

Article

Pattern Dynamics of Cross Diffusion Predator–Prey System with Strong Allee Effect and Hunting Cooperation

Meng Zhu ¹, Jing Li ² and Xinze Lian ^{2,*}¹ College of Mathematics and Physics, Wenzhou University, Wenzhou 325000, China² School of Data Science and Artificial Intelligence, Wenzhou University of Technology, Wenzhou 325000, China

* Correspondence: xinzelian@163.com or xinzelian@wzu.edu.cn

Abstract: In this paper, we consider a Leslie–Gower cross diffusion predator–prey model with a strong Allee effect and hunting cooperation. We mainly investigate the effects of self diffusion and cross diffusion on the stability of the homogeneous state point and processes of pattern formation. Using eigenvalue theory and Routh–Hurwitz criterion, we analyze the local stability of positive equilibrium solutions. We give the conditions of Turing instability caused by self diffusion and cross diffusion in detail. In order to discuss the influence of self diffusion and cross diffusion, we choose self diffusion coefficient and cross diffusion coefficient as the main control parameters. Through a series of numerical simulations, rich Turing structures in the parameter space were obtained, including hole pattern, strip pattern and dot pattern. Furthermore, We illustrate the spatial pattern through numerical simulation. The results show that the dynamics of the model exhibits that the self diffusion and cross diffusion control not only form the growth of dots, stripes, and holes, but also self replicating spiral pattern growth. These results indicate that self diffusion and cross diffusion have important effects on the formation of spatial patterns.

Keywords: predator–prey; strong Allee effect; hunting cooperation; self diffusion; cross diffusion

MSC: 39A30; 35G31; 65M20



Citation: Zhu, M.; Li, J.; Lian, X. Pattern Dynamics of Cross Diffusion Predator–Prey System with Strong Allee Effect and Hunting Cooperation. *Mathematics* **2022**, *10*, 3171. <https://doi.org/10.3390/math10173171>

Academic Editor: Youssef Raffoul

Received: 19 July 2022

Accepted: 29 August 2022

Published: 3 September 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

The study of population dynamics has an important application value in the sustainable utilization of biological resources, diversity protection, infectious disease control, and so on. The best way to understand the dynamic properties of predator–prey model is to establish a mathematical model. The most classical predator–prey models are Leslie–Gower model [1] and Holling–Tanner model [2,3]. In the real world, we can establish a mathematical model to evaluate the pathogenicity and transmissibility of VOC and control the spread of diseases [4]. Ho et al. [5] proposed an effective damage detection method by using the marine predator algorithm. In this method, the learning ability of feed-forward neural networks is improved by adopting the optimal foraging strategy and marine memory. In [6], Hamadneh et al. predicted the number of COVID-19 cases in Brazil and Mexico in the coming days by using artificial neural networks and predator–prey algorithms.

In most studies on the predator–prey model, it was assumed that the prey grew in a Logistic model. Due to the effects of factors such as reproduction promotion, mate restriction, cooperative defense, and environmental restriction, Allee [7] proposed that the prey may have the growth rate of Allee effect [8–14]. The Allee effect is mainly divided into a strong Allee effect [12–14] and a weak Allee effect [9]. Ni and Wang [14] considered the influence of strong Allee effect and proposed Leslie–Gower predator–prey system with a strong Allee effect on prey. It is well known that a key factor affecting predator–prey interactions is the functional response [15–23]. In order to effectively capture isolated prey or groups of prey, predators usually adopt a cooperative approach during hunting, such

as ants and birds. Therefore, Alves and Hilker [24] proposed the functional response of hunting cooperation. The influence of hunting cooperation in predator–prey system has attracted the interest of many scholars [25–28].

Combining the contents of literature [14,24], Ye and Wu [29] proposed a Leslie–Gower predator–prey model with strong Allee effect and hunting cooperation. The specific form is as follows:

$$\begin{cases} \frac{du}{dt} = u(1-u)(u-b) - (1+av)uv, \\ \frac{dv}{dt} = sv(1 - \frac{v}{cu}), \end{cases} \quad (1)$$

where u and v are prey and predator densities, respectively; a is a parameter describing the cooperation of predators in hunting; b indicates the strength of the prey affected by Allee effect, where $0 < b < 1$; c is a parameter to measure the energy required for the conversion of the prey into the predator; s indicates the growth rate of the predator, and all parameters are positive.

In Ref. [13], Shang and Qiao discussed the existence, dissipation and bifurcation of the solution of the predator–prey model with the Allee effect. In Ref. [24], the authors added a cooperation term to the attack rate of the predator population, and studied the equilibrium stability in bifurcation diagrams and phase plane. In [25], Yan et al. studied the pattern structure of a predator–prey model with hunting cooperation. Wu and Zhao [27] studied the influence of hunting cooperation on predator population and the influence of diffusion coefficient on Turing instability. In Ref. [29], Ye and Wu discussed the stability of degenerate equilibrium and non-negative equilibrium in Leslie–Gower predator–prey model with a strong Allee effect and hunting cooperation, and deduced the existence of saddle node bifurcation.

So far, no scholars have studied the conditions of Turing instability after adding self diffusion term and cross diffusion term to the model (1). No one has studied the influence of self diffusion and cross diffusion on the formation of spatial pattern in model (1). Hence, the purpose of this paper is to improve the contents of [29] and study the effects of self diffusion and cross diffusion on the pattern formation of the model (1). Referring to the stability analysis of Leslie–Gower predator–prey system, we analysis the stability of Leslie–Gower cross diffusion predator–prey system with strong Allee effect and hunting cooperation.

In the real world, species may spread in search of food and shelter. This diffusion will lead to some charming spatial patterns, such as the Turing pattern [25,30–33]. Therefore, by incorporating diffusion into the model (1), the following predator–prey reaction–diffusion model is obtained:

$$\begin{cases} \frac{\partial u}{\partial t} = u(1-u)(u-b) - (1+av)uv + d_1\Delta u, \\ \frac{\partial v}{\partial t} = sv(1 - \frac{v}{cu}) + d_2\Delta v, \end{cases} \quad (2)$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is Laplacian operator in two-dimensional space. d_1 is the diffusion coefficient of prey and d_2 is the diffusion coefficient of predator.

However, the predator–prey system mainly describes the phenomenon of pursuit and evasion. In order to obtain the prey, the predator will continue to shorten the distance with the prey, and the prey will continue to expand the spatial distance with the predator in order to avoid the predator [34,35]. Therefore, more and more scholars begin to devote themselves to the research of the predator–prey model with cross diffusion term, and made

some significant achievements [36–41]. The cross diffusion term was incorporated into the model (2), and the following model was obtained:

$$\begin{cases} \frac{\partial u}{\partial t} = u(1-u)(u-b) - (1+av)uv + d_{11}\Delta u + d_{12}\Delta v, \\ \frac{\partial v}{\partial t} = sv(1-\frac{v}{cu}) + d_{21}\Delta u + d_{22}\Delta v, \end{cases} \quad (3)$$

the parameters d_{11} , d_{22} are called self diffusion coefficient, and d_{12} , d_{21} are cross diffusion coefficient. d_{12} represents the tendency of prey population u to avoid predators, while d_{21} represents the tendency of predator v to pursue prey u . d_{12} and d_{21} can be positive or negative. A positive cross diffusion coefficient indicates that the population moves towards a low-density population, while a negative cross diffusion coefficient indicates that it moves towards a high-density population.

We consider this model in the square domain and impose non-zero initial boundary value conditions and zero boundary conditions:

$$\begin{cases} u(x, y) > 0, (x, y) \in \Omega, \\ v(x, y) > 0, (x, y) \in \Omega, \\ \frac{\partial u}{\partial \bar{n}}|_{(x,y)} = \frac{\partial v}{\partial \bar{n}}|_{(x,y)} = 0, (x, y) \in \partial\Omega, \end{cases} \quad (4)$$

where $(x, y) \in \Omega = [0, L] \times [0, L]$, $\Omega \subset \mathbb{R}^2$ is a bounded region with smooth boundary $\partial\Omega$. L represents the size of the system in the square domain, the vector $\bar{n}$ is the normal vector outside the unit of $\partial\Omega$. The zero boundary condition indicates that the system has no external input, that is to say, we can determine that the model is independent. The objective of this paper is to study the dynamic properties of the model (3), and finally obtain the spatiotemporal pattern through numerical simulation.

The layout of this paper is as follows: in Section 2, we analyze the existence of equilibrium point and the stability of positive equilibrium point of model (2). We use the method of linear stability analysis to deduce the conditions for Turing instability of the system. Finally, the evolution process of the prey is shown by numerical simulation. In Section 3, we obtain the Turing instability conditions of model (3) and draw the Turing pattern induced by cross diffusion of model (3). In addition to Turing pattern, we also obtain the conditions for the generation of spiral pattern, and draw the spiral pattern through a series of numerical simulations. In Section 4, we give our conclusions and remarks.

2. Analysis of Self Diffusion Model

2.1. Existence of Equilibrium Point

It is easy to see that model (1) and model (2) have the same equilibrium point. Now, we consider the case of spatial homogeneous solution and simplify the reaction–diffusion model (2) to an ordinary differential equation model:

$$\begin{cases} \frac{du}{dt} = u(1-u)(u-b) - (1+av)uv = f(u, v), \\ \frac{dv}{dt} = sv(1-\frac{v}{cu}) = g(u, v), \end{cases} \quad (5)$$

Theorem 1. Let $a, c, s > 0$, $0 < b < 1$, $\Omega \subset \mathbb{R}^2$ is a bounded region with smooth boundary $\partial\Omega$, then:

- (1) The model (5) has a semi trivial equilibrium point $E_{01}(b, 0)$.
- (2) The model (5) has a semi trivial equilibrium point $E_{02}(1, 0)$.

- (3) When $c < 1 + b - 2\sqrt{b}$ and $a = \frac{c^2 - 2(1+b)c + (1+b)^2}{4bc^2}$, the model (5) has a unique positive equilibrium point $E^* = (u^*, v^*) = (\frac{2b}{1+b-c}, \frac{2bc}{1+b-c})$.
- (4) When $c < 1 + b - 2\sqrt{b}$ and $a < \frac{c^2 - 2(1+b)c + (1+b)^2}{4bc^2}$, the model (5) has two positive equilibrium points $E_1(u_1, cu_1)$ and $E_2(u_2, cu_2)$. Where $u_1 = \frac{(1+b-c) + \sqrt{\Delta}}{2(1+ac^2)}$, $u_2 = \frac{(1+b-c) - \sqrt{\Delta}}{2(1+ac^2)}$, $\Delta = (1+b-c)^2 - 4b - 4abc^2$.
- (5) When $c < 1 + b - 2\sqrt{b}$ and $a > \frac{c^2 - 2(1+b)c + (1+b)^2}{4bc^2}$ or $c > 1 + b$, the model (5) has no positive equilibrium point.

Proof. First, (1) and (2) are obviously valid. The positive equilibrium point of model (5) is proved below. Obviously, in order to obtain the positive equilibrium point of model (5), the following formula must be satisfied:

$$(1-u)(u-b) - (1+av)v = 0, \quad (6)$$

and

$$1 - \frac{v}{cu} = 0. \quad (7)$$

Substituting (7) into (6), we obtain

$$(1+ac^2)u^2 - (1+b-c)u + b = 0, \quad (8)$$

and

$$\Delta = (1+b-c)^2 - 4b - 4abc^2. \quad (9)$$

Note that when $1+b < c$, Equation (8) has no positive root. Equation (8) has positive roots if, and only if, $0 < c < 1+b$. Let

$$\hat{a} = \frac{c^2 - 2(1+b)c + (1-b)^2}{4bc^2}, f(c) = c^2 - 2(1+b)c + (1-b)^2,$$

then $f(c) = 0$ must have two roots, as shown below:

$$c_1 = 1 + b + 2\sqrt{b}, c_2 = 1 + b - 2\sqrt{b}.$$

If $\hat{a} > 0$, it is satisfied $f(c) > 0$, then there is $0 < c < 1 + b - 2\sqrt{b}$ or $c > 1 + b + 2\sqrt{b}$.

To sum up, Equation (8) has positive roots if, and only if, $0 < c < 1 + b - 2\sqrt{b}$. At this time, there are three situations:

- (a) When $\Delta = 0$, that is $a = \frac{c^2 - 2(1+b)c + (1-b)^2}{4bc^2}$, Equation (8) has a positive root:

$$u = \frac{2b}{1+b-c}.$$

- (b) When $\Delta > 0$, that is $0 < a < \frac{c^2 - 2(1+b)c + (1-b)^2}{4bc^2}$, Equation (8) has two positive roots:

$$u_1 = \frac{(1+b-c) + \sqrt{\Delta}}{2(1+ac^2)}, u_2 = \frac{(1+b-c) - \sqrt{\Delta}}{2(1+ac^2)}.$$

- (c) When $a > \frac{c^2 - 2(1+b)c + (1-b)^2}{4bc^2}$, Equation (8) has no roots.

The proof is completed. $\square$

2.2. Stability of Equilibrium Point

From the perspective of biology, this section considers only the behavior of the positive equilibrium point of the model (5). Taking the positive equilibrium point E_1 as an example, we consider the stability of the positive equilibrium point E_1 of the model. The Jacobian matrix of the model (5) at $E_1(u_1, cu_1)$ is shown as follows:

$$J_{E_1} = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} = \begin{pmatrix} u_1(1+b-2u_1) & -u_1(1+2acu_1) \\ sc & -s \end{pmatrix}. \quad (10)$$

and the trace and determinant of matrix J_{E_1} are as follows:

$$\text{tr}(J_{E_1}) = f_u + g_v,$$

and

$$\det(J_{E_1}) = f_u g_v - f_v g_u.$$

Using the Routh–Hurwitz basis, the following theorem can be obtained.

Theorem 2. Let $s > 0$, $0 < c < 1 + b - 2\sqrt{b}$, $0 < b < 1$ and $0 < a < \frac{c^2 - 2(1+b)c + (1-b)^2}{4bc^2}$, $\Omega \subset \mathbb{R}^2$ is a bounded region with smooth boundary $\partial\Omega$, and if the following conditions hold

$$\text{tr}(J_{E_1}) = f_u + g_v < 0, \quad (11)$$

and

$$\det(J_{E_1}) = f_u g_v - f_v g_u > 0, \quad (12)$$

then the sufficient condition for the positive equilibrium E_1 of the model (5) to be locally asymptotically stable.

Proof. For the positive equilibrium E_1 , there is always $\det(J_{E_1}) = su_1\sqrt{\Delta} > 0$. Therefore, the stability of the positive equilibrium point E_1 is determined by the sign of $\text{tr}(J_{E_1})$. If $\text{tr}(J_{E_1}) > 0$, this means that at least one of the eigenvalues of the matrix J_{E_1} has a positive real part, so the positive equilibrium point E_1 is unstable; If $\text{tr}(J_{E_1}) < 0$, this means that eigenvalues of the matrix J_{E_1} are all negative real parts, that is, the positive equilibrium point E_1 is stable under condition $\text{tr}(J_{E_1}) < 0$.

The proof is completed. $\square$

2.3. Turing Instability

Turing [30] showed that when diffusion occurs, it will break this uniform equilibrium state, make it become non-uniform equilibrium state, and lead to a non-uniform spatial pattern, which is Turing instability. In this case, the pattern formed by the biological model is called the Turing pattern. In other words, diffusion will destabilize the uniform steady state of the coupled reaction–diffusion equation and lead to the generation of spatial heterogeneous patterns. Over the years, Turing's ideas have inspired many biologists, chemists, and mathematicians [31,32,41–44]. Therefore, the purpose of this section is to study the influence of self diffusion on model (5).

Next, we consider the diffusion of the population in space and analyze the stability of the model (2) at the positive equilibrium. It is well known that Turing instability means that the non-spatial model (5) is stable at the positive equilibrium, but for the spatial model (2) is unstable at the positive equilibrium. Obviously, it is known from the previous section that the model (5) is stable at the positive equilibrium point E_1 when the condition (11) and (12) are satisfied. Therefore, in this section, we consider the conditions for the Turing instability of the positive equilibrium E_1 .

The characteristic equation of the model (2) at the positive equilibrium point $E_1(u_1, cu_1)$ is shown as follows:

$$|J_k - \lambda I| = 0, \quad (13)$$

where $J_k = J_{E_1} - \text{diag}(d_1, d_2)k^2$, k is the wave number, d_1, d_2 is the diffusion coefficient, and J_{E_1} is given by the following formula:

$$J_{E_1} = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} = \begin{pmatrix} u_1(1+b-2u_1) & -u_1(1+2acu_1) \\ sc & -s \end{pmatrix}.$$

Solve Equation (13) and obtain the characteristic equation of the original model (2) as follows:

$$\lambda^2 - \text{tr}(J_k)\lambda + \det(J_k) = 0, \quad (14)$$

where

$$\text{tr}(J_k) = \text{tr}(J_{E_1}) - (d_1 + d_2)k^2, \quad (15)$$

and

$$\det(J_k) = \det(J_{E_1}) + d_1d_2k^4 - (d_2f_u + d_1g_v)k^2. \quad (16)$$

Additionally, the roots of Equation (14) produce dispersion relationship:

$$\lambda_{1,2}(J_k) = \frac{1}{2}[\text{tr}(J_k) \pm \sqrt{\text{tr}(J_k)^2 - 4\det(J_k)}].$$

Thus, Turing instability will occur when at least one of (15) and (16) is violated. Obviously, when the condition $f_u + g_v < 0$ is satisfied, the first condition $\text{tr}(J_k) < 0$ is not violated. Therefore, only the violation of the condition $\det(J_k) > 0$ can cause diffusion driven instability. Hence, the instability caused by diffusion drive is given by the following formula:

$$\det(J_k) = \det(J_{E_1}) + d_1d_2k^4 - (d_2f_u + d_1g_v)k^2 < 0,$$

general linear analysis shows that when the following two conditions are met, there is $\det(J_k) < 0$.

$$d_2f_u + d_1g_v > 0, \quad (17)$$

$$(d_2f_u + d_1g_v)^2 > 4d_1d_2(f_ug_v - f_vg_u). \quad (18)$$

Based on the above analysis, We find that the condition of Turing instability in the model (2) is:

$$f_u + g_v < 0, \quad (19)$$

$$f_ug_v - f_vg_u > 0, \quad (20)$$

$$d_2f_u + d_1g_v > 0, \quad (21)$$

$$(d_2f_u + d_1g_v)^2 > 4d_1d_2(f_ug_v - f_vg_u). \quad (22)$$

In Figure 1, we show the bifurcation diagram for model (2) in the $c - d_1$ parameter space. This blue curve represents the Turing bifurcation curve, which satisfies $\det(J_k) = 0$. Use Maple software to analyze the data and draw the picture. The Turing curve separates the $c - d_1$ parameter space into two domains. The point above the curve satisfies $\det(J_k) > 0$, which is the stable region; The lower point satisfies $\det(J_k) < 0$, which is the unstable region.

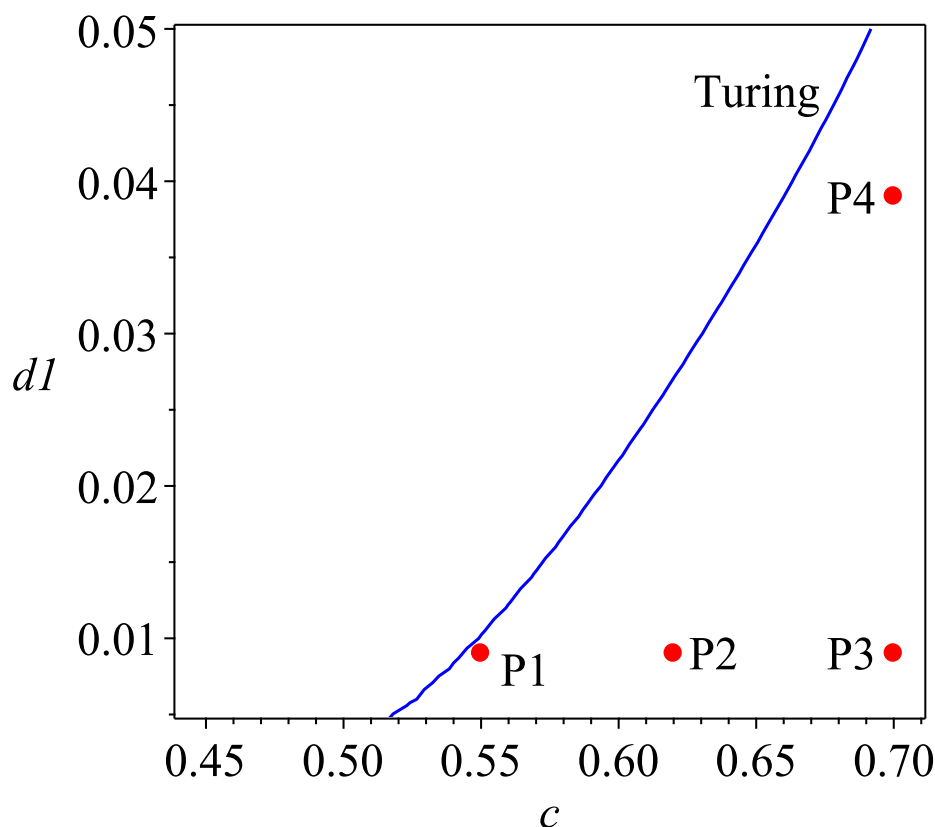


Figure 1. Bifurcation diagram for model (2) in the $c - d_1$ parameter space. The parameters are taken as $a = 0.14$, $b = 0.02$, $s = 0.23$, and $d_2 = 0.24$. The blue curve represents the Turing bifurcation curve, which divides the plane into two regions. The upper part of the blue curve is the stable region and the lower part is the unstable region. P_1 , P_2 , P_3 , and P_4 correspond to the parameter spaces of Figure 2, Figure 3, Figure 4, and Figure 5, respectively.

2.4. Diffusion Induced Pattern

In this section, we use the mathematical software MATLAB to carry out some numerical simulations to verify our theoretical analysis. Our numerical simulation uses a model size of 200×200 , the space step size is $\Delta x = 0.2$, the time step size is $\Delta t = 0.01$, and all the simulations obey Neumann boundary conditions. In this part, we select appropriate parameters that meet conditions (19)–(22), and the pattern generated in this case is called Turing pattern. The fixed parameter values are:

$$a = 0.14, b = 0.02, s = 0.23, d_1 = 0.01, d_2 = 0.24, \quad (23)$$

then, we choose three sets of values for parameter c and observe the effect of c on the pattern. The chosen values of c are 0.55, 0.62, and 0.7, respectively. Obviously, they all satisfy conditions (19)–(22). Taking the random disturbance around the positive equilibrium point as the initial data, a spatial non-uniform steady state is obtained, as shown in Figure 2.

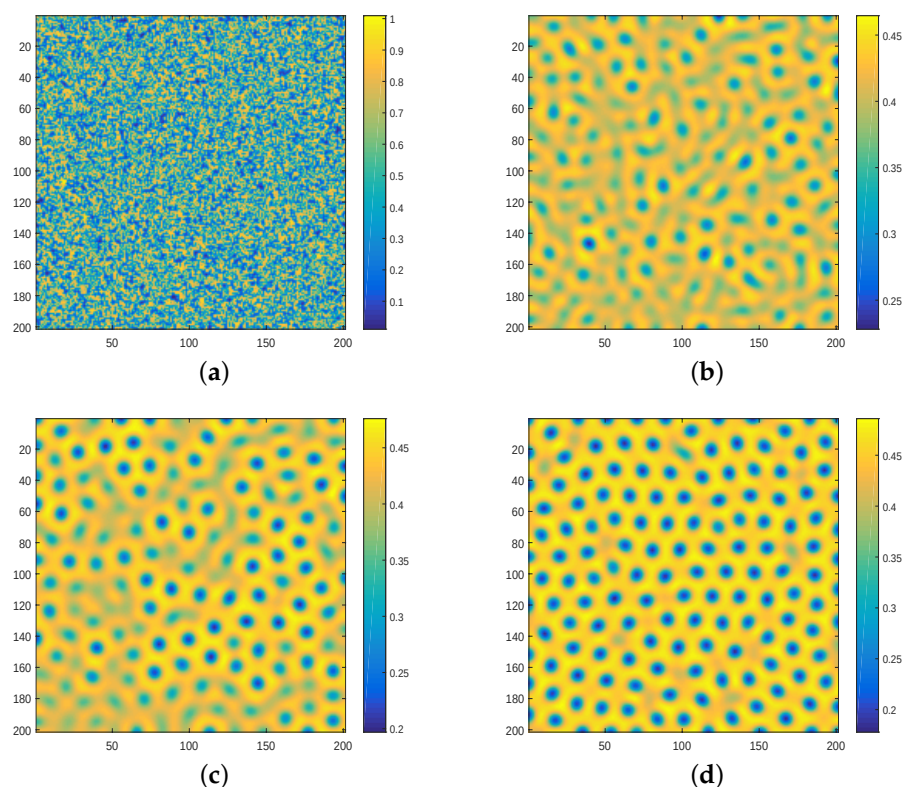


Figure 2. The process of formation of the prey patterns at $c = 0.55$, the other parameters are fixed as in (23). Times: (a) $t = 0$, (b) $t = 150$, (c) $t = 300$, and (d) $t = 1000$.

Figure 2 shows the process of formation of the prey patterns at different moments when $c = 0.55$. Figure 2a–d, respectively, show the evolution pattern of u when $t = 0, 150, 300$, and 1000. The numerical simulation results show that after a period of time, the holes occupy the whole region, the whole model is stable, and the model (2) will eventually converge to a hole distribution.

Figure 3 sets the parameter $c = 0.62$ and keeps other parameter values unchanged. The numerical simulation results are shown in Figure 3. The hole pattern is no longer stable, but converges into the coexistence of hole and strip patterns.

In Figure 4, we show the pattern formation process of the prey population u in the model (2) at $t = 0, 50, 200$, and 500 when the parameter $c = 0.7$. Starting from the random disturbance of the equilibrium point E_1 of the uniform equilibrium state, with the passage of time, the model self-organized to form “strip” patterns (see Figure 4c,d). In Figure 4, we observe that the pattern diffuses from the initial “dot” pattern into a “strip” pattern, and then the line breaks into a “dot” pattern, and, finally, the similar “dot” are gradually connected and merged to form a “linear” pattern, which tends to be stable.

In Figures 2 and 4, we show in detail the evolution process of these different patterns when c is given different values. As can be seen from Figures 2–4, the model gradually tends to be stable after a period of time. It is noted that with the increase in c , the pattern will evolve from stable hole patterns to hole-strip patterns, and finally become stable strip patterns.

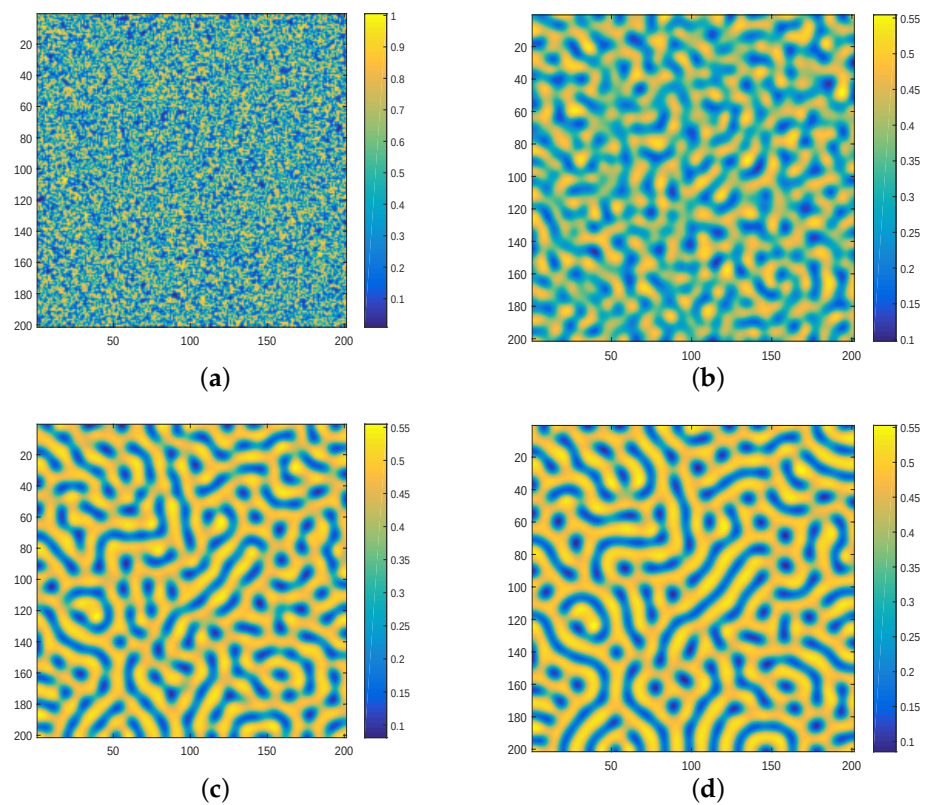


Figure 3. The patterns of u at $c = 0.62$, the other parameters are fixed as in (23). Times: (a) $t = 0$, (b) $t = 50$, (c) $t = 200$, and (d) $t = 500$.

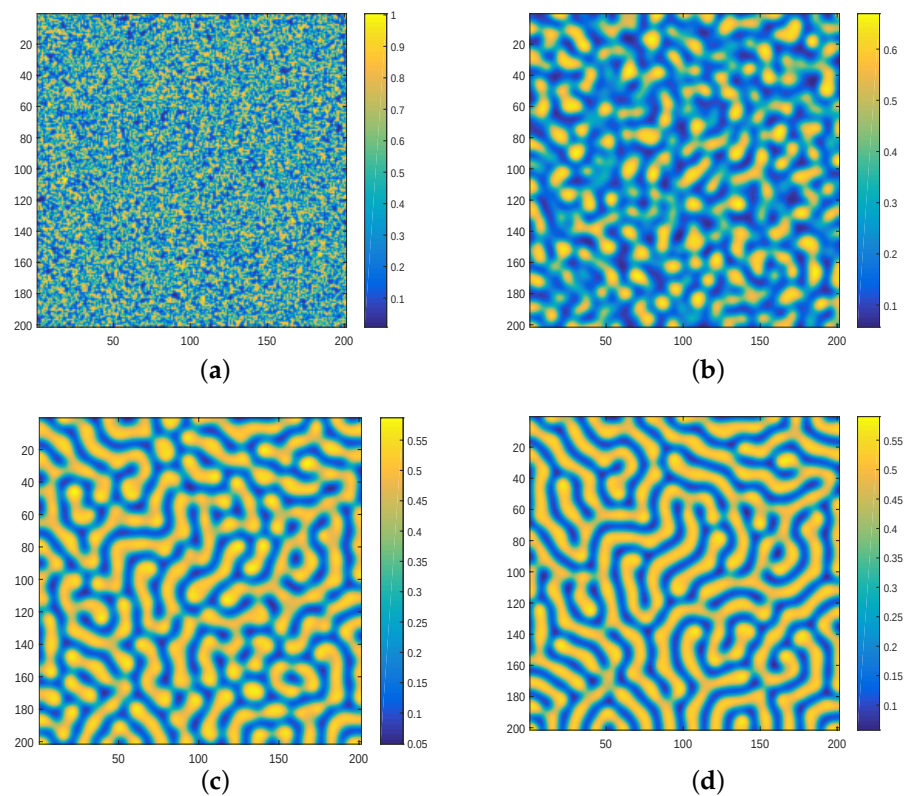


Figure 4. The patterns of u at $c = 0.7$, the other parameters are fixed as in (23). Times: (a) $t = 0$, (b) $t = 50$, (c) $t = 200$, (d) $t = 500$.

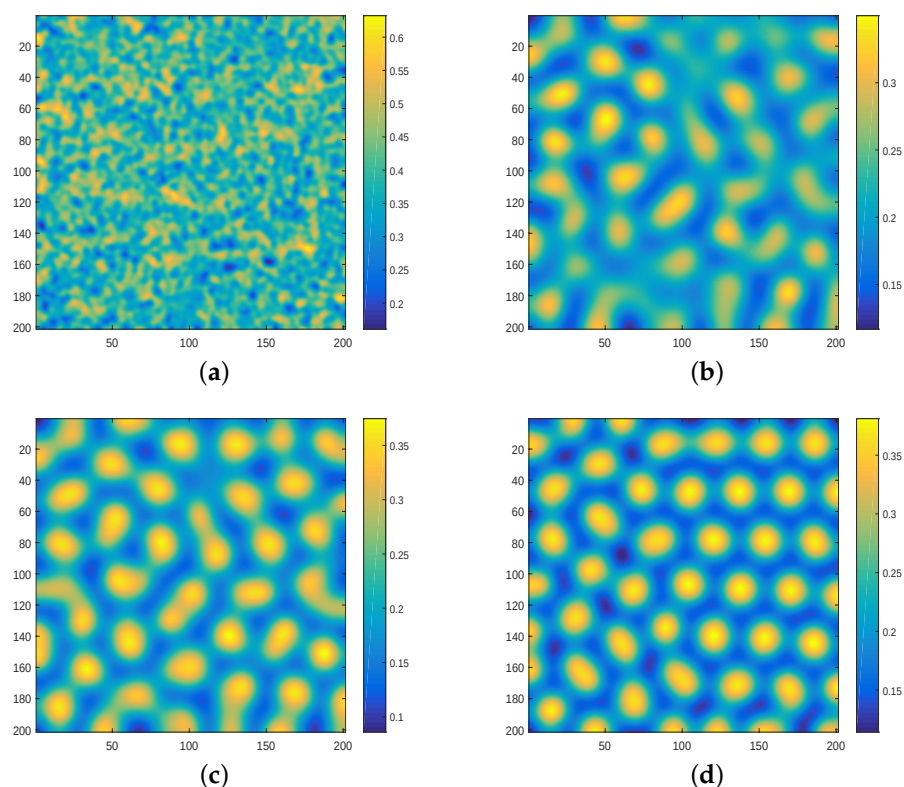


Figure 5. The patterns of prey evolution. Parameters: $a = 0.14$, $b = 0.02$, $c = 0.7$, $s = 0.23$, $d_1 = 0.04$, $d_2 = 0.24$. Times: (a) $t = 0$, (b) $t = 150$, (c) $t = 300$, and (d) $t = 1000$.

On the basis of Figure 4, we only change the value of diffusion coefficient d_1 , let $d_1 = 0.04$, and other parameter values remain unchanged. The numerical simulation results show that when the diffusion coefficient d_1 increases, the Turing pattern of prey will change from the original strip patterns to dot patterns (see Figure 5). By observing Figure 5, we found that with the evolution of time, the model will gradually form a stable “dot” pattern (see Figure 5d). Finally, it will not change with the change of time. From Figure 5, we can also see that the area of the color of the prey population was becoming larger and larger, and the color concentration was increasing, which indicated that the prey population gathers together in a “dot” pattern over time.

This section focuses on the influence of diffusion on the model, and observes different types of dynamics in numerical simulation. Therefore, the fixed parameter values are:

$$a = 0.14, b = 0.02, s = 0.23, c = 0.7, d_2 = 0.73. \quad (24)$$

In order to facilitate the research, we only study the influence of one diffusion term on the model. Therefore, we select four different values for the parameter d_1 , which are $d_1 = 0.002$, $d_1 = 0.01$, $d_1 = 0.04$, $d_1 = 0.12$. The numerical simulation results are shown in Figure 6.

In Figure 6, we can see that with the increase in diffusion coefficient d_1 , the color area of the prey population was becoming smaller and smaller, which indicated that the density of the prey population was gradually decreasing. In Figure 6, we show three typical Turing patterns of the model (2). The numerical results show that with the increase in the diffusion coefficient d_1 , the initial irregular pattern eventually evolved into regular hole-strip pattern, strip pattern and dot pattern. With the evolution of time, the whole model tends to be stable and the dynamic behavior of the model will not change.

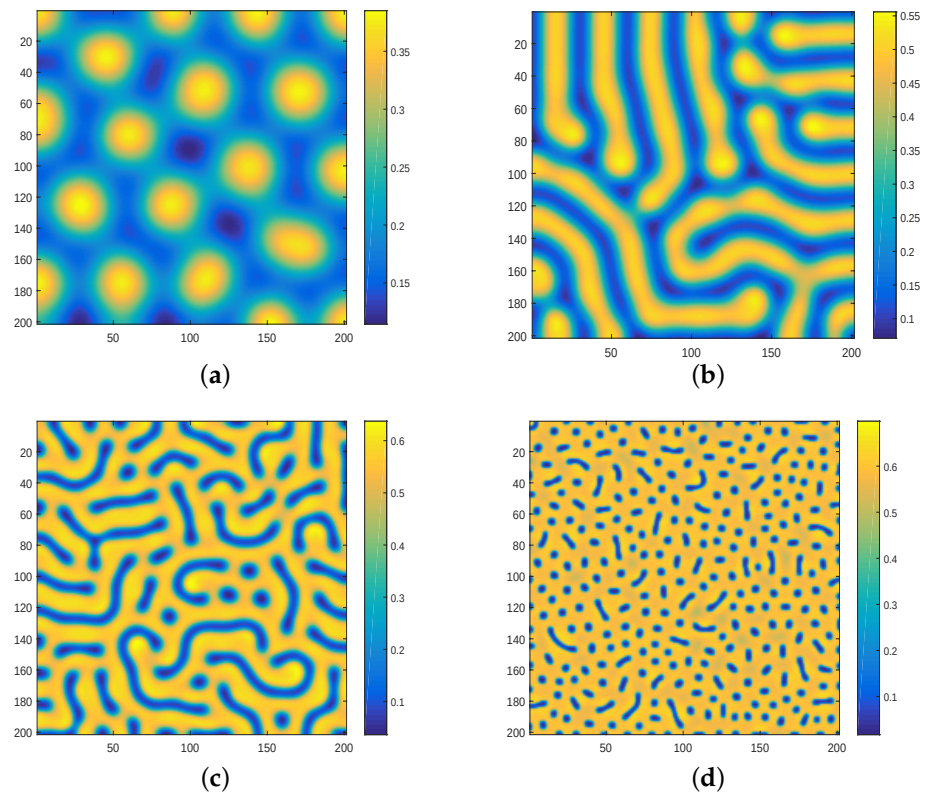


Figure 6. The patterns of the prey with different values at d_1 , the other parameters are fixed as in (24). Types: (a) $d_1 = 0.12$, (b) $d_1 = 0.04$, (c) $d_1 = 0.01$, (d) $d_1 = 0.002$.

3. Analysis of Cross Diffusion Model

3.1. Turing Instability Induced by Cross Diffusion

In order to study the influence of cross diffusion on the positive equilibrium point E_1 of the model (3), a small disturbance is introduced at $E_1(u_1, cu_1)$:

$$\begin{cases} u(x, y, t) = u_1 + \tilde{u}(x, y, t), \\ v(x, y, t) = v_1 + \tilde{v}(x, y, t), \end{cases}$$

where $|\tilde{u}(x, y, t)|, |\tilde{v}(x, y, t)| \ll 1$, under this disturbance, the linear model corresponding to the model (3) is:

$$\begin{cases} \frac{\partial \tilde{u}}{\partial t} = f_u \tilde{u} + f_v \tilde{v} + d_{11} \Delta \tilde{u} + d_{12} \Delta \tilde{v}, \\ \frac{\partial \tilde{v}}{\partial t} = g_u \tilde{u} + g_v \tilde{v} + d_{21} \Delta \tilde{u} + d_{22} \Delta \tilde{v}, \end{cases} \quad (25)$$

any solution of the model (25) can be expanded by Fourier series:

$$\begin{cases} \tilde{u}(r, t) = \sum_{n,m=0}^{\infty} u_{nm}(r, t) = \sum_{n,m=0}^{\infty} \alpha_{nm} \sin(kr), \\ \tilde{v}(r, t) = \sum_{n,m=0}^{\infty} v_{nm}(r, t) = \sum_{n,m=0}^{\infty} \beta_{nm} \sin(kr), \end{cases} \quad (26)$$

where $r = (x, y)$, $0 < x < L_x$, $0 < y < L_y$, $k = (k_n, k_m)$, and $k_n = \frac{n\pi}{L_x}$, $k_m = \frac{m\pi}{L_y}$ is the corresponding wavelength. Substitute (26) into (25) to obtain:

$$\begin{cases} \frac{d\alpha_{nm}}{dt} = (f_u - d_{11}k^2)\alpha_{nm} + (f_v - d_{12}k^2)\beta_{nm}, \\ \frac{d\beta_{nm}}{dt} = (g_u - d_{21}k^2)\alpha_{nm} + (g_v - d_{22}k^2)\beta_{nm}, \end{cases} \quad (27)$$

where $k^2 = k_n^2 + k_m^2$. The form of the solution of Equation (27) is $C_1 \exp(\lambda_1 t) + C_2 \exp(\lambda_2 t)$, where C_1 and C_2 are determined by the initial condition (4), and $\lambda_{1,2}$ is the eigenvalue of the following matrix:

$$J = \begin{pmatrix} f_u - d_{11}k^2 & f_v - d_{12}k^2 \\ g_u - d_{21}k^2 & g_v - d_{22}k^2 \end{pmatrix}, \quad (28)$$

then $\lambda_{1,2}$ is the solution of the following equation:

$$\lambda^2 - \text{tr}(J)\lambda + \det(J) = 0, \quad (29)$$

where:

$$\text{tr}(J) = \text{tr}(J_{E_1}) - (d_{11} + d_{22})k^2, \quad (30)$$

$$\det(J) = \det(D)k^4 - (d_{11}g_v + d_{22}f_u - d_{21}f_v - d_{12}g_u)k^2 + \det(J_{E_1}), \quad (31)$$

where

$$D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

is the diffusion coefficient matrix.

Therefore, when at least one of (30) and (31) is violated, Turing instability will occur. Obviously, when condition (11) is satisfied, the first condition $\text{tr}(J) < 0$ is not violated. Therefore, only violation of condition $\det(J) > 0$ will cause diffusion driven instability. Therefore, the instability caused by diffusion drive is given by the following formula:

$$H(k^2) \equiv \det(D)k^4 - (d_{11}g_v + d_{22}f_u - d_{21}f_v - d_{12}g_u)k^2 + \det(J_{E_1}) < 0,$$

if the function $H(k^2)$ has a minimum value and its value is negative, we can calculate the region of k^2 and define the lower bound as k_-^2 and the upper bound as k_+^2 . when $k^2 \in (k_-^2, k_+^2)$, $H(k^2) < 0$. The expression of k_-^2 and k_+^2 are as follows:

$$k_-^2 = \frac{F - \sqrt{F^2 - 4 \det(D) \det(J_{E_1})}}{2 \det(D)},$$

$$k_+^2 = \frac{F + \sqrt{F^2 - 4 \det(D) \det(J_{E_1})}}{2 \det(D)},$$

where $F = d_{11}g_v + d_{22}f_u - d_{21}f_v - d_{12}g_u$. At this time, the minimum value of $H(k^2)$ is:

$$H_{\min} = \det(J_{E_1}) - \frac{F^2}{4 \det(D)},$$

the minimum point is:

$$K_c = k^2 = \frac{F}{2 \det(D)},$$

the following equivalent conditions are obtained:

$$\det(J_k) < 0 \Leftrightarrow \max(k_-^2, 0) < k^2 < k_+^2.$$

Through the above analysis, it can be seen that the conditions for Turing instability in the model (3) are:

$$f_u + g_v < 0, \quad (32)$$

$$f_u g_v - f_v g_u > 0, \quad (33)$$

$$d_{11} g_v + d_{22} f_u - d_{21} f_v - d_{12} g_u > 0, \quad (34)$$

$$d_{11} g_v + d_{22} f_u - d_{21} f_v - d_{12} g_u^2 > 4 \det(D) \det(J_{E_1}). \quad (35)$$

3.2. Pattern Formation Induced by Self Diffusion and Cross Diffusion

In this section, our numerical simulation uses a model size of 200×200 , the space step size is $\Delta x = 0.2$, the time step size is $\Delta t = 0.01$, and all the simulations obey Neumann boundary conditions. In this part, appropriate parameters that meet conditions (32)–(35) are selected, and the pattern generated in this case is called Turing pattern. The fixed parameter values are:

$$b = 0.04, s = 0.2, d_{11} = 0.04, d_{22} = 0.9. \quad (36)$$

In order to better understand the influence of cross diffusion on the pattern formation, we first give the pattern formation without cross diffusion (that is, $d_{12} = d_{21} = 0$), take the parameter value $a = 0.1$, $c = 0.56$, and the other parameters are given by (36). In this case, Figure 7 illustrates the pattern formation process of the prey population at $t = 0$, $t = 200$, $t = 500$, and $t = 1000$ when there is no cross diffusion term in the model (3). From Figure 7, it can be seen that with the evolution of time, the model finally forms hole patterns.

In order to study the effect of cross diffusion on model (3), on the basis of Figure 7, we add the value of the cross diffusion term d_{12} , d_{21} , take $d_{12} = 0.2$, $d_{21} = 0.02$, and the other parameters remain unchanged. In this case, Turing instability occurs after checking calculation. Figure 8 illustrates the patterns formation process of the prey population u of the model (3) when $t = 0$, $t = 1000$, $t = 3000$, and $t = 5000$. As can be seen from Figure 8, as time evolves, the strip pattern will gradually become clear, and, finally, the model will form a stable strip pattern.

On the basis of Figure 8, we appropriately change the value of a , c , and d_{12} , taking $a = 0.4$, $c = 0.6$, $d_{12} = 0.5$, and the other parameters remain unchanged. In this case, Turing instability occurs after checking calculation. At this time, Figure 9 illustrates the formation process of the dot pattern of the prey population u of the model (3) when $t = 0$, $t = 200$, $t = 500$, and $t = 2000$. From the formation process of the patch pattern in Figure 9, we can see that the prey populations gather together in the form of dots with the evolution of time, and finally form a stable dot pattern.

It can be seen from the figure that the evolutionary pattern of the prey changes with the change of the cross diffusion coefficient of the prey. With the passage of time, the initial irregular pattern eventually evolves into regular hole pattern, dot pattern and strip pattern. The whole model tends to be stable and the dynamic behavior of the model will not change.

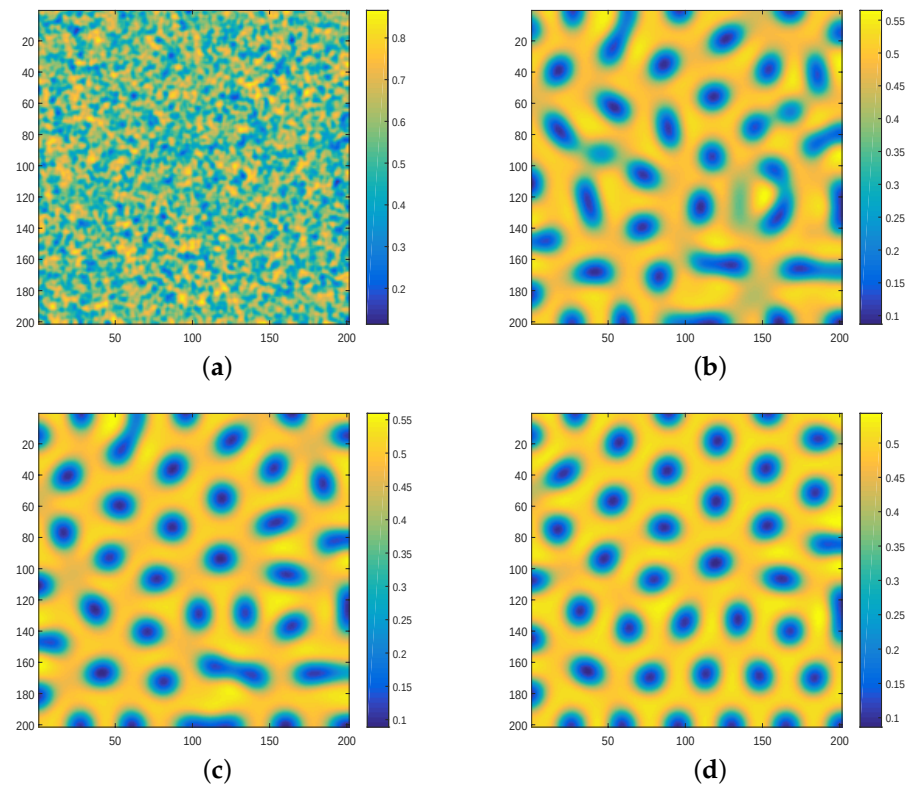


Figure 7. The patterns formation process of the prey population at $(a, c, d_{12}, d_{21}) = (0.1, 0.56, 0, 0)$, the other parameters are fixed as in (36). Times: (a) $t = 0$, (b) $t = 200$, (c) $t = 500$, and (d) $t = 1000$.

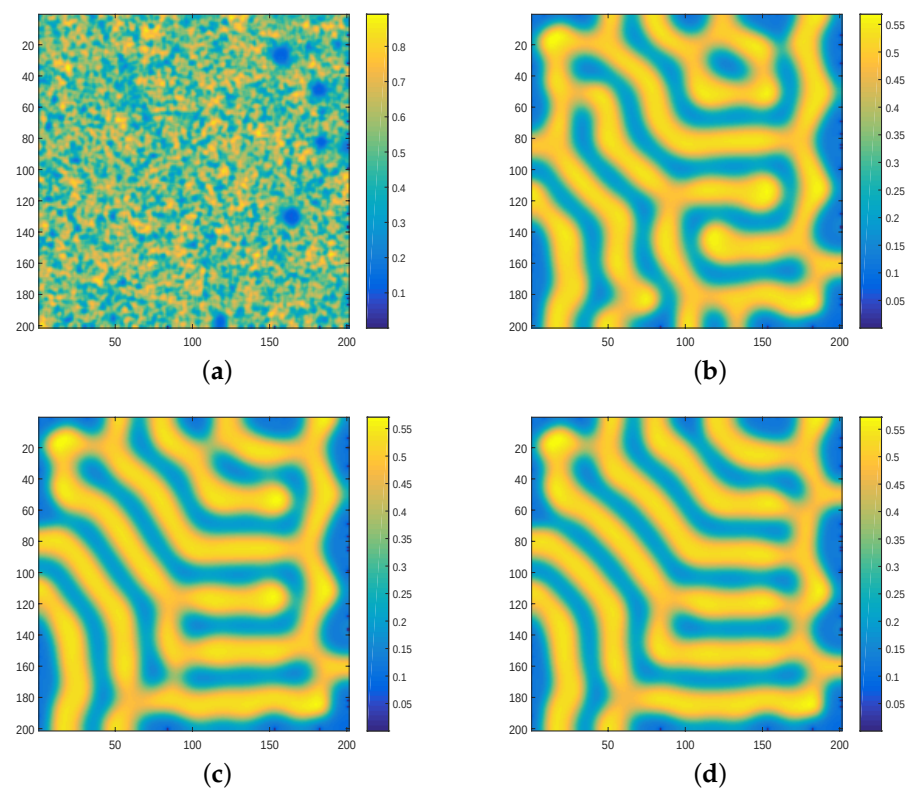


Figure 8. The patterns of u at $(a, c, d_{12}, d_{21}) = (0.1, 0.56, 0.2, 0.02)$, the other parameters are fixed as in (36). Times: (a) $t = 0$, (b) $t = 1000$, (c) $t = 3000$, and (d) $t = 5000$.

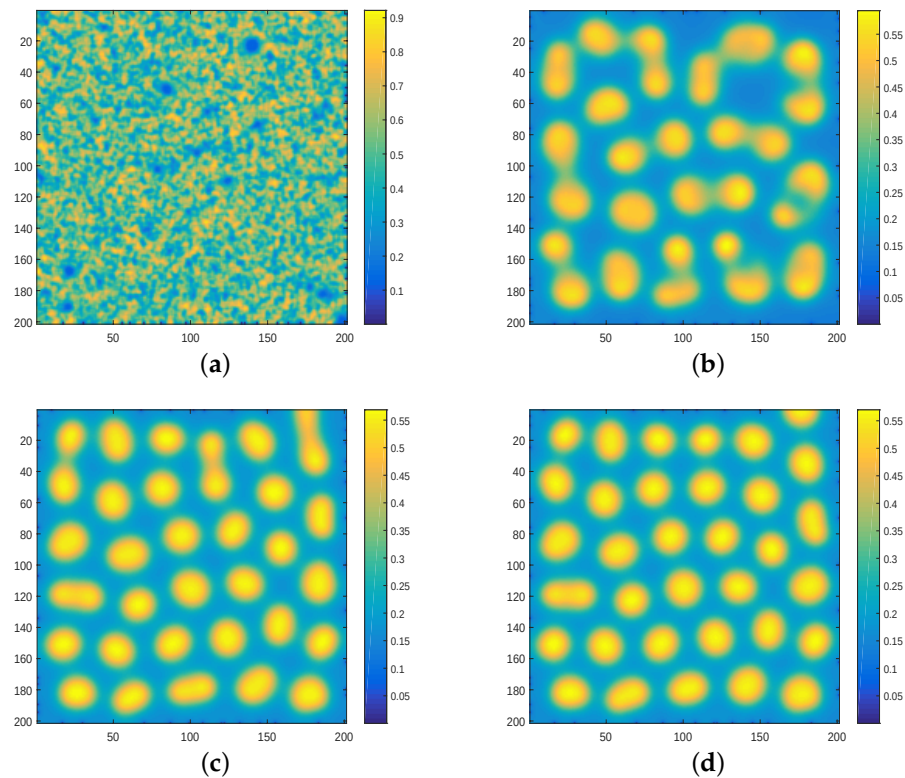


Figure 9. The patterns of u at $(a, c, d_{12}, d_{21}) = (0.4, 0.6, 0.5, 0.02)$, the other parameters are fixed as in (36). Times: (a) $t = 0$, (b) $t = 200$, (c) $t = 500$, and (d) $t = 2000$.

In the pattern study of various non-linear models, non-linear scientists pay more attention to the study of spiral wave dynamics. Therefore, in addition to Turing pattern, we next consider the formation of spiral pattern. Through verification, we can see that the generation of spiral pattern must meet the following conditions:

$$f_u + g_v > 0, \quad (37)$$

$$f_u g_v - f_v g_u > 0, \quad (38)$$

$$f_u + g_v - (d_{11} + d_{22})k^2 < 0, \quad (39)$$

$$d_{11}g_v + d_{22}f_u - d_{21}f_v - d_{12}g_u^2 < 4 \det(D) \det(J_{E_1}). \quad (40)$$

In this numerical simulation, select the parameters that meet the above conditions, and we take the parameter value as:

$$a = 0.92, b = 0.04, c = 0.5, s = 0.11, d_{11} = 0.82, d_{12} = 0.93, d_{21} = 0.36, d_{22} = 0.83. \quad (41)$$

At this point, the equilibrium point $(u_1, v_1) = (0.3447, 0.17235)$, Through Ref. [32], we consider the formation of spiral pattern of model (3) under three different initial conditions. In the case of these three initial values, in order to see the pattern more clearly, we increase the grid size to 400×400 , the space step size is $\Delta x = 10$, the time step size is $\Delta t = 0.1$.

In the first case, special initial values are used as follows:

$$\begin{cases} u(x, y, 0) = u_1 - \varepsilon_1(x - 200), \\ v(x, y, 0) = v_1 - \varepsilon_2(y - 200), \end{cases} \quad (42)$$

where $\varepsilon_1 = 2 \times 10^{-4}$, $\varepsilon_2 = 3 \times 10^{-4}$.

Figure 10 shows the formation process of the spiral wave pattern of the model (3). It can be seen from the figure that when the time $t = 1000$, the spiral wave begins to incubate,

and the center position of the spiral wave appears (see Figure 10b). Then when $t = 5000$, the prototype of the spiral wave of the prey population is basically formed (see Figure 10c). Finally, when $t = 10,000$, the model is basically stable, and we can see that the spiral wave pattern becomes very clear.

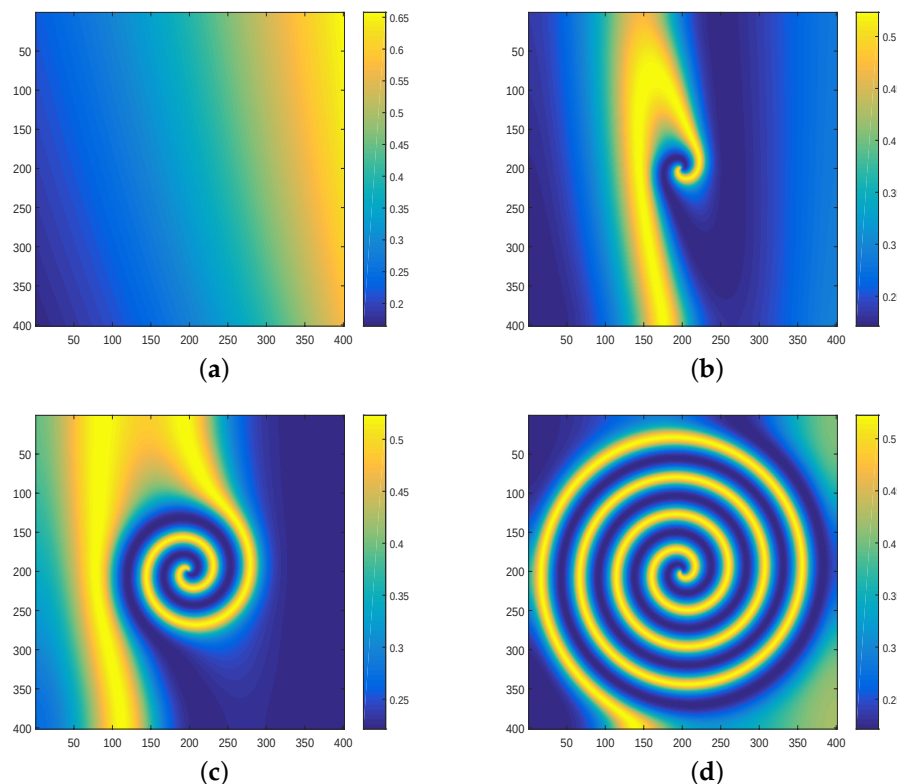


Figure 10. The spiral patterns formation process of the model (3), the parameters are given by (41). Times: (a) $t = 0$, (b) $t = 1000$, (c) $t = 5000$, and (d) $t = 10,000$.

The appearance of the spiral wave is not determined by the initial conditions, but the initial conditions will determine the structure of the spiral wave. Therefore, we can use the initial value to control the number of spiral waves and the location of the spiral wave center. In the previous content, We have obtained the pattern with one spiral wave. Next, we consider the case where two and four spiral waves appear in the pattern.

In the second case, the initial values are as follows:

$$\begin{cases} u(x, y, 0) = u_1 - \varepsilon_1(x - 200), \\ v(x, y, 0) = v_1 - \varepsilon_2(y - 100)(y - 300), \end{cases} \quad (43)$$

where $\varepsilon_1 = 1 \times 10^{-4}$, $\varepsilon_2 = 3 \times 10^{-7}$.

Figure 11 shows the patterns of spiral wave at time $t = 0, 1000, 5000$, and $10,000$. The initial condition is given by Formula (43). When $t = 1000$, spiral wave begins to appear, and the center position of spiral wave appears (see Figure 11b). Under the given special initial value, the spiral wave pattern is slowly formed with the deduction of time.

In the third case, the initial values are as follows:

$$\begin{cases} u(x, y, 0) = u_1 - \varepsilon_1(x - 80)(x - 320), \\ v(x, y, 0) = v_1 - \varepsilon_2(y - 80)(y - 320), \end{cases} \quad (44)$$

where $\varepsilon_1 = 2 \times 10^{-6}$, $\varepsilon_2 = 3 \times 10^{-6}$.

Figure 12 evolves the spiral pattern under the initial value condition (44). We find that with the evolution of time, its evolution process gradually becomes regular, and the spiral pattern gradually becomes clear.

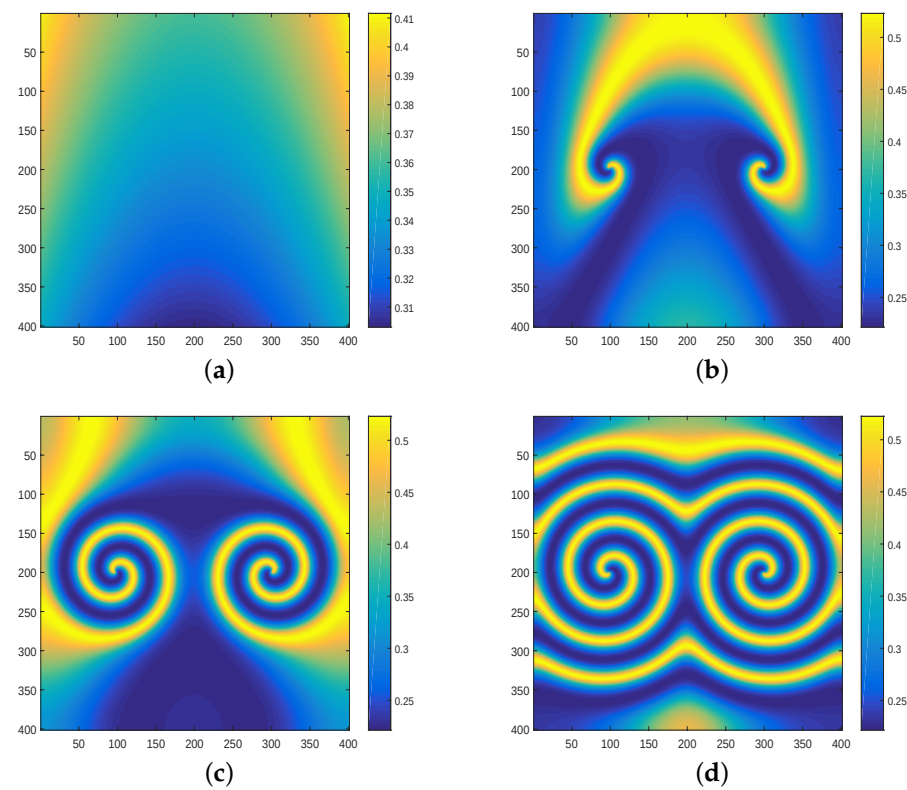


Figure 11. The spiral pattern evolution process of the model (3), the parameters are given by (41). Times: (a) $t = 0$, (b) $t = 1000$, (c) $t = 5000$, (d) $t = 10,000$.

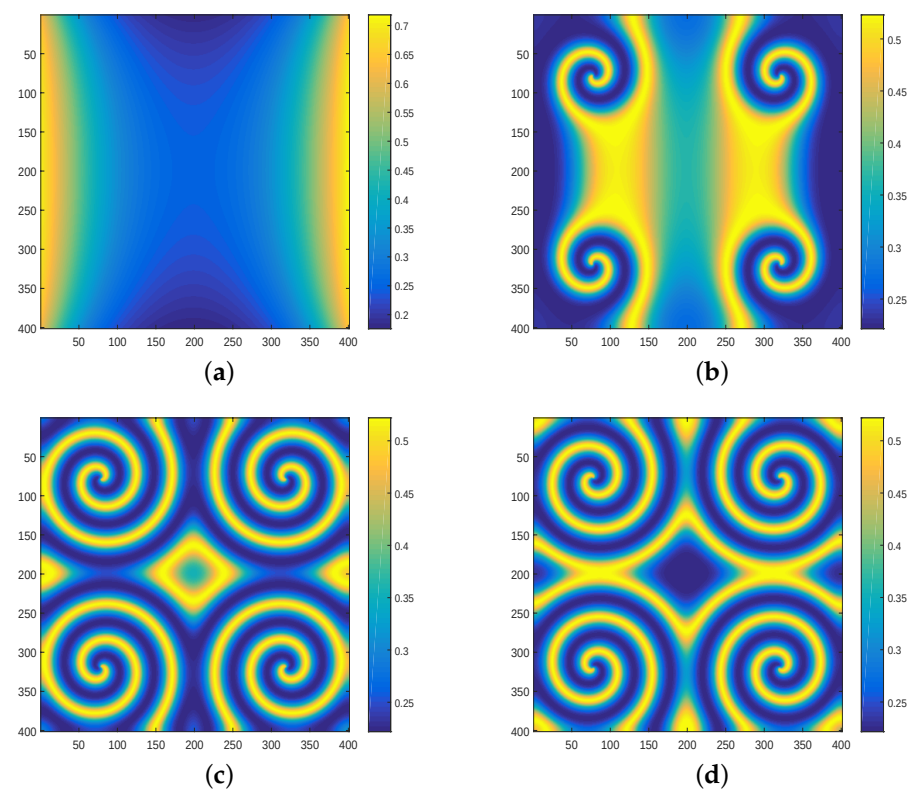


Figure 12. The spiral pattern formation process of the model (3), the parameters are given by (41). Times: (a) $t = 0$, (b) $t = 3000$, (c) $t = 8000$, and (d) $t = 10,000$.

4. Conclusions and Remarks

In this paper, we study the spatiotemporal dynamics of Leslie–Gower cross diffusion predator–prey model with strong Allee effect in prey and hunting cooperation among predators. First, we analyze the existence of the equilibrium point and the stability of the positive equilibrium point of the model (2), and deduce the Turing instability condition of the model (2). The results show that with the increase in c , the pattern will change from “hole” pattern to “hole-strip” pattern, and then to “strip” pattern. In order to study the influence of self diffusion on the formation of spatial patterns of the model (2), we choose self diffusion coefficient as the main control parameter. The numerical simulation results show that with the increase in diffusion coefficient d_1 , the pattern will change from “hole-strip” pattern to “strip” pattern, and then to “dot” pattern. Second, we study the effects of self diffusion and cross diffusion on Turing instability of the model (3), and analyze the effects of self diffusion and cross diffusion on the formation of spatiotemporal patterns. Through a series of numerical simulations, the Turing structures in the parameter space are obtained, including hole pattern, strip pattern, and dot pattern. Non-linear scientists pay more attention to the study of spiral wave dynamics in the study of speckle patterns of various non-linear models. Therefore, we consider the spiral pattern formation of the model (3) under three different initial conditions. The numerical simulation results show that the dynamics of the model exhibits that the self diffusion and cross diffusion control not only form the growth of dots, stripes, and holes, but also self replicating spiral pattern growth.

In Ref. [29], Ye and Wu deduced the existence of saddle-node bifurcation by using Sotomayor’s theorem, and determined the stability of limit cycles arising from the Hopf bifurcation by calculating the first Lyapunov number. Compared with the results of [29], this article discusses two types of instability and analyzes the influence of diffusion coefficient on the formation of spatiotemporal patterns. These contents were not considered in literature [29]. First, we analyze Turing instability caused by self diffusion; in this case, $\det(J_k) > 0$ is violated and the Turing patterns appear in model (2). Secondly, we study the Turing instability induced by self diffusion and cross diffusion, and obtain the Turing pattern induced by self diffusion and cross diffusion through numerical simulation. Finally, referring to the content of [32], we verify the conditions for generating the spiral pattern of the model (3), and obtain the spiral pattern under three different initial conditions by using numerical simulation. We find that the appearance of spiral waves does not change with the change of initial conditions. In addition, these spiral waves (Figures 10–12) have obvious periodicity.

We hope that our results will be helpful for future research on the spatial-temporal complexity and dynamic properties of ecosystems with Allee effect and hunting cooperation.

Author Contributions: Formal analysis, X.L.; Investigation, M.Z. and J.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by NSSF of China with No. 19BTQ056.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are grateful to the anonymous referee for his/her careful reading of the article and many pertinent remarks that lead to various improvements to this article.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Leslie, P.H.; Gower, J.C. The properties of a stochastic model for the predator-prey type of interaction between two species. *Biometrika* **1960**, *47*, 219–234. [[CrossRef](#)]
2. Holling, C.S. The functional response of invertebrate predators to prey density. *Mem. Entomol. Soc. Can.* **1966**, *98*, 1–86. [[CrossRef](#)]

3. Tanner, J.T. The stability and the intrinsic growth rates of prey and predator populations. *Ecology* **1975**, *56*, 855–867. [\[CrossRef\]](#)
4. Ding, Z.; Chen, T.; Lan, J.; Wong, G. Application of animal models to compare and contrast the virulence of current and future potential SARS-CoV-2 variants. *Biosaf. Health* **2022**, *4*, 154–160. [\[CrossRef\]](#) [\[PubMed\]](#)
5. Ho, L.V.; Nguyen, D.H.; Mousavi, M.; De Roeck, G.; Bui-Tien, T.; Gandomi, A.H.; Wahab, M.A. A hybrid computational intelligence approach for structural damage detection using marine predator algorithm and feedforward neural networks. *Comput. Struct.* **2021**, *252*, 106568. [\[CrossRef\]](#)
6. Hamadneh, N.N.; Tahir, M.; Khan, W.A. Using Artificial Neural Network with Prey Predator Algorithm for Prediction of the COVID-19: The Case of Brazil and Mexico. *Mathematics* **2021**, *9*, 180. [\[CrossRef\]](#)
7. Allee, W.C. Animal Aggregations. *Q. Rev. Biol.* **1927**, *2*, 367–398. [\[CrossRef\]](#)
8. Courchamp, F.; Berec, L.; Gascoigne, J. Allee effects in ecology and conservation. *Environ. Conserv.* **2018**, *36*, 80–85.
9. Zhang, C.; Yang, W. Dynamic behaviors of a predator-prey model with weak additive Allee effect on prey. *Nonlinear Anal. Real World Appl.* **2020**, *55*, 103137. [\[CrossRef\]](#)
10. Rebelo, C.; Soaresina, C. Coexistence in seasonally varying predator-prey systems with Allee effect. *Nonlinear Anal. Real World Appl.* **2020**, *55*, 103140. [\[CrossRef\]](#)
11. Shi, Y.; Wu, J.; Cao, Q. Analysis on a diffusive multiple Allee effects predator-prey model induced by fear factors. *Nonlinear Anal. Real World Appl.* **2021**, *59*, 103249. [\[CrossRef\]](#)
12. Liu, P.; Yang, B.W. Dynamics analysis of a reaction-diffusion system with Beddington-DeAngelis functional response and strong Allee effect. *Nonlinear Anal. Real World Appl.* **2020**, *51*, 102953. [\[CrossRef\]](#)
13. Shang, Z.C.; Qiao, Y.H. Bifurcation analysis of a Leslie-type predator-prey system with simplified Holling type IV functional response and strong Allee effect on prey. *Nonlinear Anal. Real World Appl.* **2022**, *64*, 103453. [\[CrossRef\]](#)
14. Ni, W.; Wang, M. Dynamical properties of a leslie-gower prey-predator model with strong allee effect in prey. *Discret. Contin. Dyn. Syst.-B* **2017**, *22*, 3409. [\[CrossRef\]](#)
15. Mukherjee, D.; Maji, C. Bifurcation analysis of a holling type II predator-prey model with Refuge. *Chin. J. Phys.* **2020**, *65*, 153–162. [\[CrossRef\]](#)
16. Aguirre, P.; Flores, J.D.; González-Olivares, E. Bifurcations and global dynamics in a predator-prey model with a strong allee effect on the prey and a ratio-dependent functional response. *Nonlinear Anal. Real World Appl.* **2014**, *16*, 235–249. [\[CrossRef\]](#)
17. Kar, T.K.; Matsuda, H. Global dynamics and controllability of a harvested prey-predator system with Holling type III functional response. *Nonlinear Anal. Hybrid Syst.* **2007**, *1*, 59–67. [\[CrossRef\]](#)
18. Zhang, Z.Z.; Yang, H.Z. Hopf bifurcation in a delayed predator-prey system with modified leslie-gower and Holling type III schemes. *Acta Autom. Sin.* **2013**, *39*, 610–616. [\[CrossRef\]](#)
19. Sarwardi, S.; Haque, M.; Mandal, P.K. Persistence and global stability of Bazykin predator-prey model with Beddington-deangelis response function. *Commun. Nonlinear Sci. Numer. Simulate* **2014**, *19*, 189–209. [\[CrossRef\]](#)
20. Chen, X.; Du, Z. Existence of positive periodic solutions for a neutral delay predator-prey model with Hassell-Varley type functional response and impulse. *Qual. Theory Dyn. Syst.* **2018**, *17*, 67–80. [\[CrossRef\]](#)
21. Dai, Y.; Zhao, Y. Hopf cyclicity and global dynamics for a predator-prey system of leslie type with simplified Holling type IV functional response. *Int. J. Bifur. Chaos* **2018**, *28*, 1850166. [\[CrossRef\]](#)
22. Shang, Z.C.; Qiao, Y.H.; Duan, L.J.; Miao, J. Stability and bifurcation analysis in a nonlinear harvested predator-prey model with simplified Holling type IV functional response. *Int. J. Bifurc. Chaos* **2020**, *30*, 2050205. [\[CrossRef\]](#)
23. Shang, Z.C.; Qiao, Y.H.; Duan, L.J.; Jun, M. Bifurcation analysis in a predator-prey system with an increasing functional response and constant-yield prey harvesting. *Math. Comput. Simul.* **2021**, *190*, 976–1002. [\[CrossRef\]](#)
24. Alves, M.T.; Hilker, F.M. Hunting cooperation and allee effects in predators. *J. Theor. Biol.* **2017**, *419*, 13–22. [\[CrossRef\]](#) [\[PubMed\]](#)
25. Yan, S.; Jia, D.; Zhang, T.; Yuan, S. Pattern dynamics in a diffusive predator-prey model with hunting cooperations. *Chaos Solitons Fractals* **2020**, *130*, 109428. [\[CrossRef\]](#)
26. Sen, D.; Ghorai, S.; Banerjee, M. Allee effect in prey versus hunting cooperation on predator-enhancement of stable coexistence. *Int. J. Bifurc. Chaos* **2019**, *29*, 1950081. [\[CrossRef\]](#)
27. Wu, D.; Zhao, M. Qualitative analysis for a diffusive predator-prey model with hunting cooperative. *Phys. A Stat. Mech. Appl.* **2019**, *515*, 299–309. [\[CrossRef\]](#)
28. Pal, S.; Pal, N.; Chattopadhyay, J. Hunting cooperation in a discrete-time predator-prey system. *Int. J. Bifurc. Chaos* **2018**, *28*, 1850083. [\[CrossRef\]](#)
29. Ye, P.H.; Wu, D.Y. Impacts of strong Allee effect and hunting cooperation for a Leslie-Gower predator-prey system. *Chin. J. Phys.* **2020**, *68*, 49–64. [\[CrossRef\]](#)
30. Turing, A.M. The chemical basis of morphogenesis. *Philos. Trans. R. Soc. Lond. Biol. Sci.* **1952**, *237*, 37–72.
31. Liu, X.; Zhang, T.; Meng, X.; Zhang, T. Turing-Hopf bifurcations in a predator-prey model with herd behavior quadratic mortality and prey-taxis. *Phys. A* **2018**, *496*, 446–460. [\[CrossRef\]](#)
32. Lian, X.Z.; Wang, H.L.; Wang, W.M. Delay-driven pattern formation in a reaction-diffusion predator-prey model incorporating a prey refuge. *J. Stat. Mech. Theory Exp.* **2013**, *2013*, P04006. [\[CrossRef\]](#)
33. Chen, M.X.; Wu, R.C.; Liu, B.; Chen, L.P. Pattern selection in a predator-prey model with Michaelis-Menten type nonlinear predator harvesting. *Ecol. Complex.* **2018**, *36*, 239–249. [\[CrossRef\]](#)

34. Biktashev, V.; Brindley, J.; Holden, A.; Tsyganov, M. Pursuit-evasion predator-prey waves in two spatial dimensions. *Chaos* **2004**, *14*, 988–994. [[CrossRef](#)] [[PubMed](#)]
35. Shukla, J.; Verma, S. Effects of convective and dispersive interactions on the stability of two species. *Bull. Math. Biol.* **1981**, *43*, 593–610. [[CrossRef](#)]
36. Kerner, E. Further considerations on the statistical mechanics of biological associations. *Bull. Math. Biophys.* **1959**, *21*, 217–255. [[CrossRef](#)]
37. Song, D.; Li, C.; Song, Y. Stability and cross-diffusion-driven instability in a diffusive predator-prey system with hunting cooperation functional response. *Nonlinear Anal. Real World Appl.* **2020**, *54*, 103106. [[CrossRef](#)]
38. Ma, T.; Meng, X.; Hayat, T.; Hobiny, A. Stability analysis and optimal harvesting control of a cross-diffusion prey-predator system. *Chaos* **2021**, *152*, 111418. [[CrossRef](#)]
39. Peng, Y.; Ling, H. Pattern formation in a ratio-dependent predator-prey model with cross-diffusion. *Appl. Math. Comput.* **2018**, *331*, 307–318. [[CrossRef](#)]
40. Meng, Q.; Yang, L.L. Steady state in a cross-diffusion predator-prey model with the Beddington-DeAngelis functional response. *Nonlinear Anal. Real World Appl.* **2019**, *45*, 401–413. [[CrossRef](#)]
41. Lian, X.Z.; Yue, Y.H.; Wang, H.L. Pattern Formation in a Cross-Diffusive Ratio-Dependent Predator-Prey Model. *Discret. Dyn. Nat. Soc.* **2012**, *2012*, 814069. [[CrossRef](#)]
42. Yang, R.Z.; Zhang, C.R. Dynamics in a diffusive modified Leslie-Gower predator-prey model with time delay and prey harvesting. *Nonlinear Dyn.* **2017**, *87*, 863–878. [[CrossRef](#)]
43. Cao, J.; Sun, H.; Hao, P.; Wang, P. Bifurcation and Turing instability for a predator-prey model with nonlinear reaction cross-diffusion. *Appl. Math. Model.* **2021**, *89*, 1663–1677. [[CrossRef](#)]
44. Yan, X.; Maimaiti, Y.; Yang, W. Stationary pattern and bifurcation of a Leslie-Gower predator-prey model with prey-taxis. *Math. Comput. Simul.* **2022**, *201*, 163–192. [[CrossRef](#)]