y^3}{(x+y)^2} \\ &\quad +\tfrac{-\frac{\sigma_1^2x^2}{2}-\frac{\sigma_2^2y^2}{2}}{(x+y)^2} \\ &= \tfrac{g(x,y)}{(x+y)^2} \end{aligned} \tag{18}$$

where

$$\begin{aligned} g(x,y) &= rx^2-\tfrac{rx^3}{k}-\tfrac{x^2y}{a+x+y}+rxy-\tfrac{rx^2y}{k}-\tfrac{xy^2}{a+x+y}+\tfrac{ux^2y}{a+x+y}+Dxy-xy^2+\tfrac{uxy^2}{a+x+y} \\ &\quad +Dy^2-y^3-\tfrac{\sigma_1^2x^2}{2}-\tfrac{\sigma_2^2y^2}{2} \\ &\le (r-\tfrac{\sigma_1^2}{2})x^2+(r+u+D)xy+(u+D-\tfrac{\sigma_2^2}{2})y^2 \\ &= f(x,y) \end{aligned} \tag{19}$$

where $f(x,y)=\begin{bmatrix} x & y \end{bmatrix}B\begin{bmatrix} x \\ y \end{bmatrix}$, it is a quadratic type, and B is a 2×2 matrix as follows

$$B_{2\times 2}=\begin{bmatrix} r-\frac{\sigma_1^2}{2} & \frac{r+u+D}{2} \\ \frac{r+u+D}{2} & u+D-\frac{\sigma_2^2}{2} \end{bmatrix}$$

because the question set conditions

$$\begin{cases} r-\dfrac{\sigma_1^2}{2}<0 \\ (r-\dfrac{\sigma_1^2}{2})(u+D-\dfrac{\sigma_2^2}{2})-\dfrac{(r+u+D)^2}{4}>0 \end{cases}$$

thus the B is a negative definite matrix, and the eigenvalues of the matrix B are all negative, then there must be a reversible matrix P, makes the $B=P^{-1}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}P$ established, so the $f(x,y)$ can also be represented by the following way

$$f(x,y)=\lambda_1x_1^2+\lambda_2y_1^2$$

where λ_1,λ_2 are the two eigenvalues of the matrix B, and $\lambda_1<0,\lambda_2<0$, in that way

$$\begin{aligned} f(x,y) &= \lambda_1x_1^2+\lambda_2y_1^2 \\ &\le \lambda_ix^2+\lambda_jy^2 \\ &\le \lambda_{max}(x^2+y^2) \end{aligned} \tag{20}$$

where $\lambda_{max}=max\{\lambda_i,\lambda_j\}<0$, i,j = 1 or 2, combined with conclusions (18)–(20), the following result holds

$$LV_4(x,y)\le\frac{\lambda_{max}(x^2+y^2)}{(x+y)^2}\le\frac{\lambda_{max}(x^2+y^2)}{2(x^2+y^2)}=\frac{\lambda_{max}}{2}<0 \tag{21}$$

bring (21) into the Formula (17), and then both sides simultaneously integrate in the interval $(0, t)$.

$$\ln(x+y) \leq \ln(x_0, y_0) + \frac{\lambda_{max}}{2}t + \int_0^t \frac{1}{x+y}\sigma_1 x \mathrm{d}B_1(t) + \int_0^t \frac{1}{x+y}\sigma_2 \mathrm{d}B_2(t)$$

then the both sides are simultaneously divided by the time t, and take the limit on t.

$$\lim_{t\to\infty}\frac{1}{t}\ln(x+y) \leq \frac{\lambda_{max}}{2} + \lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{1}{x+y}\sigma_1 x \mathrm{d}B_1(t) + \lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{1}{x+y}\sigma_2 y \mathrm{d}B_2(t) \quad (22)$$

according to the strong number law [36] of the martingale

$$\begin{aligned} \lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{1}{x+y}\sigma_1 x \mathrm{d}B_1(t) = 0 \quad a.s. \\ \lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{1}{x+y}\sigma_1 y \mathrm{d}B_1(t) = 0 \quad a.s. \end{aligned} \quad (23)$$

bring (23) into Formula (22)

$$\lim_{t\to\infty} sup\frac{1}{t}\ln(x+y) \leq \frac{\lambda_{max}}{2} < 0 \quad a.s.$$

since both $x(t)$ and $y(t)$ are non-negative, so

$$\begin{aligned} \lim_{t\to\infty} x(t) = 0 \quad a.s. \\ \lim_{t\to\infty} y(t) = 0 \quad a.s. \end{aligned} \quad (24)$$

□

Brief summary: Through the discussion and study of this subsection, when the parameters meet the above conditions, both the predator and the predation become extinct with probability 1. We observe that when other parameters are determined if the two-parameter perturbation coefficient σ_1 and σ_2 are large enough, both groups become extinct, which is the same as the conclusion of Theorem 4.

4.3. Numerical Simulation

Theorem 5 shows that the system populations perish when the parameters meet the above conditions. In order to verify the correctness of the theorem, take the parameter set $x_0 = 8, y_0 = 5, r = 0.1, k = 10, \sigma_1 = 0.8, u = 0.1, a = 10, D = 0.2, \sigma_2 = 1, \Delta = \frac{200}{2^{16}}$, it has $0.1 - \frac{0.8^2}{2} = -0.22 < 0$ and $(0.4 - \frac{0.8^2}{2}) \times (0.1 + 0.2 - \frac{0.8^2}{2}) - \frac{(0.1+0.1+0.2)^2}{4} = 0.004 > 0$, conditions satisfy the Theorem 5, numerical simulations are shown in Figure 2.

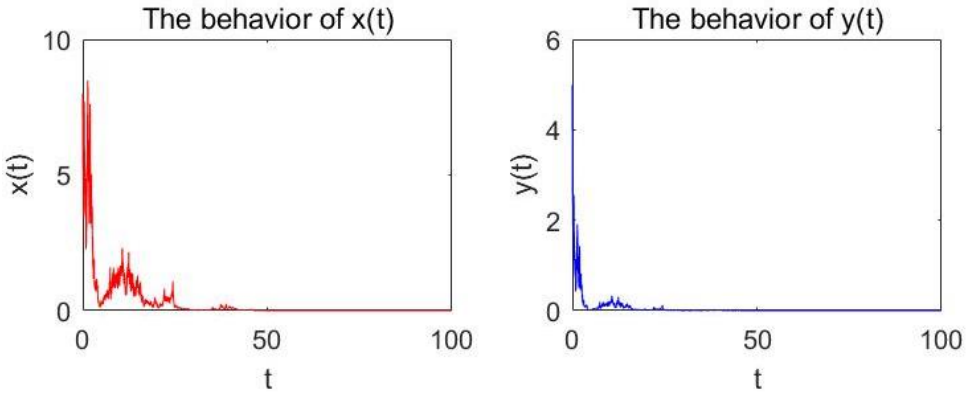


Figure 2. Model (2) population extinction behavior.

5. Conclusions

In the course of the research on biological mathematics, population size does play an important role in ecosystems. The paper studies a non-linear predation–prey model, in which the number of populations will be affected by mortality rate, growth rate and many other factors, and the population will be affected by the environmental factors, because the environment is always changing, in this paper, we consider the influence of white noise on model.

The paper studies a predation–prey model with Brownian motion, and the article is carried out in three parts. In the first part, the thesis proves that the positive solution is present and unique according to the Lipshiz condition, then we use the proof by contradiction to show that the solution of the system exists globally. In the second part, the article studies the long-term properties of the systematic solution. Firstly, we study the existing conditions of stationary distribution, then we prove that the stationary distribution is ergodic. Secondly, we study the progressive properties of the stochastic systems, we study the progressive behavior of the solution of a stochastic system near the internal equilibrium of its corresponding deterministic system. In the third part, using two methods explores the extinction behavior of the system populations. We also verify the correctness of the conclusions through numerical simulation.

This paper proves the properties of the system through the construction of the Lyapunov function and mathematical derivation according to Ito's formula, and the conclusions are reached by numerical simulation, making the conclusions scientific and rigorous. Moreover, two different methods are used to prove the population extinction conditions. However, the paper discusses the effect of white noise on the populations except for the delay factors and the effect of Levy noise. I sincerely hope that I can make great progress in the future work.

Author Contributions: Formal analysis, P.L. and W.L.; Funding acquisition, X.T.; Software, H.C. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Natural Science Foundation of China Tianyuan Mathematical Foundation (no. 12126363).

Data Availability Statement: Data openly available in a public repository.

Conflicts of Interest: The authors declare no conflict of interest.

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