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Article

New Generalized Contractions by Employing Two Control Functions and Coupled Fixed-Point Theorems with Applications

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Abstract: In this study, we obtain certain coupled fixed-point results for generalized contractions involving two control functions in a controlled metric space. Additionally, we establish some coupled fixed-point results in graph-enabled controlled metric spaces. Many well-known results from the literature will be expanded upon and modified by our results. In order to demonstrate the validity of the stated results, we also offer some examples. Finally, we apply the theoretical results to obtain the solution to a system of integral equations.

Keywords: coupled fixed-point technique; control function; contraction mapping; controlled metric space; directed graph; integral equation

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1. Introduction and Preliminaries

It is possible to find analogous fixed-point (FP) solutions to many nonlinear functional equation-based problems in science and engineering. The operator equation $\Gamma\theta = 0$ can be stated as an FP equation $\Gamma\theta = \theta$, where Γ is self-mapping with an appropriate domain. In order to solve issues that arise in various branches of mathematical analysis, such as split feasibility problems, variational inequality issues, nonlinear optimization issues, equilibrium issues, complementarity issues, and problems proving the existence of solutions to integral and differential equations, FP theory provides crucial tools.

FP theory has spread widely because of its entry into many vital disciplines, such as topology, game theory, artificial intelligence, dynamical systems (and chaos), logic programming, economics, and optimal control. Furthermore, it has become an essential pillar of nonlinear analysis, where it is used to study the existence and uniqueness of the solutions for many differential and nonlinear integral equations [1–5].

In 1906, M. Fréchet fundamentally presented the manifest evolution of a metric space. Many researchers have generalized and expanded this approach in recent years as complex-valued metric space, cone metric space, $F-$metric space, $M-$metric space, orthogonal metric space, extended $b-$metric space, $b-$metric space, and controlled metric space (CM-space), among others, motivated by this contemporary concept.

In 1993, the idea of a $b-$metric space was presented by Czerwik [6], as follows:

Definition 1. *Assume that $\Omega \neq \emptyset$, $s \geq 1$ and $\eta : \Omega \times \Omega \to [0, \infty)$. If for all $\theta, \vartheta, \ell \in \Omega$, the assertions below hold:*

(i) $\eta(\theta, \vartheta) = 0$ *iff* $\theta = \vartheta$;

(ii) $\eta(\theta,\vartheta)=\eta(\vartheta,\theta)$;
(iii) $\eta(\theta,\ell)\leq s[\eta(\theta,\vartheta)+\eta(\vartheta,\ell)]$,

then the pair (Ω,η) *is called a* $b-$*metric space.*

In 2017, the parameter s was replaced with the function $\varphi:\Omega\times\Omega\rightarrow[1,\infty)$ by Kamran et al. [7] to obtain the notion of an extended $b-$metric space as the following:

Definition 2. *Suppose that* $\Omega\neq\emptyset$, $\varphi:\Omega\times\Omega\rightarrow[1,\infty)$ *and* $\eta:\Omega\times\Omega\rightarrow[0,\infty)$. *If for all* $\theta,\vartheta,\ell\in\Omega$, *the hypotheses below hold:*

(i) $\eta(\theta,\vartheta)=0$ *iff* $\theta=\vartheta$;
(ii) $\eta(\theta,\vartheta)=\eta(\vartheta,\theta)$;
(iii) $\eta(\theta,\ell)\leq\varphi(\theta,\ell)[\eta(\theta,\vartheta)+\eta(\vartheta,\ell)]$,

then the pair (Ω,η) *is called an extended* $b-$*metric space.*

Mlaiki et al. [8] presented a novel type of extended $b-$metric space in 2018:

Definition 3. *Suppose that* $\Omega\neq\emptyset$, $\varphi:\Omega\times\Omega\rightarrow[1,\infty)$ *and* $\eta:\Omega\times\Omega\rightarrow[0,\infty)$. *If for all* $\theta,\vartheta,\ell\in\Omega$, *the hypotheses below hold:*

(i) $\eta(\theta,\vartheta)=0$ *iff* $\theta=\vartheta$;
(ii) $\eta(\theta,\vartheta)=\eta(\vartheta,\theta)$;
(iii) $\eta(\theta,\ell)\leq\varphi(\theta,\vartheta)\eta(\theta,\vartheta)+\varphi(\vartheta,\ell)\eta(\vartheta,\ell)$,

then (Ω,φ,η) *is called a* $CM-$*space.*

Example 1 ([8]). *Consider* $\Omega=\{1,2,\dots\}$. *Let* $\eta:\Omega\times\Omega\rightarrow[0,\infty)$ *be so that*

$$\eta(\theta,\vartheta)=\begin{cases}0, & \textit{if } \theta=\vartheta,\\ \frac{1}{\theta}, & \textit{If } \vartheta \textit{ is odd and } \theta \textit{ is even},\\ \frac{1}{\vartheta}, & \textit{If } \theta \textit{ is odd and } \vartheta \textit{ is even},\\ 1, & \textit{otherwise}.\end{cases}$$

Take $\varphi:\Omega\times\Omega\rightarrow[1,\infty)$ as

$$\varphi(\theta,\vartheta)=\begin{cases}\theta, & \text{If } \vartheta \text{ is odd and } \theta \text{ is even},\\ \vartheta, & \text{If } \theta \text{ is odd and } \vartheta \text{ is even},\\ 1, & \textit{otherwise}.\end{cases}$$

Then, $\eta(\theta,\vartheta)$ is CM and (Ω,φ,η) is a CM-space.

Theorem 1 ([8]). *Let* (Ω,φ,η) *be a complete controlled metric space (CCM-space) and let* $Z:\Omega\rightarrow\Omega$ *be so that*

$$\eta(Z\theta,Z\vartheta)\leq\varpi(\eta(\theta,\vartheta)),$$

for all $\theta,\vartheta\in\Omega$, *where* $0\leq\varpi<1$. *For* $\theta_0\in\Omega$, *consider* $\theta_k=Z^k\theta_0$. *Suppose that*

$$\sup_{j\geq1}\lim_{i\rightarrow\infty}\frac{\varphi(\theta_{i+1},\theta_{i+2})\varphi(\theta_{i+1},\theta_j)}{\varphi(\theta_i,\theta_{i+1})}<\frac{1}{\varpi}.$$

In addition, assume that for every $\theta\in\Omega$, *we obtain* $\lim_{i\rightarrow\infty}\varphi(\theta_i,\theta)$ *and* $\lim_{i\rightarrow\infty}\varphi(\theta,\theta_i)$, *which exist and are finite. Then, there is* $\widehat{\theta}\in\Omega$ *so that* $\widehat{\theta}=Z\widehat{\theta}$ *which is unique.*

In the context of a CM-space, Lateef [9] developed FP theorems of the sort discovered by Kannan [10], as follows:

Theorem 2 ([9]). *Let (Ω, φ, η) be a CCM-space and let $Z : \Omega \to \Omega$ be such that*

$$\eta(Z\theta, Z\vartheta) \leq \varpi(\eta(\theta, Z\theta) + \eta(\vartheta, Z\vartheta)),$$

for all $\theta, \vartheta \in \Omega$, where $0 < \varpi < \frac{1}{2}$. For $\theta_0 \in \Omega$, take $\theta_k = Z^k\theta_0$. Suppose that

$$\sup_{j\geq 1} \lim_{i\to\infty} \frac{\varphi(\theta_{i+1}, \theta_{i+2})\varphi(\theta_{i+1}, \theta_j)}{\varphi(\theta_i, \theta_{i+1})} < \frac{1}{\varpi}.$$

Furthermore, assume that for every $\theta \in \Omega$, we obtain $\lim_{i\to\infty} \varphi(\theta_i, \theta)$ and $\lim_{i\to\infty} \varphi(\theta, \theta_i)$, which exist and are finite. Then, there is $\widehat{\theta} \in \Omega$ so that $\widehat{\theta} = Z\widehat{\theta}$ which is unique.

The following FP theorem of the Reich type was developed by Ahmad [11] in the same space:

Theorem 3 ([11]). *Let (Ω, φ, η) be a CCM-space and let $Z : \Omega \to \Omega$ be self-mapping. Suppose that there are $\varpi_1, \varpi_2, \varpi_3 \in (0, 1)$ so that $\varpi = \varpi_1 + \varpi_2 + \varpi_3 < 1$ and*

$$\eta(Z\theta, Z\vartheta) \leq \varpi_1\eta(\theta, \vartheta) + \varpi_2\eta(\theta, Z\theta) + \varpi_3\eta(\vartheta, Z\vartheta),$$

for all $\theta, \vartheta \in \Omega$. For $\theta_0 \in \Omega$, take $\theta_k = Z^k\theta_0$. Suppose that

$$\sup_{j\geq 1} \lim_{i\to\infty} \frac{\varphi(\theta_{i+1}, \theta_{i+2})\varphi(\theta_{i+1}, \theta_j)}{\varphi(\theta_i, \theta_{i+1})} < \frac{1}{\varpi}.$$

Moreover, assume that for every $\theta \in \Omega$, $\lim_{i\to\infty} \varphi(\theta_i, \theta)$ and $\lim_{i\to\infty} \varphi(\theta, \theta_i)$ are exist and are finite. Then, there is a unique FP of Z.

Different FP conclusions for single and multivalued mappings have been produced by numerous authors after studying CM-spaces. We urge readers to [12–17] for additional information in this direction.

On the other hand, Bhaskar and Lakhsmikantham [18] introduced and examined another direction, the coupled fixed point (CFP). Many scholars in this field have shown interest in it since they researched the CFP results using appropriate contraction mappings and used their findings to demonstrate the existence of solutions for periodic boundary value problems (see, for example, [19–25]).

In the context of CM-spaces, we obtain several CFP results in this study for generalized contractions using specific control functions. We also proved the main CFP theorem endowed with a graph in the mentioned space. To demonstrate the reliability of the established results, some examples are provided. Finally, we look into the solution of integral equations as an application of our main finding.

2. Main Results

We begin directly with the first main result.

Theorem 4. *Assume that (Ω, φ, η) is a CCM-space and $Z : \Omega \times \Omega \to \Omega$. Let $\xi, \varrho : \Omega \times \Omega \to [0, 1)$ be such that the postulates below hold:*

(p_1) *$\xi(Z(\theta, \vartheta), Z(\ell, \upsilon)) \leq \xi(\theta, \vartheta)$ and $\varrho(Z(\theta, \vartheta), Z(\ell, \upsilon)) \leq \varrho(\theta, \vartheta)$;*
(p_2) *$\xi(\theta, \ell) = \xi(\ell, \theta)$ and $\varrho(\theta, \ell) = \varrho(\ell, \theta)$ with $(\xi + \varrho)(\theta, \vartheta) < 1$;*
(p_3)

$$\eta(Z(\theta, \vartheta), Z(\ell, \upsilon)) \leq \frac{\xi(\theta, \vartheta)}{2}(\eta(\theta, \ell) + \eta(\vartheta, \upsilon)) + \frac{\varrho(\theta, \vartheta)}{2}\left(\frac{[\eta(\theta, Z(\theta, \vartheta)) + \eta(\vartheta, Z(\vartheta, \theta))][\eta(\ell, Z(\ell, \upsilon)) + \eta(\upsilon, Z(\upsilon, \ell))]}{1 + \eta(\theta, \ell) + \eta(\vartheta, \upsilon)}\right), \quad (1)$$

for all $\theta, \vartheta, \ell, v \in \Omega$. For $\theta_0, \vartheta_0 \in \Omega$, we put $\frac{\xi(\theta_0,\vartheta_0)}{1-\varrho(\theta_0,\vartheta_0)} = \hbar$. Let $\varphi(\vartheta_i, \vartheta_{i+1}) \leq \varphi(\theta_i, \theta_{i+1})$ and

$$\sup_{j\geq 1} \lim_{i\to\infty} \frac{\varphi(\theta_{i+1}, \theta_{i+2})\varphi(\theta_{i+1}, \theta_j)}{\varphi(\theta_i, \theta_{i+1})} < \frac{1}{\hbar}, \tag{2}$$

where $\theta_{i+1} = Z(\theta_i, \vartheta_i)$ for each $i \geq 0$. Furthermore, if for every $\theta, \vartheta \in \Omega$, $\lim_{i\to\infty} \varphi(\theta_i, \theta)$ and $\lim_{i\to\infty} \varphi(\theta, \theta_i)$ exist and are finite, then the mapping Z has a unique CFP.

Proof. Assume that $(\theta_0, \ell_0) \in \Omega \times \Omega$. Construct sequences $\{\theta_i\}$ and $\{\vartheta_i\}$ in Ω by $\theta_{i+1} = Z(\theta_i, \vartheta_i)$ and $\vartheta_{i+1} = Z(\vartheta_i, \theta_i)$, for all $i \geq 0$. From stipulation (1), we obtain

$$\begin{aligned}
& \eta(\theta_{i+1}, \theta_{i+2}) \\
= \; & \eta(Z(\theta_i, \vartheta_i), Z(\theta_{i+1}, \vartheta_{i+1})) \\
\leq \; & \frac{\xi(\theta_i, \vartheta_i)}{2}(\eta(\theta_i, \theta_{i+1}) + \eta(\vartheta_i, \vartheta_{i+1})) \\
& + \frac{\varrho(\theta_i, \vartheta_i)}{2} \frac{[\eta(\theta_i, Z(\theta_i, \vartheta_i)) + \eta(\vartheta_i, Z(\vartheta_i, \theta_i))][\eta(\theta_{i+1}, Z(\theta_{i+1}, \vartheta_{i+1})) + \eta(\vartheta_{i+1}, Z(\vartheta_{i+1}, \theta_{i+1}))]}{1 + \eta(\theta_i, \theta_{i+1}) + \eta(\vartheta_i, \vartheta_{i+1})} \\
= \; & \frac{\xi(\theta_i, \vartheta_i)}{2}(\eta(\theta_i, \theta_{i+1}) + \eta(\vartheta_i, \vartheta_{i+1})) \\
& + \frac{\varrho(\theta_i, \vartheta_i)}{2} \left(\frac{[\eta(\theta_i, \theta_{i+1}) + \eta(\vartheta_i, \vartheta_{i+1})][\eta(\theta_{i+1}, \theta_{i+2}) + \eta(\vartheta_{i+1}, \vartheta_{i+2})]}{1 + \eta(\theta_i, \theta_{i+1}) + \eta(\vartheta_i, \vartheta_{i+1})} \right) \\
\leq \; & \frac{\xi(\theta_i, \vartheta_i)}{2}(\eta(\theta_i, \theta_{i+1}) + \eta(\vartheta_i, \vartheta_{i+1})) + \frac{\varrho(\theta_i, \vartheta_i)}{2}(\eta(\theta_{i+1}, \theta_{i+2}) + \eta(\vartheta_{i+1}, \vartheta_{i+2})) \\
= \; & \frac{\xi(Z(\theta_{i-1}, \vartheta_{i-1}), Z(\vartheta_{i-1}, \theta_{i-1}))}{2}(\eta(\theta_i, \theta_{i+1}) + \eta(\vartheta_i, \vartheta_{i+1})) \\
& + \frac{\varrho(Z(\theta_{i-1}, \vartheta_{i-1}), Z(\vartheta_{i-1}, \theta_{i-1}))}{2}(\eta(\theta_{i+1}, \theta_{i+2}) + \eta(\vartheta_{i+1}, \vartheta_{i+2})).
\end{aligned}$$

It follows that

$$\begin{aligned}
\eta(\theta_{i+1}, \theta_{i+2}) \leq \; & \frac{\xi(\theta_{i-1}, \vartheta_{i-1})}{2}(\eta(\theta_i, \theta_{i+1}) + \eta(\vartheta_i, \vartheta_{i+1})) + \frac{\varrho(\theta_{i-1}, \vartheta_{i-1})}{2}(\eta(\theta_{i+1}, \theta_{i+2}) + \eta(\vartheta_{i+1}, \vartheta_{i+2})) \\
= \; & \frac{\xi(Z(\theta_{i-2}, \vartheta_{i-2}), Z(\vartheta_{i-2}, \theta_{i-2}))}{2}(\eta(\theta_i, \theta_{i+1}) + \eta(\vartheta_i, \vartheta_{i+1})) \\
& + \frac{\varrho(Z(\theta_{i-2}, \vartheta_{i-2}), Z(\vartheta_{i-2}, \theta_{i-2}))}{2}(\eta(\theta_{i+1}, \theta_{i+2}) + \eta(\vartheta_{i+1}, \vartheta_{i+2})) \\
\leq \; & \frac{\xi(\theta_{i-2}, \vartheta_{i-2})}{2}\left(\frac{\eta(\theta_i, \theta_{i+1}) + \eta(\vartheta_i, \vartheta_{i+1})}{2}\right) + \frac{\varrho(\theta_{i-2}, \vartheta_{i-2})}{2}(\eta(\theta_{i+1}, \theta_{i+2}) + \eta(\vartheta_{i+1}, \vartheta_{i+2})) \\
& \vdots \\
\leq \; & \frac{\xi(\theta_0, \vartheta_0)}{2}(\eta(\theta_i, \theta_{i+1}) + \eta(\vartheta_i, \vartheta_{i+1})) + \frac{\varrho(\theta_0, \vartheta_0)}{2}(\eta(\theta_{i+1}, \theta_{i+2}) + \eta(\vartheta_{i+1}, \vartheta_{i+2})).
\end{aligned} \tag{3}$$

Similarly

$$\begin{aligned}
& \eta(\vartheta_{i+1},\vartheta_{i+2}) \\
= \;& \eta(Z(\vartheta_i,\theta_i),Z(\vartheta_{i+1},\theta_{i+1})) \\
\leq \;& \frac{\xi(\vartheta_i,\theta_i)}{2}(\eta(\vartheta_i,\vartheta_{i+1})+\eta(\theta_i,\theta_{i+1})) \\
& +\frac{\varrho(\vartheta_i,\theta_i)}{2}\left(\frac{[\eta(\vartheta_i,Z(\vartheta_i,\theta_i))+\eta(\theta_i,Z(\theta_i,\vartheta_i))][\eta(\vartheta_{i+1},Z(\vartheta_{i+1},\theta_{i+1}))+\eta(\theta_{i+1},Z(\theta_{i+1},\vartheta_{i+1}))]}{1+\eta(\theta_i,\theta_{i+1})+\eta(\vartheta_i,\vartheta_{i+1})}\right) \\
= \;& \frac{\xi(\theta_i,\vartheta_i)}{2}(\eta(\theta_i,\theta_{i+1})+\eta(\vartheta_i,\vartheta_{i+1})) \\
& +\frac{\varrho(\theta_i,\vartheta_i)}{2}\left(\frac{[\eta(\theta_i,\theta_{i+1})+\eta(\vartheta_i,\vartheta_{i+1})][\eta(\theta_{i+1},\theta_{i+2})+\eta(\vartheta_{i+1},\vartheta_{i+2})]}{1+\eta(\theta_i,\theta_{i+1})+\eta(\vartheta_i,\vartheta_{i+1})}\right) \\
\leq \;& \frac{\xi(\theta_{i-1},\vartheta_{i-1})}{2}(\eta(\theta_i,\theta_{i+1})+\eta(\vartheta_i,\vartheta_{i+1}))+\frac{\varrho(\theta_{i-1},\vartheta_{i-1})}{2}(\eta(\theta_{i+1},\theta_{i+2})+\eta(\vartheta_{i+1},\vartheta_{i+2})) \\
\leq \;& \frac{\xi(\theta_{i-2},\vartheta_{i-2})}{2}(\eta(\theta_i,\theta_{i+1})+\eta(\vartheta_i,\vartheta_{i+1}))+\varrho(\theta_{i-2},\vartheta_{i-2})(\eta(\theta_{i+1},\theta_{i+2})+\eta(\vartheta_{i+1},\vartheta_{i+2})) \\
& \vdots \\
\leq \;& \frac{\xi(\theta_0,\vartheta_0)}{2}(\eta(\theta_i,\theta_{i+1})+\eta(\vartheta_i,\vartheta_{i+1}))+\frac{\varrho(\theta_0,\vartheta_0)}{2}(\eta(\theta_{i+1},\theta_{i+2})+\eta(\vartheta_{i+1},\vartheta_{i+2})).
\end{aligned} \tag{4}$$

Combining (3) and (4), we have

$$\eta(\theta_{i+1},\theta_{i+2})+\eta(\vartheta_{i+1},\vartheta_{i+2})\leq\xi(\theta_0,\vartheta_0)(\eta(\theta_i,\theta_{i+1})+\eta(\vartheta_i,\vartheta_{i+1}))+\varrho(\theta_0,\vartheta_0)(\eta(\theta_{i+1},\theta_{i+2})+\eta(\vartheta_{i+1},\vartheta_{i+2})),$$

which implies that

$$\begin{aligned}
\eta(\theta_i,\theta_{i+1})+\eta(\vartheta_i,\vartheta_{i+1}) &\leq \frac{\xi(\theta_0,\vartheta_0)}{1-\varrho(\theta_0,\vartheta_0)}(\eta(\theta_{i-1},\theta_i)+\eta(\vartheta_{i-1},\vartheta_i)) \\
&= \hbar(\eta(\theta_{i-1},\theta_i)+\eta(\vartheta_{i-1},\vartheta_i)) \\
&\leq \hbar^2(\eta(\theta_{i-2},\theta_i)+\eta(\vartheta_{i-1},\vartheta_i)) \\
&\;\;\vdots \\
&\leq \hbar^i(\eta(\theta_0,\theta_1)+\eta(\vartheta_0,\vartheta_1)).
\end{aligned} \tag{5}$$

For $i,j\in\mathbb{N}$, $(i<j)$, we obtain

$$\begin{aligned}
\eta(\theta_i,\theta_j) &\leq \varphi(\theta_i,\theta_{i+1})\eta(\theta_i,\theta_{i+1})+\varphi(\theta_{i+1},\theta_j)\eta(\theta_{i+1},\theta_j) \\
&\leq \varphi(\theta_i,\theta_{i+1})\eta(\theta_i,\theta_{i+1})+\varphi(\theta_{i+1},\theta_j)\varphi(\theta_{i+1},\theta_{i+2})\eta(\theta_{i+1},\theta_{i+2})+\varphi(\theta_{i+1},\theta_j)\varphi(\theta_{i+2},\theta_j)\eta(\theta_{i+2},\theta_j) \\
&\leq \varphi(\theta_i,\theta_{i+1})\eta(\theta_i,\theta_{i+1})+\varphi(\theta_{i+1},\theta_j)\varphi(\theta_{i+1},\theta_{i+2})\eta(\theta_{i+1},\theta_{i+2}) \\
&\quad +\varphi(\theta_{i+1},\theta_j)\varphi(\theta_{i+2},\theta_j)\varphi(\theta_{i+1},\theta_{i+3})\eta(\theta_{i+1},\theta_{i+3}) \\
&\quad +\varphi(\theta_{i+1},\theta_j)\varphi(\theta_{i+2},\theta_j)\varphi(\theta_{i+3},\theta_j)\eta(\theta_{i+3},\theta_j) \\
&\leq \cdots \\
&\leq \varphi(\theta_i,\theta_{i+1})\eta(\theta_i,\theta_{i+1})+\sum_{k=i+1}^{j-2}\left(\prod_{u=i+1}^{k}\varphi(\theta_u,\theta_j)\right)\varphi(\theta_k,\theta_{k+1})\eta(\theta_k,\theta_{k+1}) \\
&\quad +\prod_{k=i+1}^{j-1}\varphi(\theta_k,\theta_j)\eta(\theta_{j-1},\theta_j),
\end{aligned}$$

which yields that

$$\begin{aligned}
\eta(\theta_i,\theta_j) \leq\;& \varphi(\theta_i,\theta_{i+1})\eta(\theta_i,\theta_{i+1})+\sum_{k=i+1}^{j-2}\left(\prod_{u=i+1}^{k}\varphi(\theta_u,\theta_j)\right)\varphi(\theta_k,\theta_{k+1})\eta(\theta_k,\theta_{k+1}) \\
& +\left(\prod_{k=i+1}^{j-1}\varphi(\theta_k,\theta_j)\right)\varphi(\theta_{j-1},\theta_j)\eta(\theta_{j-1},\theta_j).
\end{aligned} \tag{6}$$

Analogously, one can obtain

$$\begin{aligned}\eta\left(\vartheta_i,\vartheta_j\right) \quad \leq \quad & \varphi(\vartheta_i,\vartheta_{i+1})\eta(\vartheta_i,\vartheta_{i+1}) + \sum_{k=i+1}^{j-2}\left(\prod_{u=i+1}^{k}\varphi\left(\vartheta_u,\vartheta_j\right)\right)\varphi(\vartheta_k,\vartheta_{k+1})\eta(\vartheta_k,\vartheta_{k+1}) \\ & + \left(\prod_{k=i+1}^{j-1}\varphi\left(\vartheta_k,\vartheta_j\right)\right)\varphi\left(\vartheta_{j-1},\vartheta_j\right)\eta\left(\vartheta_{j-1},\vartheta_j\right). \end{aligned} \tag{7}$$

Adding (6) to (7) and using the assumption $\varphi(\vartheta_i,\vartheta_{i+1}) \leq \varphi(\theta_i,\theta_{i+1})$ and (5), we obtain

$$\begin{aligned} & \eta\left(\theta_i,\theta_j\right) + \eta\left(\vartheta_i,\vartheta_j\right) \\ \leq \quad & \varphi(\theta_i,\theta_{i+1})[\eta(\theta_i,\theta_{i+1}) + \eta(\vartheta_i,\vartheta_{i+1})] \\ & + \left(\sum_{k=i+1}^{j-2}\left(\prod_{u=i+1}^{k}\varphi(\theta_u,\theta_j)\right)\varphi(\theta_k,\theta_{k+1})\right)[\eta(\theta_k,\theta_{k+1}) + \eta(\vartheta_k,\vartheta_{k+1})] \\ & + \left(\left(\prod_{k=i+1}^{j-1}\varphi\left(\vartheta_k,\vartheta_j\right)\right)\varphi\left(\vartheta_{j-1},\vartheta_j\right)\right)\left[\eta\left(\theta_{j-1},\theta_j\right) + \eta\left(\vartheta_{j-1},\vartheta_j\right)\right] \\ \leq \quad & \varphi(\theta_i,\theta_{i+1})\hbar^i(\eta(\theta_0,\theta_1) + \eta(\vartheta_0,\vartheta_1)) \\ & + \left(\sum_{k=i+1}^{j-2}\left(\prod_{u=i+1}^{k}\varphi(\theta_u,\theta_j)\right)\varphi(\theta_k,\theta_{k+1})\right)\hbar^k(\eta(\theta_0,\theta_1) + \eta(\vartheta_0,\vartheta_1)) \\ & + \left(\left(\prod_{k=i+1}^{j-1}\varphi\left(\vartheta_k,\vartheta_j\right)\right)\varphi\left(\vartheta_{j-1},\vartheta_j\right)\right)\hbar^{j-1}(\eta(\theta_0,\theta_1) + \eta(\vartheta_0,\vartheta_1)) \\ = \quad & \varphi(\theta_i,\theta_{i+1})\hbar^i(\eta(\theta_0,\theta_1) + \eta(\vartheta_0,\vartheta_1)) \\ & + \left(\sum_{k=i+1}^{j-1}\left(\prod_{u=i+1}^{k}\varphi(\theta_u,\theta_j)\right)\varphi(\theta_k,\theta_{k+1})\right)\hbar^k(\eta(\theta_0,\theta_1) + \eta(\vartheta_0,\vartheta_1)). \end{aligned} \tag{8}$$

Put

$$\Theta_g = \left(\sum_{k=0}^{g}\left(\prod_{u=0}^{k}\varphi(\theta_u,\theta_j)\right)\varphi(\theta_k,\theta_{k+1})\right)\hbar^k(\eta(\theta_0,\theta_1) + \eta(\vartheta_0,\vartheta_1)). \tag{9}$$

Applying (9) in (8), one can write

$$\eta\left(\theta_i,\theta_j\right) + \eta\left(\vartheta_i,\vartheta_j\right) \leq (\eta(\theta_0,\theta_1) + \eta(\vartheta_0,\vartheta_1))\left[\hbar^i\varphi(\theta_i,\theta_{i+1}) + \left(\Theta_{j-1} - \Theta_i\right)\right]. \tag{10}$$

Because $\varphi(\theta,\vartheta) \geq 1$ and from the ratio test, we have that $\lim_{i\to\infty}\Theta_i$ exists. Thus, the sequence $\{\Theta_i\}$ is a Cauchy sequence. Letting $i,j \to \infty$ in (10), we have

$$\lim_{i,j\to\infty}\left[\eta\left(\theta_i,\theta_j\right) + \eta\left(\vartheta_i,\vartheta_j\right)\right] = 0,$$

which implies that $\lim\limits_{i,j\to\infty}\eta\left(\theta_i,\theta_j\right) = \lim\limits_{i,j\to\infty}\eta\left(\vartheta_i,\vartheta_j\right) = 0$. Thus, there are $\theta^*,\vartheta^* \in \Omega$ so that

$$\lim_{i\to\infty}\eta(\theta_i,\theta^*) = 0 \text{ and } \lim_{i\to\infty}\eta(\vartheta_i,\vartheta^*) = 0. \tag{11}$$

Using postulates (p_2) and (p_3), we obtain

$$
\begin{aligned}
\eta(\theta^*, Z(\theta^*, \vartheta^*)) &\leq \varphi(\theta^*, \theta_{i+1})\eta(\theta^*, \theta_{i+1}) + \varphi(\theta_{i+1}, Z(\theta^*, \vartheta^*))\eta(\theta_{i+1}, Z(\theta^*, \vartheta^*)) \\
&= \varphi(\theta^*, \theta_{i+1})\eta(\theta^*, \theta_{i+1}) + \varphi(\theta_{i+1}, Z(\theta^*, \vartheta^*))\eta(Z(\theta_i, \vartheta_i), Z(\theta^*, \vartheta^*)) \\
&\leq \varphi(\theta^*, \theta_{i+1})\eta(\theta^*, \theta_{i+1}) + \varphi(\theta_{i+1}, Z(\theta^*, \vartheta^*))\bigg\{ \frac{\xi(\theta_i, \vartheta_i)}{2}(\eta(\theta_i, \theta^*) + \eta(\vartheta_i, \vartheta^*)) \\
&\quad + \frac{\varrho(\theta, \vartheta)}{2}\left(\frac{[\eta(\theta_i, Z(\theta_i, \vartheta_i)) + \eta(\vartheta_i, Z(\vartheta_i, \theta_i))][\eta(\theta^*, Z(\theta^*, \vartheta^*)) + \eta(\vartheta^*, Z(\vartheta^*, \theta^*))]}{1 + \eta(\theta_i, \theta^*) + \eta(\vartheta_i, \vartheta^*)} \right) \bigg\} \\
&= \varphi(\theta^*, \theta_{i+1})\eta(\theta^*, \theta_{i+1}) + \varphi(\theta_{i+1}, Z(\theta^*, \vartheta^*))\bigg\{ \frac{\xi(\theta_i, \vartheta_i)}{2}(\eta(\theta_i, \theta^*) + \eta(\vartheta_i, \vartheta^*)) \\
&\quad + \frac{\varrho(\theta, \vartheta)}{2}\left(\frac{[\eta(\theta_i, \theta_{i+1}) + \eta(\vartheta_i, \vartheta_{i+1})][\eta(\theta^*, Z(\theta^*, \vartheta^*)) + \eta(\vartheta^*, Z(\vartheta^*, \theta^*))]}{1 + \eta(\theta_i, \theta^*) + \eta(\vartheta_i, \vartheta^*)} \right) \bigg\}.
\end{aligned} \tag{12}
$$

Similarly, one can write

$$
\begin{aligned}
&\eta(\vartheta^*, Z(\vartheta^*, \theta^*)) \\
\leq\ & \varphi(\vartheta^*, \vartheta_{i+1})\eta(\vartheta^*, \vartheta_{i+1}) + \varphi(\vartheta_{i+1}, Z(\vartheta^*, \theta^*))\bigg\{ \frac{\xi(\vartheta_i, \theta_i)}{2}(\eta(\vartheta_i, \vartheta^*) + \eta(\theta_i, \theta^*)) \\
&+ \frac{\varrho(\vartheta_i, \theta_i)}{2}\left(\frac{[\eta(\vartheta_i, \vartheta_{i+1}) + \eta(\theta_i, \theta_{i+1})][\eta(\vartheta^*, Z(\vartheta^*, \theta^*)) + \eta(\theta^*, Z(\theta^*, \vartheta^*))]}{1 + \eta(\theta_i, \theta^*) + \eta(\vartheta_i, \vartheta^*)} \right) \bigg\}.
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
&\eta\left(\theta^*, \widehat{\theta}\right) \\
=\ & \eta\left(Z(\theta^*, \vartheta^*), Z\left(\widehat{\theta}, \widehat{\vartheta}\right)\right) \leq \frac{\xi(\theta^*, \vartheta^*)}{2}\left(\eta\left(\theta^*, \widehat{\theta}\right) + \eta\left(\vartheta^*, \widehat{\vartheta}\right)\right) \\
&+ \frac{\varrho(\theta^*, \vartheta^*)}{2}\left(\frac{[\eta(\theta^*, Z(\theta^*, \vartheta^*)) + \eta(\vartheta^*, Z(\vartheta^*, \theta^*))]\left[\eta\left(\widehat{\theta}, Z\left(\widehat{\theta}, \widehat{\vartheta}\right)\right) + \eta\left(\widehat{\vartheta}, Z\left(\widehat{\vartheta}, \widehat{\theta}\right)\right)\right]}{1 + \eta\left(\theta^*, \widehat{\theta}\right) + \eta\left(\vartheta^*, \widehat{\vartheta}\right)} \right) \\
=\ & \frac{\xi(\theta^*, \vartheta^*)}{2}\left(\eta\left(\theta^*, \widehat{\theta}\right) + \eta\left(\vartheta^*, \widehat{\vartheta}\right)\right).
\end{aligned} \tag{14}
$$

In the same way, we can obtain

$$
\eta\left(\vartheta^*, \widehat{\vartheta}\right) \leq \frac{\xi(\vartheta^*, \theta^*)}{2}\left(\eta\left(\theta^*, \widehat{\theta}\right) + \eta\left(\vartheta^*, \widehat{\vartheta}\right)\right). \tag{15}
$$

Combining (14) and (15), we have

$$
\eta\left(\theta^*, \widehat{\theta}\right) + \eta\left(\vartheta^*, \widehat{\vartheta}\right) \leq \xi(\vartheta^*, \theta^*)\left(\eta\left(\theta^*, \widehat{\theta}\right) + \eta\left(\vartheta^*, \widehat{\vartheta}\right)\right).
$$

Because $\xi \in [0,1)$, we conclude that $\eta\left(\theta^*, \widehat{\theta}\right) + \eta\left(\vartheta^*, \widehat{\vartheta}\right) = 0$. This is only achieved if $\eta\left(\theta^*, \widehat{\theta}\right) = 0$ and $\eta\left(\vartheta^*, \widehat{\vartheta}\right) = 0$. Hence, $\theta^* = \widehat{\theta}$ and $\vartheta^* = \widehat{\vartheta}$. Therefore, there is a unique CFP of Z on Ω. $\square$

Now, we present some direct results for Theorem 4, as follows:
Putting $\xi(\theta, \vartheta) = \xi$ and $\varrho(\theta, \vartheta) = \varrho$ in Theorem 4, we obtain the result below

Corollary 1. *Let (Ω, φ, η) be a CCM-space and $Z : \Omega \times \Omega \to \Omega$ be such that*

$$
\begin{aligned}
&\eta(Z(\theta, \vartheta), Z(\ell, \upsilon)) \\
\leq\ & \frac{\xi}{2}(\eta(\theta, \ell) + \eta(\vartheta, \upsilon)) + \frac{\varrho}{2}\left(\frac{[\eta(\theta, Z(\theta, \vartheta)) + \eta(\vartheta, Z(\vartheta, \theta))][\eta(\ell, Z(\ell, \upsilon)) + \eta(\upsilon, Z(\upsilon, \ell))]}{1 + \eta(\theta, \ell) + \eta(\vartheta, \upsilon)} \right),
\end{aligned}
$$

for all $\theta, \vartheta, \ell, v \in \Omega$. For $\theta_0, \vartheta_0 \in \Omega$, we put $\frac{\xi(\theta_0,\vartheta_0)}{1-\varrho(\theta_0,\vartheta_0)} = \hbar$. Let $\varphi(\vartheta_i, \vartheta_{i+1}) \leq \varphi(\theta_i, \theta_{i+1})$ and

$$\sup_{j\geq 1} \lim_{i\to\infty} \frac{\varphi(\theta_{i+1}, \theta_{i+2})\varphi\left(\theta_{i+1}, \theta_j\right)}{\varphi(\theta_i, \theta_{i+1})} < \frac{1}{\hbar},$$

where $\theta_{i+1} = Z(\theta_i, \vartheta_i)$ for each $i \geq 0$ and $\xi + \varrho < 1$. Furthermore, if for every $\theta, \vartheta \in \Omega$, $\lim_{i\to\infty} \varphi(\theta_i, \theta)$ and $\lim_{i\to\infty} \varphi(\theta, \theta_i)$ exist and are finite, then the mapping Z has a unique CFP.

Setting $\varrho(.,.) = 0$ in Theorem 4, we have the following result.

Corollary 2. *Assume that (Ω, φ, η) is a CCM-space and $Z : \Omega \times \Omega \to \Omega$. Let $\xi : \Omega \times \Omega \to [0, 1)$ be such that*

- $\xi(Z(\theta, \vartheta), Z(\ell, v)) \leq \xi(\theta, \vartheta)$;
- $\eta(Z(\theta, \vartheta), Z(\ell, v)) \leq \frac{\xi(\theta,\vartheta)}{2}(\eta(\theta, \ell) + \eta(\vartheta, v))$ *for all $\theta, \vartheta, \ell, v \in \Omega$ and for $\theta_0, \vartheta_0 \in \Omega$, assume that*

$$\sup_{j\geq 1} \lim_{i\to\infty} \frac{\varphi(\theta_{i+1}, \theta_{i+2})\varphi\left(\theta_{i+1}, \theta_j\right)}{\varphi(\theta_i, \theta_{i+1})} < \frac{1}{\xi(\theta_0, \vartheta_0)},$$

where $\theta_{i+1} = Z(\theta_i, \vartheta_i)$ for each $i \geq 0$. Additionally, assume that $\lim_{i\to\infty} \varphi(\theta_i, \theta)$ and $\lim_{i\to\infty} \varphi(\theta, \theta_i)$ exist and are finite for each $\theta, \vartheta \in \Omega$, then Z has a unique CFP.

Theorem 5. *Assume that (Ω, φ, η) is a CCM-space and $Z : \Omega \times \Omega \to \Omega$. If there are $\xi, \varrho : \Omega \times \Omega \to [0, 1)$ so that for $\theta, \vartheta, \ell, v \in \Omega$ and for some $k \in \mathbb{N}$, we have*

(1) $\xi\left(Z^k(\theta, \vartheta), Z^k(\ell, v)\right) \leq \xi(\theta, \vartheta)$ *and* $\varrho\left(Z^k(\theta, \vartheta), Z^k(\ell, v)\right) \leq \varrho(\theta, \vartheta)$;

(2) $\xi(\theta, \ell) = \xi(\ell, \theta)$ *and* $\varrho(\theta, \ell) = \varrho(\ell, \theta)$ *with* $(\xi + \varrho)(\theta, \vartheta) < 1$;

(3)

$$\begin{aligned}\eta\left(Z^k(\theta, \vartheta), Z^k(\ell, v)\right) &\leq \frac{\xi(\theta, \vartheta)}{2}(\eta(\theta, \ell) + \eta(\vartheta, v)) \\ &+ \frac{\varrho(\theta, \vartheta)}{2}\left(\frac{\left[\eta\left(\theta, Z^k(\theta, \vartheta)\right) + \eta\left(\vartheta, Z^k(\vartheta, \theta)\right)\right]\left[\eta\left(\ell, Z^k(\ell, v)\right) + \eta\left(v, Z^k(v, \ell)\right)\right]}{1 + \eta(\theta, \ell) + \eta(\vartheta, v)}\right).\end{aligned}$$

For $\theta_0, \vartheta_0 \in \Omega$, we set $\frac{\xi(\theta_0,\vartheta_0)}{1-\varrho(\theta_0,\vartheta_0)} = \hbar$. Let $\varphi(\vartheta_i, \vartheta_{i+1}) \leq \varphi(\theta_i, \theta_{i+1})$ and

$$\sup_{j\geq 1} \lim_{i\to\infty} \frac{\varphi(\theta_{i+1}, \theta_{i+2})\varphi\left(\theta_{i+1}, \theta_j\right)}{\varphi(\theta_i, \theta_{i+1})} < \frac{1}{\hbar},$$

where $\theta_{i+1} = Z(\theta_i, \vartheta_i)$ for each $i \geq 0$. Further, if for every $\theta, \vartheta \in \Omega$, $\lim_{i\to\infty} \varphi(\theta_i, \theta)$ and $\lim_{i\to\infty} \varphi(\theta, \theta_i)$ are exist and finite. Then, the mapping Z has a unique CFP.

Proof. Based on Theorem 4, we have Z^k has a unique CFP $(\theta^*, \vartheta^*) \in \Omega \times \Omega$. Because

$$Z^k(Z(\theta^*, \vartheta^*), Z(\vartheta^*, \theta^*)) = Z\left(Z^k(\theta^*, \vartheta^*), Z^k(\vartheta^*, \theta^*)\right) = Z(\theta^*, \vartheta^*),$$

then $Z(\theta^*, \vartheta^*)$ is a CFP of Z^k. Hence, $Z(\theta^*, \vartheta^*) = \theta^*$ and $Z(\vartheta^*, \theta^*) = \vartheta^*$. By the uniqueness of a CFP of Z^k and because the CFP of Z is also CFP of Z^k, then (θ^*, ϑ^*) is a unique CFP of Z. □

The examples below support Theorem 4.

Example 2. *Consider* $\Omega = [0,1]$. *Describe* $\eta : \Omega \times \Omega \to [0,\infty)$ *and* $\varphi : \Omega \times \Omega \to [1,\infty)$ *as*

$$\eta(\theta,\vartheta) = (\theta+\vartheta)^2 \text{ and } \varphi(\theta,\vartheta) = 1+\theta+\vartheta, \text{ for all } \theta,\vartheta \in \Omega.$$

Clearly, (Ω,φ,η) *is a CCM-space. Define the mapping* $Z : \Omega\times\Omega \to \Omega$ *by* $Z(\theta,\vartheta) = \frac{5\theta+4\vartheta}{25}$, *for* $\theta,\vartheta \in \Omega$. *Select* $\xi,\varrho : \Omega\times\Omega \to [0,1)$ *by*

$$\xi(\theta,\vartheta) = \frac{25+\theta+\vartheta}{128} \text{ and } \varrho(\theta,\vartheta) = \frac{21+\theta+\vartheta}{128}.$$

Take $(\theta_0,\vartheta_0) = (0,0)$, *so the stipulation (2) is fulfilled. Consider* $\theta,\vartheta,\ell,\upsilon \in \Omega$. *Then,*

$$\begin{aligned}
\eta(Z(\theta,\vartheta),Z(\ell,\upsilon)) &= \left(\frac{5\theta+4\vartheta}{16}+\frac{5\ell+4\upsilon}{16}\right)^2 \\
&\le \frac{((5\theta+5\ell)+(4\vartheta+4\upsilon))^2}{256} \\
&= \frac{(5\theta+5\ell)^2+(4\vartheta+4\upsilon)^2+2(5\theta+5\ell)(4\vartheta+4\upsilon)}{256} \\
&\le \frac{(5\theta+5\ell)^2+(5\vartheta+5\upsilon)^2+2(5\theta+5\ell)(4\vartheta+4\upsilon)}{256} \\
&\le \frac{25+(\theta+\vartheta)}{128}\left(\frac{(\theta+\ell)^2+(\vartheta+\upsilon)^2}{2}\right)+\frac{5(\theta+\ell)(\vartheta+\upsilon)}{32} \\
&\le \frac{25+(\theta+\vartheta)}{128}\left(\frac{(\theta+\ell)^2+(\vartheta+\upsilon)^2}{2}\right) \\
&\quad +\frac{21+(\theta+\vartheta)}{128}\cdot\frac{\left((21\theta+4\vartheta)^2+(21\vartheta+4\theta)^2\right)\left((21\ell+4\upsilon)^2+(21\upsilon+4\ell)^2\right)}{2\left(1+(\theta+\ell)^2+(\vartheta+\upsilon)^2\right)} \\
&= \frac{25+(\theta+\vartheta)}{128}\left(\frac{(\theta+\ell)^2+(\vartheta+\upsilon)^2}{2}\right) \\
&\quad +\frac{21+(\theta+\vartheta)}{2\times 128}\frac{\left(\left(\theta+\frac{5\theta+4\vartheta}{16}\right)^2+\left(\vartheta+\frac{5\vartheta+4\theta}{16}\right)^2\right)\left(\left(\ell+\frac{5\ell+4\upsilon}{16}\right)^2+\left(\upsilon+\frac{5\upsilon+4\ell}{16}\right)^2\right)}{\left(1+(\theta+\ell)^2+(\vartheta+\upsilon)^2\right)} \\
&= \frac{\xi(\theta,\vartheta)}{2}(\eta(\theta,\ell)+\eta(\vartheta,\upsilon)) \\
&\quad +\frac{\varrho(\theta,\vartheta)}{2}\left(\frac{[\eta(\theta,Z(\theta,\vartheta))+\eta(\vartheta,Z(\vartheta,\theta))][\eta(\ell,Z(\ell,\upsilon))+\eta(\upsilon,Z(\upsilon,\ell))]}{1+\eta(\theta,\ell)+\eta(\vartheta,\upsilon)}\right).
\end{aligned}$$

Hence, all requirements of Theorem 4 are fulfilled. Therefore, Z has a unique CFP, which is $(0,0)$.

Example 3. *Consider* $\Omega = \{0,1,2\}$. *Describe* $\varphi : \Omega\times\Omega \to [1,\infty)$ *and* $\eta : \Omega\times\Omega \to [0,\infty)$ *as* $\varphi(\theta,\vartheta) = 2+\theta\vartheta$, *for all* $\theta,\vartheta \in \Omega$ *and*

$$\begin{aligned}
&\eta(0,0)=\eta(1,1)=\eta(2,2)=0, && \eta(0,1)=\eta(1,0)=25, \\
&\eta(0,2)=\eta(2,0)=10, && \eta(1,2)=\eta(2,1)=20.
\end{aligned}$$

Obviously, (Ω,φ,η) *is a CCM-space. Now, define* $Z : \Omega\times\Omega \to \Omega$, *and* $\xi,\varrho : \Omega\times\Omega \to [0,1)$ *by*

$$\begin{aligned}
&Z(0,0)=Z(1,1)=Z(2,2)=0, && Z(0,1)=Z(1,0)=2, \\
&Z(0,2)=Z(2,0)=1, && Z(1,2)=Z(2,1)=1.
\end{aligned}$$

and

$$\xi(0,0)=\xi(1,1)=\xi(2,2)=0 \quad \xi(0,1)=\xi(1,0)=\tfrac{8}{17} \quad \xi(0,2)=\xi(2,0)=\tfrac{1}{5} \quad \xi(1,2)=\xi(2,1)=\tfrac{1}{7}$$
$$\varrho(0,0)=\varrho(1,1)=\varrho(2,2)=0 \quad \varrho(0,1)=\varrho(1,0)=\tfrac{1}{2} \quad \varrho(0,2)=\varrho(2,0)=\tfrac{1}{8} \quad \varrho(1,2)=\varrho(2,1)=\tfrac{1}{9}$$

To verify the condition (1), we consider the following cases:

- *If* $(\theta,\vartheta)=(\ell,\upsilon)=(0,0)$ *or* $(1,1)$ *or* $(2,2)$. *It is a trivial case.*
- *If* $(\theta,\vartheta)=(\ell,\upsilon)=((0,1)$ *or* $(1,0))$, $((0,2)$ *or* $(2,0))$, *and* $((1,2)$ *or* $(2,1))$, *we have* $\eta(Z(\theta,\vartheta),Z(\ell,\upsilon))=0$, *and it is a trivial case too.*
- *If* $((\theta,\vartheta)=(0,1)$ *and* $(\ell,\upsilon)=(0,2))$ *or* $((\theta,\vartheta)=(1,0)$ *and* $(\ell,\upsilon)=(2,0))$, *we obtain*

$$\begin{aligned}
\eta(Z(\theta,\vartheta),Z(\ell,\upsilon)) &= \eta(Z(0,1),Z(0,2))=\eta(2,1)=20\\
&< 20.77731092\\
&= \frac{8}{17\times 2}(20)+\frac{1}{2\times 2}\left(\frac{30\times 45}{1+20}\right)\\
&= \frac{\xi(0,1)}{2}(0+20)+\frac{\varrho(0,1)}{2}\left(\frac{[\eta(0,2)+\eta(1,2)][\eta(0,1)+\eta(2,1)]}{1+\eta(0,0)+\eta(1,2)}\right)\\
&= \frac{\xi(0,1)}{2}(\eta(0,0)+\eta(1,2))\\
&\quad+\frac{\varrho(0,1)}{2}\left(\frac{[\eta(0,Z(0,1))+\eta(1,Z(1,0))][\eta(0,Z(0,2))+\eta(2,Z(2,0))]}{1+\eta(0,0)+\eta(1,2)}\right)\\
&= \frac{\xi(\theta,\vartheta)}{2}(\eta(\theta,\ell)+\eta(\vartheta,\upsilon))\\
&\quad+\frac{\varrho(\theta,\vartheta)}{2}\left(\frac{[\eta(\theta,Z(\theta,\vartheta))+\eta(\vartheta,Z(\vartheta,\theta))][\eta(\ell,Z(\ell,\upsilon))+\eta(\upsilon,Z(\upsilon,\ell))]}{1+\eta(\theta,\ell)+\eta(\vartheta,\upsilon)}\right).
\end{aligned}$$

- *If* $((\theta,\vartheta)=(0,2)$ *and* $(\ell,\upsilon)=(1,2))$ *or* $((\theta,\vartheta)=(2,0)$ *and* $(\ell,\upsilon)=(2,1))$, *we have* $\eta(Z(\theta,\vartheta),Z(\ell,\upsilon))=\eta(1,1)=0$. *It is satisfied for any value of* ξ *and* ϱ.
- *If* $((\theta,\vartheta)=(1,2)$ *and* $(\ell,\upsilon)=(0,2))$ *or* $((\theta,\vartheta)=(2,1)$ *and* $(\ell,\upsilon)=(1,0))$, *we obtain* $\eta(Z(\theta,\vartheta),Z(\ell,\upsilon))=\eta(Z(1,2),Z(0,2))=\eta(1,1)=0$. *It is fulfilled for any value of* ξ *and* ϱ.

Moreover, in the above cases, we can find for each $\theta,\vartheta,\ell,\upsilon\in\Omega$ *that* $\xi(Z(\theta,\vartheta),Z(\ell,\upsilon))\leq\xi(\theta,\vartheta)$ *and* $\varrho(Z(\theta,\vartheta),Z(\ell,\upsilon))\leq\varrho(\theta,\vartheta)$ *and* $\xi+\varrho<1$. *Hence, all requirements of Theorem 4 are fulfilled. Therefore,* Z *has a unique CFP, which is* $(0,0)$.

3. Fixed-Point Techniques on Graphs

Motivated by the results of Jachymski [26], assume that (Ω,φ,η) is a CCM-space, ∇ is diagonal of $\Omega\times\Omega$, and $\partial=(\wp(\partial),\Im(\partial))$ is a directed graph (DG), where $\wp(\partial)$ is the set of vertices that coincide with Ω and $\Im(\partial)$ is the set of edges of the graph so that $\nabla\subseteq\Im(\partial)$. Furthermore, if ∂ does not have any parallel edges, then ∂ can be identified by the pair $(\wp(\partial),\Im(\partial))$. In this essay, ∂ will be portrayed as a graph that satisfies the criteria listed above. Let us represent the graph that we obtained from ∂ by flipping the edges' directions as ∂^{-1}. Thus,

$$\partial^{-1}=\{(\vartheta,\theta)\in\Omega\times\Omega:(\theta,\vartheta)\in\Im(\partial)\}.$$

Definition 4. *The set* Ξ *stands for the set of all CFPs of a nonlinear mapping* $Z:\Omega\times\Omega\to\Omega$, *that is*

$$\Xi=\left\{\left(\widehat{\theta},\widehat{\vartheta}\right)\in\Omega\times\Omega:\widehat{\theta}=Z\left(\widehat{\theta},\widehat{\vartheta}\right)\text{ and }\widehat{\vartheta}=Z\left(\widehat{\vartheta},\widehat{\theta}\right)\right\}.$$

Definition 5. *Let* $Z:\Omega\times\Omega\to\Omega$ *be self-mapping on a CCM-space* (Ω,φ,η) *endowed with a DG* ∂. *We say that* Z *is* $\partial-$*orbital, if for any* $\theta,\vartheta\in\Omega$, *we have*

$$(\theta,Z(\theta,\vartheta)),(\vartheta,Z(\vartheta,\theta))\in\Im(\partial)\Longrightarrow(Z(\theta,\vartheta),Z(Z(\theta,\vartheta),Z(\vartheta,\theta))),(Z(\vartheta,\theta),Z(Z(\vartheta,\theta),Z(\theta,\vartheta)))\in\Im(\partial).$$

For simplicity, we consider

$$\Omega^Z = \{(\theta, \vartheta) \in \Omega \times \Omega : (\theta, Z(\theta, \vartheta)) \in \Im(\eth) \text{ and } (\vartheta, Z(\vartheta, \theta)) \in \Im(\eth)\}.$$

Now, our first main result in this part is as follows:

Theorem 6. *Suppose that (Ω, φ, η) is a CCM-space endowed with a DG $\eth$. Let $Z : \Omega \times \Omega \to \Omega$ be a $\eth-$orbital mapping so that*

(a) $\Omega^Z \neq \emptyset$,

(b) *for each $\theta, \vartheta, \ell, v \in \Omega^Z$ and for $\zeta_1, \zeta_2, \zeta_3 > 0$ with $\zeta_1 + \zeta_2 + \zeta_3 < 1$, we have*

$$\begin{aligned} \eta(Z(\theta, \vartheta), Z(\ell, v)) \leq{} & \frac{\zeta_1}{2}(\eta(\theta, \ell) + \eta(\vartheta, v)) + \frac{\zeta_2}{2}(\eta(\theta, Z(\theta, \vartheta)) + \eta(\vartheta, Z(\vartheta, \theta))) \\ & + \frac{\zeta_3}{2}(\eta(\ell, Z(\ell, v)) + \eta(v, Z(v, \ell))), \end{aligned}$$

(c) *for any sequences $\{\theta_i\}, \{\vartheta_i\} \subset \Omega$ with $(\theta_i, \theta_{i+1}), (\vartheta_i, \vartheta_{i+1}) \in \Im(\eth)$, we obtain*

$$\sup_{j \geq 1} \lim_{i \to \infty} \frac{\varphi(\theta_{i+1}, \theta_{i+2}) \varphi(\theta_{i+1}, \theta_j)}{\varphi(\theta_i, \theta_{i+1})} < \frac{1}{\hbar},$$

where $\hbar = \max\left\{\frac{\zeta_1+\zeta_2}{1-\zeta_3}, \frac{\zeta_1+\zeta_3}{1-\zeta_2}\right\}$,

(d) *Z is continuous, or for any sequences $\{\theta_i\}_{i\in\mathbb{N}}, \{\vartheta_i\}_{i\in\mathbb{N}} \subset \Omega$ with $\lim_{i\to\infty} \theta_i = \theta$, $\lim_{i\to\infty} \vartheta_i = \vartheta$ and $(\theta_i, \theta_{i+1}), (\vartheta_i, \vartheta_{i+1}) \in \Im(\eth)$, we have $\Xi \neq \emptyset$, i.e., there is at least one CFP $\left(\widehat{\theta}, \widehat{\vartheta}\right) \in \Omega \times \Omega$ of Z,*

(e) *for every $\theta, \vartheta \in \Omega$, we have $\lim_{i\to\infty} \varphi(\theta_i, \theta)$ and $\lim_{i\to\infty} \varphi(\theta, \theta_i)$ exist and are finite,*

(f) *assume that $\left(\widehat{\theta}, \widehat{\vartheta}\right) \in \Xi$ then, we have $\widehat{\theta}, \widehat{\vartheta} \in \Omega^Z$ and Z has a unique CFP.*

Proof. Let $\theta_0, \vartheta_0 \in \Omega^Z$. Thus, $(\theta_0, Z(\theta_0, \vartheta_0)) \in \Im(\eth)$ and $(\vartheta_0, Z(\vartheta_0, \theta_0)) \in \Im(\eth)$. Because Z is $\eth-$orbital, we obtain

$$(Z(\theta_0, \vartheta_0), Z(Z(\theta_0, \vartheta_0), Z(\vartheta_0, \theta_0))) \in \Im(\eth) \text{ and } (Z(\vartheta_0, \theta_0), Z(Z(\vartheta_0, \theta_0), Z(\theta_0, \vartheta_0))) \in \Im(\eth).$$

Putting $Z(\theta_0, \vartheta_0) = \theta_1$ and $Z(\vartheta_0, \theta_0) = \vartheta_1$, we have

$$(\theta_1, Z(\theta_1, \vartheta_1)) \in \Im(\eth) \text{ and } (\vartheta_1, Z(\vartheta_1, \theta_1)) \in \Im(\eth),$$

which implies that

$$(Z(\theta_1, \vartheta_1), Z(Z(\theta_1, \vartheta_1), Z(\vartheta_1, \theta_1))) \in \Im(\eth) \text{ and } (Z(\vartheta_1, \theta_1), Z(Z(\vartheta_1, \theta_1), Z(\theta_1, \vartheta_1))) \in \Im(\eth).$$

Taking $Z(\theta_1, \vartheta_1) = \theta_2$ and $Z(\vartheta_1, \theta_1) = \vartheta_2$, we obtain

$$(\theta_2, Z(\theta_2, \vartheta_2)) \in \Im(\eth) \text{ and } (\vartheta_2, Z(\vartheta_2, \theta_2)) \in \Im(\eth).$$

Continuing with the same approach, we construct sequences $\{\theta_i\}_{i\in\mathbb{N}}$ and $\{\vartheta_i\}_{i\in\mathbb{N}}$ in Ω by $\theta_{i+1} = Z(\theta_i, \vartheta_i)$ and $\vartheta_{i+1} = Z(\vartheta_i, \theta_i)$ so that $(\theta_i, \theta_{i+1}) \in \Im(\eth)$ and $(\vartheta_i, \vartheta_{i+1}) \in \Im(\eth)$. Now, if there is $i_0 \in \mathbb{N}$ so that $\theta_{i_0} = \theta_{i_0+1}$ and $\vartheta_{i_0} = \vartheta_{i_0+1}$. Because $\nabla \subset \Im(\eth)$, then we obtain $\left(\theta_{i_0}, \theta_{i_0+1}\right) \in \Im(\eth)$ and $\left(\vartheta_{i_0}, \vartheta_{i_0+1}\right) \in \Im(\eth)$ and hence $\widehat{\theta} = \theta_{i_0}, \widehat{\vartheta} = \vartheta_{i_0}$ is a CFP of Z.

In other words, if i_0 is even, then $i_0 = 2i$, $\theta_{2i} = \theta_{2i+1} = Z(\theta_{2i}, \vartheta_{2i})$ and $\vartheta_{2i} = \vartheta_{2i+1} = Z(\vartheta_{2i}, \theta_{2i})$. Let $\theta = \theta_{2i} \in \Omega^Z$ and $\vartheta = \vartheta_{2i+1} \in \Omega^Z$, and using the condition (b), we obtain

$$\begin{aligned}
\eta(\theta_{2i+1},\theta_{2i+2}) &= \eta(Z(\theta_{2i},\vartheta_{2i}),Z(\theta_{2i+1},\vartheta_{2i+1})) \\
&\leq \frac{\zeta_1}{2}(\eta(\theta_{2i},\theta_{2i+1})+\eta(\vartheta_{2i},\vartheta_{2i+1}))+\frac{\zeta_2}{2}(\eta(\theta_{2i},Z(\theta_{2i},\vartheta_{2i}))+\eta(\vartheta_{2i},Z(\vartheta_{2i},\theta_{2i}))) \\
&\quad +\frac{\zeta_3}{2}(\eta(\theta_{2i+1},Z(\theta_{2i+1},\vartheta_{2i+1}))+\eta(\vartheta_{2i+1},Z(\vartheta_{2i+1},\theta_{2i+1}))) \\
&= \frac{\zeta_3}{2}(\eta(\theta_{2i+1},\theta_{2i+2})+\eta(\vartheta_{2i+1},\vartheta_{2i+2})). \qquad (16)
\end{aligned}$$

Similarly, one can write

$$\eta(\vartheta_{2i+1},\vartheta_{2i+2}) = \eta(Z(\vartheta_{2i},\theta_{2i}),Z(\vartheta_{2i+1},\theta_{2i+1})) \leq \frac{\zeta_3}{2}(\eta(\theta_{2i+1},\theta_{2i+2})+\eta(\vartheta_{2i+1},\vartheta_{2i+2})). \qquad (17)$$

Adding (16) and (17), we have

$$\eta(\theta_{2i+1},\theta_{2i+2})+\eta(\vartheta_{2i+1},\vartheta_{2i+2}) \leq \zeta_1(\eta(\theta_{2i+1},\theta_{2i+2})+\eta(\vartheta_{2i+1},\vartheta_{2i+2})),$$

which is a contradiction. Similarly, if we take i_0 is odd, then there exists an FP of Z. Thus, suppose $\theta_i \neq \theta_{i+1}$ and $\vartheta_i \neq \vartheta_{i+1}$ for $i \in \mathbb{N}$.

Next, we claim that $\{\theta_i\}$ and $\{\vartheta_i\}$ are Cauchy sequences. In this regard, we realize the following two cases:

- If $\theta = \theta_{2i} \in \Omega^Z$ and $\vartheta = \vartheta_{2i+1} \in \Omega^Z$, then based on the condition (b), we have

$$\begin{aligned}
\eta(\theta_{2i+1},\theta_{2i+2}) &= \eta(Z(\theta_{2i},\vartheta_{2i}),Z(\theta_{2i+1},\vartheta_{2i+1})) \\
&\leq \frac{\zeta_1}{2}(\eta(\theta_{2i},\theta_{2i+1})+\eta(\vartheta_{2i},\vartheta_{2i+1}))+\frac{\zeta_2}{2}(\eta(\theta_{2i},Z(\theta_{2i},\vartheta_{2i}))+\eta(\vartheta_{2i},Z(\vartheta_{2i},\theta_{2i}))) \\
&\quad +\frac{\zeta_3}{2}(\eta(\theta_{2i+1},Z(\theta_{2i+1},\vartheta_{2i+1}))+\eta(\vartheta_{2i+1},Z(\vartheta_{2i+1},\theta_{2i+1}))) \\
&= \frac{\zeta_1}{2}(\eta(\theta_{2i},\theta_{2i+1})+\eta(\vartheta_{2i},\vartheta_{2i+1}))+\frac{\zeta_2}{2}(\eta(\theta_{2i},\theta_{2i+1})+\eta(\vartheta_{2i},\vartheta_{2i+1})) \\
&\quad +\frac{\zeta_3}{2}(\eta(\theta_{2i+1},\theta_{2i+2})+\eta(\vartheta_{2i+1},\vartheta_{2i+2})). \qquad (18)
\end{aligned}$$

Similarly, one can obtain

$$\begin{aligned}
\eta(\vartheta_{2i+1},\vartheta_{2i+2}) &\leq \frac{\zeta_1}{2}(\eta(\vartheta_{2i},\vartheta_{2i+1})+\eta(\theta_{2i},\theta_{2i+1}))+\frac{\zeta_2}{2}(\eta(\vartheta_{2i},\vartheta_{2i+1})+\eta(\theta_{2i},\theta_{2i+1})) \\
&\quad +\frac{\zeta_3}{2}(\eta(\vartheta_{2i+1},\vartheta_{2i+2})+\eta(\theta_{2i+1},\theta_{2i+2})). \qquad (19)
\end{aligned}$$

Combining (18) and (19), we can write

$$\begin{aligned}
\eta(\theta_{2i+1},\theta_{2i+2})+\eta(\vartheta_{2i+1},\vartheta_{2i+2}) &\leq \zeta_1(\eta(\theta_{2i},\theta_{2i+1})+\eta(\vartheta_{2i},\vartheta_{2i+1}))+\zeta_2(\eta(\theta_{2i},\theta_{2i+1})+\eta(\vartheta_{2i},\vartheta_{2i+1})) \\
&\quad +\zeta_3(\eta(\theta_{2i+1},\theta_{2i+2})+\eta(\vartheta_{2i+1},\vartheta_{2i+2})),
\end{aligned}$$

that is,

$$\eta(\theta_{2i+1},\theta_{2i+2})+\eta(\vartheta_{2i+1},\vartheta_{2i+2}) \leq \left(\frac{\zeta_1+\zeta_2}{1-\zeta_3}\right)_1(\eta(\theta_{2i},\theta_{2i+1})+\eta(\vartheta_{2i},\vartheta_{2i+1})). \qquad (20)$$

- If $\theta = \theta_{2i} \in \Omega^Z$ and $\vartheta = \vartheta_{2i-1} \in \Omega^Z$, then by the condition (b), we obtain

$$\begin{aligned}\eta(\theta_{2i+1},\theta_{2i}) &= \eta(Z(\theta_{2i},\vartheta_{2i}),Z(\theta_{2i-1},\vartheta_{2i-1}))\\ &\leq \frac{\zeta_1}{2}(\eta(\theta_{2i},\theta_{2i-1})+\eta(\vartheta_{2i},\vartheta_{2i-1}))+\frac{\zeta_2}{2}(\eta(\theta_{2i},Z(\theta_{2i},\vartheta_{2i}))+\eta(\vartheta_{2i-1},Z(\vartheta_{2i-1},\theta_{2i-1})))\\ &\quad+\frac{\zeta_3}{2}(\eta(\theta_{2i-1},Z(\theta_{2i-1},\vartheta_{2i-1}))+\eta(\vartheta_{2i-1},Z(\vartheta_{2i-1},\theta_{2i-1})))\\ &= \frac{\zeta_1}{2}(\eta(\theta_{2i},\theta_{2i-1})+\eta(\vartheta_{2i},\vartheta_{2i-1}))+\frac{\zeta_2}{2}(\eta(\theta_{2i},\theta_{2i+1})+\eta(\vartheta_{2i-1},\vartheta_{2i}))\\ &\quad+\frac{\zeta_3}{2}(\eta(\theta_{2i-1},\theta_{2i})+\eta(\vartheta_{2i-1},\vartheta_{2i})).\end{aligned}\tag{21}$$

Obviously, we can obtain

$$\begin{aligned}\eta(\vartheta_{2i+1},\vartheta_{2i+2}) &\leq \frac{\zeta_1}{2}(\eta(\vartheta_{2i},\vartheta_{2i-1})+\eta(\theta_{2i},\theta_{2i-1}))+\frac{\zeta_2}{2}(\eta(\vartheta_{2i},\vartheta_{2i+1})+\eta(\theta_{2i-1},\theta_{2i}))\\ &\quad+\frac{\zeta_3}{2}(\eta(\vartheta_{2i-1},\vartheta_{2i})+\eta(\theta_{2i-1},\theta_{2i})).\end{aligned}\tag{22}$$

Adding (21) and (22), one can write

$$\begin{aligned}\eta(\theta_{2i+1},\theta_{2i})+\eta(\vartheta_{2i+1},\vartheta_{2i}) &\leq \zeta_1(\eta(\theta_{2i},\theta_{2i-1})+\eta(\vartheta_{2i},\vartheta_{2i-1}))+\zeta_2(\eta(\theta_{2i},\theta_{2i+1})+\eta(\vartheta_{2i-1},\vartheta_{2i}))\\ &\quad+\zeta_3(\eta(\theta_{2i-1},\theta_{2i})+\eta(\vartheta_{2i-1},\vartheta_{2i})),\end{aligned}$$

that is,

$$\eta(\theta_{2i+1},\theta_{2i})+\eta(\vartheta_{2i+1},\vartheta_{2i}) \leq \left(\frac{\zeta_1+\zeta_3}{1-\zeta_2}\right)(\eta(\theta_{2i},\theta_{2i-1})+\eta(\vartheta_{2i},\vartheta_{2i-1})).\tag{23}$$

It follows from (20), (23), and $\hbar=\max\left\{\frac{\zeta_1+\zeta_2}{1-\zeta_3},\frac{\zeta_1+\zeta_3}{1-\zeta_2}\right\}$ that

$$\eta(\theta_i,\theta_{i+1})+\eta(\vartheta_i,\vartheta_{i+1}) \leq \hbar(\eta(\theta_{i-1},\theta_i)+\eta(\vartheta_{i-1},\vartheta_i)).$$

Hence, we have

$$\begin{aligned}\eta(\theta_i,\theta_{i+1})+\eta(\vartheta_i,\vartheta_{i+1}) &\leq \hbar(\eta(\theta_{i-1},\theta_i)+\eta(\vartheta_{i-1},\vartheta_i))\\ &\leq \hbar^2(\eta(\theta_{i-2},\theta_i)+\eta(\vartheta_{i-1},\vartheta_i))\\ &\quad\vdots\\ &\leq \hbar^i(\eta(\theta_0,\theta_1)+\eta(\vartheta_0,\vartheta_1)).\end{aligned}$$

In the same manner as in the proof of Theorem 4, we can find that $\{\theta_i\}$ and $\{\vartheta_i\}$ are Cauchy sequences. Thus, there are $\theta^*,\vartheta^*\in\Omega$ such that

$$\lim_{i\to\infty}\eta(\theta_i,\theta^*)=0 \text{ and } \lim_{i\to\infty}\eta(\vartheta_i,\vartheta^*)=0.$$

Clearly

$$\lim_{i\to\infty}\theta_{2i}=\lim_{i\to\infty}\theta_{2i+1}=\theta^* \text{ and } \lim_{i\to\infty}\vartheta_{2i}=\lim_{i\to\infty}\vartheta_{2i+1}=\vartheta^*.\tag{24}$$

Now, the continuity of Z leads to

$$\begin{aligned}\theta^* &= \lim_{i\to\infty}\theta_{2i+1}=\lim_{i\to\infty}Z(\theta_{2i},\vartheta_{2i})=Z(\theta^*,\vartheta^*),\\ \vartheta^* &= \lim_{i\to\infty}\vartheta_{2i+1}=\lim_{i\to\infty}Z(\vartheta_{2i},\theta_{2i})=Z(\vartheta^*,\theta^*).\end{aligned}$$

Hence, (θ^*, ϑ^*) is a CFP of Z. Therefore, $\Xi \neq \emptyset$. Otherwise, let $\theta = \theta^* \in \Omega^Z$ and $\vartheta = \vartheta_{2i+2} \in \Omega^Z$, then we obtain

$$\begin{aligned}
0 &< \eta(Z(\theta^*, \vartheta^*), \theta_{2i+2}) = \eta(Z(\theta^*, \vartheta^*), Z(\theta_{2i+1}, \vartheta_{2i+1})) \\
&\leq \frac{\zeta_1}{2}(\eta(\theta^*, \theta_{2i+1}) + \eta(\vartheta^*, \vartheta_{2i+1})) + \frac{\zeta_2}{2}(\eta(\theta^*, Z(\theta^*, \vartheta^*)) + \eta(\vartheta^*, Z(\vartheta^*, \theta^*))) \\
&\quad + \frac{\zeta_3}{2}(\eta(\theta_{2i+1}, Z(\theta_{2i+1}, \vartheta_{2i+1})) + \eta(\vartheta_{2i+1}, Z(\vartheta_{2i+1}, \theta_{2i+1}))) \\
&= \frac{\zeta_1}{2}(\eta(\theta^*, \theta_{2i+1}) + \eta(\vartheta^*, \vartheta_{2i+1})) + \frac{\zeta_2}{2}(\eta(\theta^*, Z(\theta^*, \vartheta^*)) + \eta(\vartheta^*, Z(\vartheta^*, \theta^*))) \\
&\quad + \frac{\zeta_3}{2}(\eta(\theta_{2i+1}, \theta_{2i+2}) + \eta(\vartheta_{2i+1}, \vartheta_{2i+2})).
\end{aligned}$$

Letting $i \to \infty$, we conclude that

$$\eta(Z(\theta^*, \vartheta^*), \theta^*) \leq \frac{\zeta_2}{2}(\eta(\theta^*, Z(\theta^*, \vartheta^*)) + \eta(\vartheta^*, Z(\vartheta^*, \theta^*))) \tag{25}$$

Similarly, if we take $\theta = \theta_{2i+1} \in \Omega^Z$ and $\vartheta = \vartheta^* \in \Omega^Z$, one can obtain

$$\eta(\vartheta^*, Z(\vartheta^*, \theta^*)) \leq \frac{\zeta_2}{2}(\eta(\vartheta^*, Z(\vartheta^*, \theta^*)) + \eta(\theta^*, Z(\theta^*, \vartheta^*))) \tag{26}$$

Combining (25) and (26), we have

$$\eta(Z(\theta^*, \vartheta^*), \theta^*) + \eta(\vartheta^*, Z(\vartheta^*, \theta^*)) \leq \zeta_2(\eta(\theta^*, Z(\theta^*, \vartheta^*)) + \eta(\vartheta^*, Z(\vartheta^*, \theta^*))).$$

Because $\zeta_2 < 1$, then the above inequality holds only if $\eta(Z(\theta^*, \vartheta^*), \theta^*) = 0$ and $\eta(\vartheta^*, Z(\vartheta^*, \theta^*)) = 0$, which implies that $Z(\theta^*, \vartheta^*) = \theta^*$ and $\vartheta^* = Z(\vartheta^*, \theta^*)$. The uniqueness follows immediately from the stipulation (f) and this finishes the proof. □

The following example supports Theorem 6.

Example 4. *Consider $\Omega = \{0, 1, 2, 3, 4\}$ equipped with the distance $\eta(\theta, \vartheta) = |\theta - \vartheta|^2$ and the function $\varphi(\theta, \vartheta) = 1 + \theta\vartheta$, for all $\theta, \vartheta \in \Omega$. Obviously, (Ω, φ, η) is a CCM-space. Define the mapping $Z : \Omega \times \Omega \to \Omega$ by*

$$Z(\theta, \vartheta) = \begin{cases} 0, & \text{if } \theta, \vartheta \in \{0, 1, 2\}, \\ 1, & \text{if } \theta, \vartheta \in \{3, 4\}. \end{cases}$$

Describe a DG as $\eth = \{(0,0), (1,1), (2,2), (3,3), (4,4), (0,1), (0,2), (1,2), (0,3), (3,4)\}$. Then, with $\zeta_1 = \frac{1}{18}$, $\zeta_2 = \frac{1}{4}$ and $\zeta_3 = \frac{3}{52}$, all requirements of Theorem 6 are satisfied and $(0,0)$ is a unique CFP.

4. Applications

In this part, we apply the theoretical results presented in Theorem 4 to discuss the existence of the solution to the following integral equations:

$$\begin{cases} \theta(\nu) = \beth(\nu) + \int_0^1 \aleph_1(\nu, \tau, \theta(\tau), \vartheta(\tau)) d\tau, \\ \vartheta(\nu) = \beth(\nu) + \int_0^1 \aleph_2(\nu, \tau, \vartheta(\tau), \theta(\tau)) d\tau, \end{cases} \tag{27}$$

where $\tau \in [0,1]$, $\theta, \vartheta, \beth \in \Omega$ and $\aleph_k : [0,1] \times [0,1] \times \Omega^2 \to \Omega$, $k = 1, 2$ are continuous functions. Assume that $\Omega = C([0,1])$ equipped with $\eta(\theta, \vartheta) = \max_{\nu \in [0,1]} (\theta(\nu) - \vartheta(\nu))^2$. Then, (Ω, φ, η) is a CCM-space.

Now, we present our hypotheses to obtain the solution of the problem (27) in the theorem below.

Theorem 7. *Via the system (27), describe the functions $\psi_\theta(\nu), \psi_\vartheta(\nu) \in \Omega$ as*

$$\begin{cases} \psi_\theta(\nu) = \int_0^1 \aleph_1(\nu, \tau, \theta(\tau), \vartheta(\tau)) d\tau, \\ \psi_\vartheta(\nu) = \int_0^1 \aleph_2(\nu, \tau, \vartheta(\tau), \theta(\tau)) d\tau, \end{cases}$$

for each $\theta, \vartheta \in \Omega$ and for $\nu \in [0,1]$. If there are $\xi, \varrho : \Omega \times \Omega \to [0,1)$ so that the following assertions are true:

(1) $\xi(\psi_\theta(\nu) + \daleth(\nu), \psi_\vartheta(\nu) + \daleth(\nu)) \leq \xi(\theta, \vartheta)$ *and* $\varrho(\psi_\theta(\nu) + \daleth(\nu), \psi_\vartheta(\nu) + \daleth(\nu)) \leq \varrho(\theta, \vartheta)$;
(2) $\xi(\theta, \ell) = \xi(\ell, \theta)$ *and* $\varrho(\theta, \ell) = \varrho(\ell, \theta)$ *with* $(\xi + \varrho)(\theta, \vartheta) < 1$;
(3) $\|\psi_\theta(\nu) - \psi_\ell(\nu)\|^2 \leq \frac{\xi(\theta,\vartheta)}{2} W_1(\theta, \vartheta, \ell, \upsilon)(\nu) + \frac{\varrho(\theta,\vartheta)}{2} W_2(\theta, \vartheta, \ell, \upsilon)(\nu)$,
for all $\theta, \vartheta, \ell, \upsilon \in \Omega$, where

$$W_1(\theta, \vartheta, \ell, \upsilon)(\nu) = \|\theta(\nu) - \ell(\nu)\|^2 + \|\vartheta(\nu) - \upsilon(\nu)\|^2,$$

and

$$\begin{aligned} W_2(\theta, \vartheta, \ell, \upsilon)(\nu) &= \left[\|\psi_\theta(\nu) + \daleth(\nu) - \theta(\nu)\|^2 + \|\psi_\vartheta(\nu) + \daleth(\nu) - \vartheta(\nu)\|^2\right] \\ &\quad \times \frac{\left[\|\psi_\ell(\nu) + \daleth(\nu) - \ell(\nu)\|^2 + \|\phi_\upsilon(\nu) + \daleth(\nu) - \upsilon(\nu)\|^2\right]}{1 + \|\theta(\nu) - \ell(\nu)\|^2 + \|\vartheta(\nu) - \upsilon(\nu)\|^2}. \end{aligned}$$

Then, there exists a unique solution to the considered problem (27).

Proof. Define the mapping $Z : \Omega \times \Omega \to \Omega$ by

$$Z(\theta, \vartheta) = \psi_\theta(\nu) + \daleth(\nu).$$

Then

$$\begin{aligned} \eta(Z(\theta, \vartheta), Z(\ell, \upsilon)) &= \max_{\nu \in [0,1]} \left(\|\psi_\theta(\nu) - \psi_\ell(\nu)\|^2\right), \\ \eta(\theta, Z(\theta, \vartheta)) + \eta(\vartheta, Z(\vartheta, \theta)) &= \max_{\nu \in [0,1]} \left(\|\psi_\theta(\nu) + \daleth(\nu) - \theta(\nu)\|^2 + \|\psi_\vartheta(\nu) + \daleth(\nu) - \vartheta(\nu)\|^2\right), \\ \eta(\ell, Z(\ell, \upsilon)) + \eta(\upsilon, Z(\upsilon, \ell)) &= \max_{\nu \in [0,1]} \left(\|\psi_\ell(\nu) + \daleth(\nu) - \ell(\nu)\|^2 + \|\phi_\upsilon(\nu) + \daleth(\nu) - \upsilon(\nu)\|^2\right). \end{aligned}$$

By simple calculations, one can verify that

- $\xi(Z(\theta, \vartheta), Z(\ell, \upsilon)) \leq \xi(\theta, \vartheta)$ and $\varrho(Z(\theta, \vartheta), Z(\ell, \upsilon)) \leq \varrho(\theta, \vartheta)$;
- $\xi(\theta, \ell) = \xi(\ell, \theta)$ and $\varrho(\theta, \ell) = \varrho(\ell, \theta)$ with $(\xi + \varrho)(\theta, \vartheta) < 1$;
- $\eta(Z(\theta, \vartheta), Z(\ell, \upsilon)) \leq \frac{\xi(\theta,\vartheta)}{2}(\eta(\theta, \ell) + \eta(\vartheta, \upsilon)) +$

$$\frac{\varrho(\theta, \vartheta)}{2} \left(\frac{[\eta(\theta, Z(\theta, \vartheta)) + \eta(\vartheta, Z(\vartheta, \theta))][\eta(\ell, Z(\ell, \upsilon)) + \eta(\upsilon, Z(\upsilon, \ell))]}{1 + \eta(\theta, \ell) + \eta(\vartheta, \upsilon)} \right),$$

for all $\theta, \vartheta, \ell, \upsilon \in \Omega$.

Based on Theorem 4, Z has a unique CFP, which is a unique solution to the proposed system (27). □

Example 5. *Under the same distance in this section, suppose that* (Ω, φ, η) *is a CCM-space and consider the following coupled problem:*

$$\begin{cases} \theta(\nu) = e^{-\nu} + \int_0^1 \left(\frac{(\theta(\tau) - \vartheta(\tau))}{6 + (\theta(\tau) - \vartheta(\tau))} \right) d\tau, \\ \vartheta(\nu) = e^{-\nu} + \int_0^1 \left(\frac{(\vartheta(\tau) - \theta(\tau))}{6 + (\vartheta(\tau) - \theta(\tau))} \right) d\tau, \end{cases} \tag{28}$$

where $\tau \in [0,1]$ *and* $\theta, \vartheta \in \Omega$. *It is clear that* $\daleth(\nu) = e^{-\nu}$ *and*

$$\begin{aligned} \psi_\theta(\nu) &= \int_0^1 \left(\frac{(\theta(\tau) - \vartheta(\tau))}{6 + (\theta(\tau) - \vartheta(\tau))} \right) d\tau, \\ \psi_\vartheta(\nu) &= \int_0^1 \left(\frac{(\vartheta(\tau) - \theta(\tau))}{6 + (\vartheta(\tau) - \theta(\tau))} \right) d\tau. \end{aligned}$$

Now, define the control functions $\xi, \varrho : \Omega \times \Omega \to [0,1)$ *by* $\xi(\theta, \vartheta) = \frac{1}{3-(\theta+\vartheta)}$ *and* $\varrho(\theta, \vartheta) = \frac{1}{3+(\theta+\vartheta)}$, *and then for all* $\theta, \vartheta, \ell, \upsilon \in \Omega$, *we have*

(1) $\xi(\psi_\theta(\nu) + \daleth(\nu), \psi_\vartheta(\nu) + \daleth(\nu)) = \frac{1}{3-(\psi_\theta(\nu)+\daleth(\nu))+(\psi_\vartheta(\nu)+\daleth(\nu))} = \frac{1}{3-(\theta(\nu)+\vartheta(\nu))} = \xi(\theta, \vartheta)$, *and* $\varrho(\psi_\theta(\nu) + \daleth(\nu), \psi_\vartheta(\nu) + \daleth(\nu)) = \frac{1}{3+(\psi_\theta(\nu)+\daleth(\nu))+(\psi_\vartheta(\nu)+\daleth(\nu))} = \frac{1}{3+(\theta(\nu)+\vartheta(\nu))} = \varrho(\theta, \vartheta)$,

(2) *clearly,* $\xi(\theta, \ell) = \frac{1}{3-(\theta+\ell)} = \frac{1}{3-(\ell+\theta)} = \xi(\ell, \theta)$ *and* $\varrho(\theta, \ell) = \frac{1}{3+\theta+\ell} = \frac{1}{3+\ell+\theta} = \varrho(\ell, \theta)$ *with* $(\xi + \varrho)(\theta, \vartheta) = \frac{6}{9-(\theta+\vartheta)^2} < 1$.

(3)

$$\begin{aligned} \|\psi_\theta(\nu) - \psi_\ell(\nu)\|^2 &= \left\| \left(\frac{(\theta(\tau) - \vartheta(\tau))}{6 + (\theta(\tau) - \vartheta(\tau))} \right) - \left(\frac{(\ell(\tau) - \upsilon(\tau))}{6 + (\ell(\tau) - \upsilon(\tau))} \right) \right\|^2 \\ &\leq \frac{1}{6} \|(\theta(\tau) - \ell(\tau)) + (\vartheta(\tau) - \upsilon(\tau))\|^2 \\ &\leq \frac{1}{2} \frac{1}{(3 - (\theta + \vartheta))} \left[\|\theta(\tau) - \ell(\tau)\|^2 + \|\vartheta(\tau) - \upsilon(\tau)\|^2 \right] \\ &= \frac{\xi(\theta, \vartheta)}{2} W_1(\theta, \vartheta, \ell, \upsilon)(\nu) \\ &\leq \frac{\xi(\theta, \vartheta)}{2} W_1(\theta, \vartheta, \ell, \upsilon)(\nu) + \frac{\varrho(\theta, \vartheta)}{2} W_2(\theta, \vartheta, \ell, \upsilon)(\nu), \end{aligned}$$

where W_1 *and* W_2 *are defined in Theorem 7. Hence, all requirements of Theorem 7 are satisfied. Therefore, the coupled problem (28) has a unique solution in* Ω.

5. Conclusions and Future Works

The fixed-point technique has become a prominent role in nonlinear analysis, especially when it has been demonstrated on perfect metric distances, where it enters into many diverse and exciting applications in several directions. Many researchers turned to different methods of generalization, either by changing the contractive condition or by generalizing the space used. By applying two control functions to the right-hand side of the inequality, we constructed a new contractive condition in a controlled metric space. Additionally, we used a specific rational expression for the contractive condition. Moreover, we have taken coupled self-mapping instead of the usual self-mapping in the contractive condition of our main results. Moreover, to demonstrate the veracity of our main findings, some examples are provided. Furthermore, in controlled metric spaces equipped with a graph, we were able to derive certain CFP solutions. Ultimately, as an application of our study, the existence of the solution to the integral equations is investigated. Our upcoming work in this field will concentrate on investigating the CFPs of multivalued and fuzzy mappings in controlled metric spaces, with applications to fractional differential inclusion problems.

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