

Article

Maximum-Profit Inventory Model with Generalized Deterioration Rate

Yu-Lan Wang ¹, Ming-Li Chen ² and Peterson Julian ^{3,*}¹ College of Teacher Education, Weifang University of Science and Technical, Weifang 262799, China² School of Intelligent Manufacturing, Weifang University of Science and Technical, Weifang 262799, China³ School of General Studies, Weifang University of Science and Technical, Weifang 262799, China

* Correspondence: una211@mail.cpu.edu.tw

Abstract: We developed a maximum profit inventory model with a generalized deterioration rate where the selling rate is dependent on the inventory level that is an extension of two published papers. A complete solution structure is provided to decide the optimal solution with reasonable conditions supported by numerical examples, and then we prove that the optimal solution is independent of the demand pattern. Numerical examples are provided to illustrate our findings. In a previously published paper, three examples had symmetric conditions to decide the local maximum solution. Our approach provides a reasonable explanation for this symmetric phenomenon. Our findings will help researchers develop new inventory models in the future.

Keywords: stock-dependent selling rate; maximum profit; inventory model

MSC: 90B05



Citation: Wang, Y.-L.; Chen, M.-L.; Julian, P. Maximum-Profit Inventory Model with Generalized Deterioration Rate. *Mathematics* **2022**, *10*, 3189. <https://doi.org/10.3390/math10173189>

Academic Editor: María del Carmen Valls Martínez

Received: 31 July 2022

Accepted: 31 August 2022

Published: 3 September 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Hill [1] is a pioneer, creating an inventory model with ramp-type demands where the demand is separated into two phases. In the first phase, the demand is an increasing function, and then in the second phase, the demand is a constant function. There has been a trend of considering these kinds of inventory models with minimum-cost and maximum-profit problems. For example, we list several articles by Mandal and Pal [2] on decay products; Wu et al. [3] on the backlogging rate related to the waiting time; Wu and Ouyang [4] on two replenishment policies starting with a shortage or without a shortage; Wu [5] on the deteriorated items following the Weibull distribution; and Giri et al. [6] on three-parameter Weibull distribution. Wu and Ouyang [4] examined the cyclic deterioration of items. Deng [7], to improve upon the work of Wu et al. [3], showed the existence of an optimal solution. Manna and Chaudhuri [8] developed inventory models where the deterioration is dependent on time. Deng et al. [9] improved upon the work of Mandal and Pal [2] to prove that an optimal solution exists. Wu et al. [10] examined the maximum-profit problem in inventory models with stock-dependent selling rates and ramp-type demand. They constructed two inventory systems which were dependent on the changing point of the ramp-type demand, and then examined the optimal solution for each case. In Wu et al. [10], the demand is a ramp type, and Chuang [11] generalized it to arbitrary demand where the selling rate is stock-dependent, and then Chuang [11] derived a complete solution procedure. Several related papers include: Lin et al. [12] with net present value, Mahajan and Diatha [13] with continuous compounding, Alim and Beullens [14] with a flexible delivery option, Khakzad and Gholamian [15] with inspection and advanced payment, Khan et al. [16] with an all-units discount environment and price-sensitive demand, Rezagholifam et al. [17] with non-instantaneous deteriorating items and capacity constraint, Sarkar et al. [18] with the time value of money, and Tiwari et al. [19] with trade credit. These papers are worth mentioning to point out the current research trend

for this kind of inventory system. Several related papers that studied inventory models with deteriorating items are worthwhile to mention. Padiyar et al. [20] solved an inventory model of decay products under a fuzzy environment and with shortages where the demand is related to the selling price. Acevedo-Chedid et al. [21] developed a four-level inventory model, consisting of the supplier, production plants, distribution, and retailers, to examine collaborative planning for decay items. Mishra et al. [22] solved a sustainable supply chain inventory system with carbon emissions and controllable deteriorating products in a greenhouse environment. Mashud et al. [23] constructed a sustainable inventory system for imperfect items with controllable emissions and decay products to reduce carbon dioxide through green technology investments. Mishra et al. [24] examined an EOQ inventory system where the demand is dependent on the selling price and inventory level with preservation technology investment, shortages, and a controllable deterioration rate. Tiwari et al. [25] developed a two-echelon inventory system with partial trade credits for decay products with an expiration date to provide solution algorithms and sensitivity analysis. Khan et al. [26] constructed inventory models of deteriorating items with an expiration date where the demand is related to the selling price. They presented a solution algorithm to find the optimal solution. Padiyar et al. [27] considered an integrated EPQ inventory system of producer and buyer, with an imperfect production process for decay items that have two different demands: the producer with exponential demand, and the buyer with triangular demand, where the production rate is related to the demand. We also point out the following two other important papers related to inventory systems. Yadav et al. [28] examined a two-level integrated inventory system of manufacturer and retailer where the lead time is crashable with a service-level constraint, and the setup cost has a learning and forgetting effect. Navarro et al. [29] studied a maximum-profit EPQ inventory model with a supply chain of three levels, manufacturers, distribution centers, and retailers, to consider the effect of the marketing effort on the demand under multiple-item environments. We provide Table 1 below to compare our paper with the above-mentioned articles to indicate the current trend of research directions. Our findings will help researchers to realize maximum-profit inventory models.

Table 1. Comparison of our present work and some related papers.

Paper	<i>D</i>	<i>II</i>	<i>PDD</i>	<i>PC</i>	<i>SP</i>	<i>SA</i>	<i>S</i>	<i>SM</i>	<i>SDD</i>	<i>SC</i>
[10]	X			X	X		X	X	X	
[11]	X			X	X		X	X	X	
[20]	X		X		X	X	X			X
[21]	X	X		X	X		X			X
[22]	X		X		X	X	X	X		X
[23]	X	X	X	X	X	X		X		
[24]	X		X		X	X	X	X	X	
[25]	X		X	X	X	X	X	X		X
[26]	X		X	X	X	X	X	X		
[27]	X	X		X		X	X			X
[28]					X	X		X		X
[29]		X		X	X	X				X
ours	X			X	X	X	X	X	X	

D deterioration; *II* imperfect items; *PDD* price-dependent demand; *PC* purchasing cost; *SP* selling price; *SA* sensitivity analysis; *S* shortages; *SM* solution method; *SDD* stock-dependent demand; *SC* supply chain.

Based on Table 1, the selling price and purchasing cost have been studied by many papers. It shows that maximum-profit models are more prevalent than minimum-cost models. Many models consider the deterioration cost, revealing that decision-makers

pay attention to decay problems. Sensitivity analysis and shortages are the third and fourth most common issues in Table 1. Sensitivity analysis examines the influence of decision variables by the variation in constant parameters. Investigation of shortages can prevent stocking too many products to provide a balance between inventory and backorders. The solution method is another important research issue, showing that besides developing new inventory systems, researchers also focus on finding the optimal solution. Our paper reviews the above-mentioned six issues to indicate that it follows the main research direction.

2. Notation and Assumptions

To develop an inventory system where the demand is dependent on the inventory level, to be compatible with Wu et al. [10] and Chuang [11], we use their assumptions and notations, except the deterioration function is generated from a constant function to an arbitrary function. Moreover, we constructed an auxiliary function, $m(t_1)$. The notations are given below:

- (1) The lead time is zero because replenishments are instantaneous with an infinite replenishment rate.
- (2) A is the ordering cost per order.
- (3) c is the purchasing cost per unit.
- (4) C_d is the deterioration cost per unit.
- (5) C_h is the inventory holding cost per unit, per unit of time.
- (6) C_s is the shortage cost per unit, per unit of time.
- (7) s is the selling price per unit.
- (8) t_1 is the time when the inventory level drops to zero.
- (9) t_1^* is the optimal solution for the variable t_1 .
- (10) T is the finite time horizon under consideration.
- (11) α is the portion of extra selling related to inventory level. We define those sales related to inventory level as “extra sells”. For a technical reason, we assume that $\alpha(s + C_d) < C_h$, which will be explained in Section 3.
- (12) μ is the time that the ramp-type demand changes from an increasing phase to a constant phase.
- (13) $\theta(t)$ is the deteriorated fraction of the on-hand inventory deterioration per unit of time.
- (14) $I(t)$ is the on-hand inventory level at the time t over the ordering cycle $[0, T]$.
- (15) The shortage is allowed and fully backlogged.
- (16) Without referring to extra sales, the demand rate $R(t)$ is a positive continuous function.
- (17) $D(t)$ is the consumption sale rate (the actual rate), which is stock-dependent with

$$D(t) = \begin{cases} R(t) + \alpha I(t), & I(t) > 0, \\ R(t), & I(t) \leq 0, \end{cases} \quad \text{where } \alpha \text{ is a positive constant.}$$
- (18) $Z(t_1)$ is the total profit that consists of sale revenue, order cost, purchase cost, holding cost, deterioration cost, and shortage cost.
- (19) $m(t_1)$ is an auxiliary function such that $m(t_1) = 0$ and $\frac{d}{dt_1}Z(t_1) = 0$ have the same solution.

We recall that (a) t_1 is the time when the inventory level drops to zero, and (b) μ is the time when the ramp-type demand changes from an increasing phase to a constant phase. Depending on the relation between t_1 and μ , Wu et al. [10] divided their model into two cases: Case 1, $t_1 \geq \mu$, and Case 2, $t_1 \leq \mu$. The deterioration rate is a constant function in Wu et al. [10]. Hence, there were two objective functions in Wu et al. [10] with different domains. Chuang [11] studied the same inventory model as Wu et al. [10], and he also considered the deterioration cost. Chuang [11] did not write down the explicit formula for the ramp-type demand, such that there is only one objective function in Chuang [11]. Our paper extended the work of Wu et al. [10] and Chuang [11] considering a deterioration rate from a constant function to a non-negative function. We followed the solution procedure of

Chuang [11]. We focus on (i) the demand function and (ii) the deterioration function, both in an implicit setting, such that there is only one objective function in our paper.

3. Our proposed Inventory Model

We recall the inventory systems with ramp-type demand. These kinds of inventories were first proposed by Hill [1], and then further investigated by Mandal and Pal [2], Deng et al. [9], Wu and Ouyang [4], and Wu et al. [10]. Replenishment occurs at the time $t = 0$ when the inventory level begins at its maximum level. From $t = 0$ to t_1 , the inventory level decreases owing to deterioration and demand, $R(t)$. At t_1 , the inventory level drops to zero, after which shortages occurred for the period (t_1, T) , and all of the unsatisfied demand during the shortage period (t_1, T) is completely backlogged. Since the actual selling rate is dependent on the inventory level, the analytical model is described by the following two differential equations:

$$\frac{d}{dt}I(t) + (\theta(t) + \alpha) I(t) = -R(t), 0 < t < t_1 \quad (1)$$

and

$$\frac{d}{dt}I(t) = -R(t), t_1 < t < T \quad (2)$$

We try to solve the differential system of Equations (1) and (2), then

$$I(t) = \int_t^{t_1} R(y) e^{\int_t^y (\theta(x) + \alpha) dx} dy \quad (3)$$

for $0 \leq t \leq t_1$, and

$$I(t) = \int_t^{t_1} R(x) dx \quad (4)$$

for $t_1 \leq t \leq T$.

The purchased quantity at $t = 0$ plus the backorder quantity at $t = T$ is the sum of $I(0)$ and $-I(T)$; then, the purchased quantity will be

$$I(0) - I(T) = \int_0^{t_1} R(y) e^{\int_0^y (\theta(x) + \alpha) dx} dy + \int_{t_1}^T R(x) dx \quad (5)$$

The holding cost during the period $[0, t_1]$ is evaluated by changing the order of integration

$$C_h \int_0^{t_1} I(t) dt = C_h \int_0^{t_1} R(y) \int_0^y e^{\int_t^y (\theta(x) + \alpha) dx} dt dy \quad (6)$$

The number of deteriorated items during the period, $[0, t_1]$, is evaluated

$$I(0) - \int_0^{t_1} D(x) dx = \int_0^{t_1} R(y) \left(e^{\int_0^y (\theta(x) + \alpha) dx} - 1 \right) dy - \alpha \int_0^{t_1} R(y) \int_0^y e^{\int_t^y (\theta(x) + \alpha) dx} dt dy \quad (7)$$

The shortage cost during the period $[t_1, T]$ is computed by changing the order of integration

$$C_s \int_{t_1}^T -I(t) dt = C_s \int_{t_1}^T (T - x) R(x) dx \quad (8)$$

We evaluate the sale revenue per replenishment to derive

$$s \left(\int_0^{t_1} D(t) dt - I(T) \right) = s\alpha \int_0^{t_1} R(y) \int_0^y e^{\int_t^y (\theta(x) + \alpha) dx} dt dy + s \int_0^T R(x) dx \quad (9)$$

We obtain the total profit per replenishment cycle to denote it as the sale revenue minus the total cost, where the total cost is consistent with deterioration cost, shortage cost, holding cost, purchasing cost, and ordering cost.

$$\begin{aligned} Z(t_1) = & \frac{1}{T} \left\{ s\alpha \int_0^{t_1} R(y) \int_0^y e^{\int_t^y (\theta(x) + \alpha) dx} dt dy + s \int_0^T R(x) dx - A \right. \\ & - c \left[\int_0^{t_1} R(y) e^{\int_0^y (\theta(x) + \alpha) dx} dy + \int_{t_1}^T R(x) dx \right] - C_h \int_0^{t_1} R(y) \int_0^y e^{\int_t^y (\theta(x) + \alpha) dx} dt dy \\ & - C_s \int_{t_1}^T (T - x) R(x) dx - C_d \left[\int_0^{t_1} R(y) \left(e^{\int_0^y (\theta(x) + \alpha) dx} - 1 \right) dy \right. \\ & \left. \left. - \alpha \int_0^{t_1} R(y) \int_0^y e^{\int_t^y (\theta(x) + \alpha) dx} dt dy \right] \right\} \quad (10) \end{aligned}$$

Based on Equation (10), we derive

$$\begin{aligned} \frac{d}{dt_1} Z(t_1) = & \frac{R(t_1)}{T} \left\{ (s\alpha - C_h + \alpha C_d) \int_0^{t_1} e^{\int_t^{t_1} (\theta(x) + \alpha) dx} dt \right. \\ & \left. - (c + C_d) \left(e^{\int_0^{t_1} (\theta(x) + \alpha) dx} - 1 \right) + C_s (T - t_1) \right\} \quad (11) \end{aligned}$$

Based on Equation (11), we assume an auxiliary function, say $m(t_1)$, to simplify the later discussion, as follows

$$m(t_1) = (s\alpha - C_h + \alpha C_d) \int_0^{t_1} e^{\int_t^{t_1} (\theta(x) + \alpha) dx} dt - (c + C_d) \left(e^{\int_0^{t_1} (\theta(x) + \alpha) dx} - 1 \right) + C_s (T - t_1) \quad (12)$$

such that

$$\frac{d}{dt_1} Z(t_1) = \frac{R(t_1)}{T} m(t_1) \quad (13)$$

for $0 \leq t_1 \leq T$. Hence, $\frac{d}{dt_1} Z(t_1)$ and $m(t_1)$ have the same roots.

From Equation (12), it yields

$$\begin{aligned} m'(t_1) = & (s\alpha - C_h + \alpha C_d) \left(1 + (\theta(t_1) + \alpha) \int_0^{t_1} e^{\int_t^{t_1} (\theta(x) + \alpha) dx} dt \right) \\ & - (c + C_d) (\theta(t_1) + \alpha) e^{\int_0^{t_1} (\theta(x) + \alpha) dx} - C_s \quad (14) \end{aligned}$$

We rewrite Equation (14) as

$$\begin{aligned} m'(t_1) = & \alpha(s + C_d) - C_h - C_s \\ & + (\theta(t_1) + \alpha) \left\{ (s\alpha + \alpha C_d - C_h) \int_0^{t_1} e^{\int_t^{t_1} (\theta(x) + \alpha) dx} dt - (c + C_d) e^{\int_0^{t_1} (\theta(x) + \alpha) dx} \right\} \quad (15) \end{aligned}$$

We also compute

$$m(0) = C_s T > 0 \quad (16)$$

to

$$m(T) = (\alpha(s + C_d) - C_h) \int_0^{t_1} e^{\int_t^{t_1} (\theta(x) + \alpha) dx} dt - (c + C_d) \left(e^{\int_0^{t_1} (\theta(x) + \alpha) dx} - 1 \right) < 0 \quad (17)$$

From Equations (15) and (17), we notice that without giving an explicit expression of the deterioration function $\theta(t)$, researchers cannot compute those integrations related to $\frac{d}{dt}m(t)$. The alternative approach is to consider numerical support to face the monotonic problem of $\frac{d}{dt}m(t)$ and $m(T) < 0$.

We can say that the most feasible way to prove that $m(t_1) = 0$ has a unique solution for $0 < t_1 < T$ is to show that $m(t_1)$ is a strictly decreasing function. Hence, we try to find conditions to guarantee $\frac{d}{dt_1}m(t_1) < 0$ and $m(T) < 0$.

We check the coefficients in Equations (15) and (17) to find that

$$\alpha(s + C_d) - C_h < 0 \quad (18)$$

and

$$\alpha(s + C_d) - C_h - C_s < 0 \quad (19)$$

Owing to the inequality in Equation (18), the inequality in Equation (19) is implied, so we will try to verify that the inequality in Equation (18) is valid.

Consequently, we can derive our desired result of the existence and uniqueness of only one root for $m(t_1)$, which is the optimal solution for $Z(t_1)$, which is the unique solution for $\frac{d}{dt_1}Z(t_1) = 0$.

Since our proposed model is new, in the literature, there are no inventory models that have considered those parameters altogether. Hence, we have to break our condition into two parts:

$$\alpha s < 0.5C_h \quad (20)$$

and

$$\alpha C_d < 0.5C_h \quad (21)$$

The reason we separate Equation (18) into two parts is explained below.

We recall that in the numerical examples of Wu et al. [10] and Chuang [11], $\alpha = 0.01$, $s = 20$, $C_h = 3$, such that

$$\alpha s = 0.2 < 0.5C_h = 1.5 \quad (22)$$

Therefore, our assertion of Equation (21) is supported by the numerical examples of Wu et al. [10] and Chuang [11].

On the other hand, Wu et al. [10] did not include the deterioration cost per unit, C_d , in their inventory models. Chuang [11] has already developed an inventory model with the deterioration cost. However, Chuang [11] did not inform us of his deterioration cost. In comparison with Wu et al. [10], in the numerical examples of Chuang [11], he still assumed that $C_d = 0$. Hence, we must refer to another paper, Lin [30], and his numerical example, $C_d = 3$ and $C_h = 10$, to imply that

$$\alpha C_d = 0.03 < 0.5C_h = 5 \quad (23)$$

Thus, our claim of Equation (22) is confirmed by the numerical example of Lin [30].

Hence, our extra condition of Equation (18) is supported by the numerical examples of Wu et al. [10], Lin [30], and Chuang [11].

Since exponential functions are positive, we imply that those two integrations of exponential functions are also positive. The two coefficients before integrations are negative. We combine our observations to derive

$$m'(t_1) < 0 \quad (24)$$

We show that $m(t_1)$ is a decreasing function from $m(0) > 0$ to $m(T) < 0$.

Hence, there is a unique point, say t_1^* , that is satisfying $m(t_1^*) = 0$. We recall that $m(t_1)$ and $Z'(t_1)$ have the same sign, such that

$$Z'(t_1) > 0, \text{ for } 0 < t_1 < t_1^* \quad (25)$$

and

$$Z'(t_1) < 0, \text{ for } t_1^* < t_1 < T \quad (26)$$

We prove that t_1^* is the optimal solution for our maximum-profit inventory model. Equation (12) reveals that the demand rate, $R(t_1)$, is not included in the expression of $m(t_1)$, so the optimal solution is independent of the demand rate. This finding is consistent with the results of Lin [30] and Hung [31], which are minimum-cost inventory models, that is, the optimal solution is independent of the demand pattern.

4. Numerical Examples

We recall numerical examples in Chuang [11] with the following data: the setup (the ordering) cost $A = 50$, the holding cost $C_h = 3$, the shortage cost $C_s = 5$, the purchasing cost $c = 15$, the actual sale rate $D_0 = 400$, the selling price $s = 20$, the planning horizon $T = 1$, the ratio of extra selling related to the inventory level $\alpha = 0.01$, and the deterioration rate $\theta = 0.05$. In Chuang [11], his inventory model did not include the deterioration cost. Hence, we refer to Lin [30] to adopt $C_d = 3$. To check the influence of the variation for parameters, we examine a detailed sensitivity analysis to alter the values from the decreasing percentages of 30%, 20%, and 10%, to increasing percentages of 10%, 20%, and 30% for the setup cost A , the holding cost C_h , the shortage cost C_s , the purchasing cost c , the selling price s , the planning horizon T , the ratio of extra selling related to the inventory level α , and the deterioration function θ . We list the results for variation in the setup cost in Table 2.

Table 2. Variation in the setup cost A .

A	−30%	−20%	−10%	base	+10%	+20%	+30%
t_1^*	0.5608	0.5608	0.5608	0.5608	0.5608	0.5608	0.5608

Our findings in Table 2 show that the optimal solution of t_1^* is independent of the setup cost. Our findings for variation in the holding cost are listed in Table 3.

Table 3. Variation in the holding cost C_h .

C_h	−30%	−20%	−10%	base	+10%	+20%	+30%
t_1^*	0.6245	0.6017	0.5805	0.5608	0.5424	0.5252	0.5090

Our results in Table 3 show that the optimal solution of t_1^* has a negative relationship with the holding cost. Our results for variation in the shortage cost are presented in Table 4.

Table 4. Variation in the shortage cost C_s .

C_s	−30%	−20%	−10%	base	+10%	+20%	+30%
t_1^*	0.4726	0.5058	0.5349	0.5608	0.5840	0.6048	0.6236

Our findings in Table 4 show the optimal solution t_1^* has a positive relationship with the shortage cost. Our findings for variation in the purchasing cost are shown in Table 5.

Table 5. Variation in the purchasing cost c .

c	−30%	−20%	−10%	base	+10%	+20%	+30%
t_1^*	0.5785	0.5725	0.5666	0.5608	0.5552	0.5497	0.5442

Our results in Table 5 they assert that the optimal solution t_1^* has a negative relationship with the purchasing cost. Our results for variation in the selling price are listed in Table 6.

Table 6. Variation in the selling price s .

s	−30%	−20%	−10%	base	+10%	+20%	+30%
t_1^*	0.5570	0.5583	0.5596	0.5608	0.5621	0.5634	0.5647

Our results in Table 6 denote that the optimal solution t_1^* has a positive relationship with the selling price. Our findings for variation in the planning horizon are expressed in Table 7.

Table 7. Variation in the planning horizon T .

T	−30%	−20%	−10%	base	+10%	+20%	+30%
t_1^*	0.3934	0.4493	0.5051	0.5608	0.6164	0.6720	0.7275

Our findings in Table 7 imply that the optimal solution t_1^* has a positive relationship with the planning horizon. Our results for variation in the ratio of extra selling are denoted in Table 8.

Table 8. Variation in the ratio of extra selling α .

θ	−30%	−20%	−10%	base	+10%	+20%	+30%
t_1^*	0.5601	0.5603	0.5606	0.5608	0.5611	0.5613	0.5616

Our results in Table 8 claim that the optimal solution t_1^* has a positive relationship with the ratio of extra selling. Our findings for the results for variation in the deterioration function are shown in Table 9.

Table 9. Variation in the deterioration function θ .

θ	−30%	−20%	−10%	base	+10%	+20%	+30%
t_1^*	0.5796	0.5732	0.5669	0.5608	0.5548	0.5489	0.5432

Our findings in Table 9 point out that the optimal solution t_1^* has a negative relationship with the deterioration function. Our results for variation in the deterioration cost are expressed in Table 10.

Table 10. Variation in the deterioration cost C_d .

C_d	−30.	−20%	−10%	base	+10%	+20%	+30%
t_1^*	0.5637	0.5627	0.5618	0.5608	0.5599	0.5589	0.5580

Our results in Table 10 claim that the optimal solution t_1^* has a negative relation with the deterioration cost.

5. Application of Our Approach

When $\theta(t)$ is degenerated to a constant function, denoted as θ , we can rewrite Equation (12) as

$$m(t_1) = \left[\frac{\alpha(s-c) - C_h - \theta(c + C_d)}{\theta + \alpha} \right] \left(e^{(\theta+\alpha)t_1} - 1 \right) + C_s(T - t_1) \quad (27)$$

which is the same result as the criterion of Chuang [11] for the optimal solution, revealing that Chuang [11] is a special case of our model.

To be comparable with Wu et al. [10], we consider the same numerical examples as their Example 1, $A = 50$, $C_h = 3$, $C_s = 5$, $D_0 = 400$, $T = 1$, $c = 15$, $s = 20$, $\alpha = 0.01$, and $\theta = 0.05$.

Wu et al. [10] did not consider the deterioration cost. In comparing our work with that of Wu et al. [10], we suppose $C_d = 0$ in the following discussion.

Wu et al. [10] studied a ramp-type demand as

$$D(t) = \begin{cases} D_0 t, & 0 \leq t \leq \mu, \\ D_0 \mu, & \mu \leq t \leq T. \end{cases} \quad (28)$$

Depending on the relationship between μ and t_1 , Wu et al. [10] partitioned their model into two cases: Case A: $t_1 \geq \mu$, and Case B: $\mu \leq t_1$. Consequently, they constructed their profit function $Z_1(t_1)$ and $Z_2(t_1)$ for Cases A, and B, respectively.

For Case A, Wu et al. [10] obtained

$$\frac{\alpha(s-c) - (C_h + \theta c)}{\theta + \alpha} \left[e^{(\theta+\alpha)t_1} - 1 \right] + C_s(T - t_1) = 0 \quad (29)$$

to decide their optimal solution for $\mu \leq t_1 \leq T$.

For Case B, Wu et al. [10] derived the same result as Equation (29) to decide their optimal solution for $0 \leq t_1 \leq \mu$.

Wu et al. [10] did not explain why their results for Cases A and B are identical under two different domains. They examined three examples for their inventory model with (i) Example 1, $\mu = 0.4$, (ii) Example 2, $\mu = 0.5953$, and (iii) Example 3, $\mu = 0.6$.

For Example 1, Wu et al. [10] claimed that the local maximum solution for $Z_1(t_1)$ is μ and the local maximum solution for $Z_2(t_1)$ is $t_1^* = 0.5953$, with

$$Z_2(t_1^* = 0.5953) > Z_2(\mu = 0.4) = Z_1(\mu = 0.4) \quad (30)$$

For Example 2, Wu et al. [10] found that the local maximum solution for $Z_1(t_1)$ is μ and the local maximum solution for $Z_2(t_1)$ is μ , with

$$Z_2(\mu = 0.5953) = Z_1(\mu = 0.5953) \quad (31)$$

For Example 3, Wu et al. [10] found that the local maximum solution for $Z_1(t_1)$ is $t_1^* = 0.5953$ and the local maximum solution for $Z_2(t_1)$ is μ , with

$$Z_2(\mu = 0.6) = Z_1(\mu = 0.6) < Z_1(t_1^* = 0.5953) \quad (32)$$

Wu et al. [10] were not aware that their three examples had the same global maximum point, $t_1^* = 0.5953$. Hence, Wu et al. [10] did not know that the maximum solution for these kinds of inventory models is independent of the demand type such that there are two divisions: the first one is to construct $Z_1(t_1)$ and $Z_2(t_1)$ for Cases A and B, respectively, concerning the relation between t_1 and μ , which is unnecessary. The second one is to develop three examples, based on the relation t_1^* and μ , which is redundant. Hence, applying our approach, researchers can solve this kind of inventory model without those lengthy derivations proposed by Wu et al. [10].

At last, but not least, Wu et al. [10] and Chuang [11] mentioned that

$$t_1^* = 0.5953 \quad (33)$$

In the following, we list four related values to clearly indicate the values of $m(t_1)$ decreasing from positive to negative. Hence, our solution is the right result.

On the other hand, we derive that $m(0.570511) = 9.922 \times 10^{-6}$, $m(0.570512) = 1.093 \times 10^{-6}$, $m(0.570513) = -7.736 \times 10^{-6}$, and $m(0.5953) = -0.219$ to indicate that the maximum solution should be revised as

$$t_1^* = 0.570512 \quad (34)$$

to show that the work of Wu et al. [10] and Chuang [11] needs revisions.

We can also apply our analytical approach to consider the deterioration rate function as a linear function such that $\theta(t) = a + bt$, and then rewrite Equation (12) as follows:

$$\begin{aligned} m(t_1) &= (s\alpha - C_h + \alpha C_d) \int_0^{t_1} e^{\int_t^{t_1} (a+\alpha+bx)dx} dt - (c + C_d) \left(e^{\int_0^{t_1} (a+\alpha+bx)dx} - 1 \right) + C_s(T - t_1) \\ &= (s\alpha - C_h + \alpha C_d) e^{(a+\alpha)t_1 + (bt_1^2/2)} \int_0^{t_1} e^{-(a+\alpha)t - (bt^2/2)} dt \\ &\quad - (c + C_d) \left(e^{(a+\alpha)t_1 + (bt_1^2/2)} - 1 \right) + C_s(T - t_1) \end{aligned} \quad (35)$$

Under the condition of $\theta(t) = a + bt$, solving $m(t_1) = 0$ will be a challenging problem because, in the integration of Equation (35), the following quadratic form denoted as E, with

$$E = \int_0^{t_1} e^{-(a+\alpha)t - (bt^2/2)} dt \quad (36)$$

appeared.

For a linear term in the exponential function, researchers can derive that

$$\int_0^{t_1} e^{-(a+\alpha)t} dt = \frac{1}{a+\alpha} \left[1 - e^{-(a+\alpha)t_1} \right] \quad (37)$$

to indicate that when the deterioration function is a constant term as $\theta(t) = a$, then the integration can be directly solved as Equation (37).

When we extend our inventory model for the deterioration from a constant term to a linear function as $\theta(t) = a + bt$, the integration problem of Equation (36) cannot be directly solved to an exact result.

Therefore, for a quadratic form in the exponential function of Equation (36), we change the expression of Equation (36) to

$$\int_0^{t_1} e^{-(a+\alpha)t - (bt^2/2)} dt = \sqrt{\frac{2}{b}} e^{(a+\alpha)^2/2b} \int_{(a+\alpha)/\sqrt{2b}}^{(\sqrt{b/2})(t_1 + ((a+\alpha)/b))} e^{-y^2} dy \quad (38)$$

with

$$y = \sqrt{\frac{b}{2}} \left(t + \frac{a+\alpha}{b} \right) \quad (39)$$

and

$$\frac{-b}{2} t^2 - (a+\alpha)t = - \left(\sqrt{\frac{b}{2}} \left(t + \frac{a+\alpha}{b} \right) \right)^2 + \frac{(a+\alpha)^2}{2b} \quad (40)$$

For the improper integration of $\int_0^\infty e^{-y^2} dy$, researchers know that

$$\int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2} \quad (41)$$

However, for another definite integration, for example, that denoted as F , with

$$F = \int_3^5 e^{-y^2} dy, \quad (42)$$

researchers cannot find the exact solution for Equation (42). To the best of our knowledge, many practitioners have applied Taylor's series expansion of the exponential function to derive an infinite series for the definite integration of Equation (42)

$$\begin{aligned} \int_3^5 e^{-y^2} dy &= \int_3^5 \sum_{k=0}^{\infty} \frac{(-y^2)^k}{k!} dy \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{5^{2k+1} - 3^{2k+1}}{2k+1} \right) \end{aligned} \quad (43)$$

Depending on the accuracy, practitioners selected a number, denoted as " m ", to accept that

$$\int_3^5 e^{-y^2} dy \approx \sum_{k=0}^m \frac{(-1)^k}{k!} \left(\frac{5^{2k+1} - 3^{2k+1}}{2k+1} \right) \quad (44)$$

where the estimation error is less than $\frac{5^{2m+3} - 3^{2m+3}}{(m+1)!(2m+3)}$, because the infinite series on the right-hand side of Equation (44) is an alternative series.

The second way to solve the value of Equation (42) is to consider the well-known fact that finite integration is the limit of the Riemann sum. It shows that

$$\int_3^5 e^{-y^2} dy = \lim_{n \rightarrow \infty} \sum_{k=1}^n e^{-(3+\frac{2k}{n})^2} \frac{2}{n}. \quad (45)$$

Based on the above discussion, in the beginning, we will adopt the second approach to solve $m(t_1) = 0$, when $\theta(t) = a + bt$.

We construct an example for a linear deterioration function with $a = 0.05$ and $b = 0.1$ to extend the deterioration function to a linear form such that $A = 50$, $C_d = 3$, $C_h = 3$, $C_s = 5$, $D_0 = 400$, $T = 1$, $c = 15$, $s = 20$, $\alpha = 0.01$, $\theta = 0.05$, and $\theta(t) = 0.05 + 0.1t$.

For a given value of t_1 , we uniformly partition the integration interval $[0, t_1]$ into twenty subintervals, and then we estimate $m(t_1)$ using the following approximation:

$$\begin{aligned} m(t_1) \approx & (s\alpha - C_h + \alpha C_d) e^{(a+\alpha)t_1 + (b t_1^2/2)} \frac{t_1}{20} \sum_{k=1}^{20} e^{-(a+\alpha)\frac{k}{20}t_1 - \frac{b}{2}(\frac{k}{20}t_1)^2} \\ & - (c + C_d) \left(e^{(a+\alpha)t_1 + (b t_1^2/2)} - 1 \right) + C_s(T - t_1) \end{aligned} \quad (46)$$

We begin to search for the optimal solution, t_1^* , which is the solution of $m(t_1) = 0$, with the knowledge that $m(0) > 0$, $m(T = 1) < 0$, and $m(t_1)$ is a decreasing function.

Many methods can locate the value of the optimal solution, t_1^* . In the following, we will use a direct numerical method to list values of $m(t_1)$ that will clearly reveal that $m(t_1)$ is a decreasing function from a positive value to a negative value, and then we can find the solution of $m(t_1) = 0$.

For $t_1 = 0, 0.1, \dots, 1$, we list the estimation result of $m(t_1)$ in the following Table 11.

Table 11. The estimation result for $m(t_1)$ with $t_1 = 0, 0.1, \dots, 1$.

t_1	0	0.1	0.2	0.3	0.4	0.5
$m(t_1)$	5.000	4.105	3.188	2.250	1.288	0.302
t_1	0.6	0.7	0.8	0.9	1	
$m(t_1)$	−0.709	−1.747	−2.814	−3.910	−5.038	

Based on Table 11, we know that the optimal solution, t_1^* , satisfies

$$0.5 < t_1^* < 0.6 \quad (47)$$

Hence, for $t_1 = 0.50, 0.51, \dots, 0.6$, we list the estimation result of $m(t_1)$ in Table 12.

Table 12. The estimation result for $m(t_1)$, with $t_1 = 0.50, 0.51, \dots, 0.6$.

t_1	0.50	0.51	0.52	0.53	0.54	0.55
$m(t_1)$	0.302	0.202	0.102	1.518×10^{-3}	−0.099	−0.200
t_1	0.56	0.57	0.58	0.59	0.60	
$m(t_1)$	−0.302	−0.403	−0.505	−0.607	−0.709	

Based on Table 12, we know that the optimal solution, t_1^* , satisfies

$$0.53 < t_1^* < 0.54 \quad (48)$$

Hence, for $t_1 = 0.530, 0.531, \dots, 0.540$, we list the estimation result of $m(t_1)$ in Table 13.

Table 13. The estimation result for $m(t_1)$, with $t_1 = 0.530, 0.531, \dots, 0.540$.

t_1	0.530	0.531	0.532	0.533	0.534	0.535
$m(t_1)$	1.518×10^{-3}	-8.54×10^{-3}	−0.019	−0.029	−0.039	−0.049
t_1	0.536	0.537	0.538	0.539	0.540	
$m(t_1)$	−0.059	−0.069	−0.079	−0.089	−0.099	

Based on Table 13, we know that the optimal solution, t_1^* , satisfies

$$0.530 < t_1^* < 0.531 \quad (49)$$

Hence, for $t_1 = 0.5300, 0.5301, \dots, 0.5310$, we list the estimation result of $m(t_1)$ in Table 14.

Table 14. The estimation result for $m(t_1)$, with $t_1 = 0.5300, 0.5301, \dots, 0.5310$.

t_1	0.5300	0.5301	0.5302	0.5303	0.5304	0.5305
$m(t_1)$	1.518×10^{-3}	5.11×10^{-4}	-4.95×10^{-4}	-1.50×10^{-3}	-2.50×10^{-3}	-3.51×10^{-3}
t_1	0.5306	0.5307	0.5308	0.5309	0.5310	
$m(t_1)$	-4.52×10^{-3}	-5.53×10^{-3}	-6.53×10^{-3}	-7.54×10^{-3}	-8.54×10^{-3}	

Based on Table 14, we know that the optimal solution, t_1^* , satisfies

$$0.5301 < t_1^* < 0.5302 \quad (50)$$

We compare

$$m(0.5301) = 5.113 \times 10^{-4} \quad (51)$$

and

$$m(0.5302) = -4.949 \times 10^{-4}, \quad (52)$$

to select

$$t_1^* = 0.5302 \quad (53)$$

6. Conclusions

In this paper, we constructed a maximum-profit inventory model to show that the optimal solution is independent of the demand such that the deterioration rate is generalized to the general expression. We show that without writing the implicit expression of demand function and deterioration function, through our approach, complicated solution procedures with several different objective functions can be avoided. Our paper is not only an extension of the work of Wu et al. [10] and Chuang [11] but also provides a complete solution structure to find the optimal solution.

Author Contributions: Conceptualization, Y.-L.W.; methodology, Y.-L.W.; software, Y.-L.W. and M.-L.C.; validation, P.J.; formal analysis, Y.-L.W.; investigation, Y.-L.W.; resources, M.-L.C.; data curation, M.-L.C.; writing—original draft preparation, Y.-L.W.; writing—review and editing, Y.-L.W.; visualization, M.-L.C.; supervision, Y.-L.W.; project administration, Y.-L.W.; funding acquisition, Y.-L.W. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external financial support.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest: The authors declare no conflict of interest.

References

- Hill, R.M. Inventory models for increasing demand followed by level demand. *J. Oper. Res. Soc.* **1995**, *46*, 1250–1259. [\[CrossRef\]](#)
- Mandal, B.; Pal, A.K. Order level inventory system with ramp type demand rate for deteriorating items. *J. Interdiscipl. Math.* **1998**, *1*, 49–66. [\[CrossRef\]](#)
- Wu, J.W.; Lin, C.; Tan, B.; Lee, W.C. An EOQ model with ramp type demand rate for items with Weibull deterioration. *Int. J. Info. Manag. Sci.* **1999**, *10*, 41–51.
- Wu, K.S.; Ouyang, L.Y. A replenishment policy for deteriorating items with ramp type demand rate. *Proc.-Natl. Sci. Counc. Repub. China Part A Phys. Sci. Eng.* **2000**, *24*, 279–286.
- Wu, K.S. An EOQ inventory model for items with Weibull distribution deterioration, ramp type demand rate and partial backlogging. *Prod. Plan. Control* **2001**, *12*, 787–793. [\[CrossRef\]](#)
- Giri, B.C.; Jalan, A.K.; Chaudhuri, K.S. Economic Order Quantity model with Weibull deterioration distribution, shortage and ramp-type demand. *Int. J. Syst. Sci.* **2003**, *34*, 237–243. [\[CrossRef\]](#)
- Deng, P.S. Improved inventory models with ramp type demand and Weibull deterioration. *Int. J. Inf. Manag. Sci.* **2005**, *16*, 79–86.
- Manna, S.K.; Chaudhuri, K.S. An EOQ model with ramp type demand rate, time dependent deterioration rate, unit production cost and shortages. *Eur. J. Oper. Res.* **2006**, *171*, 557–566. [\[CrossRef\]](#)
- Deng, P.S.; Lin, R.; Chu, P. A note on the inventory models for deteriorating items with ramp type demand rate. *Eur. J. Oper. Res.* **2007**, *178*, 112–120. [\[CrossRef\]](#)
- Wu, K.S.; Ouyang, L.Y.; Yang, C.T. Retailer's optimal ordering policy for deteriorating items with ramp-type demand under stock-dependent consumption rate. *Int. J. Inf. Manag. Sci.* **2008**, *19*, 245–262.
- Chuang, J.P.C. Inventory model with stock dependent selling rate. *Afr. J. Bus. Manag.* **2012**, *6*, 780–786.
- Lin, R.; Lin, J.S.J.; Chen, K.; Julian, P.C. Note on inventory model with net present value. *J. Interdiscip. Math.* **2007**, *10*, 587–592. [\[CrossRef\]](#)
- Mahajan, S.; Diatha, K. Minimizing the Discounted Average Cost Under Continuous Compounding in the EOQ Models with a Regular Product and a Perishable Product. *Am. J. Oper. Manag. Inf. Syst.* **2018**, *3*, 52–60. [\[CrossRef\]](#)
- Alim, M.; Beullens, P. Joint inventory and distribution strategy for online sales with a flexible delivery option. *Int. J. Prod. Econ.* **2020**, *222*, 107487. [\[CrossRef\]](#)
- Khakzad, A.; Gholamian, M.R. The effect of inspection on deterioration rate: An inventory model for deteriorating items with advanced payment. *J. Clean. Prod.* **2020**, *254*, 120117. [\[CrossRef\]](#)

16. Khan, M.A.A.; Ahmed, S.; Babu, M.S.; Sultana, N. Optimal lot-size decision for deteriorating items with price-sensitive demand, linearly time-dependent holding cost under all-units discount environment. *Int. J. Syst. Sci. Oper. Logist.* **2020**, *9*, 61–74. [\[CrossRef\]](#)
17. Rezagholifam, M.; Sadjadi, S.J.; Heydari, M.; Karimi, M. Optimal pricing and ordering strategy for non-instantaneous deteriorating items with price and stock sensitive demand and capacity constraint. *Int. J. Syst. Sci. Oper. Logist.* **2020**, *9*, 121–132. [\[CrossRef\]](#)
18. Sarkar, S.; Giri, B.C.; Sarkar, A.K. A vendor-buyer inventory model with lot-size and production rate dependent lead time under time value of money. *RAIRO-Oper. Res.* **2020**, *54*, 961–979. [\[CrossRef\]](#)
19. Tiwari, S.; Cárdenas-Barrón, L.E.; Shaikh, A.A.; Goh, M. Retailer's optimal ordering policy for deteriorating items under order-size dependent trade credit and complete backlogging. *Comput. Ind. Eng.* **2020**, *139*, 105559. [\[CrossRef\]](#)
20. Padiyar, S.V.S.; Singh, S.R.; Punetha, N. Inventory system with price dependent consumption for deteriorating items with shortages under fuzzy environment. *Int. J. Sust. Agric. Manag. Inf.* **2021**, *7*, 218–231. [\[CrossRef\]](#)
21. Acevedo-Chedid, J.; Salas-Navarro, K.; Ospina-Mateus, H.; Villalobo, A.; Sana, S.S. Production system in a collaborative supply chain considering deterioration. *Int. J. Appl. Comput. Math.* **2021**, *7*, 69. [\[CrossRef\]](#)
22. Mishra, U.; Mashud, A.H.M.; Tseng, M.-L.; Wu, J.-Z. Optimizing a sustainable supply chain inventory model for controllable deterioration and emission rates in a greenhouse farm. *Mathematics* **2021**, *9*, 495. [\[CrossRef\]](#)
23. Mashud, A.H.M.; Roy, D.; Daryanto, Y.; Ali, M.H. A sustainable inventory model with imperfect products, deterioration, and controllable emissions. *Mathematics* **2020**, *8*, 2049. [\[CrossRef\]](#)
24. Mishra, U.; Cárdenas-Barrón, L.E.; Tiwari, S.; Shaikh, A.A.; Graza, G.T. An inventory model under price and stock dependent demand for controllable deterioration rate with shortages and preservation technology investment. *Ann. Oper. Res.* **2017**, *254*, 165–190. [\[CrossRef\]](#)
25. Tiwari, S.; Cárdenas-Barrón, L.E.; Goh, M.; Shaikh, A.A. Joint pricing and inventory model for deteriorating items with expiration dates and partial backlogging under two-level partial trade credits in supply chain. *Int. J. Prod. Econ.* **2018**, *200*, 16–36. [\[CrossRef\]](#)
26. Khan, A.A.; Shaikh, A.A.; Panda, G.C.; Konstantaras, I.; Taleizadeh, A.A. Inventory system with expiration date: Pricing and replenishment decisions. *Comput. Ind. Eng.* **2019**, *132*, 232–247. [\[CrossRef\]](#)
27. Padiyar, S.V.S.; Kuraie, V.C.; Bhagat, N.; Singh, S.R.; Chaudhary, R. An integrated inventory model for imperfect production process having preservation facilities under fuzzy and inflationary environment. *Int. J. Math. Model. Numer. Optim.* **2022**, *12*, 252–286. [\[CrossRef\]](#)
28. Yadav, D.; Singh, S.R.; Kumar, S.; Cárdenas-Barrón, L.E. Manufacturer-retailer integrated inventory model with controllable lead time and service level constraint under the effect of learning-forgetting in setup cost. *Sci. Iran.* **2022**, *29*, 800–815.
29. Navarro, K.S.; Chedid, J.A.; Florez, W.F.; Mateus, H.O.; Cárdenas-Barrón, L.E.; Sana, S.S. A collaborative EPQ inventory model for a three-echelon supply chain with multiple products considering the effect of marketing effort on demand. *J. Ind. Manag. Optim.* **2020**, *16*, 1613–1633. [\[CrossRef\]](#)
30. Lin, S.W. Inventory models with managerial policy independent of demand. *Eur. J. Oper. Res.* **2011**, *211*, 520–524. [\[CrossRef\]](#)
31. Hung, K.C. An inventory model with generalized type demand, deterioration and backorder rates. *Eur. J. Oper. Res.* **2011**, *208*, 239–242. [\[CrossRef\]](#)